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Irfan Mahmood

^{a}*University of Angers*, 2*Bd Lavoisier* 49145 *Angers*, *France*.
^{b}*University of the Punjab*, 54590 *Lahore*, *Pakistan*
*mahirfan@yahoo.com**

Abstract: In this paper, I construct the Darboux transformations for the non-commutative Toda solutions at $n = 1$ with the help of linear systems whose compatibility condition yields zero curvature representation of associated systems of non-linear differential equations. I also derive the quasideterminant solutions of the non-commutative Painlevé II equation by taking the Toda solutions at $n = 1$ as a seed solution in its Darboux transformations. Further by iteration, I generalize the Darboux transformations of the seed solutions to the N-th form. At the end I describe the zero curvature representation of quantum Painlevé II equation that involves Planck constant $\hbar$ explicitly and system reduces to the classical Painlevé II when $\hbar \to 0$.

Keywords: Integrable systems; Painlevé II equation; Toda equations; zero curvature condition; Riccati equation; Darboux transformation.

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1. Introduction

The Painlevé equations were discovered by Painlevé and his colleagues when they have classified the nonlinear second-order ordinary differential equations with respect to their solutions [1].The study of Painlevé equations is important from mathematical point of view because of their frequent appearance in the various areas of physical sciences including plasma physics, fiber optics, quantum gravity and field theory, statistical mechanics, general relativity and nonlinear optics. The classical Painlevé equations are regarded as completely integrable equations and obeyed the Painlevé test [2,3,4]. These equations admit some properties such as linear representations, hierarchies, they possess Darboux transformations(DTs) and Hamiltonian structure. These equations also arise as ordinay differential equations (ODEs) reduction of some integrable systems, i.e, the ODE reduction of the KdV equation is Painlevé II (PII) equation[5,6].

The noncommutative(NC) and quantum extension of Painlevé equations is quite interesting in order to explore the properties which they possess with respect to usual Painlevé systems on ordinary spaces. NC spaces are characterized by the noncommutativity of the spatial co-ordinates. For example, if x^{μ} are the space co-ordinates then the noncommutativity is defined by $[x^{\mu}, x^{\nu}]_{\star} = i\theta^{\mu\nu}$ where parameter $\theta^{\mu\nu}$ is anti-symmetric tensor and Lorentz invariant and $[x^{\mu}, x^{\nu}]_{\star}$ is commutator

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under the star product. NC field theories on flat spaces are given by the replacement of ordinary products with the Moyal-products and realized as deformed theories from the commutative ones. Moyal product for ordinary fields $f(x)$ and $g(x)$ is explicitly defined by

$$f(x)\star g(x) = exp(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x'^{\mu}}\frac{\partial}{\partial x''^{\nu}})f(x')g(x'')_{x=x'=x''}$$

$$= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\frac{\partial f}{\partial x'^{\mu}}\frac{\partial g}{\partial x''^{\nu}} + \mathscr{O}(\theta^2).$$

this product obeys associative property $f\star(g\star h) = (f\star g)\star h$, if we apply the commutative limit $\theta^{\mu\nu}\to 0$ then above expression will reduce to ordinary product as $f\star g = f.g$. In our case the NC product is the Moyal-product and we consider the non-commutativity between space variable and its function.

We are familiar with Lax equations as a nice representation of integrable systems. The Lax equation and zero curvature condition both have same form on deformed spaces as they possess on ordinary space. These representations involve two linear operators, these operators may be differential operators or matrices [7-12]. If A and B are the linear operators then Lax equation is given by $A_t = [B,A]$ where $[B,A]$ is commutator under the star product or quantum product, this Lax pair formalism is also helpful to construct the DT, Riccati equation and BT of integrable systems. The compatibility condition of inverse scattering problem $\Psi_x = A(x,t)\Psi$ and $\Psi_t = B(x,t)\Psi$ yields $A_t - B_x = [B,A]$ which is called the zero curvature representation of integrable systems [13-16. Further we will denote the commutator and anti-commutator by $[,]_-$ and $[,]_+$ respectively. Now the Lax equation and zero curvature condition can be expressed as $A_t = [B,A]_-$ and $A_t - B_x = [B,A]_-$.

The Painlevé equations can be represented by the Noumi-Yamada systems [25], these systems are discovered by Noumi and Yamada while studying symmetry of Painlevé equations and these systems also possess the affine Weyl group symmetry of type A_l^1. For example Noumi-Yamada system for Painlevé II equation is given by

$$\begin{cases} u_0' = u_0u_2 + u_2u_0 + \alpha_0 \\ u_1' = -u_1u_2 - u_2u_1 + \alpha_1 \\ u_2' = u_1 - u_0 \end{cases} \tag{1.1}$$

where $u_i' = \frac{du_i}{dz}$ and α_0 , α_1 are constant parameters. Above system 1.1 also also a unique representation of NC and quantum PII equation, for NC derivation of PII equation [19] the dependent functions u_0, u_1, u_2 obey a kind of star product and in case of quantum derivation these functions are subjected to some quantum commutation relations [26] and [27].
In this paper, I construct the Darboux transformations for the solutions of Toda equations at $n = 1$, $u_1 = \phi'\phi^{-1}$ and its negative counterpart $u_{-1} = \psi'\psi^{-1}$, with the help of linear systems whose compatibility condition yields zero curvature representation of their associated systems of non-linear differential equations. I also derive the quasideterminant solutions of the non-commutative Painlevé II equation by taking the Toda solutions at $n = 1$ as a seed solution in its Darboux transformations. Further by iteration I generalize the Darboux transformations of the seed solutions of the NC PII equation to the N-th form. I also describe an equivalent zero-curvature representation of quantum

PII equation that involves Planck constant $\hbar$ explicitly. Further, I construct the quantum PII Riccati form with the help of its linear system by using the method of Konno and Wadati [28].

2. Brief introduction of Non-commutative Painlevé II equation

The following NC analogue of classical Painlevé II equation

$$u_2^{''} = 2u^3 - 2[z,u]_+ + C \tag{2.1}$$

where $[z,u]_+ = zu + uz$ and constant $C = 4(\beta + \frac{1}{2})$ was obtained by eliminating u_0 and u_1 from (1.1), here $u = u_2$ [19]. Further it was shown by V. Retakh and V. Roubtsov that with the following identities

$$\phi^{''}\phi^{-1} = 2z - 2\phi\psi \tag{2.2}$$

$$\psi^{-1}\psi^{''} = 2z - 2\phi\psi \tag{2.3}$$

and

$$\psi\phi^{'} - \psi^{'}\phi = 2\beta \tag{2.4}$$

the solutions $u_n = \theta_n^{\prime}\theta_n^{-1}$ of the Toda equation

$$(\theta_n^{\prime}\theta_n^{-1})^{\prime} = \theta_{n+1}\theta_n^{-1} - \theta_n\theta_{n-1}^{-1} \text{ for } n \geqslant 1 \tag{2.5}$$

satisfies the NC PII$(z, \beta + n - 1)$ equation and the solutions $u_{-m} = \eta_{-m}^{\prime}\eta_{-m}^{-1}$ of the negative counter part of (2.5)

$$(\eta_{-m}^{-1}\eta_{-m}^{\prime})^{\prime} = \eta_{-m}^{-1}\eta_{-m-1} - \eta_{-m} + 1^{-1}\eta_{-m} \text{ for } m \geqslant 1 \tag{2.6}$$

satisfies the NC PII $(z, \beta - n)$ equation, here $\theta_1 = \phi, \theta_0 = \psi^{-1}$ and $\eta_0 = \phi^{-1}, \eta_{-1} = \psi$. In the following section we review the zero curvature representation of NC PII equation (2.1). Further in proposition 1.1, we construct the linear representation of (2.2) and (2.3) that will be helpfull to derive an explicit expression of the Darboux transformations for ϕ and ψ..

2.1. *Zero curvature representation of NC PII equation*

The NC PII equation (2.1) can be derived from inverse scattering problems with zero constant $C = 0$ [20] and in general form with non zero constant $C \neq 0$ [21]. Let consider the following linear system [21]

$$\Psi_\lambda = A(z;\lambda)\Psi \;\; \Psi_z = B(z;\lambda)\Psi \tag{2.7}$$

with Lax matrices

$$\begin{cases} A = (8i\lambda^2 + iu^2 - 2iz)\sigma_3 + u^{'}\sigma_2 + (\frac{1}{4}C\lambda^{-1} - 4\lambda u)\sigma_1 \\ B = -2i\lambda\sigma_3 + u\sigma_1 \end{cases} \tag{2.8}$$

where σ_1, σ_2 and σ_3 are Pauli spin matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

here λ is spectral parameter. The linear system (2.7) is a equivalent representation of NC PII equation the compatibility condition of that system yields NC PII equation (2.1). The following proposition contains the derivation of NC PII Riccati form by using the method of Konno and Wadat [28]
Proposition 1.1.
The linear system (2.7) with eigenvector $\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ and setting $\Gamma = \chi\Phi^{-1}$ can be reduced to the following NC PII Riccati form

$$\Gamma^{'} = -4i\lambda\Gamma + u - \Gamma u\Gamma$$

Proof:
In order to derive the NC PII Riccati we consider following the eigenvector $\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ in linear systems (2.7) and we obtain

$$\begin{cases} \dfrac{d\chi}{d\lambda} = (8i\lambda^2 + iu^2 - 2iz)\chi + (-iu_z + \frac{1}{4}C\lambda^{-1} - 4\lambda u)\Phi \\ \dfrac{d\Phi}{d\lambda} = (iu_z + \frac{1}{4}C\lambda^{-1} - 4\lambda u)\chi + (-8i\lambda^2 - iu^2 + 2iz)\Phi \end{cases} \tag{2.9}$$

and

$$\begin{cases} \chi^{'} = -2i\lambda\chi + u\Phi \\ \Phi^{'} = u\chi + 2i\lambda\Phi \end{cases} \tag{2.10}$$

where $\chi^{'} = \frac{d\chi}{dz}$ and from system (2.10) we can evaluate the following expressions

$$\chi^{'}\Phi^{-1} = -2i\lambda\chi\Phi^{-1} + u \tag{2.11}$$

$$\Phi^{'}\Phi^{-1} = -2i\lambda + u\chi\Phi^{-1}. \tag{2.12}$$

Now let consider the following substitution

$$\Gamma = \chi\Phi^{-1} \tag{2.13}$$

and after taking the derivation of above equation with respect to z we get

$$\Gamma^{'} = \chi^{'}\Phi^{-1} - \chi\Phi^{-1}\Phi^{'}\Phi^{-1}.$$

Finally by making use of Γ and $\Gamma^{'}$ in (2.11) and (2.12) and after simplification we obtain the following expression

$$\Gamma^{'} = -4i\lambda\Gamma + u - \Gamma u\Gamma \tag{2.14}$$

the above equation (2.14) is NC PII Riccati form in Γ where u is the solution of NC PII equation (2.1). As Γ has been expressed in terms of χ and Φ, the components of eigenvector of NC PII system (2.7).
Remark1.1.

We can easily show that the NC PII Riccati form (2.14) can be satisfied by taking the solutions of u and Γ as follow

$$u[1] = -8i\lambda_1(1-e^{-8i\lambda_1 z})^{-1}e^{-4i\lambda_1 z}$$

$$\Gamma = e^{4i\lambda_1 z}$$

in that equation. The following proposition 1.2. involves the zero curvature representation of non-linear differential equations (2.2) and (2.3)

Proposition 1.2.

The compatibility condition of linear systems

$$\begin{cases} \Psi_\lambda = L\Psi \\ \Psi_z = M\Psi \end{cases} \tag{2.15}$$

with the Lax matrices

$$\begin{cases} L = 2\lambda^2 I - q' i\sigma_2 + (-q^2 - 2\phi\psi)\sigma_3 - 4z\Sigma \\ M = q\sigma_1 + \lambda I \end{cases} \tag{2.16}$$

yields equation (2.2) when $q = \phi$ and for $q = \psi$ the compatibility condition gives equation (2.3). Here σ_1, σ_2, σ_3 are Pauli spin matrices, I is identity matrix of order 2 and $\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof:

We can easily evaluate the following values

$$A_z - B_\lambda = \begin{pmatrix} -2(\phi\psi)' - (q^2)' - 1 & -q'' \\ q'' & 2(\phi\psi)' + (q^2)' - 5 \end{pmatrix} \tag{2.17}$$

$$BA - AB = \begin{pmatrix} qq' + q'q & \omega_+ \\ \omega_- & -qq' - q'q \end{pmatrix} \tag{2.18}$$

where $\omega_+ = 2q\phi\psi + 2q^3 - 4qz + 2\phi\psi q$ and $\omega_- = -2q\phi\psi - 2q^3 + 4zq - 2\phi\psi q$. Finally from zero curvature condition we get

$$2(\phi\psi)' + 2(q^2)' + 1 = 0 \tag{2.19}$$

$$q'' = -2q\phi\psi - 2q^3 - 2\phi\psi q + 4qz \tag{2.20}$$

$$2(\phi\psi)' + 2(q^2)' - 5 = 0 \tag{2.21}$$

$$q'' = -2q\phi\psi - 2q^3 - 2\phi\psi q + 4zq \tag{2.22}$$

Now adding (2.19) and (2.21) , we get

$$(\phi\psi)' + (q^2)' - 1 = 0 \tag{2.23}$$

on integrating above emuation with respect to z we get

$$\phi\psi + q^2 - z = D \tag{2.24}$$

where D is constant of integration, set $D = 0$ in above equation then

$$\phi\psi + q^2 - z = 0 \tag{2.25}$$

Now after combining equation (2.20) and equation (2.22) we obtain

$$q^{''} = -2q\phi\psi - 2q^3 - 2\phi\psi q + 2qz + 2zq \tag{2.26}$$

For $q = \phi$ above expression (2.26)

$$\phi^{''} = -2\phi(\phi\psi + \phi^2 - z) - 2\phi\psi\phi + 2z\phi. \tag{2.27}$$

Now after using equation (2.25) for $q = \phi$ in equation (2.27) we obtain following expression

$$\phi^{''} = 2z\phi - 2\phi\psi\phi. \tag{2.28}$$

When $q = \psi$ then the (2.26) can be written as

$$\psi^{''} = -2\psi\phi\psi - 2(\psi^2 + \phi\psi - z)\psi + 2\psi z \tag{2.29}$$

again using equation (2.25) for $q = \psi$ in above (2.30, we get

$$\psi^{''} = 2\psi z - 2\psi\phi\psi \tag{2.30}$$

In next proposition 1.3., we derive the explicit expressions of Darboux transformations for ϕ and ψ with the help of linear systems given in (2.15).

Proposition 1.3.

For the column vector $\Psi = \begin{pmatrix} X \\ Y \end{pmatrix}$ in linear systems (2.15) with the standard transformations on its components X and Y

$$X \to X[1] = \lambda Y - \lambda_1 Y_1(\lambda_1) X_1^{-1}(\lambda_1) X \tag{2.31}$$

$$Y \to Y[1] = \lambda X - \lambda_1 X_1(\lambda_1) Y_1^{-1}(\lambda_1) Y \tag{2.32}$$

we can construct the Darboux transformations for ϕ and ψ as follow

$$\phi[1] = Y_1 X_1^{-1} \phi Y_1 X_1^{-1} \tag{2.33}$$

and

$$\psi[1] = Y_1 X_1^{-1} \psi Y_1 X_1^{-1} \tag{2.34}$$

respectively, where X , Y are arbitrary solutions at λ and $X_1(\lambda_1)$, $Y_1(\lambda_1)$ are the particular solutions at $\lambda = \lambda_1$.

Proof:
Let us write the second expression of (2.15) in the form of

$$\begin{pmatrix} X \\ Y \end{pmatrix}_z = \begin{pmatrix} \lambda & q \\ q & \lambda \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{2.35}$$

Now under the transformations (2.31) and (2.32) above equation (2.35) becomes

$$\begin{pmatrix} X[1] \\ Y[1] \end{pmatrix}_z = \begin{pmatrix} \lambda & q[1] \\ q[1] & \lambda \end{pmatrix} \begin{pmatrix} X[1] \\ Y[1] \end{pmatrix}. \tag{2.36}$$

From (2.35) and (2.36) we obtain the following systems of equations

$$\begin{cases} X' = \lambda X + qY \\ Y' = \lambda Y + qX \end{cases} \tag{2.37}$$

and

$$\begin{cases} X'[1] = \lambda X[1] + q[1]Y[1] \\ Y'[1] = \lambda Y[1] + q[1]X[1] \end{cases} \tag{2.38}$$

Now after substituting the transformed values $X[1]$ and $Y[1]$ from (2.31) and (2.32) in equation (2.38) and then using (2.37) in resulting equation, we get one fold Darboux transformation for q.

$$q[1] = Y_1 X_1^{-1} q Y_1 X_1^{-1}. \tag{2.39}$$

It is obvious that by taking $q = \phi$ in (2.39) we obtain (2.33) and for $q = \psi$ we get transformation (2.34) on ψ. In upcoming section after taking a brief review of quasideterminant, we will substitute ϕ and ψ as seed solutions in NC PII Darboux transformations [21]. Finally we generalize the Darboux transformations (2.33) and (2.34) to the N-th form.

3. A Brief Introduction of Quasideterminants

This section is devoted to a brief review of quasideterminants introduced by Gelfand and Retakh [22]. Quasideterminants are the replacement for the determinant for matrices with noncommutative entries and these determinants plays very important role to construct the multi-soliton solutions of NC integrable systems [23,24], by applying the Darboux transformation. Quasideterminants are not just a noncommutative generalization of usual commuta- tive determinants but rather related to inverse matrices, quasideterminants for the square matrices are defined as if $A = a_{ij}$ be a $n \times n$ matrix and $B = b_{ij}$ be the inverse matrix of A. Here all matrix elements are supposed to belong to a NC ring with an associative product. Quasideterminants of A are defined formally as the inverse of the elements of $B = A^{-1}$

$$|A|_{ij} = b_{ij}^{-1}$$

this expression under the limit $\theta^{\mu\nu} \to 0$, means entries of A are commuting, will reduce to

$$|A|_{ij} = (-1)^{i+j} \frac{detA}{detA^{ij}}$$

where A^{ij} is the matrix obtained from A by eliminating the i-th row and the j-th column. We can write down more explicit form of quasideterminants. In order to see it, let us recall the following formula for a square matrix

$$A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{3.1}$$

where A and D are square matrices, and all inverses are supposed to exist. We note that any matrix can be decomposed as a 2×2 matrix by block decomposition where the diagonal parts are square matrices, and the above formula can be applied to the decomposed 2×2 matrix. So the explicit forms of quasideterminants are given iteratively by the following formula

$$|A|_{ij} = a_{ij} - \Sigma_{p \neq i, q \neq j} a_{iq} |A^{ij}|_{pq}^{-1} a_{pj}$$

the number of quasideterminant of a given matrix will be equal to the numbers of its elements for example a matrix of order 3 has nine quasideterminants. It is sometimes convenient to represent the quasi-determinant as follows

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{in} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}. \tag{3.2}$$

Let us consider examples of matrices with order 2 and 3, for 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

now the quasideterminats of this matrix are given below

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} a_{22}^{-1} a_{21}$$

$$|A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{22} a_{21}^{-1} a_{12}$$

$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{11} a_{12}^{-1} a_{22}$$

$$|A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21} a_{11}^{-1} a_{12}.$$

The number of quasideterminant of a given matrix will be equal to the numbers of its elements for example a matrix of order 3 has nine quasideterminants. Now we consider the example of 3×3

matrix, its first quasidetermints can be evaluated in the following way

$$|A|_{11}=\begin{vmatrix}\boxed{a_{11}} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}-a_{12}Ma_{21}-a_{13}Ma_{21}-a_{12}Ma_{31}-a_{13}Ma_{31}$$

where $M=\begin{vmatrix}\boxed{a_{22}} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}^{-1}$, similarly we can evaluate the other eight quasideterminants of this matrix.

4. Quasideterminant representation of Darboux transformation

4.1. *Darboux transformations of NC PII equation*

In the theory of integrable systems the applications of Darboux transformations (DTs) are quite interesting to construct the multi-soliton solutions of these systems. These transformations consist the particular solutions of corresponding linear systems of the integrable equations and their seed (initial) solutions. For example the NC PII equation (2.1) possesses following N fold DT

$$u[N+1]=\Pi_{k=1}^{N}\Theta_k[k]u\Pi_{j=N}^{1}\Theta_j[j]\quad \text{for } N\geq 0 \tag{4.1}$$

with

$$\Theta_N[N]=\Lambda_N^{\phi}[N]\Lambda_N^{\chi}[N]^{-1}$$

where $u[1]$ is seed solution and $u[N+1]$ are the new solutions of NC PII equation [21]. In above transformations (4.2) $\Lambda_N^{\phi}[N]$ and $\Lambda_N^{\chi}[N]$ are the quasideterminants of the particular solutions of NC PII linear system (2.7). Here the odd order quasideterminant representationS of $\Lambda_{2N+1}^{\phi}[2N+1]$ and $\Lambda_{2N+1}^{\chi}[2N+1]$ are presented below

$$\Lambda_{2N+1}^{\phi}[2N+1]=\begin{vmatrix}\Phi_{2N} & \Phi_{2N-1} & \cdots & \Phi_1 & \Phi\\ \lambda_{2N}\chi_{2N} & \lambda_{2N-1}\chi_{2N-1} & \cdots & \lambda_1\chi_1 & \lambda\chi\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ \lambda_{2N}^{2N-1}\chi_{2N} & \lambda_{2N-1}^{2N-1}\chi_{2N-1} & \cdots & \lambda_1^{2N-1}\chi_1 & \lambda^{2N-1}\chi\\ \lambda_{2N}^{2N}\Phi_{2N} & \lambda_{2N-1}^{2N}\Phi_{2N-1} & \cdots & \lambda_1^{2N}\Phi_1 & \boxed{\lambda^{2N}\Phi}\end{vmatrix}$$

and

$$\Lambda_{2N+1}^{\chi}[2N+1]=\begin{vmatrix}\chi_{2N} & \chi_{2N-1} & \cdots & \chi_1 & \chi\\ \lambda_{2N}\Phi_{2N} & \lambda_{2N-1}\Phi_{2N-1} & \cdots & \lambda_1\Phi_1 & \lambda\Phi\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ \lambda_{2N}^{2N-1}\Phi_{2N} & \lambda_{2N-1}^{2N-1}\Phi_{2N-1} & \cdots & \lambda_1^{2N-1}\Phi_1 & \lambda^{2N-1}\Phi\\ \lambda_{2N}^{2N}\chi_{2N} & \lambda_{2N-1}^{2N}\chi_{2N-1} & \cdots & \lambda_1^{2N}\chi_1 & \boxed{\lambda^{2N}\chi}\end{vmatrix}$$

with

$$\Lambda_1^{\phi}[1]=\Phi_1,\quad \Lambda_1^{\chi}[1]=\chi_1$$

where $\chi_1,\chi_2,\chi_3,...,\chi_N$ and $\Phi_1,\Phi_2,\Phi_3,...,\Phi_N$ are the solutions of system (2.10) at $\lambda_1,\lambda_2,\lambda_3,...,\lambda_N$.

Proposition 1.4.

By taking $u = u_1 = \phi^{'}\phi^{-1}$ as a seed solution in (4.1) then the solitonic solutions $u[N]$ of NC PII equation can be expressed in terms of N-fold quasideterminant Darboux transformation of ϕas follow

$$u[N+1] = \Pi_{k=1}^{N}\Theta_k[k]\phi^{'}[N]\phi^{-1}[N]\Pi_{j=N}^{1}\Theta_j[j] \quad \text{for } N \geq 1 \tag{4.2}$$

and $\phi[N]$ is given by

$$\phi[N] = \Pi_{k=0}^{N-1}\Theta_{N-k}[N-k]\phi\Pi_{j=N-1}^{0}\Theta_{N-j}[N-j]$$

where $\Theta_N[N] = \Omega_N^Y[N]\Omega_N^X[N]^{-1}$ and $\Omega_N^Y[N]$, $\Omega_N^X[N]$ represent quasideterminants of order N.

Proof:

The one fold NC PII Darboux transformation with seed solution $u_1 = \phi^{'}\phi^{-1}$ can be written as

$$u[1] = \Phi_1\chi_1^{-1}\phi^{'}\phi^{-1}\Phi_1\chi_1^{-1} \tag{4.3}$$

where Φ and χ are the eigenvector components of linear systems associated to NC PII equation [21]. Now we can express the two fold Darboux transformation as follow

$$u[2] = \Phi_1[1]\chi_1^{-1}[1]\phi^{'}[1]\phi^{-1}[1]\Phi_1[1]\chi_1^{-1}[1]. \tag{4.4}$$

here $\phi[1]$ is given in equation (2.33). Now we consider the third solitonic solution of NC PII equation as under

$$u[3] = \Phi_1[2]\chi_1^{-1}[2]\phi^{'}[2]\phi^{-1}[2]\Phi_1[2]\chi_1^{-1}[2]. \tag{4.5}$$

where

$$\phi[2] = Y[1]X^{-1}[1]\phi[1]Y[1]X^{-1}[1].$$

In order express $\phi[2]$ in terms of quasideterminant, fisrt we write the transformations (2.31) and (2.32) by using the definition (3.2)as under

$$X[1] = \begin{vmatrix} X_1 & X_0 \\ \lambda_1 Y_1 & \boxed{\lambda_0 Y_0} \end{vmatrix} = \delta_X^e[1] \tag{4.6}$$

and

$$Y[1] = \begin{vmatrix} Y_1 & Y_0 \\ \lambda_1 X_1 & \boxed{\lambda_0 X_0} \end{vmatrix} = \delta_Y^e[1] \tag{4.7}$$

We have taken $\lambda = \lambda_0$, $X = X_0$ and $Y = Y_0$ in order to generalize the transformations in Nth form. Further, we can represent the transformations $X[2]$ and $Y[2]$ by quasideterminants

$$X[2] = \begin{vmatrix} X_2 & X_1 & X_0 \\ \lambda_2 Y_2 & \lambda_1 Y_1 & \lambda_0 Y_0 \\ \lambda_2^2 X_2 & \lambda_1^2 X_1 & \boxed{\lambda_0^2 X_0} \end{vmatrix} = \Upsilon_X^o[2]$$

and

$$Y[2]=\begin{vmatrix} Y_2 & Y_1 & Y_0 \\ \lambda_2 X_2 & \lambda_1 X_1 & \lambda_0 X_0 \\ \lambda_2^2 Y_2 & \lambda_1^2 Y_1 & \boxed{\lambda_0^2 Y_0} \end{vmatrix}=\Upsilon_Y^o[2].$$

here superscripts e and o of Υ represent the even and odd order quasideterminants. The Nth transformations for $\Upsilon_X^o[N]$ and $\Upsilon_Y^o[N]$ in terms of quasideterminants are given below

$$\Upsilon_X^o[N]=\begin{vmatrix} X_N & X_{N-1} & \cdots & X_1 & X_0 \\ \lambda_N Y_N & \lambda_{N-1} Y_{N-1} & \cdots & \lambda_1 X_1 & \lambda_0 Y_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_N^{N-1} Y_N & \lambda_{N-1}^{N-1} Y_{N-1} & \cdots & \lambda_1^{N-1} X_1 & \lambda_0^{N-1} Y_0 \\ \lambda_N^N X_N & \lambda_{N-1}^N X_{N-1} & \cdots & \lambda_1^N X_1 & \boxed{\lambda_0^N X_0} \end{vmatrix}$$

and

$$\Upsilon_Y^o[N]=\begin{vmatrix} Y_N & Y_{N-1} & \cdots & Y_1 & Y_0 \\ \lambda_N X_N & \lambda_{N-1} X_{N-1} & \cdots & \lambda_1 X_1 & \lambda_0 X_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_N^{N-1} X_N & \lambda_{N-1}^{N-1} X_{N-1} & \cdots & \lambda_1^{N-1} X_1 & \lambda_0^{N-1} X_0 \\ \lambda_N^N Y_N & \lambda_{N-1}^N Y_{N-1} & \cdots & \lambda_1^N Y_1 & \boxed{\lambda_0^N Y_0} \end{vmatrix}$$

here N is to be taken as even. in the same way we can write Nth quasideterminant representations of $\Upsilon_X^e[N]$ and $\Upsilon_Y^e[N]$.

$$\Upsilon_X^e[N]=\begin{vmatrix} X_N & X_{N-1} & \cdots & X_1 & X_0 \\ \lambda_N Y_N & \lambda_{N-1} Y_{N-1} & \cdots & \lambda_1 X_1 & \lambda_0 Y_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_N^{N-1} X_N & \lambda_{N-1}^{N-1} X_{N-1} & \cdots & \lambda_1^{N-1} X_1 & \lambda_0^{N-1} X_0 \\ \lambda_N^N Y_N & \lambda_{N-1}^N Y_{N-1} & \cdots & \lambda_1^N Y_1 & \boxed{\gamma_0^N Y_0} \end{vmatrix}$$

and

$$\Upsilon_Y^e[N]=\begin{vmatrix} Y_N & Y_{N-1} & \cdots & Y_1 & Y_0 \\ \lambda_N X_N & \lambda_{N-1} X_{N-1} & \cdots & \lambda_1 X_1 & \lambda_0 X_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_N^{N-1} Y_N & \lambda_{N-1}^{N-1} Y_{N-1} & \cdots & \lambda_1^{N-1} Y_1 & \lambda_0^{N-1} Y_0 \\ \lambda_N^N X_N & \lambda_{N-1}^N X_{N-1} & \cdots & \lambda_1^N X_1 & \boxed{\gamma_0^N X_0} \end{vmatrix}.$$

Similarly, we can derive the expression for N-fold Darboux transformations for ϕ by applying the transformation (2.33) iteratively, for this pupose let us consider

$$\phi[1]=\Omega_1^Y[1]\Omega_1^X[1]^{-1}\phi\Omega_1^Y[1]\Omega_1^X[1]^{-1}$$

where

$$\Omega_1^Y[1]=Y_1$$

$$\Omega_1^X[1] = X_1$$

this is one fold Darboux transformation. The two fold Darboux transformation is given by

$$\phi[2] = Y[1]X^{-1}[1]\phi[1]Y[1]X^{-1}[1]. \tag{4.8}$$

We may rewrite the equation (4.6) and equation (4.7) in the following forms

$$X[1] = \begin{vmatrix} X_1 & X_0 \\ \lambda_1 Y_1 & \boxed{\lambda_0 Y_0} \end{vmatrix} = \Omega_2^X[2]$$

$$Y[1] = \begin{vmatrix} Y_1 & Y_0 \\ \lambda_1 X_1 & \boxed{\lambda_0 X_0} \end{vmatrix} = \Omega_2^Y[2].$$

and equation (4.8) may be written as

$$\phi[2] = \Omega_2^Y[2]\Omega_2^X[2]^{-1}\Omega_1^Y[1]\Omega_1^X[1]^{-1}\phi\Omega_1^Y[1]\Omega_1^X[1]^{-1}\Omega_2^Y[2]\Omega_2^X[2]^{-1}$$

We can show that the fourth solitonic solution $u[4]$ will take the following form

$$u[4] = \Phi_1[3]\chi_1^{-1}[3]\phi^{'}[3]\phi^{-1}[3]\Phi_1[3]\chi_1^{-1}[] \tag{4.9}$$

where the three fold Darboux transform $\phi[3]$ can be expressed as

$$\phi[3] = \Omega_3^Y[3]\Omega_3^X[3]^{-1}\Omega_2^Y[2]\Omega_2^X[2]^{-1}\Omega_1^Y[1]\Omega_1^X[1]^{-1}\phi\Omega_1^Y[1]\Omega_1^X[1]^{-1}\Omega_2^Y[2]\Omega_2^X[2]^{-1}\Omega_3^Y[3]\Omega_3^X[3]^{-1}.$$

here

$$X[2] = \begin{vmatrix} X_2 & X_1 & X_0 \\ \lambda_2 Y_2 & \lambda_1 Y_1 & \lambda_0 Y_0 \\ \lambda_2^2 X_2 & \lambda_1^2 X_1 & \boxed{\lambda_0^2 X_0} \end{vmatrix} = \Omega_3^X[3]$$

and

$$Y[2] = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ \lambda_2 X_2 & \lambda_1 X_1 & \lambda_0 X_0 \\ \lambda_2^2 Y_2 & \lambda_1^2 Y_1 & \boxed{\lambda_0^2 Y_0} \end{vmatrix} = \Omega_3^Y[3].$$

Finaly, by applying the transformtion iteratively we can construct the N-th solitonic solution of NC PII $(z, \beta + n - 1)$ equation in the following form

$$u[N+1] = \Pi_{k=1}^{N}\Theta_k[k]\phi^{'}[N]\phi^{-1}[N]\Pi_{j=N}^{1}\Theta_j[j] \quad \text{for } N \geq 0 \tag{4.10}$$

the N fold Darboux transformation $\phi[N]$ given by

$$\phi[N] = \Theta_N[N]\Theta_{N-1}[N-1]...\Theta_2[2]\Theta_1[1]\phi\Theta_1[1]\Theta_2[2]...\Theta_{N-1}[N-1]\Theta_N[N]$$

or

$$\phi[N] = \Pi_{k=0}^{N-1}\Theta_{N-k}[N-k]\phi\Pi_{j=N-1}^{0}\Theta_{N-j}[N-j]$$

where

$$\Theta_N[N] = \Omega^Y_N[N]\Omega^X_N[N]^{-1}.$$

Here we present only the Nth expression for odd order quasideterminants $\Omega^Y_{2N+1}[2N+1]$ and $\Omega^X_{2N+1}[2N+1]$

$$\Omega^Y_{2N+1}[2N+1] = \begin{vmatrix} Y_{2N} & Y_{2N-1} & \cdots & Y_1 & Y_0 \\ \lambda_{2N}X_{2N} & \lambda_{2N-1}X_{2N-1} & \cdots & \lambda_1 X_1 & \lambda_0 X_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2N}^{2N-1}X_{2N} & \lambda_{2N-1}^{2N-1}X_{2N-1} & \cdots & \lambda_1^{2N-1}X_1 & \lambda_0^{2N-1}X_0 \\ \lambda_{2N}^{2N}Y_{2N} & \lambda_{2N-1}^{2N}Y_{2N-1} & \cdots & \lambda_1^{2N}Y_1 & \boxed{\lambda_0^{2N}Y_0} \end{vmatrix}$$

and

$$\Omega^X_{2N+1}[2N+1] = \begin{vmatrix} X_{2N} & X_{2N-1} & \cdots & X_1 & X_0 \\ \lambda_{2N}Y_{2N} & \lambda_{2N-1}Y_{2N-1} & \cdots & \lambda_1 X_1 & \lambda_0 Y_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{2N}^{2N-1}Y_{2N} & \lambda_{2N-1}^{2N-1}Y_{2N-1} & \cdots & \lambda_1^{2N-1}X_1 & \lambda_0^{2N-1}Y_0 \\ \lambda_{2N}^{2N}X_{2N} & \lambda_{2N-1}^{2N}X_{2N-1} & \cdots & \lambda_1^{2N}X_1 & \boxed{\lambda_0^{2N}X_0} \end{vmatrix}.$$

Similarly, we can derive an explicit expression of N-fold Darboux transformation for ψ in the following form

$$\psi[N] = \Pi_{k=0}^{N-1} K_{N-k}[N-k]\psi\Pi_{j=N-1}^{0} K_{N-j}[N-j]$$

where $K_N[N] = \Xi^Y_N[N]\Xi^X_N[N]^{-1}$ and $\Xi^Y_N[N]$, $\Xi^X_N[N]$ represent quasideterminants of order N. Similarly, we can construct the N-th soliton solution of NC PII$(z, \beta - n)$ equation in terms of quasideterminant by taking $u = u_{-1} = \psi'\psi^{-1}$ as a seed solution in its Darboux transformation (4.2). In next sections we some basic quantum commutation relations [26], we will observe that in section 6 how these commutation relations are helpful to derive quantum PII equation from its Lax representation which involves Planck constant $\hbar$ explicitly.

5. Quantum Painlevé II equation

The quantum extension of classical Painlevé equations involves the symmetrical form of Painlevé equations proposed in [26] to noncommuting objects. For the quantum Painlevé II equation let us replace the function u_0 , u_1, u_2 by f_0 , f_1, f_2 respectively in system (1.1) , further parameters α_0 and α_1 belong to the complex number field $\mathbb{C}$. The operators f_0, f_1 and f_2 obey the following commutation rules

$$[f_1, f_0]_- = 2\hbar f_2, \;\; [f_0, f_2]_- = [f_2, f_1]_- = \hbar \tag{5.1}$$

where $\hbar$ is Planck constant, the derivation ∂_z preserves the commutation relations (5.1) [26] . The NC differential system (1.1) admits the affine Weyl group actions of type $\mathscr{A}_l^{(1)}$ and quantum PII equation

$$f_2'' = 2f_2^3 - zf_2 + \alpha_1 - \alpha_0. \tag{5.2}$$

can be obtained by elimination f_0 and f_1 from system (1.1) with the help of commutation relations (5.1). The above equation (5.2) is called quantum PII equation because after eliminating f_0 and f_2 from same system (1.1) we obtain P_{34} that involves Planck constant $\hbar$ [26] and [27]. In next section I construct a linear systems whose compatibility condition yields quantum PII equation with quantum commutation relation between function f_2 and independent variable z, further we show that under the classical limit when $\hbar \to 0$ this system will reduce to classical PII equation.

6. Zero curvature representation of quantum PII equation

Proposition 1.5.
The compatibility condition of following linear system

$$\Psi_\lambda = A(z;\lambda)\Psi, \quad \Psi_z = B(z;\lambda)\Psi \tag{6.1}$$

with Lax matrices

$$\begin{cases} A = (8i\lambda^2 + if_2^2 - 2iz)\sigma_3 + f_2'\sigma_2 + (\frac{1}{4}c\lambda^{-1} - 4\lambda f_2)\sigma_1 + i\hbar\sigma_2 \\ B = -2i\lambda\sigma_3 + f_2\sigma_1 + f_2 I \end{cases} \tag{6.2}$$

yields quantum PII equation, here I is 2×2 identity matrix and λ is spectral parameter and c is constant.
Proof:
The compatibility condition of system (6.1) yields zero curvature condition

$$A_z - B_\lambda = [B,A]_-. \tag{6.3}$$

We can easily evaluate the values for A_z, B_λ and $[B,A]_- = BA - AB$ from the linear system (6.2) as follow

$$A_z = (if_2'f_2 + if_2f_2' - 2i)\sigma_3 + f_2''\sigma_2 - 4\lambda f_2'\sigma_1 \tag{6.4}$$

$$B_\lambda = -2i\sigma_3 \tag{6.5}$$

and

$$[B,A] = \begin{pmatrix} if_2'f_2 + if_2f_2' + [f_2,z]_- - i\hbar & \delta \\ \lambda & -if_2'f_2 - if_2f_2'[z,f_2]_- + i\hbar \end{pmatrix} \tag{6.6}$$

where

$$\delta = -if_2'' + 2if_2^3 - 2i[z,f_2]_+ + ic + i[f_2',f_2]_- + 4i\lambda\hbar$$

and

$$\lambda = if_2'' - 2if_2^3 + 2i[z, f_2]_+ - ic + i[f_2, f_2']_- - 4i\lambda\hbar.$$

now after substituting these values from (6.4), (6.5) and (6.6) in equation (6.3) we get

$$\begin{pmatrix} [f_2, z]_- - i\hbar & \delta \\ \lambda & [z, f_2]_- + i\hbar \end{pmatrix} = 0 \tag{6.7}$$

and the above result (6.7) yields the following expressions

$$[f_2, z] = \frac{1}{2} i\hbar f_2 \tag{6.8}$$

and

$$if_2'' - 2if_2^3 + 2i[z, f_2]_+ - ic + i[f_2, f_2']_- - 4i\lambda\hbar = 0 \tag{6.9}$$

equation (6.8) shows quantum relation between the variables z and f_2. In equation (6.9) the term $i[f_2, f_2']_- - 2i\lambda\hbar$ can be eliminated by using equation $f_2' = f_1 - f_0$ from (1.1) and quantum commutation relations (5.1). For this purpose let us replace f_2 by $-\frac{1}{2}\lambda^{-1} f_2$ in (5.1), then commutation relations become

$$[f_0, f_2]_- = [f_2, f_1]_- = -2\lambda\hbar. \tag{6.10}$$

Now let us take the commutator of the both side of the equation $f_2' = f_1 - f_0$ with f_2 from right side, we get

$$[f_2', f_2]_- = [f_1, f_2]_- - [f_0, f_2]_-$$

above equation with the commutation relations (6.10) can be written as

$$[f_2', f_2]_- = -4\lambda\hbar. \tag{6.11}$$

Now after substituting the value of $[f_2', f_2]_-$ from (6.11) in (6.9) we get

$$if_2'' - 2if_2^3 + 2i[z, f_2]_+ - ic = 0.$$

Finally, we can say that the compatibility of condition of linear system (6.1) yields the following expressions

$$\begin{cases} f_2'' = 2f_2^3 - 2[z, f_2]_+ + c \\ zf_2 - f_2 z = i\hbar f_2 \end{cases} \tag{6.12}$$

in above system (6.12) the first equation can be considered as a pure version of quantum Painlevé II equation that is equipped with a quantum commutation relation $[z, f_2]_- = -i\hbar$ and this equation can be reduced to the classical PII equation under the classical limit when $\hbar \to 0$.

Remark 1.2.

The linear system (6.1) with eigenvector $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and setting $\Delta = \psi_1 \psi_2^{-1}$ can be reduced to the following quantum PII Riccati form

$$\Delta^{'} = -4i\Delta + f_2 + [f_2, \Delta]_- - \Delta f_2 \Delta$$

Proof:

Here we apply the method of Konno and Wadati [28] to the linear system (6.12) of quantum PII equation. For this purpose let us substitute the eigenvector $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in linear systems (6.1) and we get

$$\begin{cases} \dfrac{d\psi_1}{d\lambda} = (8i\lambda^2 + if_2^2 - 2iz)\psi_1 + (-if_2^{'} + \dfrac{1}{4}C_0\lambda^{-1} - 4\lambda f_2 + \hbar)\psi_2 \\ \dfrac{d\psi_2}{d\lambda} = (if_2^{'} + \dfrac{1}{4}C_0\lambda^{-1} - 4\lambda f_2 - \hbar)\psi_1 + (-8i\lambda^2 - if_2^2 + 2iz)\psi_2 \end{cases} \tag{6.13}$$

and

$$\begin{cases} \psi_1^{'} = (-2i\lambda + f_2)\psi_1 + f_2\psi_2 \\ \psi_2^{'} = f_2\psi_1 + (2i\lambda + f_2)\psi_2 \end{cases} \tag{6.14}$$

where $\psi_1^{'} = \frac{d\psi_1}{dz}$ and now from system (6.14) we can derive the following expressions

$$\psi_1^{'}\psi_2^{-1} = (-2i\lambda + f_2)\psi_1\psi_2^{-1} + f_2 \tag{6.15}$$

$$\psi_2^{'}\psi_2^{-1} = -2i\lambda + f_2 + f_2\psi_1\psi_2^{-1}. \tag{6.16}$$

Let consider the following substitution

$$\Delta = \psi_1\psi_2^{-1} \tag{6.17}$$

now taking the derivation of above equation with respect to z

$$\Delta^{'} = \psi_1^{'}\psi_2^{-1} - \psi_1\psi_2^{-1}\psi_2^{'}\psi_2^{-1}$$

after using the (6.15), (6.16) and (6.17) in above equation we obtain

$$\Delta^{'} = -4i\Delta + f_2 + [f_2, \Delta]_- - \Delta f_2 \Delta \tag{6.18}$$

the above expression (6.18) can be considered as quantum Riccati equation in Δ because it involves commutation $[f_2, \Delta]_- = f_2\Delta - \Delta f_2$ that has been derived from the linear system (6.14).

7. Conclusion

In this paper, I have derived non-vacuum solutions of NC PII equation taking the solutions of Toda equations at $n = 1$ as seed solutions in its Darboux transformation. I have also generalized the Darboux transformations of these seed solutions to N-th form. Further, I derived a zero curvature representation of quantum Painlevé II equation with its associated Riccati form and also we have derived an explicit expression of NC PII Riccati equation from the linear system of NC PII equation by using the method of Konno and Wadati [28]. Further, one can derive Bc̆klund transformations for NC PII equation with the help of our NC PII linear system and its Riccati form by using the technique described in [28, 29], these transformations may be helpful to construct the nonlinear principle of superposition for NC PII solutions. It also seems interesting to construct the connection of NC PII equation to the known integrable systems such as its connection to NC nonlinear Schrödinger equation and to NC KdV equation as it possesses this property in classical case. Further it is quite interesting symmetrically to construct zero curvature representations for Quantum Painlevé equation PIV, PV in such a way to derive the similar results that have been described in (6.12) for quantum PII by using the quantum commutation relations of [26].

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