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A HIGHER-DIMENSIONAL VERSION OF THE VANDERMONDE DETERMINANT IDENTITY

ITAÏ BEN YAACOV

ABSTRACT. We generalise the (projective) Vandermonde determinant identity to projective dimension higher than one.

The Vandermonde determinant identity asserts that in any commutative ring A ,

$$(1) \quad \det \begin{pmatrix} 1 & x_0 & \dots & x_0^d \\ 1 & x_1 & \dots & x_1^d \\ \vdots & \vdots & & \vdots \\ 1 & x_d & \dots & x_d^d \end{pmatrix} = \prod_{i < j \leq d} (x_j - x_i).$$

We may state it equivalently by replacing each x_i with a formal unknown X_i and work in $\mathbf{Z}[X]$. We shall say that this instance of the identity is in degree d , and since each row depends on a single unknown, it is in (affine) dimension one. Homogenising we get the projective dimension one version, namely

$$(2) \quad \det \begin{pmatrix} X_0^d & X_0^{d-1}Y_0 & \dots & Y_0^d \\ X_1^d & X_1^{d-1}Y_1 & \dots & Y_1^d \\ \vdots & \vdots & & \vdots \\ X_d^d & X_d^{d-1}Y_d & \dots & Y_d^d \end{pmatrix} = \prod_{i < j \leq d} (X_i Y_j - X_j Y_i) = \prod_{i < j \leq d} \det \begin{pmatrix} X_i & Y_i \\ X_j & Y_j \end{pmatrix}.$$

What about a version of (2) in higher projective dimension? In dimension n one may seek an identity of the form

$$(3) \quad \det(A) = \prod_{i_0 < \dots < i_n} \det \begin{pmatrix} X_{i_0,0} & \dots & X_{i_0,n} \\ \vdots & & \vdots \\ X_{i_n,0} & \dots & X_{i_n,n} \end{pmatrix},$$

where the columns of A should correspond to monomials of degree d in $n + 1$ variables.

Notation 1. Let R be a commutative ring, $n, m, d \in \mathbf{N}$. Order monomials of total degree d in $n + 1$ variables lexicographically, and define $\nu^d: R^{n+1} \rightarrow R^{\binom{n+d}{n}}$ as the corresponding Veronese embedding, that is, $\nu_i(x)$ is the i th monomial applied to x . We extend ν^d to a map $M_{m \times (n+1)}(R) \rightarrow M_{m \times \binom{n+d}{n}}(R)$ by applying ν^d to each row. Notice that n is determined by the argument so we shall use the same notation for different values of n . When d is clear from the context we just write ν .

Thus one may expect A in (3) to be νX where X is a matrix of size $\binom{n+d}{n} \times (n + 1)$ (and the right hand side is the product of minors of X of order $n + 1$). This, however, seems to be irremediably false. The obstacle is that in the dimension one case, the matrix X is invariant under an operation which therefore passes unnoticed, although it plays a crucial role for $n > 1$.

Notation 2. Let R be a commutative ring, $n, m, d \in \mathbf{N}$.

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- (i) We define $\mu: M_{m \times (n+1)}(R) \rightarrow M_{\binom{m}{n} \times (n+1)}(R)$ by sending a matrix X to the matrix of minors of X of order n . Minors are ordered by the lexicographic ordering on the sequences rows/columns which are taken.
- (ii) We define $\mu': M_{n \times (n+1)}(R) \rightarrow R$ by sending a matrix X to the product of minors of X of order $n + 1$.

Again, n is determined by the arguments.

In dimension one we have

$$X = \begin{pmatrix} X_0 & Y_0 \\ \vdots & \vdots \\ X_d & Y_d \end{pmatrix} = \mu X,$$

and (2) asserts that $\det(\nu X) = \det(\nu \mu X) = \mu' X$. In higher dimension we have the following.

Theorem 3. *Let R be a commutative ring, $n, d \in \mathbf{N}$, and let $X \in M_{(n+d) \times (n+1)}(R)$. Then $\nu \mu X$ is a square matrix of order $\binom{n+d}{n}$, and the Vandermonde identity of degree d and dimension n holds:*

$$(4) \quad \det(\nu \mu X) = (\mu' X)^n.$$

Definition 4. Let $P \in \mathbf{C}[X]$ be a polynomials in k unknowns. Say that $a \in \mathbf{C}^k$ is a *root of order n* of P if P as well as all its differentials up to order $n - 1$ vanish at a , but its differential of order n does not. A root of order one is *simple*.

Fact 5. *Let $P, Q \in \mathbf{C}[X]$ be polynomials in k unknowns such that Q is irreducible, and every root $a \in \mathbf{C}^k$ of Q is also a root of P of order at least n . Then Q^n divides P .*

Proof. When $n = 1$, we know by Hilbert's Nullstellensatz that Q divides P^ℓ for some ℓ . Since Q is irreducible and $\mathbf{C}[X]$ is a unique factorisation domain, Q divides P . Our assertion follows by induction, applying the case $n = 1$ to $P, P/Q, P/Q^2$ and so on. ■

Lemma 6. *Let $B \in M_{n+1}(R)$. Then there exist matrices $B' \in M_{n+1}(R)$ and $B'' \in M_{\binom{n+d}{n}}(R)$ such that for all $X \in M_{m \times (n+1)}(R)$ we have $\mu(XB) = (\mu X)B'$ and $\nu(XB) = (\nu X)B''$, and if B is invertible then so are B' and B'' .*

Proof. When B is an elementary matrix (i.e., corresponding to an elementary column operation) this is easy, and the general case follows. ■

Proof of Theorem 3. It will be enough to prove this in the case where $X = (X_{i,j})_{i < n+d, j \leq n}$ are formal unknowns and $R = \mathbf{C}[X]$ is the polynomial ring in these unknowns.

Consider a matrix $A = (a_{i,j}) \in M_{(n+d) \times (n+1)}(\mathbf{C})$ such that $a_{i,0} = 0$ for $i \leq n$:

$$(5) \quad A = X(A) = \begin{pmatrix} 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \end{pmatrix},$$

with the zeros in the first $n + 1$ rows. Next,

$$\mu X(A) = \mu A = \begin{pmatrix} 0 & \dots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & * \\ * & \dots & * & * \\ \vdots & & \vdots & \vdots \\ * & \dots & * & * \end{pmatrix}, \quad \nu \mu X(A) = \nu \mu A = \begin{pmatrix} 0 & \dots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & * \\ * & \dots & * & * \\ \vdots & & \vdots & \vdots \\ * & \dots & * & * \end{pmatrix},$$

again with the zeros in the first $n + 1$ rows (the two matrices have a different number of columns). Since the derivative of a determinant is the sum of determinants with each row derived, the polynomial $\det(\nu\mu X)$, as well as its derivatives up to order $n - 1$, vanish at A .

Consider now an arbitrary A whose first $n + 1$ rows are linearly dependent. Then there is an invertible matrix B such that $A' = AB$ is as in (5). It follows by Lemma 6 that $\nu\mu A' = (\nu\mu A)B'$, where B' is again invertible. It follows that $\det(\nu\mu X) = c \det(\nu\mu XB)$, where $c \in \mathbf{C}^\times$, and has a zero of order n at A . Let $P = \det(X_{i,j})_{i,j \leq n}$ denote the first minor of order $n + 1$. By Fact 5, P^n divides $\det(\nu\mu X)$. By symmetry, $(\mu'X)^n$ divides $\det(\nu\mu X)$ and a quick calculation reveals they have the same degrees, so they only differ by a constant factor c .

We are left with proving that $c = 1$. We do this by induction on (n, d) . When $n = 1$, we have already seen that (4) is a special case of the usual Vandermonde identity. When $d = 1$, (4) is just the fact that for a square matrix A of order $n + 1$, the determinant of the cofactor matrix of A is $\det A^n$.

When $n, d > 1$, let us substitute $(1, 0, \dots, 0)$ for the first row of X . Let Y be X without this row and let Z be X with both first row and column dropped. Let also $\nu' = \nu^{d-1}: R^{n+1} \rightarrow R^{\binom{n+d-1}{n}}$. Then by the $(n, d - 1)$ and $(n - 1, d)$ cases, respectively,

$$\det(\nu\mu Y) = (\mu'Y)^n, \quad \det(\nu\mu Z) = (\mu'Z)^{n-1}.$$

Now,

$$\mu X = \begin{pmatrix} 0 \\ \mu Z \\ \vdots \\ 0 \\ \mu Y \end{pmatrix}, \quad \nu\mu X = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \nu\mu Z & & \nu\mu Y \end{pmatrix},$$

where zeros appear in $\binom{n+d-1}{n-1}$ rows, and in $\nu\mu Y$ in all columns corresponding to monomials in which the last unknown appears. The matrices $\nu\mu Z$ and $\nu\mu Y$ have dimensions $\binom{n+d-1}{n-1} \times \binom{n+d-1}{n-1}$ and $\binom{n+d-1}{n} \times \binom{n+d-1}{n-1}$, respectively, and the right block of $\nu\mu Y$ consists of $\nu'\mu Y$ with the i th row multiplied by $(\mu Y)_{i,n}$. Thus

$$\det(\nu\mu X) = \det(\nu\mu Z) \det(\nu'\mu Y) \prod_i (\mu Y)_{i,n} = (\mu'Y)^n (\mu'Z)^{n-1} (\mu'Z) = (\mu'X)^n.$$

Since the right hand side does not vanish even in this special case, the coefficient is indeed 1 and we are done. ■

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