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# ON THE NONLINEAR DYNAMICS OF THE TRAVELING-WAVE SOLUTIONS OF THE SERRE EQUATIONS

DIMITRIOS MITSOTAKIS\*, DENYS DUTYKH, AND JOHN D. CARTER

**ABSTRACT.** In this paper, we study numerically nonlinear phenomena related to the dynamics of the traveling wave solutions of the Serre equations including their stability, their persistence, resolution into solitary waves, and wave breaking. Other forms of solutions such as Dispersive Shock Waves (DSWs), are also considered. Some differences between the solutions of the Serre equations and the full Euler equations are also studied. Euler solitary waves propagate without large variations in shape when they are used as initial conditions in the Serre equations. The nonlinearities seem to play a crucial role in the generation of small-amplitude waves and appear to cause a recurrence phenomenon in linearly unstable solutions. The numerical method used in the paper utilizes a high order Finite Element Method (FEM) with smooth, periodic splines in space and explicit Runge-Kutta methods in time. The solutions of the Serre system are compared with the corresponding ones of the asymptotically-related Euler system whenever is possible.

**Key words and phrases:** Finite element methods; solitary waves; cnoidal waves; Green–Naghdi system; stability

**MSC:** [2010]76B15 (primary), 76B25, 65M08 (secondary)

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## 1. Introduction

The Serre equations (also known as the Green–Naghdi or Su–Gardner equations) [55, 58, 32, 33] approximate the Euler equations of water wave theory and model the one-dimensional, two-way propagation of long waves. If  $a$  denotes a typical amplitude of a wave,  $d$  the mean depth of the fluid, and  $\lambda$  a typical wavelength, then the Serre equations are characterized by the parameter  $\varepsilon \doteq a/d = \mathcal{O}(1)$  and the shallow water condition,  $\sigma \doteq d/\lambda \ll 1$ , contrary to the Boussinesq equations which model the propagation of small-amplitude, long waves, *i.e.*  $\varepsilon \ll 1$  and  $\sigma \ll 1$ , when the Stokes number is  $S \doteq \varepsilon/\sigma^2 = \mathcal{O}(1)$ . The Boussinesq equations are often called weakly nonlinear, weakly dispersive equations while the Serre equations are often called fully-nonlinear shallow-water equations. In dimensionless and scaled form, the Serre equations take the form:

$$\eta_t + u_x + \varepsilon(\eta u)_x = 0, \quad (1.1a)$$

$$u_t + \eta_x + \varepsilon u u_x - \frac{\sigma^2}{3h} [h^3(u_{xt} + \varepsilon u u_{xx} - \varepsilon(u_x)^2)]_x = 0, \quad (1.1b)$$

for  $x \in \mathbb{R}$ ,  $t > 0$ , along with the initial conditions

$$\eta(x, 0) = \eta_0(x), \quad u(x, 0) = u_0(x). \quad (1.2)$$

Here  $\eta = \eta(x, t)$  is the free surface displacement, while

$$h \doteq 1 + \varepsilon \eta, \quad (1.3)$$

is the total fluid depth,  $u = u(x, t)$  is the depth-averaged horizontal velocity, and  $\eta_0, u_0$  are given real functions, such that  $1 + \varepsilon \eta_0 = h_0 > 0$  for all  $x \in \mathbb{R}$ . In these variables, the location of the horizontal bottom is given by  $y = -1$ . This system was originally derived by F. SERRE [56] while the same system was later re-derived (independently) by SU & GARDNER [58]. Moreover, GREEN & NAGHDI [33] proposed an alternative method for the approximation of the waves in the same regime called the direct approximation of fluid motion with a

free surface. Recently, LANNES & BONNETON [41] derived and justified several asymptotic models for surface wave propagation including the Serre system of equations. For a review of the derivation and the basic properties of this system we also refer to [4].

The Euler equations along with the model system (1.1) admit traveling wave solutions, *i.e.* waves that propagate without change in shape or speed. Solitary waves is a special case of traveling wave solutions of these systems. These waves are of classical form, [20], *i.e.* they consist of a single hump that decay to zero exponentially. The other important class of traveling wave solutions is the class of cnoidal wave solutions which can be thought of as the periodic generalization of solitary waves.

Many Boussinesq equations are known to possess solitary and cnoidal wave solutions, but they do not admit nontrivial, closed-form solutions [16, 15]. In contrast, the Serre equations admit closed-form solitary and cnoidal wave solutions. The solitary wave solutions of the Serre system traveling with constant speed  $c_s$  are given by the formulas

$$h_s(\xi) \doteq h_s(x, t) = (a_0 + a_1 \operatorname{sech}^2(K_s \xi)) / \sigma, \quad (1.4a)$$

$$u_s(\xi) \doteq u_s(x, t) = c_s \left( 1 - \frac{a_0}{\sigma h_s(\xi)} \right) / \epsilon, \quad (1.4b)$$

where  $\xi = x - c_s t$ ,

$$K_s = \sqrt{\frac{3a_1}{4\sigma a_0^2 c_s^2}}, \quad c_s = \sqrt{\frac{a_0 + a_1}{\sigma}},$$

and  $a_0 > 0$  and  $a_1 > 0$ . By taking  $a_0 = \sigma$  and  $a_1 = \epsilon \sigma A_s$  we obtain the formulas for the classical solitary waves that are homoclinic to the origin.

The cnoidal waves of the Serre system traveling with constant speed  $c_c$  is given by

$$h_c(\xi) \doteq h_s(x, t) = (a_0 + a_1 \operatorname{dn}^2(K_c \xi, k)) / \sigma, \quad (1.5a)$$

$$u_c(\xi) \doteq u_s(x, t) = c_c \left( 1 - \frac{h_0}{h(\xi)} \right) / \epsilon, \quad (1.5b)$$

where

$$h_0 = a_0 + a_1 \frac{E(m)}{K(m)},$$

$$K_c = \frac{\sqrt{3a_1}}{2\sqrt{a_0(a_0 + a_1)(a_0 + (1 - k^2)a_1)}}, \quad c_c = \sqrt{\frac{a_0(a_0 + a_1)(a_0 + (1 - k^2)a_1)}{\sigma h_0^2}},$$

$k \in [0, 1]$ ,  $m = k^2$ ,  $a_0 > 0$ , and  $a_1 > 0$ . Here  $K$  and  $E$  are the complete elliptic integrals of the first and second kind respectively. Note that (1.4) is the  $k = 1$  limit of (1.5).

Another fundamental property of the Serre system is the conservation of the energy which plays also the role of the Hamiltonian,

$$\mathcal{H}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[ \epsilon h u^2 + \frac{\epsilon \sigma^2}{3} h^3 u_x^2 + \frac{1}{\epsilon} h^2 \right] dx, \quad (1.6)$$

in the sense that  $\mathcal{H}(t) = \mathcal{H}(0)$  for all  $t > 0$  up to the maximal time  $T$  of the existence of the solution.

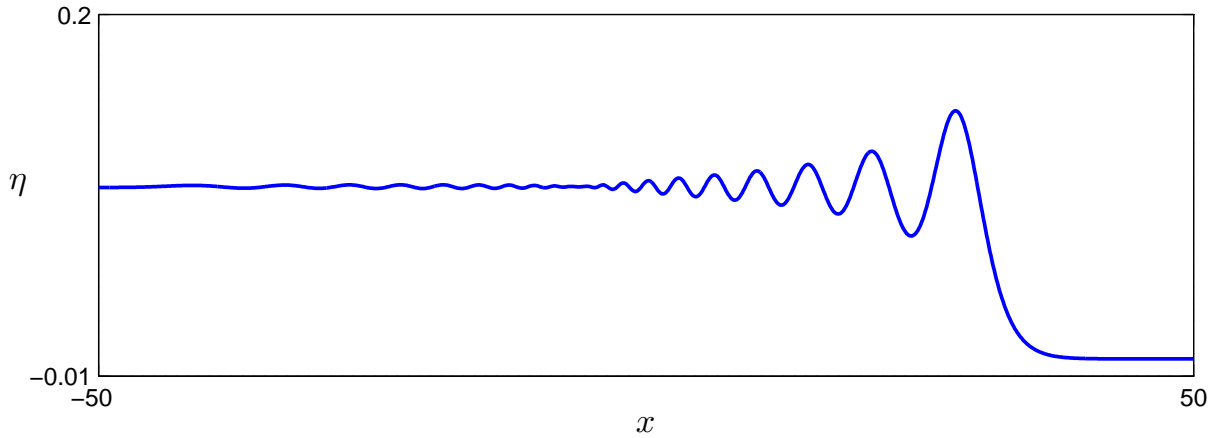
A very important property of the traveling wave solutions is their stability. In the theory of traveling waves there are two basic kinds of stability: the orbital and the asymptotic stability. Both notions predicate that perturbations of an exact traveling-wave solutions will forever remain close in some sense to the exact solution. Somewhat more precisely, a traveling-wave solution  $\phi_c(x-ct)$  is said to be orbitally stable if given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\inf_{\gamma} \|\phi(x,t) - \phi_c(x-ct+\gamma)\| < \epsilon$  for  $t > 0$  whenever  $\inf \|\phi(x,0) - \phi_c(x)\| < \delta$ . This means that if the perturbed solution  $\phi$  is initially close to the exact solution  $\phi_c$  then it will remain close to the set of translates of the solution,  $\phi_c(x-ct+\gamma)$ , forever. This notion of stability establishes that the *shape* of the wave is stable, but does not fully resolve the question of what the asymptotic behavior of the system is. The property of asymptotic stability describes the long-time asymptotic behavior of the solutions which are initially close to the traveling wave. Specifically, this notion of stability establishes that if  $\inf \|\phi(x,0) - \phi_c(x)\| < \delta$  then

$$\inf \|\phi(x,t) - \phi_c(x-ct)\| \rightarrow 0, \quad \text{as } t \rightarrow +\infty. \quad (1.7)$$

If this property holds, then the traveling wave is said to be asymptotically stable. We refer to the papers [6, 7, 51, 52] for more information about stability in general and results related to the stability of solitary waves of some model equations of the water wave theory.

Due to their complex structure, both the Serre and Euler equations are difficult to study theoretically and/or numerically. The Euler equations admit no known nontrivial closed-form solutions and only a few accurate numerical solutions have been computed. Although the situation for the Serre equations is slightly better because of the known analytical formulas for the families of traveling wave solutions, (1.4) and (1.5), there is no known proof of the orbital or asymptotic stability of these solutions. The only known analytical results are related to solitary waves and are based on the linearization of the Serre system. As is noted in [44, 45] the solitary wave solutions of the Serre equations are not minimizers of the Hamiltonian functionals (this is also true for the full Euler system) and therefore techniques applied for other model equations [6, 7, 34] fail to prove the stability of the traveling waves. For this reason the theoretical and numerical studies of the stability of the traveling waves are focused on their linear (or spectral) stability. The linear stability analysis is based on the study of the eigenvalue problem related to the system linearized about a traveling wave, [52, 2, 38]. The linear stability of the solitary waves of the Serre system has been studied in [44, 45] where it is shown that the solitary waves of small amplitude are linearly stable since the problem has no eigenvalues of positive real part and the Evans function is nonvanishing everywhere except the origin. Even more fascinating are the results related to the linear stability of the cnoidal waves. Specifically, it was shown by numerical means that cnoidal waves of sufficiently small amplitude and steepness are spectrally stable and solutions with sufficiently large amplitude or steepness are spectrally unstable [13]. Although the linear stability results for traveling waves are very useful, they do not necessarily provide information about nonlinear stability.

Other properties related to the nonlinear dynamics are the ability of solitary waves to be spawned by random initial conditions and the ability of one soliton to pass through another without changing form. Although these properties are fundamental for nonlinear,



**Figure 1.** Sketch of a dispersive shock wave.

dispersive waves, it is very difficult to establish a theory and therefore only a few results are known about these properties of the Serre and Euler equations. The justification of the Serre equations has been studied theoretically [46, 40]. Specifically, it has been proven that if the Serre and Euler equations share the same initial data, then there is an interval  $0 \leq t \leq T$  during which the solutions of the Euler and Serre systems remain close. Although analogous results exist for Boussinesq models, it is not clear whether the dynamics of the traveling wave solutions of the Serre equations are close to those of the Euler equations during their propagation. One example where a shallow water model has totally different dynamical properties compared to the Euler system is the KdV–KdV system of Boussinesq equations. This system possesses generalized solitary waves that decay to periodic orbits instead of classical solitary waves, [9, 10].

In this paper we study the problem of the nonlinear stability (orbital and asymptotic) of the traveling waves of the Serre system by using numerical techniques. We provide numerical evidence of stability with respect to certain classes of perturbations. Phenomena such as perturbations of the traveling waves, perturbations of the Serre system and interactions of traveling waves are studied analyzing the stability properties of the waves at hand. The physical relevance of the Serre equations is addressed whenever possible.

When the dispersive terms in the Serre system are neglected, the resulting nondispersive nonlinear shallow water equations admit shock waves, *i.e.* discontinuous solutions. When such shocks are regularized by dissipative effects, this gives rise to classical or viscous shocks, which are characterized by a rapid and monotonic change in the flow properties. On the other hand, in systems where dissipation is negligible compared with dispersion, the dispersive effects give rise to dispersive shock waves. DSWs are characterized by an expanding train of rapidly oscillating waves. The leading edge of a DSW consists of large amplitude waves which decay to linear waves at the trailing edge, *cf.* Fig. 1.

The DSWs of the Serre equations have been studied by asymptotic and numerical methods, *cf.* [28, 29, 43, 49]. Although some of the asymptotic characteristics of these waves are

known due to the work in [28] the behavior of the DSWs during their interactions remains unknown.

The numerical method of preference is a high-order accurate and non-dissipative Galerkin / Finite element method (FEM) for the spatial discretization combined with the classical fourth-order explicit Runge–Kutta scheme in time. In some cases adaptive time-stepping methods, such as the Runge–Kutta–Fehlberg, the Cash–Karp and the Dormand–Prince methods [35] were employed to verify that there are no spurious solutions or blow-up phenomena. Because the results obtained by all methods were identical to the fourth-order Runge–Kutta we don't present their results here. This numerical scheme has been previously analyzed and it has been shown that it is highly accurate and stable since there is no need for a restrictive condition on the step-size but only mild conditions of the form  $\Delta t \leq C\Delta x$ , while the conservation properties such as the conservation of the Hamiltonian  $\mathcal{H}(t)$  are very satisfactory and ideal for studying stability issues, *cf.* [49]. In contrast, other numerical methods such as finite volume [14, 25, 26], finite differences or spectral methods are not ideal due to the artificial, spurious dissipative or dispersive effects polluting the numerical solutions by either suppressing or triggering instabilities. In order to ensure the accuracy of the numerical results obtained with the FEM we compared them with the analogous results obtained with the pseudo-spectral method described and analyzed in [24].

The paper is organized as follows. The numerical method is presented briefly in Section 2. The compatibility of the solitary waves of the Serre and the Euler systems is examined in Section 3. The head-on collision of solitary waves is studied in Section 4. A number of issues related to the stability of the traveling waves are discussed in Sections 5. The interaction of DSWs is presented in Section 6. Finally, the major conclusions are summarized in Section 7. Additionally, Appendix A presents the numerical scheme for the numerical solution of the full Euler equations as well as the generation of solitary wave solutions in the finite depth case.

## 2. The numerical method

In this section we present a FEM for the initial-boundary value problem (IBVP) comprised of system (1.1) subject to periodic boundary conditions. For simplicity we use  $\varepsilon = \sigma = 1$ . We also rewrite the system (1.1) in terms of  $(h, u)$  rather than  $(\eta, \mathbf{u})$ . This is done by using (1.3) and yields the IBVP

$$h_t + (hu)_x = 0, \quad (2.1a)$$

$$u_t + h_x + uu_x - \frac{1}{3h} [h^3(u_{xt} + uu_{xx} - (u_x)^2)]_x = 0, \quad (2.1b)$$

$$\partial_x^i h(a, t) = \partial_x^i h(b, t), \quad i = 0, 1, 2, \dots, \quad (2.1c)$$

$$\partial_x^i u(a, t) = \partial_x^i u(b, t), \quad i = 0, 1, 2, \dots, \quad (2.1d)$$

$$h(x, 0) = h_0(x), \quad (2.1e)$$

$$u(x, 0) = u_0(x), \quad (2.1f)$$





### 3. Solitary waves

Solitary waves are fundamental solutions of nonlinear, dispersive equations including the Euler equations. It is therefore important for all approximate wave models to possess solitary wave solutions that are similar to the Euler solitary waves. The Serre system and the Euler equations both possess solitary waves that decay to zero at infinity. Although the justification of the Serre equations ensures that the Serre solutions will remain close to Euler solutions, it is not known whether an Euler solitary wave propagates as a Serre solitary wave that differs only by a small amount from the original one, when it is used as initial condition to the Serre system.

While the Serre system admits solitary wave solutions of the form given in equation (1.4), there are not known formulas for solitary waves of the Euler system. For this reason we compute Euler solitary waves numerically. The numerical method is a Petviashvili iteration applied to the Babenko equation [53, 18, 23]. This method is presented in detail in Appendix A.

By comparing the solitary waves of the Euler system with the solitary waves of the Serre system, one may deduce that they are not identical. To demonstrate the ability of the Serre equations to approximate the Euler equations, we first compare the characteristics of two solitary waves with speeds  $c_s = 1.1$  and  $c_s = 1.2$ . Figure 2 includes plots of the Serre and Euler solitary wave solutions with each of these speeds.

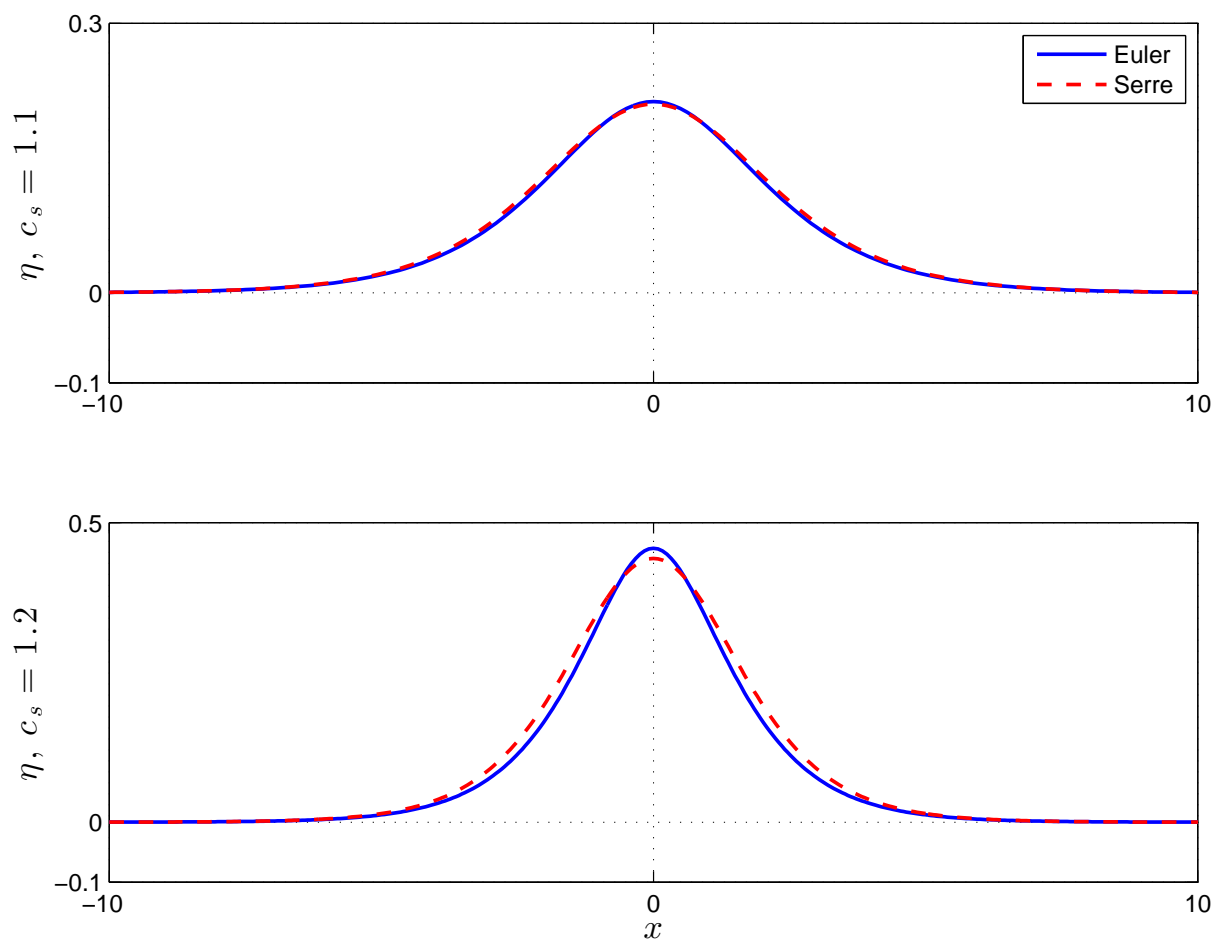
In Figure 2 and when  $c_s = 1.1$  it is difficult to observe any significant differences between the Euler and Serre solitary waves. However, there are obvious differences in the shape and the amplitude of the two solitary waves when  $c_s = 1.2$ .

In order to emphasize the differences between the Euler and Serre solitary waves, Figure 3 contains plots of the normalized difference

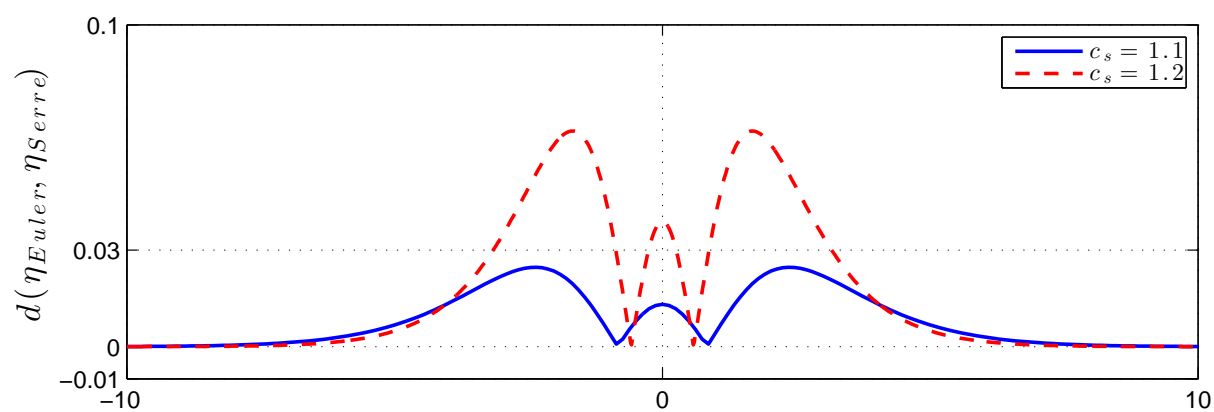
$$d(\eta_{\text{Euler}}, \eta_{\text{Serre}}) = \left| \frac{\eta_{\text{Euler}} - \eta_{\text{Serre}}}{\max(\eta_{\text{Serre}})} \right| \quad (3.1)$$

of the solitary waves shown in Figure 2. The difference of the solutions with  $c_s = 1.2$  is larger than the difference between the solutions of  $c_s = 1.1$ . Specifically, a 10% increase in the phase speed leads to an increase in solitary wave amplitude of almost 50% while the normalized difference between the Euler and Serre solitary waves increases from under 3% to over 6%. The amplitude of several solitary waves are presented in Table 1. It is easily observed that the difference in amplitude (and consequently in shape) is larger for larger values of the speed  $c_s$ .

Next, we examine how the solitary waves of the Euler system propagate when they are used as initial conditions to the Serre system. Specifically, we use the numerically generated solitary wave solutions of the Euler equations and the exact formula  $u = c_s \eta / (1 + \eta)$  to define the initial conditions  $\eta_0$  and  $u_0$  for the Serre equations. Then, we integrate the Serre system using the fully-discrete scheme presented in Section 2. Figures 4 and 5 contain plots of the solutions at  $t = 150$  obtained using the Euler solitary waves with  $c_s = 1.1$  and  $c_s = 1.2$ . These figures demonstrate that the difference between the Euler solitary wave and numerical Serre solution is greater when  $c_s = 1.2$  than when  $c_s = 1.1$ . We note that the



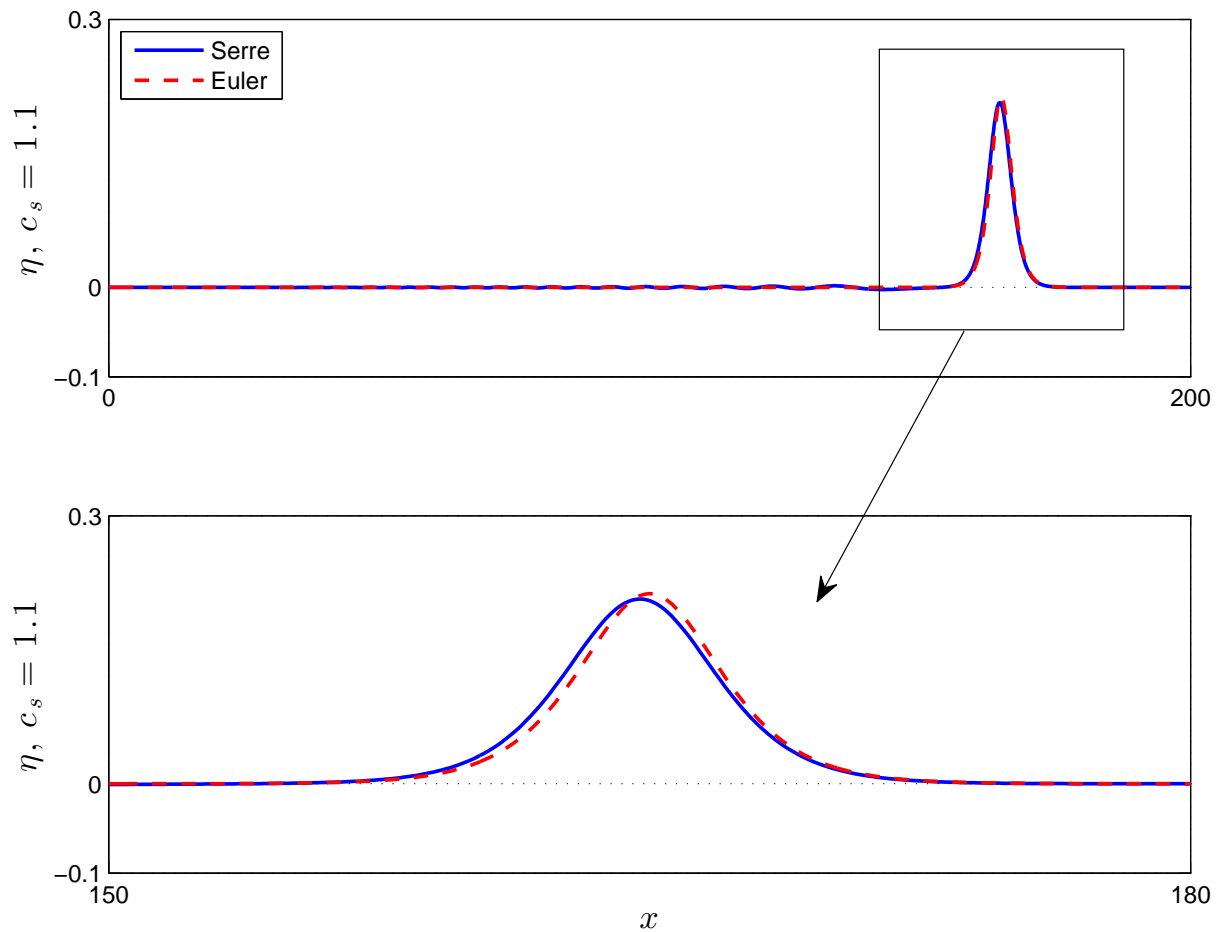
**Figure 2.** Solitary waves for the Serre system and Euler equations.



**Figure 3.** The difference between the solitary waves of Figure 2.

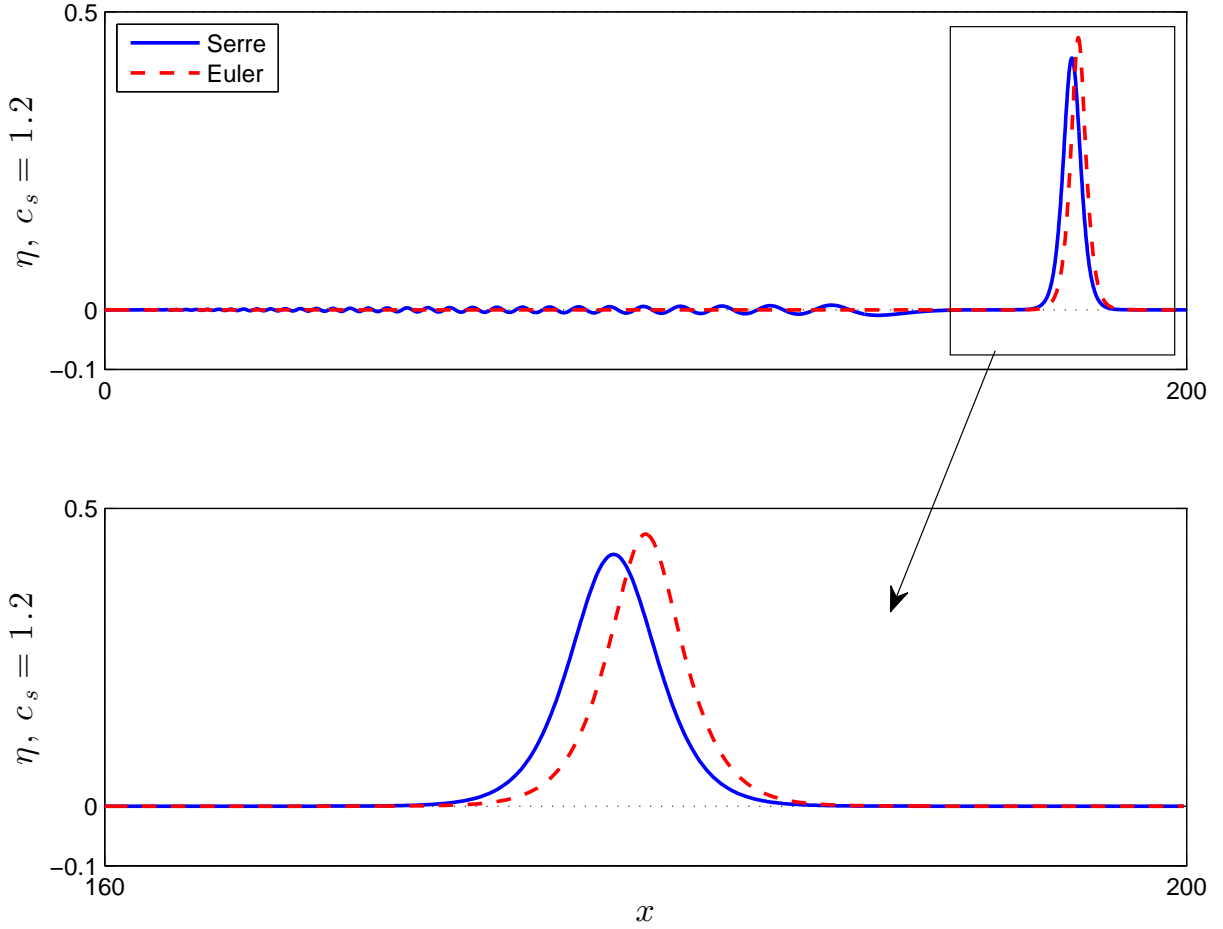
$c_s$	Euler	Serre
1.01	0.02012	0.0201
1.05	0.10308	0.1025
1.1	0.21276	0.2100
1.15	0.33007	0.3225
1.2	0.45715	0.4400
1.28	0.70512	0.6384

**Table 1.** Amplitudes of the Euler and Serre solitary waves corresponding to different speeds.



**Figure 4.** The evolution of an Euler solitary wave with  $c_s = 1.1$  when used as initial condition in the Serre system.

value  $c_s = 1.2$  is a relatively large value since the largest value we can use to generate an Euler solitary wave is  $c_s = 1.29421$ . For smaller values of the  $c_s$ , the agreement is better.



**Figure 5.** The evolution of an Euler solitary wave with  $c_s = 1.2$  when used as initial condition in the Serre system.

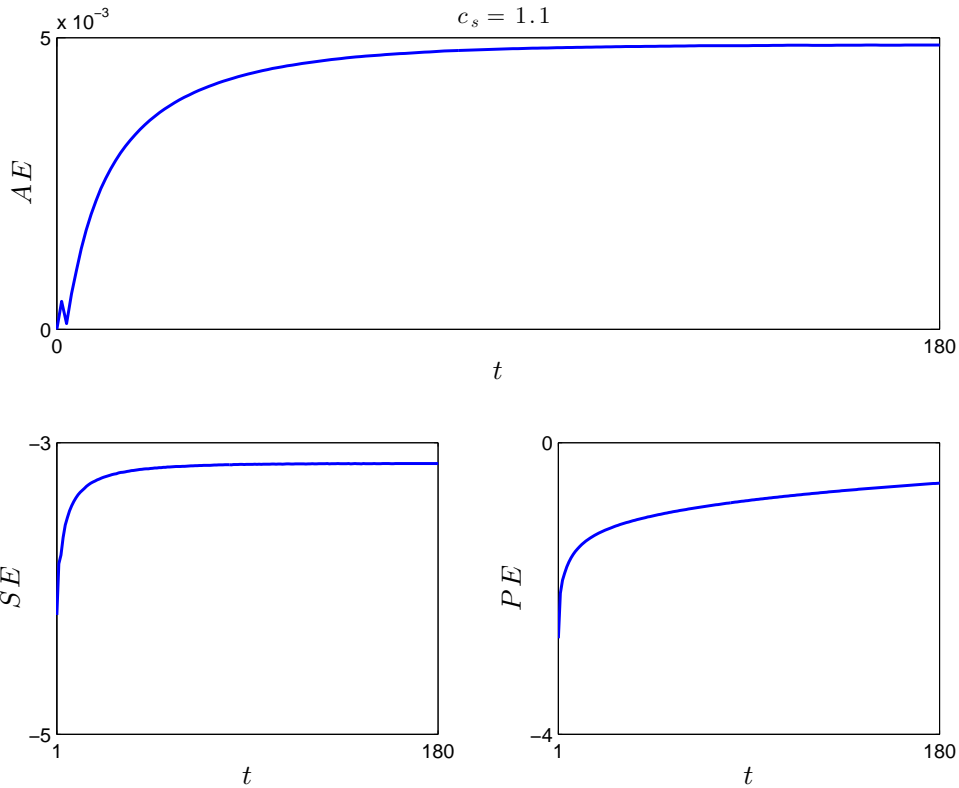
To study further the differences between the Euler and Serre solitary waves, we consider three quantities pertinent to the propagation of the solitary waves: the *amplitude*, *shape* and *phase*. First, we define the normalized peak amplitude error as

$$AE[F] \doteq \frac{|F(x^*(t), t) - F(0, 0)|}{|F(0, 0)|}, \quad (3.2)$$

where  $x^*(t)$  is the curve along which the computed solution  $F(x, t)$  achieves its maximum. Monitoring  $AE$  as a function of time, we observe that even though the Euler solitary waves do not propagate as traveling waves to the Serre system, their amplitude asymptotically tends toward a constant indicating that they evolve into a solitary wave solution of the Serre equation, see Figures 6 and 7.

We define the normalized shape error as

$$SE[F] \doteq \log_{10}(\min_{\tau} \zeta(\tau)), \quad \zeta(\tau) \doteq \frac{\|F(x, t^n) - F_{\text{exact}}(x, \tau)\|}{\|F_{\text{exact}}(x, 0)\|}. \quad (3.3)$$



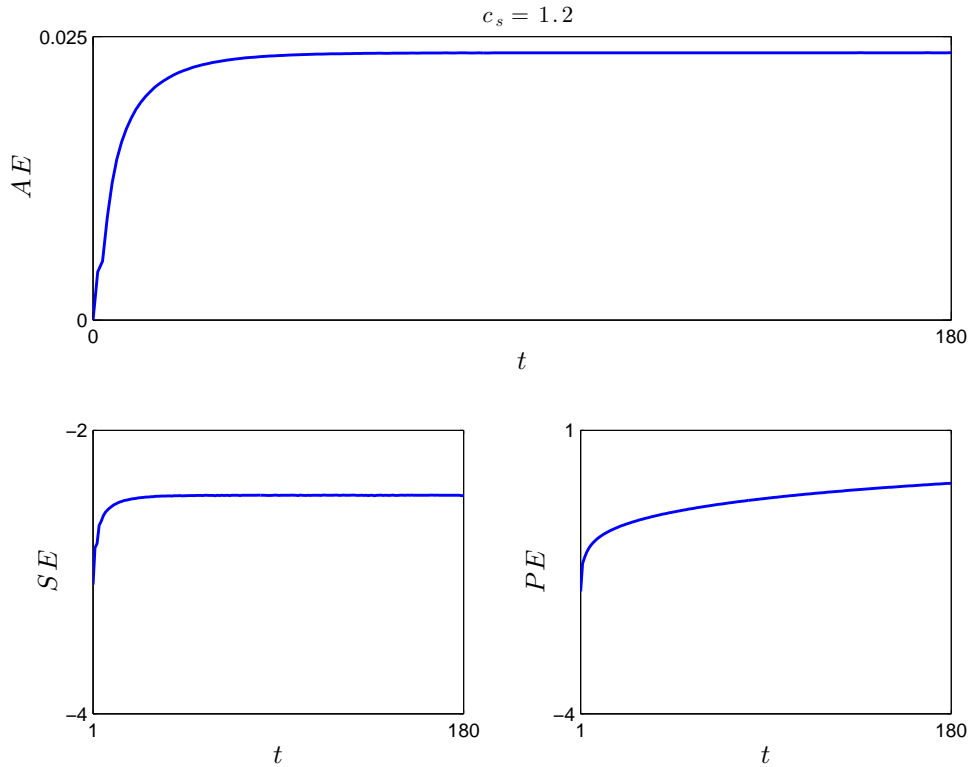
**Figure 6.** The amplitude, shape and phase error of the Euler's solitary wave of  $c_s = 1.1$  propagating with Serre equations. See also Figure 4.

The minimum in (3.3) is attained at some critical  $\tau = \tau^*(t^n)$ . This, in turn, is used to define the phase error as

$$PE[F] \doteq \log_{10}(|\tau^* - t^n|). \quad (3.4)$$

In order to find  $\tau^*$ , we use Newton's method to solve the equation  $\zeta''(\tau) = 0$ . The initial guess for Newton's method is chosen as  $\tau^0 = t^n - \Delta t$ . Having computed  $\tau^*$ , the shape error (3.3) is then  $SE[F] = \log_{10}(\zeta(\tau^*))$ . These error norms are closely related to the *orbit* of the solitary wave. In Figures 6 and 7 we present the respective shape and phase errors as a function of the temporal variable  $t$ . We observe that the shape error is of  $\mathcal{O}(10^{-3})$  in the case where  $c_s = 1.1$  and of  $\mathcal{O}(10^{-2})$  in the case where  $c_s = 1.2$ . The phase error increases since the solitary waves propagate with different speeds. It is remarkable that the phase speeds of the new solitary waves of the Serre system are almost the same as the phase speeds of the Euler's solitary waves. For example, the speeds are  $c_s \approx 1.09$  and  $c_s \approx 1.19$ .

Similar comparisons have been performed for other model equations such as the classical Boussinesq system, [8], and the results are comparable *cf.* [22]. It is also shown that the agreement between the Euler and Boussinesq solitary waves does not depend only on the order of the asymptotic model or on the linear properties (such as the linear dispersion relation). The conclusion of this analysis is that Euler solitary waves evolve into different solitary waves with very similar shape when they are used as initial conditions in the Serre



**Figure 7.** *The amplitude, shape and phase errors of the Euler's solitary wave of  $c_s = 1.1$  propagating with Serre equations. See also Figure 5.*

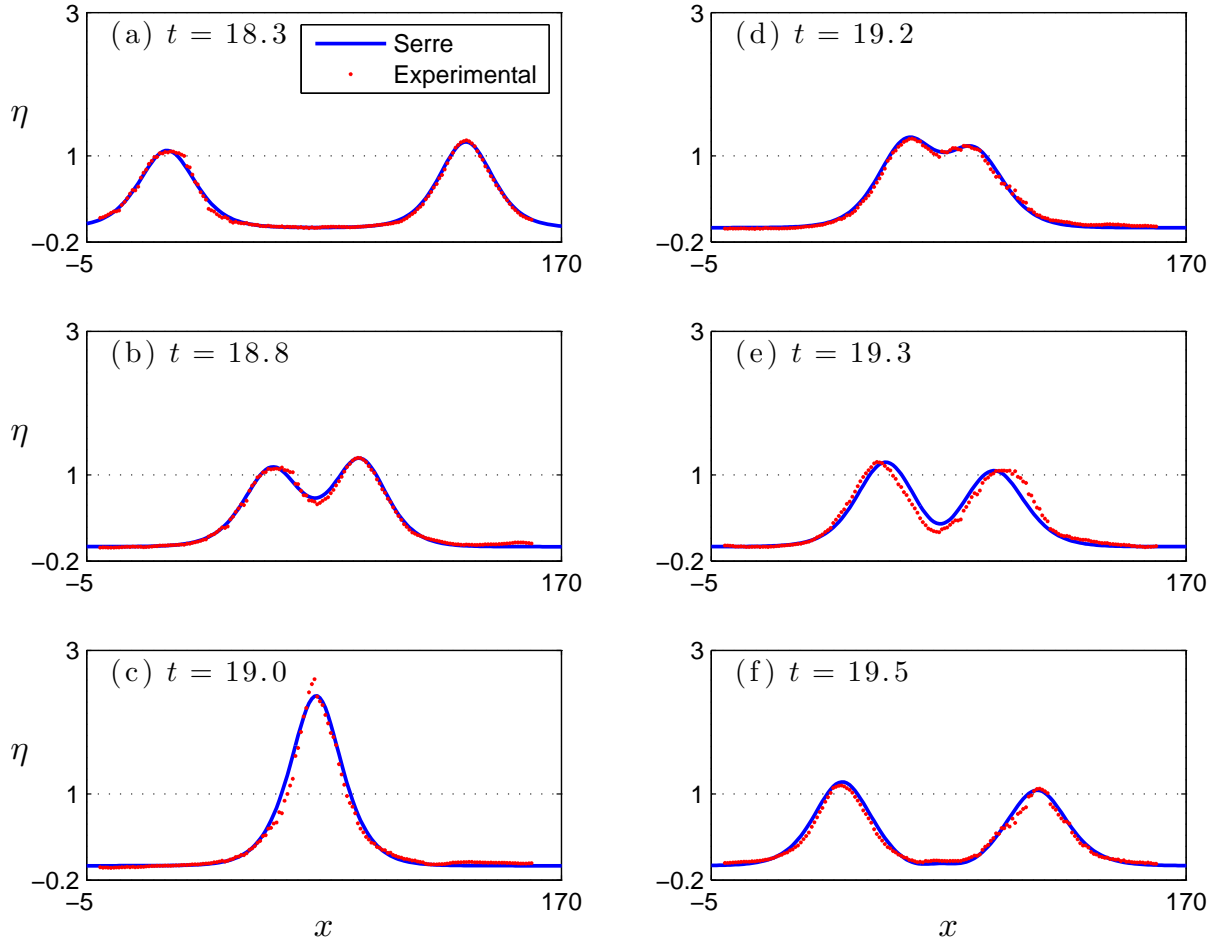
system. This indicates that the Serre equations are consistent with the Euler system and extends the justification of the Serre equations of [40] to the propagation of solitary waves.

## 4. Head-on collision of solitary waves

The head-on collision of two solitary waves of the Serre system has been studied numerically before in [49, 24, 47]. It has been observed that the interaction is more inelastic than in other weakly nonlinear models such as the classical Boussinesq system [8]. Highly nonlinear interactions result in the generation of large amplitude dispersive tails. The dispersive tails are usually considered linear waves, however in practice their generation is due to both nonlinear and dispersive interactions of the solitary waves. In this section we study the physical relevance of the Serre equation during the collision of two solitary waves with different amplitudes. We also perform a comparison between the collision of solitary waves of the Euler and the Serre equations.

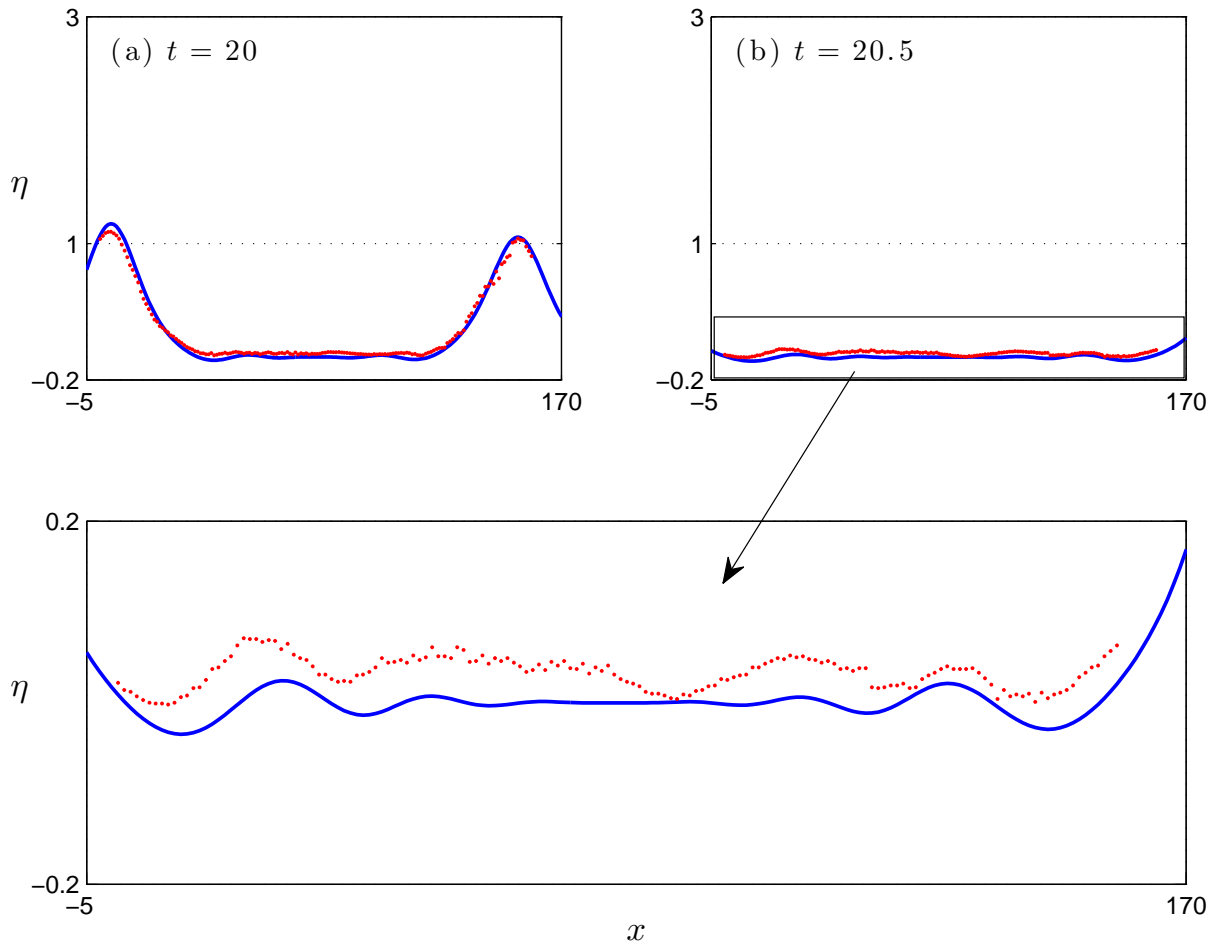
### 4.1. Physical relevance of the Serre system

In order to study the physical relevance of the head-on collision of two Serre solitary waves, we compare the Serre numerical solution with the experimental data of [19]. In this



**Figure 8.** Comparison of the head-on collision of two solitary waves of the Serre system with experimental data.

experiment, the Serre system is written in dimensional and unscaled form with an initial condition that includes two counter-propagating solitary waves in the interval  $[-5, 5]$ . The speeds of these solitary waves are  $c_{s,1} = 0.7721$  m/s and  $c_{s,2} = 0.7796$  m/s. Their amplitudes are  $A_1 = 0.0108$  m and  $A_2 = 0.0120$  m respectively. At  $t = 18.3$  s these solitary waves achieved their maximum values at  $x_1 = 0.247$  m and  $x_2 = 1.348$  m respectively. Figures 8 and 9 include comparisons between the numerical solution and experimental data. The agreement between the numerical results and the experimental data is impressive. The agreement in the generated dispersive tails in Figure 9 is even more impressive. Such agreement cannot be found in the case of head-on collisions of solitary waves of Boussinesq type models, [25], indicating that the high-order nonlinear terms are important in studying even these small-amplitude solutions. Finally, we mention that the maximum amplitude of the solution observed in Figure 8(c) during the collision is smaller than the real amplitude, possibly, because of a splash phenomenon that cannot be described by any model (see also [25]).

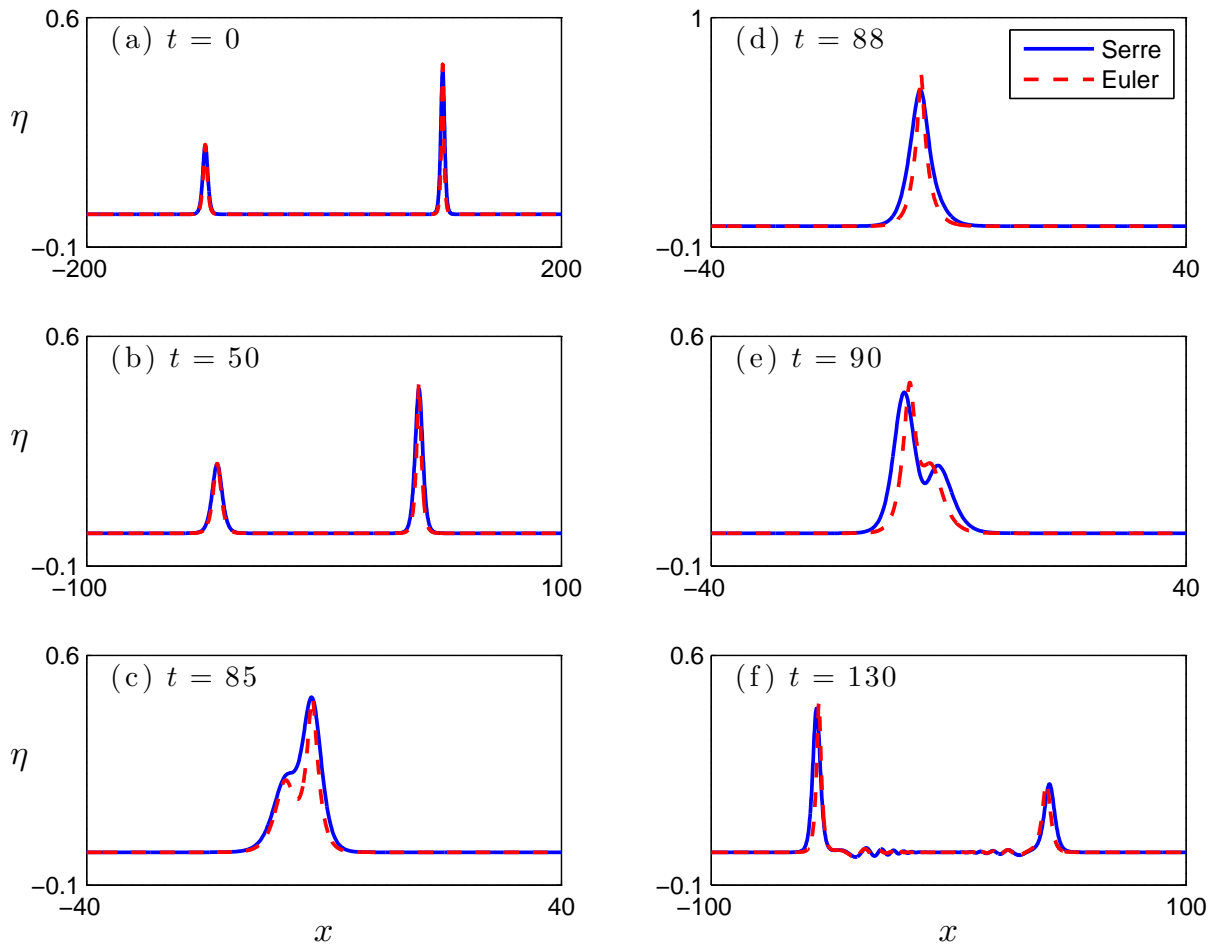


**Figure 9.** (Cont'd) Comparison of the head-on collision of two solitary waves of the Serre system with experimental data.

## 4.2. Comparison with the Euler equations

The previous experiments showed that the Serre equations provide an accurate prediction for the head-on collision of two solitary waves. Next, we compare a head-on collision of two unequal solitary waves via numerical solutions of the Serre and Euler equations. For both models, we consider a right-traveling solitary wave with  $c_s = 1.1$  and a left-traveling with  $c_s = 1.2$ . These solitary waves are initially translated so that the maximum peak amplitudes are achieved at  $x = -100$  and  $x = 100$  respectively. Results from both numerical simulations are included in Figures 10 and 11. Both models show similar behavior, however the maximum amplitude observed during the collision using the Euler equations is larger than in the Serre system. Also the interaction in the Euler equations lasts longer and therefore a larger phase shift can be observed. Figure 11 contains magnifications of the solitary waves and generated trailing dispersive tails after the interaction. The leading waves of the dispersive tails are almost identical in the two models, but the amplitude



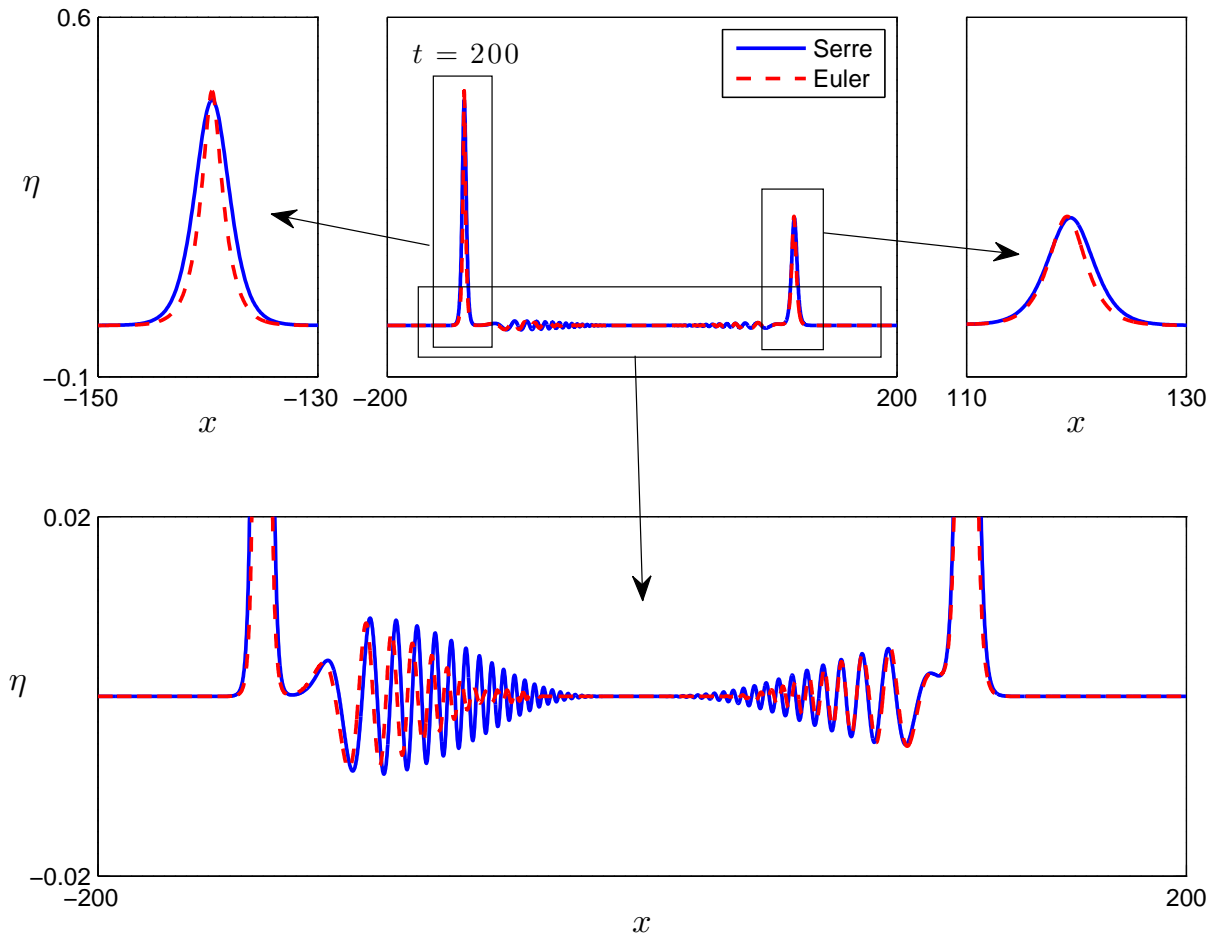


**Figure 10.** Comparison between the head-on collisions of two solitary waves for the Serre and Euler systems.

of the tails in the Euler system decay to zero slower than the amplitude of the tails in the Serre system. These numerical simulations verify the ability of the Serre system to accurately model head-on collisions of solitary waves. They also show that the Serre system is consistent with the Euler equations during and after the head-on collision with almost identical solutions.

## 5. Stability of traveling waves

The previous experiment of the head-on collision of two solitary waves indicates that the solitary waves are robust. Other head-on collision experiments with larger amplitude solitary waves show further robustness. In this section, we present the behavior of a solitary wave under small perturbations. We also explore the effects of the modification of some of the high-order terms of the Serre system on the persistence of the solitary waves. We also show that modifying one such term one can produce *regularized* shock waves, as opposed



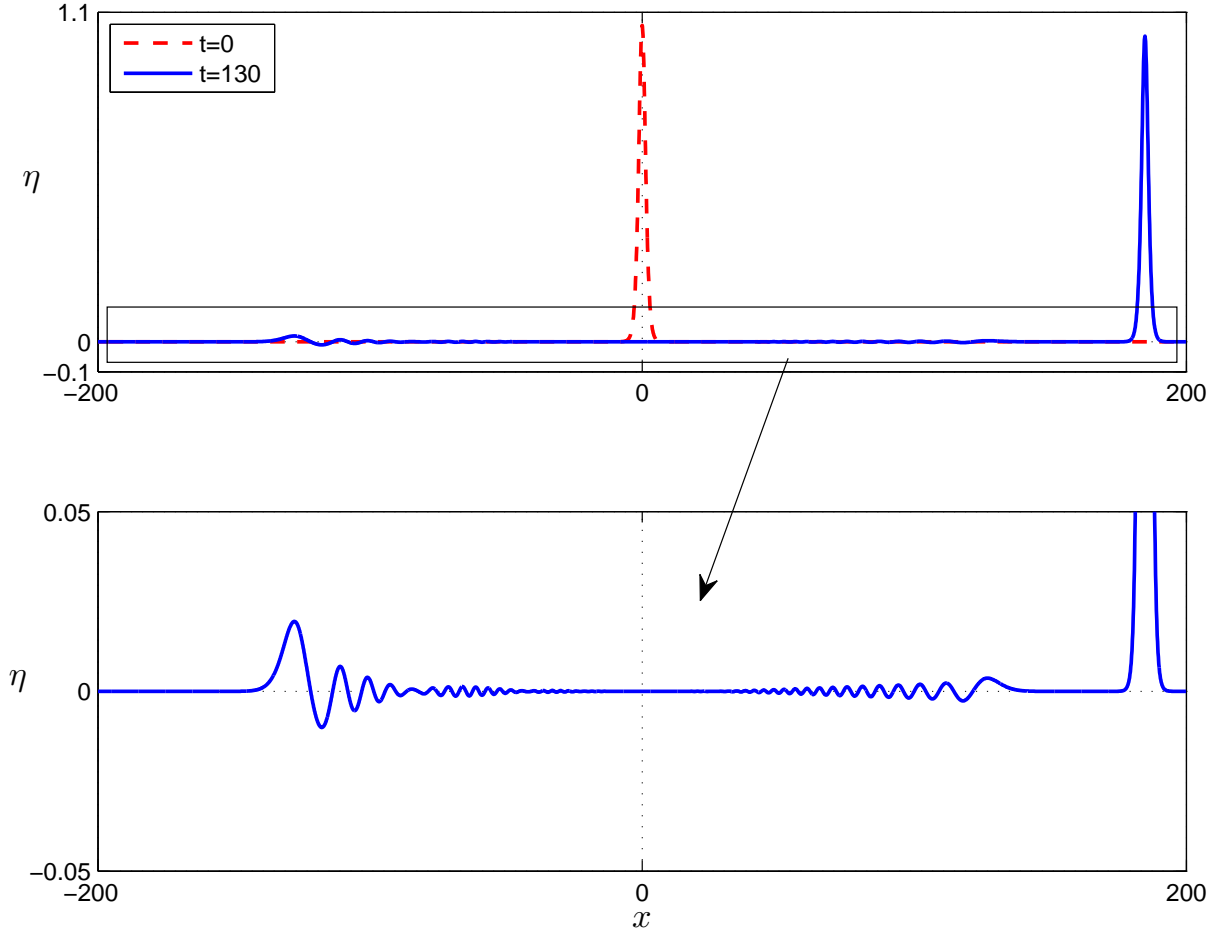
**Figure 11.** (Cont'd) Comparison between the head-on collisions of two solitary waves for the Serre and Euler systems.

to classical dispersive shock waves. Finally, we examine the stability of the cnoidal waves solutions.

### 5.1. Stability of solitary waves

We consider three different types of perturbations: perturbations of the amplitude, perturbations of the wavelength, and perturbations of the shape by random-noise. These perturbations alter the shape of the solitary wave by a small amount. If the solitary wave is orbitally stable with respect to the given perturbation, then as time evolves it will not differ significantly from appropriate translations of the unperturbed solitary wave. If the solitary wave is asymptotically stable with respect to a given perturbation, then the solution will limit to a very similar to that of the unperturbed solution as  $t$  increases. All of the solitary waves we tested were stable to all of the perturbations we considered.

We chose a solitary wave with speed  $c_s = 1.4$  and amplitude  $A = 0.96$  for all numerical simulations in this section. We perturb the amplitude of the solitary wave by multiplying



**Figure 12.** The perturbed solution generated by the perturbation of the amplitude of the solitary wave with  $c_s = 1.4$ ,  $A = 0.96$ , and perturbation parameter  $p = 1.1$ .

the pulse by a parameter  $p$  such that

$$h_p(x, 0) = 1 + p \cdot a_1 \operatorname{sech}^2(K_s x) , \quad (5.1)$$

while keeping the velocity component of the solution unperturbed as in (1.4b). By taking  $p = 1.1$ , *i.e.* by perturbing the solitary wave uniformly by 10% the initial condition sheds a small-amplitude dispersive tail and results in a new solitary wave with amplitude  $A = 1.02050$ . Figure 12 presents the initial condition and the resulting solution at  $t = 130$ . Similar observations resulted in all cases we tested. We also consider perturbations of the wavelength  $K_s$  by taking the initial condition for  $h$  to be

$$h_p(x, 0) = 1 + a_1 \operatorname{sech}^2(p \cdot K_s x) . \quad (5.2)$$

We note that when we perturb the wavelength the results were very similar to the results we got when we perturbed the amplitude of the solitary waves and so we don't show the results here. Table 2 shows the amplitudes of the solitary waves that result from various amplitude and wavelength perturbations.

$p$	Amplitude perturbation	Wavelength perturbation
0.8	0.83860	1.02710
0.9	0.89936	0.99225
1.1	1.02050	0.93017
1.2	1.08087	0.90260

**Table 2.** Amplitudes of the uniformly perturbed solitary waves.

$p$	Amplitude
-0.2	0.90110
-0.1	0.93055
0.1	0.98942
0.2	1.01883
0.5	1.10694

**Table 3.** Amplitudes of the randomly perturbed solitary waves.

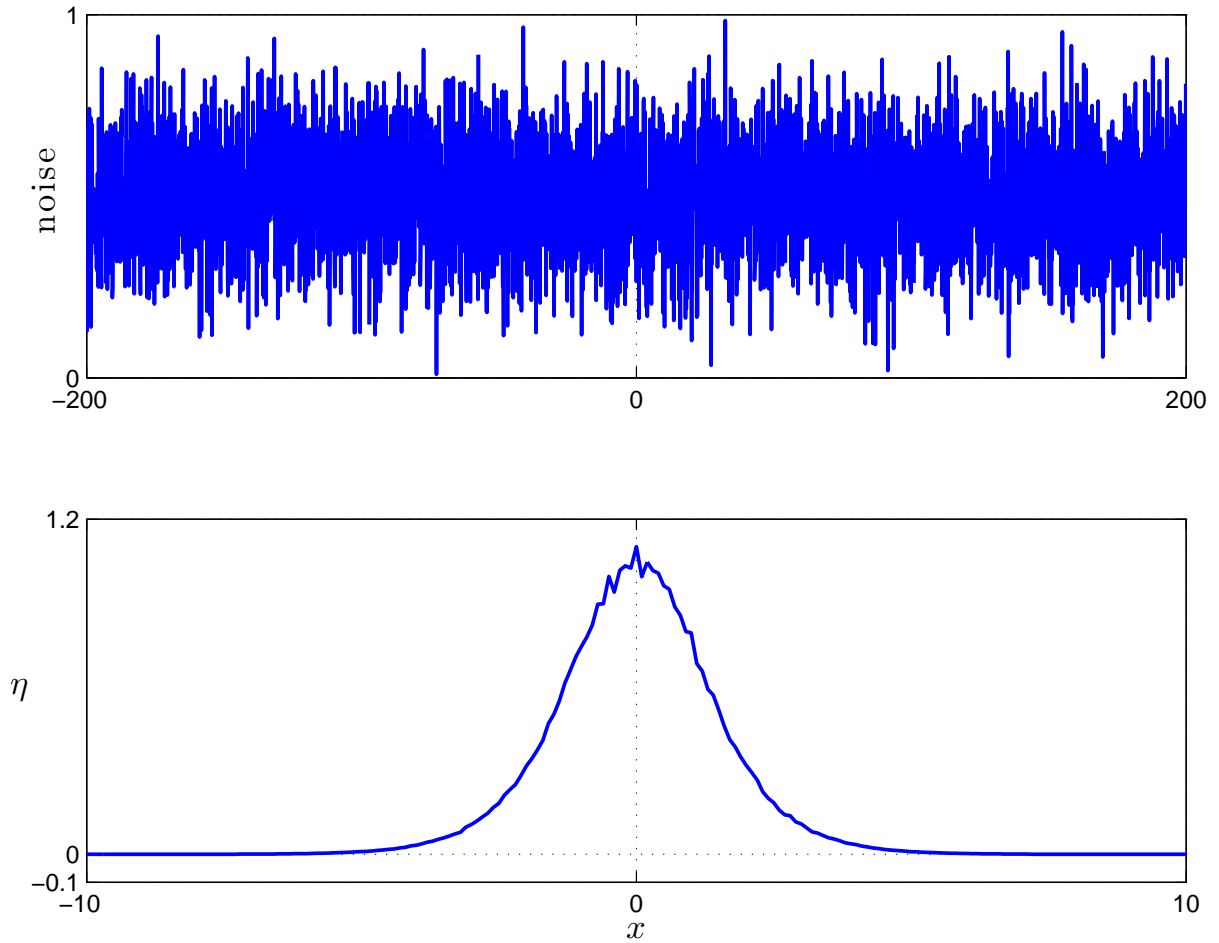
Similar results were obtained when extreme non-uniform perturbations were used. In order to consider non-uniform perturbations, we used pseudo-random noise distributed uniformly in  $[0, 1]$ . Denoting the noise function by  $N(x)$ , the perturbed solitary wave is given by

$$h_p(x, t) = 1 + (1 - p N(x)) \cdot a_1 \operatorname{sech}^2(K_s x - c_s t), \quad (5.3)$$

where the parameter  $p$  determines the magnitude of the noise. Figure 13 shows the noise  $N(x)$  and the perturbed solitary wave with  $p = 0.2$ . This type of perturbation is not only non-uniform, but is also non-smooth. Nevertheless the initial condition (shown in Figure 13) is the  $L^2$ -projection of the actual solution which ensures the required by the FEM smoothness. Figure 14 shows the evolution of a randomly perturbed solitary wave with  $p = 0.2$ . The solution does not differ qualitatively from the solution shown in Figure 12. The solution consists of a new solitary wave and a small-amplitude dispersive tail. In this case, the solitary wave has amplitude  $A = 1.01883$  while the dispersive tail contains part of the noise initially applied to the solitary pulse. The values of the amplitudes of the emerging solitary waves are presented in Table 3. These results suggest that the solitary waves of the Serre system are orbitally stable with respect to this class of perturbations.

## 5.2. Persistence of the solitary waves

One other aspect related to the stability of the solitary waves is their ability to persist when we perturb some high-order terms of the system. In this section, we examine if a Serre solitary wave persists when some of the terms of the Serre system are perturbed. Introducing the parameters  $\alpha$ ,  $\beta$ , and  $\gamma$ , we rewrite the Serre equations in the following



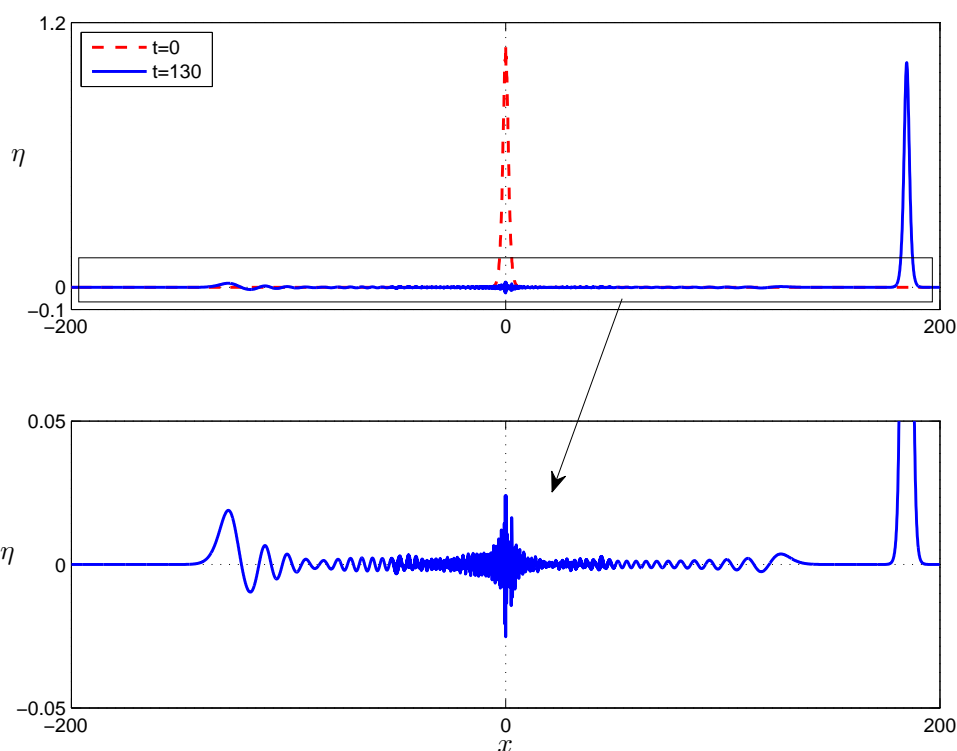
**Figure 13.** *The noise function  $N(x)$  and the perturbed solitary wave with  $p = 0.2$ .*

form

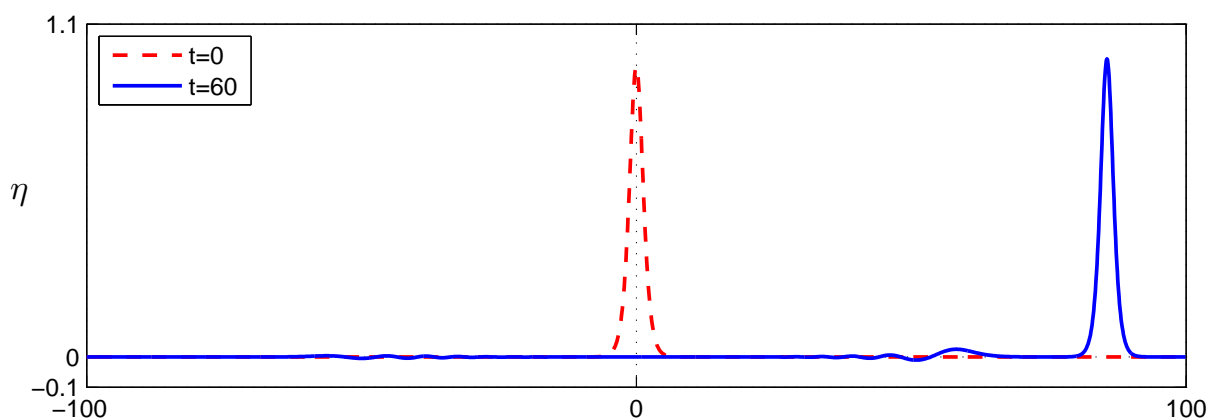
$$\eta_t + u_x + (\eta u)_x = 0, \quad (5.4a)$$

$$u_t + \eta_x + uu_x - \frac{1}{3h} [h^3 (\alpha u_{xt} + \beta uu_{xx} - \gamma (u_x)^2)]_x = 0. \quad (5.4b)$$

The unperturbed Serre equations correspond to  $\alpha = \beta = \gamma = 1$ . We study the persistence of the solitary waves when the system is perturbed by perturbing the parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  and considering a solitary wave of the unperturbed system as an initial condition. In this section we use the solitary wave (1.4) with  $c_s = 1.4$  as an initial condition. Taking  $\alpha = 0.9$  the solitary wave persists and the perturbation of the parameter  $\alpha$  acts in a manner similar to the amplitude perturbations in Section 5.1. Figure 15 includes the evolution of the solitary wave of the perturbed system. The new solitary wave is very similar to the unperturbed solitary wave further indicating that the solitary waves of the Serre system are stable. The results for the  $\beta = 0.9$  and  $\gamma = 0.9$  cases are very similar and are not presented here.



**Figure 14.** Evolution of the initial condition shown in Figure 13 with  $p = 0.2$ .



**Figure 15.** The evolution of a solitary wave of a perturbed system ( $\alpha = 0.9$ ,  $\beta = \gamma = 1$ ).

More interesting phenomena including the persistence of the solitary waves and wave breaking results can be observed when the solitary waves are used as initial conditions to a system with small values of the parameters  $\alpha$ ,  $\beta$  and  $\gamma$ . When all the three parameters are very small, the solutions tend to break into dispersive shock waves or other forms of undular bores. In the first numerical simulation, we consider  $\alpha = \beta = \gamma = 0.01$ . This is similar to the case of the small dispersion limit where the weakly nonlinear terms are dominant. As expected, the solitary wave breaks into a dispersive shock wave. Figure 16

presents the formation of the dispersive shock. This phenomenon has been previously observed in dispersive systems, *cf.* [25, 28, 42]. Unexpectedly, taking the parameters  $\beta$  and  $\gamma$  to be very small, *i.e.*  $\beta = \gamma = 0.001$  and keeping the parameter  $\alpha = 1$ , the solitary wave appears to persist and appears to evolve into a new solitary wave which is very similar to the unperturbed solitary wave. This persistence is remarkable because the solitary wave remains almost the same even if two of the most important terms have been almost eliminated. Figure 17 shows the generation of the new solitary wave if  $\alpha = 1$ ,  $\beta = \gamma = 0.001$ . If  $\alpha = \beta = 1$  and  $\gamma = 0.001$  or if  $\alpha = \gamma = 1$  and  $\beta = 0.001$  the behavior and persistence of the solitary wave are similar in all cases.

The behavior changes completely when large perturbations of the parameter  $\alpha$  are considered. For example, when  $\alpha = 0.001$  and  $\beta = \gamma = 1$ , the time evolution is very different, but remains interesting. In this case, as it is shown in Figure 18, the initial condition breaks into other solutions but instead of forming a dispersive shock wave, it forms a new kind of regularized shock wave. This suggests a new breaking mechanism by the elimination of the  $u_{xt}$  term. Similar dissipative behavior has been observed in nonlinear KdV-type equations where some high-order nonlinear terms introduce dissipation to the system [12].

When a solitary wave is perturbed by a large amount, the solitary wave is resolved into a series of solitary waves. This property is related to the ability of the solitary waves to be generated from other general initial conditions and is known to as the *resolution* property of the solitary waves. Many of the previous experiments are also related to this property and additional experiments on the resolution of solitary waves can be found in [49].

### 5.3. Stability of Cnoidal waves

We follow the work of CARTER & CIENFUEGOS [13] in order to study the linear stability of the solutions given in equation (1.5). We enter a coordinate frame moving with the speed of the solutions by defining  $\chi = x - c_c t$  and  $\tau = t$ . In this moving frame, the Serre equations are given by

$$h_\tau - c_c h_\chi + (hu)_\chi = 0, \quad (5.5a)$$

$$u_\tau - c_c u_\chi + uu_\chi + h_\chi - \frac{1}{3h} \left( h^3 (u_{\chi\tau} - c_c u_{\chi\chi} + uu_{\chi\chi} - (u_\chi)^2) \right)_\chi = 0, \quad (5.5b)$$

and the solution given in equation (1.5) simplifies to the following time-independent solution

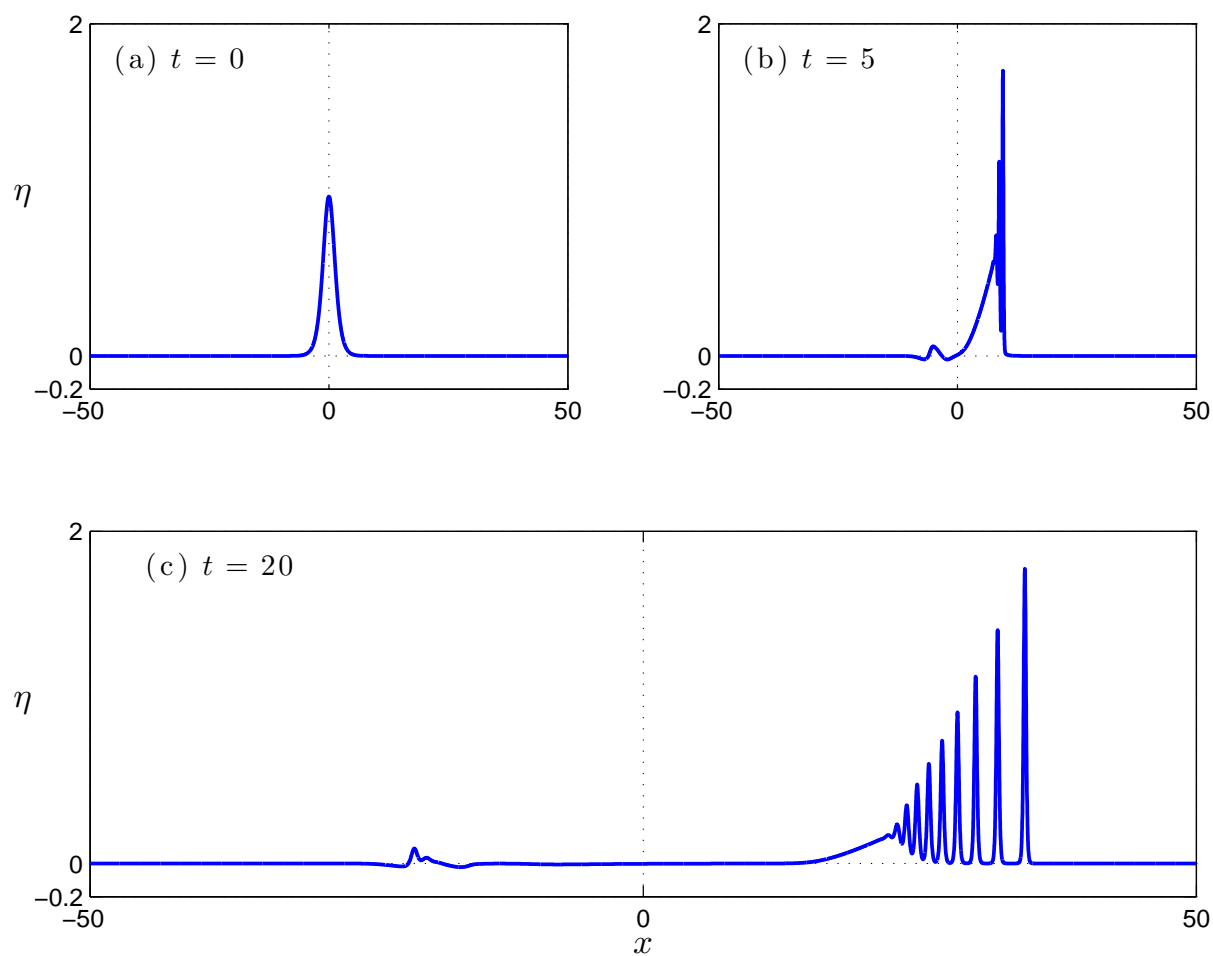
$$h = h_0(\chi) = a_0 + a_1 \operatorname{dn}^2(K_c \chi, k), \quad (5.6a)$$

$$u = u_0(\chi) = c_c \left( 1 - \frac{h_0}{h(\chi)} \right). \quad (5.6b)$$

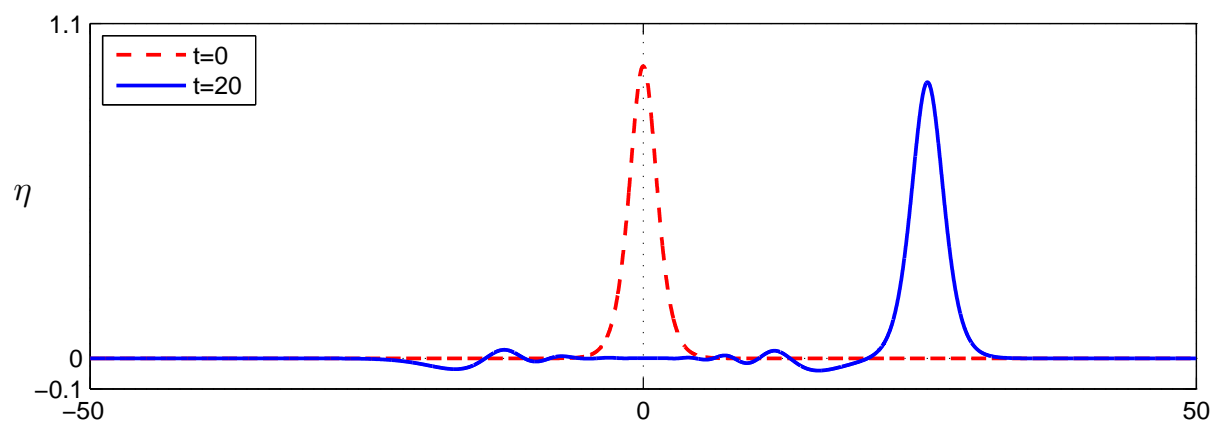
We consider perturbed solutions of the form

$$h_{\text{pert}}(\chi, \tau) = h_0(\chi) + \mu h_1(\chi, \tau) + \mathcal{O}(\mu^2), \quad (5.7a)$$

$$u_{\text{pert}}(\chi, \tau) = u_0(\chi) + \mu u_1(\chi, \tau) + \mathcal{O}(\mu^2), \quad (5.7b)$$

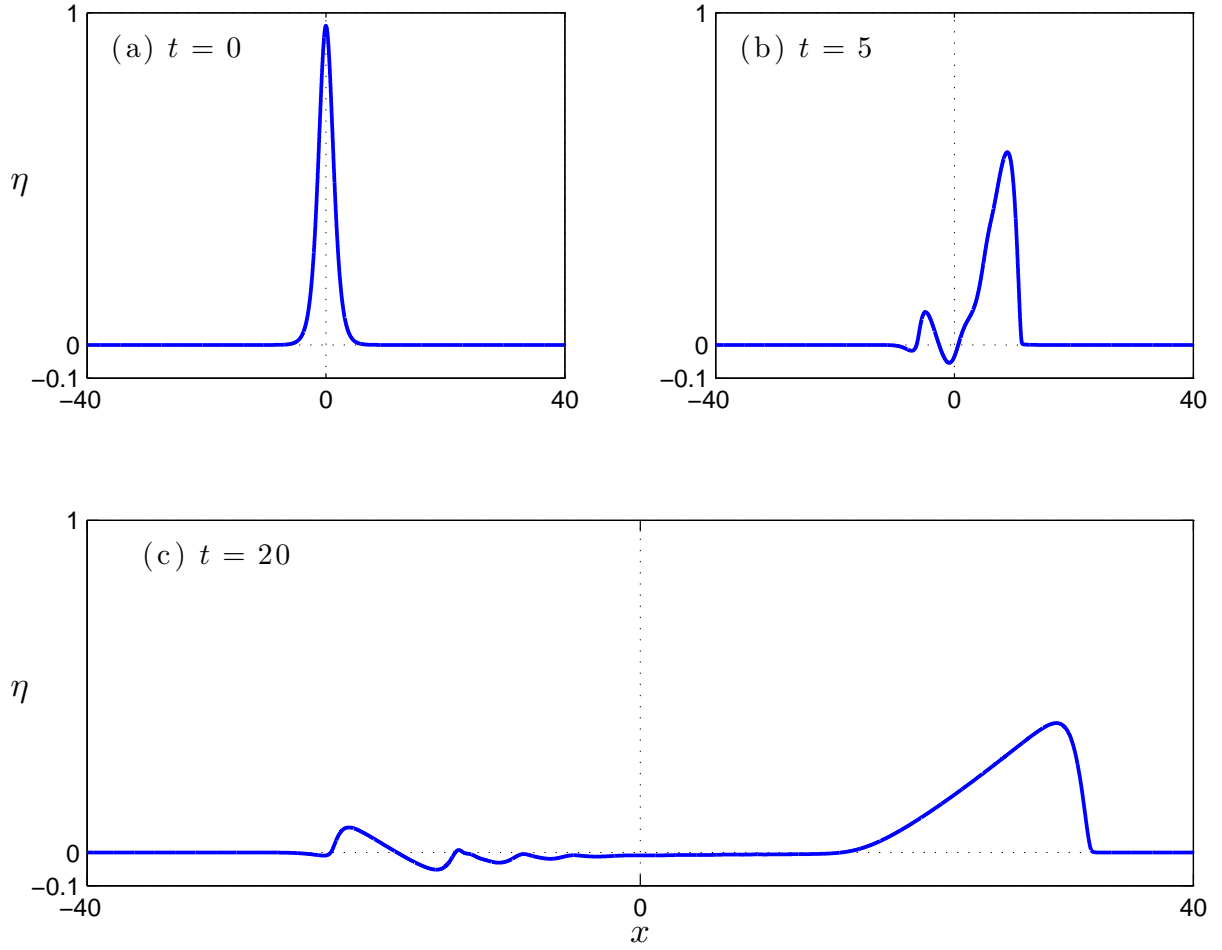


**Figure 16.** The evolution of a solitary wave of a perturbed system ( $\alpha = \beta = \gamma = 0.01$ ).



**Figure 17.** The evolution of a solitary wave of a perturbed system ( $\alpha = 1$ ,  $\beta = \gamma = 0.001$ ).





**Figure 18.** The evolution of a solitary wave of a perturbed system and the generation of a regularized shock ( $\alpha = 0.001$ ,  $\beta = \gamma = 1$ ).

where  $h_1$  and  $u_1$  are real-valued functions and  $\mu$  is a small real parameter. Substituting (5.7) into (5.5) and linearizing leads to a pair of coupled, linear partial differential equations that are constant coefficient in  $\tau$ . Without loss of generality, assume

$$h_1(\chi, \tau) = H(\chi)e^{\Omega\tau} + \text{c.c.}, \quad (5.8a)$$

$$u_1(\chi, \tau) = U(\chi)e^{\Omega\tau} + \text{c.c.}, \quad (5.8b)$$

where  $H(\chi)$  and  $U(\chi)$  are complex-valued functions,  $\Omega$  is a complex constant, and c.c. denotes complex conjugate. If  $\Omega$  has a positive real part, *i.e.* if  $\text{Re}(\Omega) > 0$ , then the perturbations  $h_1$  and  $u_1$  grow exponentially in  $\tau$  and the solution is said to be unstable.

Substituting (5.8) into the linearized PDEs gives

$$\mathcal{L} \begin{pmatrix} H \\ U \end{pmatrix} = \Omega \mathcal{M} \begin{pmatrix} H \\ U \end{pmatrix}, \quad (5.9)$$

where  $\mathcal{L}$  and  $\mathcal{M}$  are the linear differential operators defined by

$$\mathcal{L} = \begin{pmatrix} -u'_0 + (c_c - u_0)\partial_\chi & -\eta'_0 - \eta_0\partial_\chi \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}, \quad (5.10a)$$

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \eta_0\eta'_0\partial_\chi - \frac{1}{3}\eta_0^2\partial_{\chi\chi} \end{pmatrix}, \quad (5.10b)$$

where prime represents derivative with respect to  $\chi$  and

$$\begin{aligned} \mathcal{L}_{21} = & -\eta'_0(u'_0)^2 - c_c\eta'_0u''_0 - \frac{2}{3}c_c\eta_0u'''_0 + \eta'_0u_0u''_0 - \frac{2}{3}\eta_0u'_0u''_0 + \\ & \frac{2}{3}\eta_0u_0u'''_0 + (\eta_0u_0u''_0 - g - \eta_0(u'_0)^2 - c_c\eta_0u''_0)\partial_\chi, \end{aligned} \quad (5.11a)$$

$$\begin{aligned} \mathcal{L}_{22} = & -u'_0 + \eta_0\eta'_0u''_0 + \frac{1}{3}\eta_0^2u'''_0 + (c_c - u_0 - 2\eta_0\eta'_0u'_0 - \frac{1}{3}\eta_0^2u''_0)\partial_\chi + \\ & (\eta_0\eta'_0u_0 - c_c\eta_0\eta'_0 - \frac{1}{3}\eta_0^2u'_0)\partial_{\chi\chi} + (\frac{1}{3}\eta_0^2u_0 - \frac{1}{3}c_c\eta_0^2)\partial_{\chi\chi\chi}. \end{aligned} \quad (5.11b)$$

The Fourier–Floquet–Hill method described in DECONINCK & KUTZ [21] is then used to solve the differential eigenvalue problem given in (5.9). This method establishes that all bounded solutions of (5.9) have the form

$$\begin{pmatrix} H \\ U \end{pmatrix} = e^{i\rho\chi} \begin{pmatrix} H^P \\ U^P \end{pmatrix}, \quad (5.12)$$

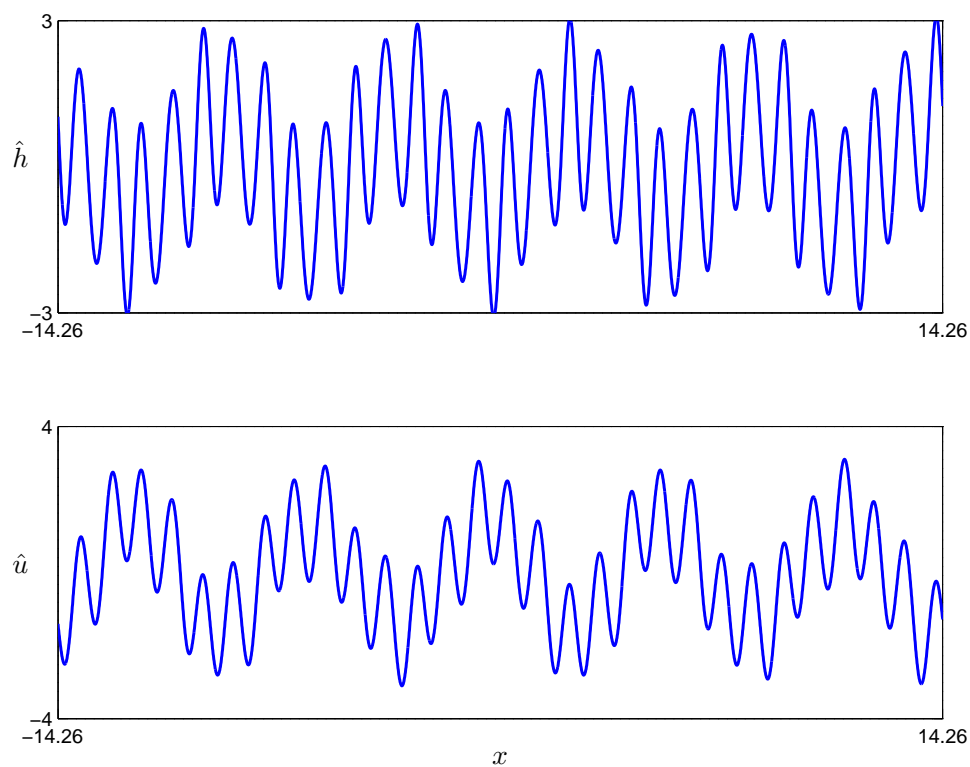
where  $H^P$  and  $U^P$  are periodic in  $\chi$  with period  $2K/K_c$  and  $\rho \in [-\pi K_c/(4K), \pi K_c/(4K)]$ .

Using this method, CARTER & CIENFUEGOS [13] established that solutions of the form given in (5.6) with sufficiently small amplitude and steepness are spectrally stable and solutions with sufficiently large amplitude or steepness are spectrally unstable. For example, the solution with  $a_0 = 0.3$ ,  $a_1 = 0.2$  and  $k = 0.75$  is unstable with respect to the perturbation shown in Figure 19. The period of this perturbation is twelve times the period of the exact solution ( $\rho = 1/12$ ). The theory establishes that the magnitude of this perturbation will grow like  $e^{0.00569t}$ . We corroborated this result by using the following perturbation-seeded solution as the initial condition in our Serre solver

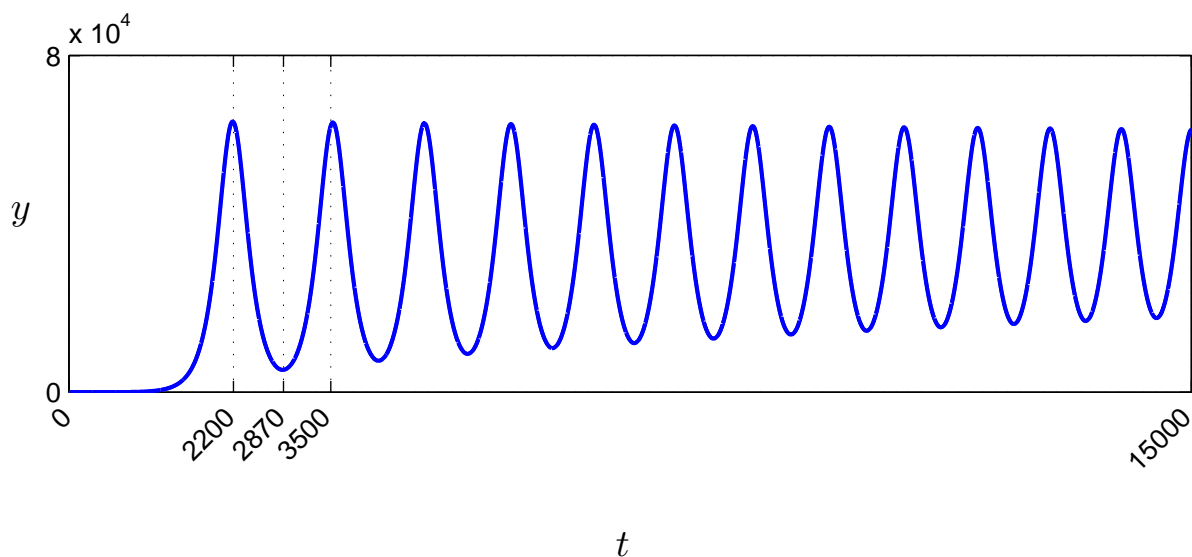
$$h_{pert}(x, 0) = h(x, 0) + 10^{-7}h_1(x), \quad (5.13a)$$

$$u_{pert}(x, 0) = u(x, 0) + 10^{-7}u_1(x). \quad (5.13b)$$

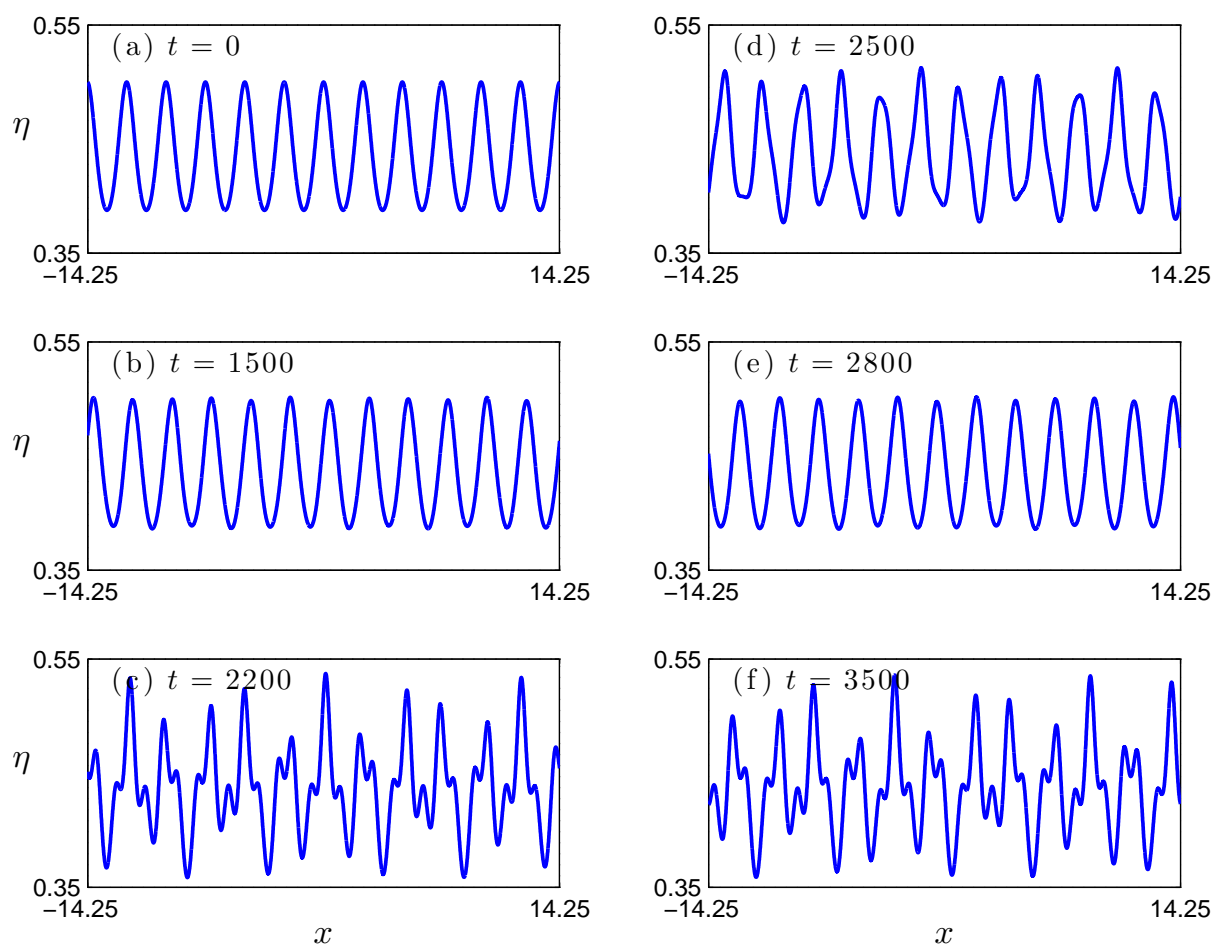
Here  $(h(x, 0), u(x, 0))$  is the solution given in equation (5.6) with  $a_0 = 0.3$ ,  $a_1 = 0.2$ ,  $k = 0.75$  and  $t = 0$  and  $(h_1(x), u_1(x))$  is the perturbation shown in Figure 19. Figure 21 contains a plot of the magnitude of the first Fourier mode of the perturbation versus time. This mode initially (up to  $t = 1500$ ) grows exponentially with a rate of 0.00577, very close to the rate predicted by the linear theory. However after more time, the solution returns to a state close to the initial one. The first portion of this recurrence phenomenon is depicted in Figure 21. Figure 22 contains plots demonstrating that the solution at  $t = 2870$  has nearly returned to its initial state. Similar behavior has been observed to other shallow water models by Ruban [54] and it is referred to as the Fermi-Pasta-Ulam (FPU) recurrence.



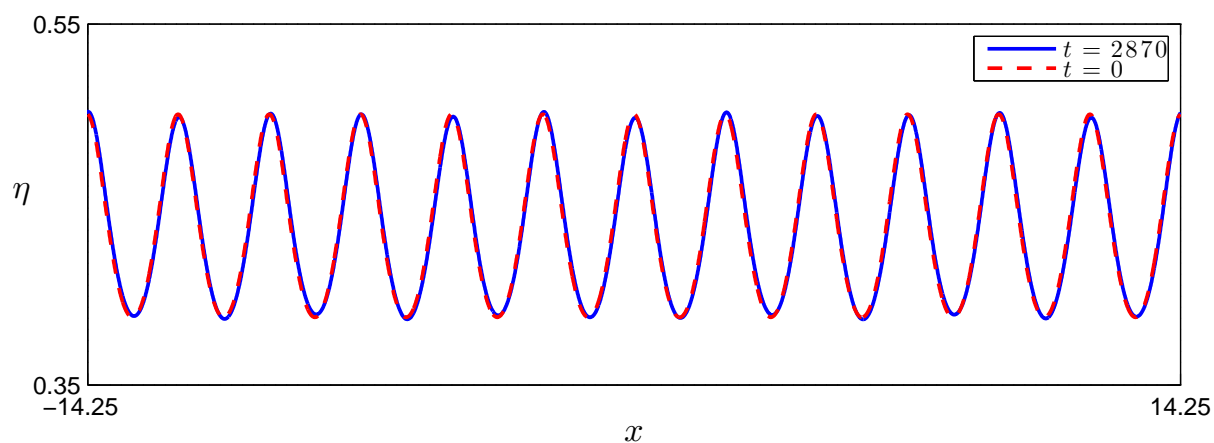
**Figure 19.** Unstable perturbations corresponding to the solution given in (1.5) with  $a_0 = 0.3$ ,  $a_1 = 0.2$  and  $k = 0.75$ .



**Figure 20.** The magnitude of the first Fourier mode of the perturbation versus time.



**Figure 21.** *The periodic instability of the perturbed cnoidal wave.*



**Figure 22.** *Recurrence of the solution to the initial state after one time-period.*

## 6. Dispersive shock waves

A simple DSW traveling to the right can be generated using the Riemann initial data, cf. [28]

$$h(x,0) = \begin{cases} h^-, & \text{for } x < 0 \\ h^+, & \text{for } x > 0 \end{cases}, \quad u(x,0) = \begin{cases} u^-, & \text{for } x < 0 \\ u^+, & \text{for } x > 0 \end{cases}, \quad (6.1)$$

with the compatibility condition (Riemann invariant)

$$\frac{u^-}{2} - \sqrt{h^-} = \frac{u^+}{2} - \sqrt{h^+}. \quad (6.2)$$

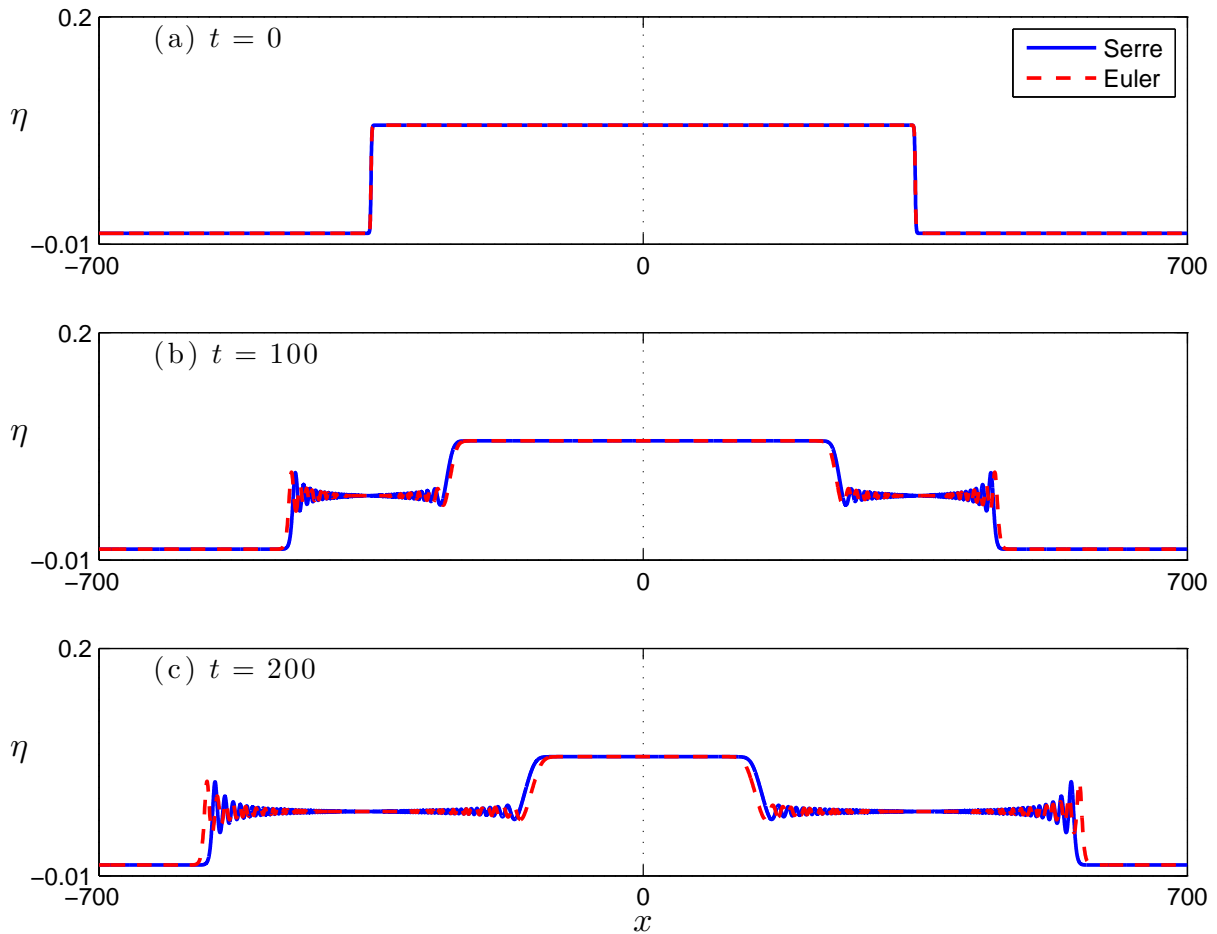
DSWs can also be generated during the dam-break problem simulation. In this case, the initial data for  $h(x,0)$  are the same as in (6.1). However, there is no flow at  $t = 0$ , i.e.,  $u(x,0) = 0$ . As shown in [28], this generates two counter-propagating DSWs, one on each side of the “dam”, and two rarefaction waves that travel toward the center. It appears that similar initial conditions trigger similar solutions. We consider the initial condition for  $h$  to be a smooth step function that decays to zero as  $|x| \rightarrow \infty$ . Specifically, we choose

$$\eta(x,0) = \frac{1}{2}\eta_0 \left[ 1 + \tanh\left(\frac{x_0 - |x|}{2}\right) \right], \quad (6.3)$$

where  $\eta_0 = 0.1$ ,  $x_0 = 350$ , and  $u(x,0) = 0$ . A plot of this initial condition is included in Fig. 23. Both the Euler and Serre equations generate two counter propagating DSWs and two rarefaction waves. Figure 23 demonstrates that the amplitude of the leading wave for both solutions is almost the same. For example, the amplitude of the Euler leading wave at  $t = 200$  is  $A = 0.06372$  while the amplitude of the Serre leading wave at  $t = 200$  is  $A = 0.06356$ . Although the leading waves have almost the same amplitudes, the phase speeds are slightly different. The difference in phase speeds can easily be observed in Figure 23.

After verifying that the Serre system has dispersive shock waves that are compatible with the full Euler equations, we examine the interactions of simple DSWs starting with the head-on collision. For the head-on collision we again consider two initial waveforms similar to (6.1) but translated as it is shown in Figure 24(a). These step functions generate two counter-propagating waves that begin to interact at approximately  $t = 27$ . The collision is inelastic. After the collision there are two DSWs propagating in different directions on the trailing edge of the DSWs.

We close this analysis with a description of the overtaking collision of DSWs. For this collision we consider two step-like initial conditions that are translated and are attached at  $x = -300$  as is shown in Figure 25. The first step has amplitude of 0.1 and it is translated to the left of a shorter step with amplitude 0.05. The two steps generate two DSWs that are propagating to the right. Because shorter DSWs propagate with smaller phase speeds than taller DSWs, the taller DSW approaches the shorter one and eventually they interact. The interaction is so strong that the symmetry of the leading wave of both DSWs is destroyed. The two waves appear to merge and propagate as one single-phase DSW. Similar behavior has been observed in NLS-type and KdV-type equations [36, 1]. We note that larger DSWs



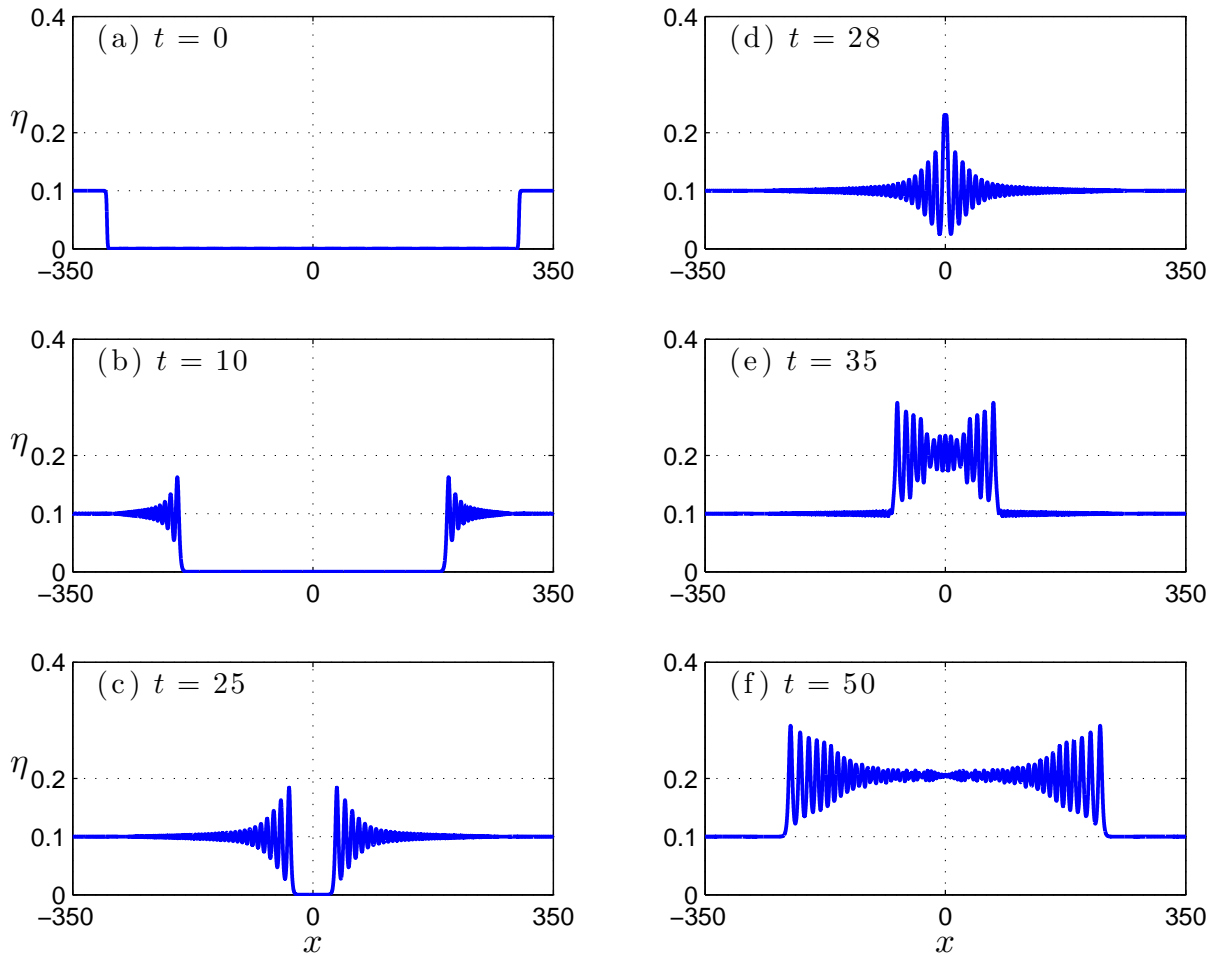
**Figure 23.** *The dam break problem.*

can lead to blow-up. These phenomena are in general unphysical and are not presented in this paper. Finally, we mention that the solutions shown in Figures 24 and 25 are magnifications of the actual solutions. The rest of the solution, not shown in these figures, consists of dispersive rarefaction waves that we do not study in this paper. For more information see [49].

## 7. Conclusions

In this paper we studied several important nonlinear and dispersive phenomena related to the Serre equations using a stable and highly accurate FEM scheme. Among other problems, we studied the stability of solitary and cnoidal waves, the persistence of the solitary waves, the head-on collision of solitary waves and DSWs, the consistency with the Euler equations, and the physical relevance of the Serre equations.

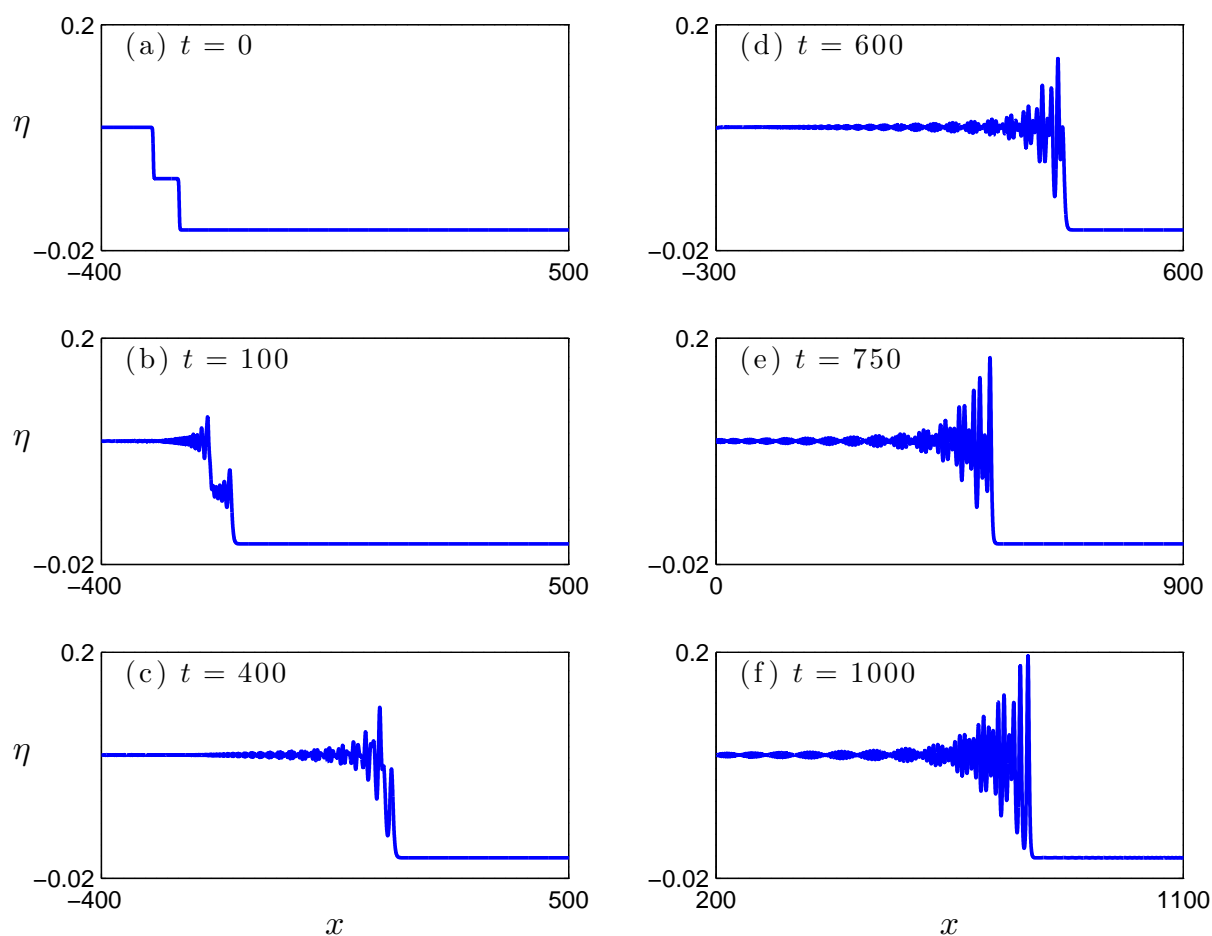
The numerical experiments presented in this paper demonstrate that solitary waves are stable with respect to large classes of perturbations and that some cnoidal waves are



**Figure 24.** *Head-on collision of two simple DSWs.*

unstable with respect to very specific perturbations. A new recurrence phenomenon also is presented. The solitary wave solutions appear to persist even if some high-order terms of the system are almost eliminated, while in one case the solitary wave was transformed into a new form of regularized shock wave. The last observation implies a new method for the description of the wave breaking.

Finally, we verified that the solutions of the Serre equations are close to the solutions of the full Euler equations in many cases. We verified the consistency of the Serre equations even when large-amplitude solitary waves were used, reaching the limits of the full Euler equations. We also showed that the generation of the dispersive tails in various interactions is a highly nonlinear process that cannot be captured accurately by weakly nonlinear models.



**Figure 25.** *Overtaking interaction of two simple DSWs.*

## Acknowledgments

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## A. Traveling wave solutions to the full Euler equations

We consider surface gravity waves in an ideal incompressible and irrotational fluid of finite depth. A two-dimensional Cartesian coordinate system  $xOy$  is chosen such that the ordinate axis  $Oy$  is directed vertically upwards and the horizontal axis  $Ox$  coincides with the still water level. The fluid layer is bounded above by the free surface  $y = \eta(x, t)$  and below by the flat bottom  $y = -h$ . The governing equations for the flow are [57, 60, 37]

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0, \quad -h < y < \eta(x, t), \quad (\text{A.1})$$

$$\eta_t + \phi_x \eta_x = \phi_y, \quad y = \eta(x, t), \quad (\text{A.2})$$

$$\phi_t + \frac{1}{2} |\nabla\phi|^2 + gy = 0, \quad y = \eta(x, t), \quad (\text{A.3})$$

$$\phi_y = 0, \quad y = -h, \quad (\text{A.4})$$

where  $\phi(x, y, t)$  is the velocity potential and  $g$  is the acceleration due to gravity. The Laplace equation (A.1) expresses the combination of fluid incompressibility and flow irrotationality. This equation is completed by the boundary conditions (A.2)-(A.4). There is one kinematic (A.2) and one dynamic isobaricity condition (A.3) on the free surface. On the solid bottom we require that the impermeability condition (A.4) is satisfied.

### A.1. Evolution equations

One of the main difficulties of the water wave problem (A.1) – (A.4) is that the fluid domain is unknown a priori and has to be determined amongst other unknowns. Consequently, in order to simplify the solution procedure, it would be advantageous to transform the dynamic physical domain into a fixed computational one. For numerical purposes the idea to use time-dependent conformal maps was formalized and implemented for the first time by A. DYACHENKO *et al.* (1996) [27]. Later this technique was exploited by other authors in the infinite and finite depth cases [17, 47, 48]. Our exposition below differs slightly from that of A. DYACHENKO *et al.* (1996) in that we manipulate the Euler equations directly and not their Hamiltonian variational formulation.

The main idea behind handling the unknown free surface computationally is to reformulate the system using a time-dependent conformal map from the physical domain on a uniform strip<sup>1</sup>. This transformation is schematically depicted in Figure 26. The new horizontal and vertical coordinates are denoted by  $\xi$  and  $\zeta$  respectively. The transformation  $x = x(\xi, \zeta, t)$ ,  $y = y(\xi, \zeta, t)$  can be found by solving the harmonic boundary value problem

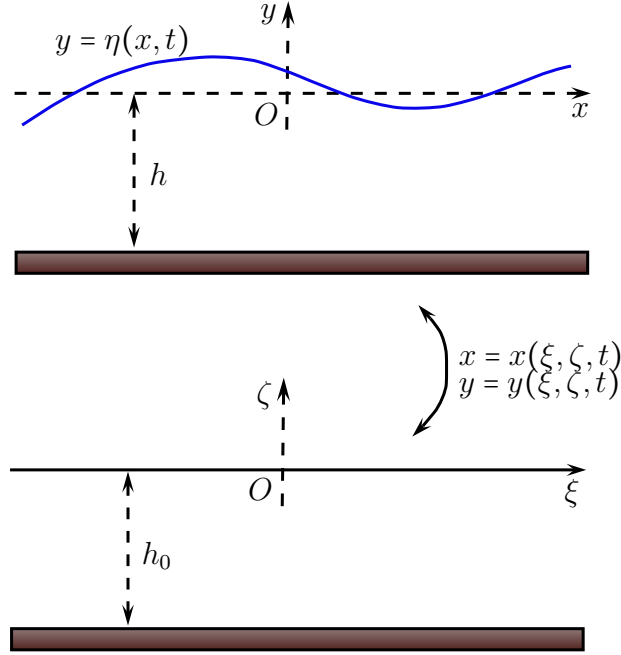
$$y_{\xi\xi} + y_{\zeta\zeta} = 0, \quad -h_0 \leq \zeta \leq 0, \quad (\text{A.5})$$

$$y = \gamma(\xi, t), \quad \zeta = 0, \quad (\text{A.6})$$

$$y = -h, \quad \zeta = -h_0, \quad (\text{A.7})$$

where  $\gamma(\xi, t) = \eta(\chi(\xi, t), t)$  and  $\chi(\xi, t) = x(\xi, 0, t)$ . We will explain below how the constant  $h_0$  is chosen. The harmonic conjugate variable  $x(\xi, \zeta, t)$  can be computed through the

<sup>1</sup>In the infinite depth case the conformal map is done on the lower half-plane [48].



**Figure 26.** Conformal map of the physical domain into a uniform strip.

Cauchy-Riemann relations for the complex function  $z(\xi, \zeta, t) = x(\xi, \zeta, t) + iy(\xi, \zeta, t)$

$$x_\xi = y_\zeta, \quad x_\zeta = -y_\xi. \quad (\text{A.8})$$

Using harmonic analysis one can solve the problem (A.5) – (A.7) to obtain

$$y(\xi, \zeta, t) = \frac{h}{h_0} \zeta + \mathcal{S}(\zeta)[\gamma(\xi, t)],$$

where the nonlocal operator  $\mathcal{S}(\zeta)$  is defined by its symbol in Fourier space

$$\hat{\mathcal{S}}(\zeta) = \frac{\sinh(|k|(\zeta + h_0))}{\sinh(|k|h_0)}.$$

The variable  $k$  is the Fourier transform parameter classically defined as

$$\hat{f}(k) = \mathcal{F}[f(\xi)] = \int_{\mathbb{R}} f(\xi) e^{-ik\xi} d\xi, \quad f(\xi) = \mathcal{F}^{-1}[\hat{f}(k)] = \int_{\mathbb{R}} \hat{f}(k) e^{ik\xi} dk.$$

By using the first Cauchy-Riemann relation (A.8) we can determine the trace of the conjugate variable at the free surface

$$\chi_\xi = 1 - \mathcal{H}[\gamma_\xi].$$

After one integration we easily recover the function  $\chi(\xi, t)$  from its derivative

$$\chi(\xi, t) = \xi - \mathcal{H}[\gamma] + x_0(t),$$

where  $x_0(t)$  is the origin of the  $x$ -coordinate in the physical domain. Without loss of generality we assume that the origins of both coordinate systems coincide, *i.e.*  $x_0(t) \equiv 0$ .

The operator  $\mathcal{H}$  has the following expressions in Fourier and physical space

$$\hat{\mathcal{H}} = i \coth(kh_0), \quad \mathcal{H}[f](x) = \frac{1}{2h_0} \int_{\mathbb{R}} f(y) \coth\left(\frac{\pi}{2h_0}(y-x)\right) dy.$$

The last improper integral has to be understood in the sense of the Cauchy principal value.

It can be shown by direct computations that in the transformed plane the velocity potential  $\phi(\xi, \zeta, t) = \phi(x(\xi, \zeta, t), y(\xi, \zeta, t), t)$  and its harmonic conjugate (the stream function)  $\Psi(\xi, \zeta, t)$  also satisfy the respective Laplace equations

$$\begin{aligned} \Psi_{\xi\xi} + \Psi_{\zeta\zeta} &= 0, & -h_0 \leq \zeta \leq 0, \\ \Psi &= \psi(\xi, t), & \zeta = 0, \\ \Psi &= -f(t), & \zeta = -h_0, \end{aligned}$$

where  $\psi(\xi, t) = \Psi(\chi(\xi, t), 0, t)$  is the stream function at the free surface. The bottom is always a streamline<sup>2</sup>. Since the only difference between the stream function at the free surface and at the bottom is physically meaningful, we choose  $f(t) \equiv 0$  without loss of generality and absorb the time dependence into  $\psi(\xi, t)$ .

By applying the same harmonic analysis technique that we used for the independent variables  $(x, y)$ , we can obtain the following relations between the traces of conjugate functions  $\psi(\xi, t)$  and  $\varphi(\xi, t) := \phi(\chi(\xi, t), 0, t)$  at the free surface

$$\psi_\xi = \mathcal{T}[\varphi_\xi], \quad \psi = \mathcal{T}[\varphi] + \psi_0(t),$$

where the operator  $\mathcal{T}$  is defined as

$$\hat{\mathcal{T}} = i \tanh(kh_0), \quad \mathcal{T}[f](x) = \frac{1}{2h_0} \int_{\mathbb{R}} f(y) \operatorname{cosech}\left(\frac{\pi}{2h_0}(y-x)\right) dy.$$

Moreover, one can show that the following relations hold for the operators  $\mathcal{H}$  and  $\mathcal{T}$

$$\mathcal{H}[\mathcal{T}[f]] = -f, \quad \mathcal{T}[\mathcal{H}[f]] = -f.$$

In other words,  $\mathcal{T}$  is the inverse operator to  $-\mathcal{H}$  and vice versa.

**Remark 1.** *If we take the deep-water limit,  $h_0 \rightarrow +\infty$ , the operators  $\mathcal{H}$  and  $\mathcal{T}$  will become the usual Hilbert transform  $\mathcal{H}_0$  that routinely arises in deep water and internal wave models [5, 50, 48]*

$$\hat{\mathcal{H}}_0 = i \operatorname{sign}(k), \quad \mathcal{H}_0[f](x) = \int_{\mathbb{R}} \frac{f(y)}{y-x} dy.$$

The evolution equations for  $\gamma$  and  $\varphi$  are found from the free surface boundary conditions (A.2) and (A.3). By using the chain rule of differentiation, the kinematic condition (A.2) becomes

$$\chi_t \gamma_\xi - \gamma_t \chi_\xi = \psi_\xi. \tag{A.9}$$

<sup>2</sup>Using the Cauchy-Riemann relations one can show that the normal derivative of the velocity potential vanishes along the streamlines. Thus, the bottom impermeability (A.4) is automatically ensured in this way.

The integration of this expression with respect to  $\xi$  leads to the mass conservation property (see also Remark 2)

$$\frac{d}{dt} \int_{\mathbb{R}} \gamma \chi_{\xi} d\xi = 0.$$

Similarly, the dynamic equation (A.3) can be rewritten as

$$\varphi_t + g\gamma + \frac{1}{J} \left[ \frac{1}{2} (\varphi_{\xi}^2 + \psi_{\xi}^2) - (\chi_t \chi_{\xi} + \gamma_t \gamma_{\xi}) \varphi_{\xi} - (\chi_t \gamma_{\xi} - \gamma_t \chi_{\xi}) \psi_{\xi} \right] = 0, \quad (\text{A.10})$$

where the Jacobian  $J$  is given by

$$J = \chi_{\xi}^2 + \gamma_{\xi}^2. \quad (\text{A.11})$$

So far, the relations (A.9) and (A.10) are implicit in the time derivative. However, we can solve them by isolating the derivatives of dependent variables. In fact,  $\chi_t$  and  $\gamma_t$  are not independent. Notice that  $z_t/z_{\xi}$  is an analytic function of  $\xi + i\zeta$ . Therefore, the real and imaginary parts of their boundary values are also related by the  $\mathcal{H}$ -transform

$$\operatorname{Re}\left(\frac{z_t}{z_{\xi}}\right) = -\mathcal{H}\left[\operatorname{Im}\left(\frac{z_t}{z_{\xi}}\right)\right].$$

After some simple computations we obtain

$$\operatorname{Im}\left(\frac{z_t}{z_{\xi}}\right)\Big|_{\zeta=0} = \frac{\gamma_t \chi_{\xi} - \chi_t \gamma_{\xi}}{J} = -\frac{\psi_{\xi}}{J}, \quad \operatorname{Re}\left(\frac{z_t}{z_{\xi}}\right)\Big|_{\zeta=0} = \frac{\gamma_t \gamma_{\xi} + \chi_t \chi_{\xi}}{J} = \mathcal{H}\left[\frac{\psi_{\xi}}{J}\right]. \quad (\text{A.12})$$

The last two relations can be seen as a linear system for  $\chi_t$  and  $\gamma_t$

$$\begin{aligned} \chi_t \gamma_{\xi} - \gamma_t \chi_{\xi} &= \psi_{\xi}, \\ \chi_t \chi_{\xi} + \gamma_t \gamma_{\xi} &= J \mathcal{H}\left[\frac{\psi_{\xi}}{J}\right]. \end{aligned}$$

By solving this system, we have explicit evolution equations for  $\chi$  and  $\gamma$

$$\begin{aligned} \chi_t &= \chi_{\xi} \mathcal{H}\left[\frac{\psi_{\xi}}{J}\right] + \gamma_{\xi} \frac{\psi_{\xi}}{J}, \\ \gamma_t &= \gamma_{\xi} \mathcal{H}\left[\frac{\psi_{\xi}}{J}\right] - \chi_{\xi} \frac{\psi_{\xi}}{J}. \end{aligned}$$

Using relations (A.12) we can rewrite the free surface kinematic and dynamic conditions (A.9), (A.10) as

$$\gamma_t = \gamma_{\xi} \mathcal{H}\left[\frac{\psi_{\xi}}{J}\right] - \chi_{\xi} \frac{\psi_{\xi}}{J}, \quad (\text{A.13})$$

$$\varphi_t = \frac{1}{2} \frac{\psi_{\xi}^2 - \varphi_{\xi}^2}{J} - g\gamma + \varphi_{\xi} \mathcal{H}\left[\frac{\psi_{\xi}}{J}\right]. \quad (\text{A.14})$$

Two evolution equations above have to be completed by two additional relations in order to close the system

$$\psi_{\xi} = \mathcal{T}[\varphi_{\xi}], \quad \chi_{\xi} = 1 - \mathcal{H}[\gamma_{\xi}].$$

We emphasize the fact that equations (A.13), (A.14) are exact evolution equations for free surface gravity waves in water of finite depth. No additional assumptions have been made to derive (A.13), (A.14) from the full Euler equations (A.1) – (A.4).

**Remark 2.** *The free surface Euler equations possess three classical conserved quantities (the mass  $M$ , horizontal momentum  $P$  and the total energy  $E$ ) which can be expressed in terms of the free surface quantities only*

$$M = \int_{\mathbb{R}} \gamma \chi_{\xi} \, d\xi, \quad P = \int_{\mathbb{R}} \gamma \varphi_{\xi} \, d\xi, \quad E = \frac{1}{2} \int_{\mathbb{R}} \psi \phi_{\xi} \, d\xi + \frac{g}{2} \int_{\mathbb{R}} \gamma^2 \chi_{\xi} \, d\xi.$$

*These quantities can be used to assess the discretization errors during numerical computations.*

In order to discretize the free surface Euler system (A.13), (A.14) one can use a Fourier-type pseudospectral method, where all derivatives along with nonlocal pseudo-differential operators are computed spectrally [11, 59]. Nonlinear products are computed in real space and antialiased using the 3/2 rule. The overall implementation is very efficient thanks to the FFT algorithm [30, 31].

**Remark 3.** *The last element we have to specify is the way in which we choose the parameter  $h_0$  which defines the depth of the transformed domain. If we assume that the length of the periodic  $\xi$ -domain is equal to  $\ell$  then  $h_0$  is determined by the following expression*

$$h_0 = h + \frac{1}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \gamma \, d\xi.$$

## A.2. Traveling wave solutions

We restrict our attention to a special class of solutions that move with a constant speed  $c$  and which are generally referred to as traveling waves. In this particular case, all functions depend only on the variable  $x - ct$

$$\eta(x, t) = \eta(x - ct), \quad \phi(x, y, t) = \phi(x - ct, y), \quad c = \text{const.}$$

For progressive solutions the dynamic boundary condition (A.14) takes a simpler form

$$g\gamma = \frac{c^2}{2} \left(1 - \frac{1}{J}\right), \tag{A.15}$$

where the Jacobian  $J$  is defined in equation (A.11). The last equation has to be completed by an additional relation  $\chi_{\xi} = 1 - \mathcal{H}[\gamma_{\xi}]$  to close the integro-differential system for finding the free surface excursion  $\gamma(\xi, t)$ . Equation (A.15) will be solved iteratively to find traveling wave solutions using the Petviashvili scheme [53, 39, 61]. To apply this scheme, we have to separate the linear and nonlinear terms. By multiplying the both sides of equation (A.15) by  $J$  and after some simple algebraic computations, one can derive the following equivalent form of equation (A.15)

$$g\gamma + c^2 \mathcal{H}[\gamma_{\xi}] = 2g\gamma \mathcal{H}[\gamma_{\xi}] + \left(\frac{c^2}{2} - g\gamma\right) \left(\gamma_{\xi}^2 + (\mathcal{H}[\gamma_{\xi}])^2\right).$$

The last equation can be rewritten in the operator form for the sake of compactness

$$\mathcal{L} \cdot \gamma = \mathcal{N}(\gamma), \quad \mathcal{L} := g + c^2 \mathcal{H}[\partial_{\xi}], \tag{A.16}$$

where  $\mathcal{N}(\gamma)$  denotes the right-hand side of equation (A.16). The iteration scheme takes the following form

$$\gamma_{n+1} = \mathcal{S}^\beta \mathcal{L}^{-1} \cdot \mathcal{N}(\gamma_n), \quad \mathcal{S} = \frac{\langle \gamma_n, \mathcal{L} \cdot \gamma_n \rangle}{\langle \gamma_n, \mathcal{N}(\gamma_n) \rangle},$$

where  $\mathcal{S}$  is the so-called stabilizing factor and the exponent  $\beta$  is usually defined as a function of the degree of nonlinearity  $p$  ( $p = 2$  for the Euler equations). The rule of thumb prescribes the following formula  $\beta = \frac{p}{p-1}$  [3]. The scalar product is defined in the  $L_2$  space. The inverse operator  $\mathcal{L}^{-1}$  can be efficiently computed in the Fourier space. To initialize the iterative process, one can use the analytical solution to the Serre equations, see for example the explicit formulas (1.4)). We point out that this method can be very efficiently implemented using the Fast Fourier Transform [30, 31].

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