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A three lines proof for traces of H^1 functions on special Lipschitz domains

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1 Introduction

It is well known (see [2, Theorem 1.2]) that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the trace operator $\operatorname{Tr}_{|\partial\Omega} : \mathscr{C}(\overline{\Omega}) \to \mathscr{C}(\partial\Omega)$ restricted to $\mathscr{C}(\overline{\Omega}) \cap H^1(\Omega)$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and the following estimate holds:

$$\|\operatorname{Tr}_{\partial\Omega} u\|_{L^{2}(\partial\Omega)} \leq C\left(\|u\|_{L^{2}(\Omega)} + \|\nabla u\|_{L^{2}(\Omega,\mathbb{R}^{n})}\right) \quad \text{for all } u \in H^{1}(\Omega),$$

$$(1.1)$$

where $C = C(\Omega) > 0$ is a constant depending on the domain Ω . This result can be proved via a simple integration by parts and Cauchy-Schwarz inequality if the domain is the upper graph of a Lipschitz function, i.e.,

$$\Omega = \left\{ x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > \omega(x_h) \right\}$$
(1.2)

where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz function.

2 The result

Let $\Omega \subset \mathbb{R}^n$ be a domain of the form (1.2). The exterior unit normal ν of Ω at a point $x = (x_h, \omega(x_h))$ on the boundary Γ of Ω :

$$\Gamma := \left\{ x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n = \omega(x_h) \right\}$$

is given by

$$\nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \left(\nabla_h \omega(x_h), -1\right)$$

 $(\nabla_h \text{ denotes the "horizontal gradient" on } \mathbb{R}^{n-1} \text{ acting on the "horizontal variable" } x_h)$. We denote by $\theta \in [0, \frac{\pi}{2})$ the angle

$$\theta = \arccos\left(\inf_{x_h \in \mathbb{R}^{n-1}} \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}}\right),\tag{2.1}$$

so that in particular for $e = (0_{\mathbb{R}^{n-1}}, 1)$ the "vertical" direction, we have

$$-e \cdot \nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \ge \cos \theta > 0, \quad \text{for all } x_h \in \mathbb{R}^{n-1}.$$
 (2.2)

Theorem 2.1. Let $\Omega \subset \mathbb{R}^n$ be as above. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Then

$$\int_{\Gamma} |\varphi|^2 \,\mathrm{d}\sigma \le \frac{2}{\cos\theta} \, \|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega,\mathbb{R}^n)},\tag{2.3}$$

where θ has been defined in (2.1).

Proof. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support, and apply the divergence theorem in Ω with $u = \varphi^2 e$ where $e = (0_{\mathbb{R}^{n-1}}, 1)$. Since div $(\varphi^2 e) = 2 \varphi (e \cdot \nabla \varphi)$, we obtain

$$\int_{\Omega} 2\varphi \left(e \cdot \nabla \varphi \right) \mathrm{d}x = \int_{\Omega} \mathrm{div} \left(\varphi^2 \, e \right) \mathrm{d}x = \int_{\Gamma} \nu \cdot \left(\varphi^2 \, e \right) \mathrm{d}\sigma.$$

Therefore using (2.2) and Cauchy-Schwarz inequality, we get

$$\cos\theta \int_{\Gamma} \varphi^2 \,\mathrm{d}\sigma \le -2 \,\int_{\Omega} \varphi \left(e \cdot \nabla \varphi \right) \,\mathrm{d}x \le 2 \, \|\varphi\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega,\mathbb{R}^n)},$$

which gives the estimate (2.3).

Corollary 2.2. There exists a unique operator $T \in \mathscr{L}(H^1(\Omega), L^2(\Gamma))$ satisfying

$$T\varphi = \operatorname{Tr}_{|_{\Gamma}}\varphi, \quad for \ all \ \varphi \in H^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$$

and

$$\|T\|_{\mathscr{L}(H^1(\Omega), L^2(\Gamma))} \le \frac{1}{\sqrt{\cos\theta}}.$$
(2.4)

Proof. The existence and uniqueness of the operator T follow from Theorem 2.1 the density of $\mathscr{C}_c^{\infty}(\overline{\Omega})$ in $H^1(\Omega)$ (see, e.g., [1, Theorem 4.7, p. 248]). Moreover, (2.3) implies

$$\|\varphi\|_{L^{2}(\Gamma, \mathrm{d}\sigma)}^{2} \leq \frac{1}{\cos\theta} \left(\|\varphi\|_{L^{2}(\Omega)}^{2} + \|\nabla\varphi\|_{L^{2}(\Omega, \mathbb{R}^{n})}^{2} \right), \quad \text{for all } \varphi \in \mathscr{C}_{c}^{\infty}(\overline{\Omega}),$$

which proves (2.4).

References

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