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A three lines proof for traces of H^1 functions on special Lipschitz domains

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1 Introduction

It is well known (see [2, Theorem 1.2]) that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the trace operator $\text{Tr}_{|\partial\Omega} : \mathcal{C}(\overline{\Omega}) \rightarrow \mathcal{C}(\partial\Omega)$ restricted to $\mathcal{C}(\overline{\Omega}) \cap H^1(\Omega)$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and the following estimate holds:

$$\|\text{Tr}_{|\partial\Omega} u\|_{L^2(\partial\Omega)} \leq C (\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega, \mathbb{R}^n)}) \quad \text{for all } u \in H^1(\Omega), \quad (1.1)$$

where $C = C(\Omega) > 0$ is a constant depending on the domain Ω . This result can be proved via a simple integration by parts and Cauchy-Schwarz inequality if the domain is the upper graph of a Lipschitz function, i.e.,

$$\Omega = \{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > \omega(x_h)\} \quad (1.2)$$

where $\omega : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a globally Lipschitz function.

2 The result

Let $\Omega \subset \mathbb{R}^n$ be a domain of the form (1.2). The exterior unit normal ν of Ω at a point $x = (x_h, \omega(x_h))$ on the boundary Γ of Ω :

$$\Gamma := \{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n = \omega(x_h)\}$$

is given by

$$\nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} (\nabla_h \omega(x_h), -1)$$

(∇_h denotes the ‘‘horizontal gradient’’ on \mathbb{R}^{n-1} acting on the ‘‘horizontal variable’’ x_h). We denote by $\theta \in [0, \frac{\pi}{2})$ the angle

$$\theta = \arccos \left(\inf_{x_h \in \mathbb{R}^{n-1}} \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \right), \quad (2.1)$$

so that in particular for $e = (0_{\mathbb{R}^{n-1}}, 1)$ the ‘‘vertical’’ direction, we have

$$-e \cdot \nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \geq \cos \theta > 0, \quad \text{for all } x_h \in \mathbb{R}^{n-1}. \quad (2.2)$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be as above. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function with compact support. Then*

$$\int_{\Gamma} |\varphi|^2 d\sigma \leq \frac{2}{\cos \theta} \|\varphi\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega, \mathbb{R}^n)}, \quad (2.3)$$

where θ has been defined in (2.1).

Proof. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function with compact support, and apply the divergence theorem in Ω with $u = \varphi^2 e$ where $e = (0_{\mathbb{R}^{n-1}}, 1)$. Since $\operatorname{div}(\varphi^2 e) = 2\varphi(e \cdot \nabla\varphi)$, we obtain

$$\int_{\Omega} 2\varphi(e \cdot \nabla\varphi) \, dx = \int_{\Omega} \operatorname{div}(\varphi^2 e) \, dx = \int_{\Gamma} \nu \cdot (\varphi^2 e) \, d\sigma.$$

Therefore using (2.2) and Cauchy-Schwarz inequality, we get

$$\cos\theta \int_{\Gamma} \varphi^2 \, d\sigma \leq -2 \int_{\Omega} \varphi(e \cdot \nabla\varphi) \, dx \leq 2 \|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega, \mathbb{R}^n)},$$

which gives the estimate (2.3). □

Corollary 2.2. *There exists a unique operator $T \in \mathcal{L}(H^1(\Omega), L^2(\Gamma))$ satisfying*

$$T\varphi = \operatorname{Tr}_{|\Gamma}\varphi, \quad \text{for all } \varphi \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$$

and

$$\|T\|_{\mathcal{L}(H^1(\Omega), L^2(\Gamma))} \leq \frac{1}{\sqrt{\cos\theta}}. \tag{2.4}$$

Proof. The existence and uniqueness of the operator T follow from Theorem 2.1 the density of $\mathcal{C}_c^\infty(\overline{\Omega})$ in $H^1(\Omega)$ (see, e.g., [1, Theorem 4.7, p. 248]). Moreover, (2.3) implies

$$\|\varphi\|_{L^2(\Gamma, d\sigma)}^2 \leq \frac{1}{\cos\theta} (\|\varphi\|_{L^2(\Omega)}^2 + \|\nabla\varphi\|_{L^2(\Omega, \mathbb{R}^n)}^2), \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\overline{\Omega}),$$

which proves (2.4). □

References

- [1] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987, Oxford Science Publications.
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