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Abstract

We prove the ergodicity and the convergence of a Fleming-Viot type particle system to the minimal quasi-stationary distribution of some birth and death processes. Our main tool is a new tractable Lyapunov–type criterion which extends recent results on the domain of attraction of the minimal quasi-stationary distribution for birth and death processes.

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1 Introduction

Let X be a stable birth and death process on $\mathbb{N} = \{0, 1, 2, \dots\}$ absorbed when it hits 0. The *minimal quasi-stationary distribution* (or *Yaglom limit*) of X, when it exists, is the unique probability measure ρ on $\mathbb{N}^* = \{1, 2, \dots\}$ such that

$$\rho(\cdot) = \lim_{t\to\infty} \mathbb{P}_x \left(X_t \in \cdot \mid t < T_0 \right), \text{ for all } x \in \mathbb{N}^*,$$

where $T_0 = \inf\{t \geq 0,\ X_t = 0\}$ is the absorption time of X. The probability measure ρ is called a quasi-stationary distribution because it is stationary

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for the conditioned process, in the sense that

$$\rho = \mathbb{P}_\rho(X_t \in \cdot \mid t < T_0),\ \text{for all } t \geq 0.$$

These notions are recalled with more details in Section 2, with important definitions and well known results on quasi-stationary distributions. We also provide a new sufficient criterion ensuring that a probability measure μ belongs to the *domain of attraction* of the minimal quasi-stationary distribution, which means that

$$\lim_{t\to\infty} \mathbb{P}_\mu\left(X_t \in \cdot \mid t < T_0\right) = \rho. \tag{1.1}$$

This new criterion allows us to extend existing studies on the long time and high number of particles limit of a *Fleming-Viot type particle system.* The particles of this system evolve as independent copies of the birth and death process X, but they undergo rebirths when they hit 0 instead of being trapped at the origin. In particular, the number of particles that are in $\mathbb{N}^*$ remains constant as time goes on. Our main result is a sufficient criterion ensuring that the empirical stationary distribution of the particle system exists and converges to the minimal quasi-stationary distribution of the underlying birth and death process.

Fix $N \geq 2$ and let us describe precisely the dynamics of this system with N particles, which we denote by $(X^1, X^2, \dots, X^N)$. The process starts at a position $(X_0^1, X_0^2, \dots, X_0^N) \in (\mathbb{N}^*)^N$ and evolves as follows:

- the particles X^i, $i = 1, \dots, N$, evolve as independent copies of the birth and death process X until one of them hits 0; this hitting time is denoted by τ_1;

- then the (unique) particle hitting 0 at time τ_1 jumps instantaneously on the position of a particle chosen uniformly among the $N-1$ remaining ones; this operation is called a *rebirth*;

- because of this rebirth, the N particles lie in $\mathbb{N}^*$ at time τ_1; then the N particles evolve as independent copies of X and so on.

We denote by $\tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ the sequence of rebirths times. Since the rate at which rebirths occur is uniformly bounded above by $N\, d_1$,

$$\lim_{n\to\infty} \tau_n = +\infty \text{ almost surely.}$$

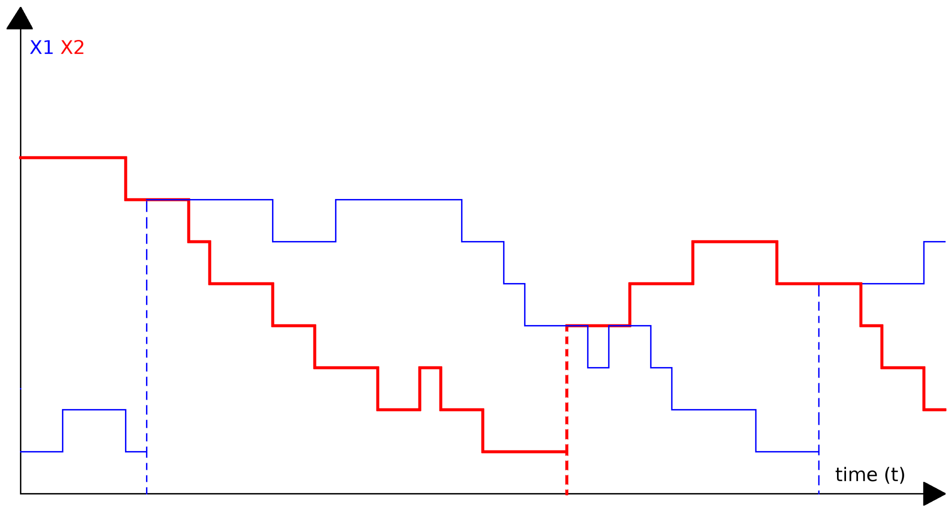

Figure 1: One path of a Fleming-Viot system with two particles.

As a consequence, the particle system $(X_t^1, X_t^2, \ldots, X_t^N)_{t\geq 0}$ is well defined for any time $t \geq 0$ in an incremental way, rebirth after rebirth (see Figure 1 for an illustration of this construction with $N = 2$ particles).

This Fleming-Viot type system has been introduced by Burdzy, Holyst, Ingermann and March in [4] and studied in [5], [11], [20], [12] for multi-dimensional diffusion processes. The study of this system when the underlying Markov process X is a continuous time Markov chain in a countable state space has been initiated in [10] and followed by [3], [1], [13] and [2], [8]. We also refer the reader to [14], where general considerations on the link between the study of such systems and front propagation problems are considered.

We emphasize that, because of the rebirth mechanism, the particle system $(X^1, X^2, \ldots, X^N)$ evolves in $(\mathbb{N}^*)^N$. For any $t \geq 0$, we denote by μ_t^N the empirical distribution of $(X^1, X^2, \ldots, X^N)$ at time t, defined by

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \in \mathcal{M}_1(\mathbb{N}^*),$$

where $\mathcal{M}_1(\mathbb{N}^*)$ is the set of probability measures on $\mathbb{N}^*$. A general convergence result obtained in [19] ensures that, if $\mu_0^N \to \mu_0$, then

$$\mu_t^N \xrightarrow[N\to\infty]{} \mathbb{P}_{\mu_0}(X_t \in \cdot \mid t < T_0).$$

The generality of this result does not extend to the long time behaviour of the particle system, which is the subject of the present study. We provide a sufficient criterion ensuring that the process $(\mu_t^N)_{t\geq 0}$ is ergodic. Denoting by $\mathcal{X}^N$ its empirical stationary distribution (a random measure whose law is the stationary distribution of μ^N), our criterion also implies that

$$\mathcal{X}^N \xrightarrow[N\to\infty]{Law} \rho, \tag{1.2}$$

where ρ is the minimal quasi-stationary distribution of the birth and death process X. Our result applies (1) to birth and death processes with a unique quasi-stationary distribution (such as logistic birth and death processes) and (2) to birth and death processes with a minimal quasi-stationary distribution satisfying an explicit Lyapunov condition (fulfilled for instance by linear birth and death processes).

In Section 2, we recall definitions and well known results on quasi-stationary distributions. We also recall and extend recent results on the domain of attraction of quasi-stationary distributions for birth and death processes. In Section 3, we state and prove the ergodicity of the Fleming-Viot process and the convergence (1.2). In Section 4, we provide a numerical study of the speed of convergence of the Fleming-Viot empirical stationary distribution expectation to the minimal quasi-stationary distribution for a linear birth and death process and a logistic birth and death process. This numerical results suggest that the bias of the approximation is surprisingly small for linear birth and death processes and even smaller for logistic birth an death processes.

2 Quasi-stationary distributions for birth and death processes

Let $(X_t)_{t\geq 0}$ be a birth and death process on $\mathbb{N} = \{0, 1, 2, \dots\}$ with birth rates $(b_i)_{i\geq 0}$ and death rates $(d_i)_{i\geq 0}$. We assume that $b_i > 0$ and $d_i > 0$ for any $i \geq 1$ and $b_0 = d_0 = 0$. The stochastic process X is a $\mathbb{N}$-valued pure jump process whose only absorption point is 0 and whose transition rates from any point $i \geq 1$ are given by

$$\begin{aligned} &i \to i+1 \text{ with rate } b_i,\\ &i \to i-1 \text{ with rate } d_i,\\ &i \to j \text{ with rate } 0, \text{ if } j \notin \{i-1, i+1\}. \end{aligned}$$

Such processes are extensively studied because of their conceptual simplicity and pertinence as demographic models. It is well known (see for instance [16, Theorem 10 and Proposition 12]) that X is stable, conservative and hits 0 in finite time almost surely (for any initial distribution) if and only if

$$\sum_{n=1}^{\infty} \frac{d_1 d_2 \cdots d_n}{b_1 b_2 \cdots b_n} = +\infty. \tag{2.1}$$

The divergence of this series will be assumed along the whole paper. In particular, for any probability measure μ on $\mathbb{N}$, the law of the process with initial distribution μ is well defined. We denote it by $\mathbb{P}_\mu$ (or by $\mathbb{P}_x$ if $\mu = \delta_x$ with $x \in \mathbb{N}$) and the associated expectation by $\mathbb{E}_\mu$ (or by $\mathbb{E}_x$ if $\mu = \delta_x$ with $x \in \mathbb{N}$). Setting $T_0 = \inf\{t \geq 0,\ X_t = 0\}$, we thus have

$$\mathbb{P}_\mu(T_0 < \infty) = 1,\ \forall \mu \in \mathcal{M}_1(\mathbb{N}),$$

where, for any subset $F \subset \mathbb{N}$, $\mathcal{M}_1(F)$ denotes the set of probability measures on F.

A *quasi-stationary distribution* for X is a probability measure ρ on $\mathbb{N}^* = \{1, 2, \ldots\}$ such that

$$\mathbb{P}_\rho\left(X_t \in \cdot \mid t < T_0\right) = \rho(\cdot),\ \forall t \geq 0.$$

The probability measure ρ is thus stationary for the conditioned process (and, as a matter of fact, was called a *stationary distribution* in the seminal work [6]). The property "ρ is a quasi-stationary distribution for X" is directly related to the long time behaviour of X conditioned to not being absorbed. Indeed (see for instance [18] or [16]), a probability measure ρ is a quasi-stationary distribution if and only if there exists $\mu \in \mathcal{M}_1(\mathbb{N}^*)$ such that

$$\rho(\cdot) = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot \mid t < T_0). \tag{2.2}$$

For a given quasi-stationary distribution ρ, the set of probability measures μ such that (2.2) holds is called the *domain of attraction of* ρ. It is nonempty since it contains at least ρ and may contains an infinite number of elements. In particular, when the limit in (2.2) exists for any $\mu = \delta_x$, $x \in \mathbb{N}^*$, and doesn't depend on the initial position x, then ρ is called the *Yaglom limit* or the *minimal quasi-stationary distribution.* Thus the minimal quasi-stationary distribution, when it exists, is the unique quasi-stationary distribution whose domain of attraction contains $\{\delta_x,\ x \in \mathbb{N}^*\}$. From a demographical point of view, the study of the minimal quasi-stationary distribution of a birth and death process aims at answer the following question: *knowing that a population isn't extinct after a long time t, what is the probability that its size is equal to n at time t?*

One of the oldest and most understood question for quasi-stationary distributions of birth and death processes concerns their existence and uniqueness. Indeed, van Doorn [18] gave the following picture of the situation: a birth and death process can have no quasi-stationary distribution, one unique quasi-stationary distribution or an infinity (in fact a continuum) of quasi-stationary distributions. In order to determine whether a birth and death process has 0, one or an infinity of quasi-stationary distributions, one define inductively the sequence of polynomials $(Q_n(x))_{n\geq 0}$ for all $x \in \mathbb{R}$ by

$$\begin{aligned}
&Q_1(x) = 1,\\
&b_1\, Q_2(x) = b_1 + d_1 - x \text{ and}\\
&b_n\, Q_{n+1}(x) = (b_n + d_n - x)\, Q_n(x) - d_{n-1}\, Q_{n-1}(x),\ \forall n \geq 2.
\end{aligned}$$

As shown in [18], one can uniquely define the non-negative number ξ_1 satisfying

$$x \leq \xi_1 \iff Q_n(x) > 0,\ \forall n \geq 1.$$

Also, the useful quantity

$$S := \sup_{x\geq 1} \mathbb{E}_x(T_1),$$

can be easily computed, since, for any $z \geq 1$,

$$\sup_{x\geq z} \mathbb{E}_x(T_z) = \sum_{k\geq z+1} \frac{1}{d_k \alpha_k} \sum_{l\geq k} \alpha_l,$$

with $\alpha_k = \left(\prod_{i=1}^{k-1} b_i\right) / \left(\prod_{i=1}^{k} d_i\right)$. The following theorem answers the question of existence and uniqueness of a QSD for birth and death processes.

Theorem 2.1 (van Doorn, 1991 [18])**.** *Let X be a birth and death process satisfying* (2.1)*.*

1. *If $\xi_1 = 0$, there is no QSD.*
2. *If $S < +\infty$, then $\xi_1 > 0$ and the Yaglom limit is the unique QSD.*
3. *If $S = +\infty$ and $\xi_1 > 0$, then there is a continuum of QSDs, given by the one parameter family $(\rho_x)_{0<x\leq\xi_1}$:*

$$\rho_x(j) = \frac{\alpha_j}{d_1}\, x\, Q_j(x),\ \forall j \geq 1,$$

and the minimal quasi-stationary distribution is given by ρ_{ξ_1}.

Remark. Theorem 2.1 gives a complete description of the set of QSDs for a BD process. However, it is not well suited for the numerical computation of the Yaglom limit of a given BD process. Indeed, the polynomials Q_n have in most cases quickly growing coefficients, so that the value of ξ_1 cannot be obtained directly by numerical computation.

Theorem 2.1 is quite remarkable since it describes completely the possible outcomes of the existence and uniqueness problem for quasi-stationary distributions. However, it only partially answers the crucial problem of finding the domain of attraction of the existing quasi-stationary distributions and in particular of the minimal quasi-stationary distribution. The two following theorems address this problem. The first one covers the case where there exists a unique quasi-stationary distribution. The second one covers the trickier case where there exist an infinity of quasi-stationary distributions.

Theorem 2.2 (Martínez, San Martín, Villemonais 2013 [15])**.** *Let* X *be a birth and death process such that*

$$S = \sup_{x \geq 1} \mathbb{E}_x(T_1) < +\infty.$$

Then there exists $\gamma \in [0, 1[$ *such that, for any probability measure* μ *on* $\mathbb{N}^*$*,*

$$\|\rho - \mathbb{P}_\mu (X_t \in \cdot \mid t < \tau_\partial)\|_{TV} \leq \gamma^{\lfloor t \rfloor},\ \forall t \geq 0,$$

where $\|\cdot\|_{TV}$ *denotes the total variation norm. In particular, the domain of attraction of the unique quasi-stationary distribution is the whole set* $\mathcal{M}_1(\mathbb{N}^*)$ *of probability measures on* $\mathbb{N}^*$*.*

A weaker form of Theorem 2.2 has also been proved in [22] but the strong form (with uniform convergence in total variation norm) is necessary to derive the results of the next section. A generalized version of Theorem 2.2 has been rencently derived in [7], with complementary results on the so-called Q-process (the process conditioned to never being absorbed).

When an infinity of quasi-stationary distributions coexist, we provide a Lyapounov–type criterion. Beyond its own interest, this theorem is a crucial step for the next section's proofs.

Theorem 2.3. *Let X be a birth and death process which admits an infinity of quasi-stationary distributions. We assume that there exist $\eta : \mathbb{N} \to \mathbb{R}_+$ such that $\eta(0) = 0$ and $\eta(i) \neq 0$ for all $i \geq 1$, which satisfy the Lyapunov–type condition*

$$\eta(i) \xrightarrow[i\to\infty]{} \infty \text{ and } \mathcal{L}\eta(i) \leq -\xi_1 \eta(i) + C, \ \forall i \geq 1,$$

where $C > 0$ is a positive constant and $\mathcal{L}$ is the infinitesimal generator of X. Then the domain of attraction of the minimal quasi-stationary distribution of X contains the set $\mathcal{D}_\eta$ defined by

$$\mathcal{D}_\eta = \left\{ \mu \in \mathcal{M}_1(\mathbb{N}^*), \sum_{i=1}^{\infty} \mu_i \eta(i) < +\infty \right\}.$$

Defining the function η_1 by $\eta_1(i) = Q_i(\xi_1)$ for all i, we have $\mathcal{L}\eta_1 \leq -\xi_1\eta_1$. In particular, $\mathcal{D}_{\eta_1}$ is a subset of the domain of attraction of the minimal quasi-stationary distribution. This particular case was proved in [21] and will be used in the proof. Note however that this last result is not explicit since neither the constant ξ_1 nor the function η_1 can be computed, but for some very specific cases. On the contrary, it is easy to find a bound for ξ_1. For instance, $d_1 \geq \xi_1$ and thus our result implies that the domain of attraction of the minimal quasi-stationary distribution contains $\mathcal{D}_\eta$, for any function η such that $\mathcal{L}\eta \leq -d_1\eta + C$. See for instance the two following examples.

Example 1. We consider the case where $b_i = b\sqrt[a]{i}$ and $d_i = d\sqrt[a]{i}$, where $b < d$ are two positive constants and $a \geq 1$ is fixed. It is easy to check here that $S = +\infty$ and that there exists at least one quasi-stationary distribution (using for instance the criteria from [9]) and thus an infinity of quasi-stationary distributions. Now, defining $\eta(0) = 0$ and

$$\eta(i) = \sqrt{d/b}^{\,i}, \ \forall i \in \mathbb{N}^*,$$

one gets

$$\begin{aligned}
\mathcal{L}\eta(i) &:= b_i(\eta(i+1) - \eta(i)) + d_i(\eta(i-1) - \eta(i)) \\
&= b\sqrt[a]{i}\left(\sqrt{d/b}^{\,i+1} - \sqrt{d/b}^{\,i}\right) + d\sqrt[a]{i}\left(\sqrt{d/b}^{\,i-1} - \sqrt{d/b}^{\,i}\right) \\
&= \sqrt[a]{i}\left[\left(\sqrt{db}\sqrt{d/b}^{\,i} - b\sqrt{d/b}^{\,i}\right) + \left(\sqrt{db}\sqrt{d/b}^{\,i} - d\sqrt{d/b}^{\,i}\right)\right] \\
&= -\sqrt[a]{i}\left(\sqrt{d} - \sqrt{b}\right)^2 \sqrt{d/b}^{\,i}.
\end{aligned}$$

Since $\sqrt[a]{i} \to \infty$ when $i \to \infty$, we immediately deduce that there exists $C > 0$ such that η satisfies $\mathcal{L}\eta \leq -\xi_1\eta + C$. Now Theorem 2.3 implies that the domain of attraction of the minimal quasi-stationary distribution contains

$$\mathcal{D}_\eta = \left\{ \mu \in \mathcal{M}_1(\mathbb{N}^*),\ \sum_{i=1}^{\infty} \mu_i \sqrt{d/b}^{\,i} < +\infty \right\}.$$

As far as we know, this is a new result for any $a > 1$.

Example 2. We consider now the case $b_i = b > 0$ for all $i \geq 1$ and $d_i = d > 0$ for all $i \geq 2$, where b, d are constants such that $\sqrt{d} - \sqrt{b} > d_1$. Once again, one easily checks that $S = +\infty$ and that there exists at least one quasi-stationary distribution. Using the same function

$$\eta(i) = \sqrt{d/b}^{\,i},\ \forall i \geq 2,$$

one gets

$$\mathcal{L}\eta(i) = -(\sqrt{d} - \sqrt{b})^2 \sqrt{d/b}^{\,i} \leq -d_1\eta(i),\ \forall i \geq 2.$$

In particular, there exists $C > 0$ such that η satisfies $\mathcal{L}\eta \leq -d_1\eta + C$. Since $d_1 \geq \xi_1$, we deduce from Theorem 2.3 that the domain of attraction of the minimal quasi-stationary distribution contains the set

$$\mathcal{D}_\eta = \left\{ \mu \in \mathcal{M}_1(\mathbb{N}^*),\ \sum_{i=1}^{\infty} \mu_i \sqrt{d/b}^{\,i} < +\infty \right\}.$$

Proof of Theorem 2.3. By Theorem 2.1, the existence of a quasi-stationary distribution implies that $\xi_1 > 0$. Since $\eta(i) \to \infty$ when $i \to \infty$, there exists $i_0 \in \mathbb{N}^*$ such that $\eta(i) \geq C/\xi_1 + 1$ for any $i \geq i_0$. Let us set

$$\eta'(i) = (\eta(i) - C/\xi_1)_+ \text{ for any } i \in \mathbb{N}^*.$$

For all $i \geq i_0$, we have $\eta'(i) \geq 1$ and

$$\begin{aligned} \mathcal{L}\eta'(i) &= \mathcal{L}\eta(i) \leq -\xi_1\eta(i) + C \\ &\leq -\xi_1(\eta'(i) + C/\xi_1) + C = -\xi_1\eta'(i). \end{aligned} \tag{2.3}$$

Setting $T_{i_0} = \inf\{n \geq 0,\ X_n = i_0\}$, we deduce from inequality (2.3) and Dynkin's formula that, for any $i \geq i_0$,

$$\eta'(i_0) = \mathbb{E}_i(\eta'(X_{T_{i_0}})) \leq \mathbb{E}_i(e^{-\xi_1 T_{i_0}})\eta'(i)$$

and, setting $\eta_1(i) = Q_i(\xi_1)$ for any $i \in \mathbb{N}^*$,

$$\eta_1(i_0) = \mathbb{E}_i(\eta_1(X_{T_{i_0}})) = \mathbb{E}_i(e^{-\xi_1 T_{i_0}})\eta_1(i),$$

since η_1 satisfies $\mathcal{L}\eta_1 = -\xi_1\eta_1$. As a consequence, for any $i \geq i_0$,

$$\eta'(i) \geq \frac{\eta'(i_0)}{\mathbb{E}_i(e^{-\xi_1 T_{i_0}})} = \frac{\eta'(i_0)}{\eta_1(i_0)}\eta_1(i).$$

Now, for any probability measure μ on $\mathbb{N}^*$ such that $\mu(\eta) < \infty$, we have $\mu(\eta') < \infty$ and thus $\mu(\eta_1) < \infty$. By [21], this implies that μ belongs to the domain of attraction of the minimal QSD for X, which concludes the proof.
□

3 Main results

This section is devoted to the study of the ergodicity and the convergence of the Fleming-Viot particle system introduce in Section 1. Assumptions H1 and H2 below covers two different situations, regarding the existence and uniqueness of the quasi-stationary distribution.

Assumption H1. The birth and death process X admits a continuum of quasi-stationary distributions ($S = +\infty$ and $\xi_1 \neq 0$) and there exist a function $\eta : \mathbb{N} \to \mathbb{R}_+$ and two constants $\lambda > d_1$ and $C \geq 0$ such that $\eta(0) = 0$, $\eta(i) > 0$ for all $i \geq 1$ and

$$\eta(x) \xrightarrow[x\to\infty]{} \infty \text{ and } \mathcal{L}\eta(i) \leq -\lambda\eta(i) + C,\ \forall i \geq 1.$$

Assumption H2. The birth and death process X admits a unique quasi-stationary distribution ($S < +\infty$).

Remark 1.

1. Assumption H1 is satisfied in the cases of Examples 1 and 2.

2. Assumption H2 is typically satisfied for processes that come fast from infinity to compact sets, as the logistic birth and death process (where $b_i = b\,i$ and $d_i = d\,i + c\,i(i-1)$ for all $i \geq 1$ with $b, c, d > 0$).

Theorem 3.1. *Assume that Assumption H1 or Assumption H2 is satisfied. Then, for any $N > \frac{\lambda}{\lambda - d_1}$ under H1 and any $N \geq 2$ under H2, the measure* *process* $(\mu_t^N)_{t\geq 0}$ *is ergodic, which means that there exists a random measure* $\mathcal{X}^N$ *on* $\mathbb{N}^*$ *such that*

$$\mu_t^N \xrightarrow[t\to\infty]{Law} \mathcal{X}^N.$$

If H1 holds, then

$$\mathbb{E}(\eta(\mathcal{X}^N)) \leq C/(\lambda - d_1 N/(N-1)).$$

Moreover, if Assumption H1 or H2 is satisfied, then

$$\mathcal{X}^N \xrightarrow[N\to\infty]{Law} \rho,$$

where ρ *is the minimal quasi-stationary distribution of* X.

Remark 2. The pure drift birth and death process (where $b_i = b$ and $d_i = d$ for all $i \geq 1$, with $b < d$ two positive constants) does not satisfy Assumption H1 nor Assumption H2. Note that this process behaves exactly - but in state 1 - as in Example 2, where Assumption H1 is fulfilled. Although this may seem surprising, the additional difficulty is not a technical one and the following proof cannot work in the pure drift situation. As a consequence, Theorem 3.1 for pure drift birth and death processes remains an open problem, as stated in [2].

Since the proof of Theorem 3.1 differs whether one assumes H1 or H2, it is split in two different subsections : in Subsection 3.1, we prove the theorem under Assumption H1 and, in Subsection 3.2, we prove the result under assumption H2.

3.1 Proof under Assumption H1: exponential ergodicity via a Foster–Lyapunov criterion

Step 1. Proof of the exponential ergodicity by a Forster–Lyapunov criterion
We define the function

$$\begin{aligned} f : \mathcal{M}_1(\mathbb{N}^*) &\to \mathbb{R} \\ \mu &\mapsto \mu(\eta), \end{aligned}$$

where η is the Lyapunov function of Assumption H1. Fix $N \geq 2$ and let us express the infinitesimal generator $\mathcal{L}^N$ of the empirical process $(\mu_t^N)_{t\geq 0}$ applied to f at a point $\mu \in \mathcal{M}_1(\mathbb{N}^*)$ given by

$$\mu = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i},$$

where $(x_1, \dots, x_N) \in (\mathbb{N}^*)^N$. In order to shorten the notations, we introduce, for any $y \in \mathbb{N}^*$, the probability measure

$$\mu^{x_j,y} = \mu + \frac{1}{N}\left(\delta_y - \delta_{x_j}\right).$$

We thus have

$$\begin{aligned}
\mathcal{L}^N f(\mu) =& \sum_{i=1}^{N} b_{x_i}\left(f(\mu^{x_i,x_{i+1}}) - f(\mu)\right) + \mathbb{1}_{x_i\neq 1} d_{x_i}\left(f(\mu^{x_i,x_{i-1}}) - f(\mu)\right)\\
&+ \sum_{i=1, x_i=1}^{N} d_1 \frac{1}{N-1}\sum_{j=1,\, j\neq i}^{N}\left(f(\mu^{1,x_j}) - f(\mu)\right)\\
=& \sum_{i=1}^{N} b_{x_i}\left(\eta(x_{i+1}) - \eta(x_i)\right)/N + \mathbb{1}_{x_i\neq 1} d_{x_i}\left(\eta(x_{i-1}) - \eta(x_i)\right)/N\\
&+ \sum_{i=1, x_i=1}^{N} d_1 \frac{1}{N-1}\sum_{j=1,\, j\neq i}^{N}\left(\eta(x_j) - \eta(1)\right)/N
\end{aligned}$$

Since $\eta(0) = 0$, one gets

$$\begin{aligned}
\mathcal{L}^N f(\mu) =& \sum_{i=1}^{N} b_{x_i}\left(\eta(x_{i+1}) - \eta(x_i)\right)/N + d_{x_i}\left(\eta(x_{i-1}) - \eta(x_i)\right)/N\\
&+ \sum_{i=1, x_i=1}^{N} d_1 \frac{1}{N-1}\sum_{j=1,\, j\neq i}^{N}\eta(x_j)/N\\
=& \frac{1}{N}\sum_{i=1}^{N}\mathcal{L}\eta(x_i) + \sum_{i=1, x_i=1}^{N} d_1 \frac{1}{N-1}\left(\sum_{j=1}^{N}\eta(x_j)/N - \eta(1)/N\right)\\
\leq& \frac{1}{N}\sum_{i=1}^{N}\mathcal{L}\eta(x_i) + \left(\frac{1}{N}\sum_{i=1, x_i=1}^{N} d_1\right)\left(\frac{1}{N-1}\sum_{j=1}^{N}\eta(x_j)\right)\\
\leq& \mu(\mathcal{L}\eta) + \frac{N}{N-1} d_1 \mu(\eta).
\end{aligned}$$

Now, using Assumption H1, we deduce that

$$\begin{aligned}\mathcal{L}^N f(\mu) &\le \mu(-\lambda\eta + C) + \frac{N}{N-1} d_1 \mu(\eta)\\ &\le -\lambda\mu(\eta) + C + \frac{N}{N-1} d_1 \mu(\eta)\\ &\le -\left(\lambda - \frac{N}{N-1} d_1\right) f(\mu) + C,\end{aligned}$$

where $\lambda - \frac{N}{N-1} d_1$ is a positive constant for any fixed $N > \frac{\lambda}{\lambda - d_1}$.
For a fixed $N > \frac{\lambda}{\lambda - d_1}$ and any constant $k > 0$, the set of probability measures $\mu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ such that $f(\mu) = \mu(\eta) \le k$ is finite because $\eta(i) \to \infty$ when $i \to \infty$. Moreover the Markov process (μ_t^N) is irreducible (this is an easy consequence of the irreducibility of the birth and death process X). Thus, using the Foster Lyapunov criterion of [17, Theorem 6.1, p.536], we deduce that the process μ^N is exponentially ergodic and, denoting by $\mathcal{X}^N$ a random measure distributed following its stationary distribution, we also have

$$\mathbb{E}(\mathcal{X}^N(\eta)) = \mathbb{E}(f(\mathcal{X}^N)) \le C / \left(\lambda - \frac{N}{N-1} d_1\right). \tag{3.1}$$

This concludes the proof of the first part of Theorem 3.1.

Step 2. Convergence to the minimal QSD
Since $\eta(i)$ goes to infinity when $i \to \infty$, we deduce from (3.1) that the family of random measures $(\mathcal{X}^N)_N$ is tight. In particular, the family admits at least one limiting random probability measure $\mathcal{X}$, which means that $\mathcal{X}^N$ converges in law to $\mathcal{X}$, up to a subsequence.

Let μ_t^N be the random position at time t of the particle system with initial (random) distribution $\mathcal{X}^N$. On the one hand, the stationarity of $\mathcal{X}^N$ implies that $\mu_t^N \sim \mathcal{X}^N$ for all $t \ge 0$, and thus

$$\mu_t^N \xrightarrow[N\to\infty]{Law} \mathcal{X},\ \forall t \ge 0.$$

On the other hand, the general convergence result of [19] implies that

$$\mu_t^N \xrightarrow[N\to\infty]{Law} \mathbb{P}_{\mathcal{X}}(X_t \in \cdot \mid t < \tau_\partial).$$

As an immediate consequence

$$\mathbb{P}_{\mathcal{X}}(X_t \in \cdot \mid t < \tau_\partial) \stackrel{Law}{=} \mathcal{X}.$$

But (3.1) also implies that $\mathbb{E}(f(\mathcal{X})) < \infty$, so that $\mathcal{X}(\eta) = f(\mathcal{X}) < \infty$ almost surely. Using Theorem 2.3, we deduce that $\mathcal{X}$ belongs to the domaine of attraction of the minimal QSD ρ almost surely, that is

$$\mathbb{P}_{\mathcal{X}}(X_t \in \cdot \mid t < \tau_\partial) \xrightarrow[t\to\infty]{almost\, surely} \rho.$$

Thus the random measure $\mathcal{X}$ converges in law to the deterministic measure ρ, which implies that

$$\mathcal{X} = \rho \text{ almost surely.}$$

In particular, ρ is the unique limiting probability measure of the family $(\mathcal{X}^N)_N$, which ends the proof of Theorem 3.1 under Assumption H1.

3.2 Proof under Assumption H2: exponential ergodicity by a Dobrushin coefficient argument

Fix $N \geq 2$ and let us prove that the process is exponentially ergodic. Under assumption (H2), it is well known (see for instance [15]) that the process X comes back in finite time from infinity to 1, which means that

$$\inf_x \mathbb{P}_x(X_1 = 1) > 0.$$

Since the particles of a FV type system are independent up to the first rebirth time, we deduce that

$$\inf_{(x_1,\ldots,x_N)\in(\mathbb{N}^*)^N} \mathbb{P}((X_1^1,\ldots,X_1^N) = (1,\ldots,1)) > 0.$$

This implies that the FV process is exponentially ergodic.

Let us now denotes by $\mathcal{X}^N$ the empirical stationary distribution of the system $(X^1,\ldots,X^N)$, for each $N \geq 2$. Theorem 2.2 implies that there exists $\gamma > 0$ such that, for any $t \geq t_\epsilon$, any initial distribution μ_0 and any function $f : \mathbb{N}^* \to \mathbb{R}_+$,

$$\mathbb{E}\left|\rho(f) - \mathbb{E}_{\mu_0}\left(f(X_t) \mid t < \tau_\partial\right)\right| \leq 2\gamma^{\lfloor t \rfloor}\|f\|_\infty.$$

But, for any $t \geq 0$, [19] implies that

$$\mathbb{E}\left|\mu_t^N(f) - \mathbb{E}_{\mu_0^N}\left(f(X_t) \mid t < \tau_\partial\right)\right| \leq \frac{2(1+\sqrt{2})e^{d_1 t}\|f\|_\infty}{\sqrt{N}}.$$

As a consequence,

$$\mathbb{E}\left|\mu_t^N(f) - \rho(f)\right| \leq \frac{2(1+\sqrt{2})e^{d_1 t}\|f\|_\infty}{\sqrt{N}} + 2\gamma^{\lfloor t \rfloor}\|f\|_\infty.$$

In particular, for any $\epsilon > 0$, there exists t_ϵ and N_ϵ such that

$$\mathbb{E}\left|\mu_t^N(f) - \rho(f)\right| \leq \epsilon\|f\|_\infty,\ \forall N \geq N_\epsilon,\ t \geq t_\epsilon.$$

But μ_t^N converges in law to $\mathcal{X}^N$, so that

$$\mathbb{E}\left|\mathcal{X}^N(f) - \rho(f)\right| \leq \epsilon\|f\|_\infty,\ \forall N \geq N_\epsilon.$$

This inequality being true for any $\epsilon > 0$, this concludes the proof of Theorem 3.1 under Assumption (H2).

4 Numerical simulation of the Fleming-Viot type particle system

In this section, we present numerical simulations of the Fleming-Viot particle system studied in Section 3. Namely, we focus on the distance in total variation norm between the expectation of the empirical stationary distribution (*i.e.* $\mathbb{E}(\mathcal{X}^N)$) and the minimal quasi-stationary distribution of the underlying Markov process X, when N goes to infinity. This means that we aim at studying the bias of the approximation method.

We start with the linear birth and death process case in Subsection 4.1. This is one of the rare situation where explicit computation of the minimal quasi-stationary distribution can be performed (see for instance [16]). In Subsection 4.2, we provide the results of numerical simulations in the logistic birth and death case.

4.1 The linear birth and death case

We assume in this section that $b_i = i$ and $d_i = 2i$ for all $i \geq 0$. This is a sub-case of Example 1 and thus one can apply Theorem 3.1: the empirical stationary distribution of the process $\mathcal{X}^N$ exists and converges in law, when the number N of particles goes to infinity, to the minimal quasi-stationary distribution ρ of the process, which is known to be given by (see [16])

$$\rho(i) = \frac{1}{2^i},\ \forall i \geq 1.$$

The results of the numerical estimations of $\|\mathbb{E}(\mathcal{X}^N) - \rho\|_{TV}$ for different values of N (from 2 to 10^4) are reproduced on Table 1. One interesting point is the confirmation that $\mathbb{E}(\mathcal{X}^N)$ is a biased estimator of ρ. A second interesting point is that the bias decreases quickly when N increases. Up to our knowledge, there exists today no theoretical justification of this fact, despite its practical implications. Indeed, one drawback of the speed of the numerical simulation is the interaction between the particles of the Fleming-Viot system: more particles in the system leads to more interaction and thus more communication between processors, which at the end slows down the simulation. A crucial optimisation problem for the approximation method is thus to keep the number of particles as small as possible. In our linear birth and death case, the numerical simulations suggest that the bias decreases as $O(N^{-1})$.

Nb of particles	$\left\|\mathbb{E}\left(\mathcal{X}^N\right) - \rho\right\|_{TV}$	Estimated error
$N = 2$	0.190	$\pm 10^{-3}$
$N = 10$	4.5×10^{-2}	$\pm 10^{-3}$
$N = 10^2$	5.0×10^{-3}	$\pm 10^{-4}$
$N = 10^3$	5.1×10^{-4}	$\pm 10^{-5}$
$N = 10^4$	2.3×10^{-5}	$\pm 10^{-5}$

Table 1: Estimation of the bias $\|\mathbb{E}(\mathcal{X}^N) - \rho\|_{TV}$ for a linear birth and death process. In the rightmost column, "Estimated error" is the order of magnitude of the error that is made when computing $\|\mathbb{E}(\mathcal{X}^N) - \rho\|_{TV}$.

4.2 The logistic birth and death case

We consider now the case where $b_i = 2i$ and $d_i = i + i(i-1)$, for all $i \geq 1$. The existence and uniqueness of a quasi-stationary distribution ρ is well known for this process, but no explicit formula for the probability measure ρ exists. Thus, in order to compute numerically the total variation distance $\|\mathbb{E}(\mathcal{X}^N) - \rho\|_{TV}$ for different values of N, we use the approximation

$$\rho \simeq \mathbb{E}(\mathcal{X}^{N_0}), \text{ where } N_0 = 10^4.$$

The histogram of the estimated quasi-stationary distribution is represented on Figure 2.

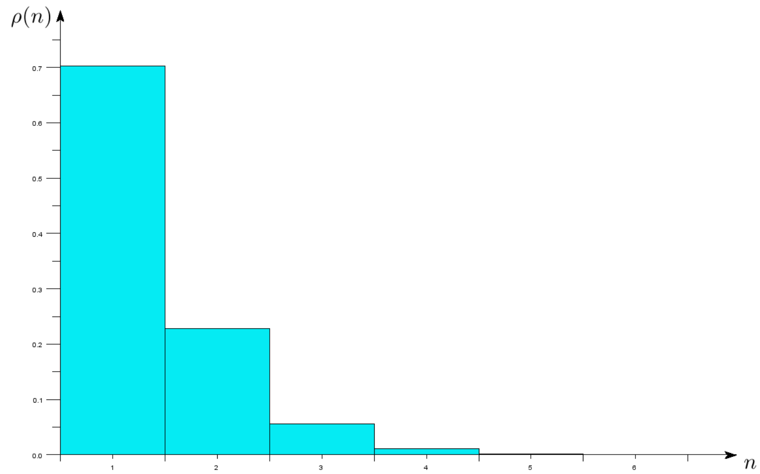

Figure 2: Estimated value of the minimal quasi-stationary distribution $\rho(n)$ for a logistic birth and death process.

The results of the numerical estimations of the bias $\|\mathbb{E}(\mathcal{X}^N) - \rho\|_{TV}$ are reproduced on Table 2. The conclusion is the same as in the linear birth and death case : $\|\mathbb{E}(\mathcal{X}^N) - \rho\|_{TV}$ declines very sharply as a function of N. In fact, the phenomenon is even more spectacular, since the estimated value of $\left\|\mathbb{E}\left(\mathcal{X}^N\right) - \rho\right\|_{TV}$ is 2.0×10^{-2}, even for $N = 2$.

References

[1] A. Asselah, P. A. Ferrari, P. Groisman, and M. Jonckheere. Fleming-Viot selects the minimal quasi-stationary distribution: The Galton-Watson case. *ArXiv e-prints*, June 2012.

[2] A. Asselah and M.-N. Thai. A note on the rightmost particle in a Fleming-Viot process. *ArXiv e-prints*, December 2012.

[3] Amine Asselah, Pablo A. Ferrari, and Pablo Groisman. Quasistationary distributions and Fleming-Viot processes in finite spaces. *J. Appl. Probab.*, 48(2):322–332, 2011.

[4] K Burdzy, R Holyst, D Ingerman, and P March. Configurational transition in a fleming-viot-type model and probabilistic interpretation of laplacian eigenfunctions. *J. Phys. A*, 29(29):2633–2642, 1996.

Nb of particles	$\left\|\mathbb{E}\left(\mathcal{X}^N\right)-\rho\right\|_{TV}$	Estimated error
$N=2$	2.0×10^{-2}	$\pm 10^{-3}$
$N=10$	3.0×10^{-3}	$\pm 10^{-4}$
$N=10^2$	3.6×10^{-4}	$\pm 10^{-5}$
$N=10^3$	2×10^{-5}	$\pm 10^{-5}$
$N=10^4$	$***$	$\pm 10^{-5}$

Table 2: Estimation of the bias $\|\mathbb{E}(\mathcal{X}^N)-\rho\|_{TV}$ for a logistic birth and death process. In the rightmost column, "Estimated error" is the order of magnitude of the error that is made when computing $\|\mathbb{E}(\mathcal{X}^N)-\rho\|_{TV}$. On the last line, there is no input for $N=N_0=10^4$ because $\rho\simeq\mathbb{E}(\mathcal{X}^{N_0})$ is the probability measure used to compute $\|\mathbb{E}(\mathcal{X}^N)-\rho\|_{TV}$.

[5] Krzysztof Burdzy, Robert Hołyst, and Peter March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, 214(3):679–703, 2000.

[6] James A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. Appl. Probab.*, 10(3):570–586, 1978.

[7] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *ArXiv e-prints*, April 2014.

[8] B. Cloez and M.-N. Thai. Quantitative results for the Fleming-Viot particle system in discrete space. *ArXiv e-prints*, December 2013.

[9] P. A. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.*, 23(2):501–521, 1995.

[10] Pablo A. Ferrari and Nevena Marić. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, 12:no. 24, 684–702 (electronic), 2007.

[11] Ilie Grigorescu and Min Kang. Hydrodynamic limit for a Fleming-Viot type system. *Stochastic Process. Appl.*, 110(1):111–143, 2004.

[12] Ilie Grigorescu and Min Kang. Immortal particle for a catalytic branching process. *Probab. Theory Related Fields*, pages 1–29, 2011. 10.1007/s00440-011-0347-6.

[13] P. Groisman and M. Jonckheere. Simulation of quasi-stationary distributions on countable spaces. *ArXiv e-prints*, June 2012.

[14] P. Groisman and M. Jonckheere. Front propagation and quasi-stationary distributions: the same selection principle? *ArXiv e-prints*, April 2013.

[15] S. Martinez, J. San Martin, and D. Villemonais. Existence and uniqueness of a quasi-stationary distribution for Markov processes with fast return from infinity. *To appear in Journal of Applied Probability*, February 2013.

[16] Sylvie Méléard and Denis Villemonais. Quasi-stationary distributions and population processes. *Probability Surveys*, 2012. To appear.

[17] Sean P. Meyn and R.L. Tweedie. Stability of Markovian processes. III: Foster-Lyapunov criteria for continuous-time processes. *Adv. Appl. Probab.*, 25(3):518–548, 1993.

[18] Erik A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23(4):683–700, 1991.

[19] D. Villemonais. General approximation method for the distribution of Markov processes conditioned not to be killed. *To appear in ESAIM: Probability and Statistics*, June 2011.

[20] D. Villemonais. Interacting particle systems and yaglom limit approximation of diffusions with unbounded drift. *Electronic Journal of Probability*, 16:1663–1692, 2011.

[21] H. Zhang, W. Liu, X. Peng, and S. Liu. Domain of attraction of the minimal quasi-stationary distribution for the birth and death process. *Applied Probability Trust*, 2011.

[22] Hanjun Zhang and Yixia Zhu. Domain of attraction of the quasistationary distribution for birth-and-death processes. *J. Appl. Probab.*, 50(1):114–126, 2013.