

# Approximation by Müntz spaces on positive intervals

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## Abstract

The so-called Bernstein operators were introduced by S.N. Bernstein in 1912 to give a constructive proof of Weierstrass' theorem. We show how to extend his result to Müntz spaces on positive intervals. *To cite this article: R. Ait-Haddou and M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ??? (200?).*

## Approximation par espaces de Müntz sur un intervalle positif. Résumé

En 1912, les opérateurs dits de Bernstein permirent à S.N. Bernstein de donner une preuve constructive du théorème de Weierstrass. Nous étendons ce résultat aux espaces de Müntz sur des intervalles positifs. *Pour citer cet article : R. Ait-Haddou and M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ??? (200?).*

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## 1. Introduction

The famous Bernstein operator  $\mathbb{B}_k$  of degree  $k$  on a given non-trivial interval  $[a, b]$ , associates with any  $F \in C^0([a, b])$  the polynomial function

$$\mathbb{B}_k F(x) := \sum_{i=0}^k F\left(\left(1 - \frac{i}{k}\right)a + \frac{i}{k}b\right) B_i^k, \quad x \in [a, b], \quad (1)$$

where  $(B_0^k, \dots, B_k^k)$  is the Bernstein basis of degree  $k$  on  $[a, b]$ , i.e.,  $B_i^k(x) := \binom{k}{i} \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{k-i}$ . It reproduces any affine function  $U$  on  $[a, b]$ , in the sense that  $\mathbb{B}_k U = U$ . In [5], S.N. Bernstein proved that, for all function  $F \in C^0([a, b])$ ,  $\lim_{k \rightarrow +\infty} \|F - \mathbb{B}_k F\|_\infty = 0$ . In Section 3 we show how this result extends to the class of Müntz spaces (i.e., spaces spanned by power functions) on a given positive interval  $[a, b]$ , see Theorem 3.1. Beforehand, in Section 2 we briefly remind the reader how to define operators of the Bernstein-type in Extended Chebyshev spaces.

## 2. Extended Chebyshev spaces and Bernstein operators

Throughout this section,  $[a, b]$  is a fixed non-trivial real interval. For any  $n \geq 0$ , a given  $(n+1)$ -dimensional space  $\mathbb{E} \subset C^n([a, b])$  is said to be an *Extended Chebyshev space* (for short, EC-space) on  $[a, b]$  when any non-zero element of  $\mathbb{E}$  vanishes at most  $n$  times on  $[a, b]$  counting multiplicities up to  $(n+1)$ .

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Let  $\mathbb{E}$  be an  $(n+1)$ -dimensional EC-space on  $[a, b]$ . Then,  $\mathbb{E}$  possesses bases  $(B_0, \dots, B_n)$  such that, for  $i = 0, \dots, n$ ,  $B_i$  vanishes exactly  $i$  times at  $a$  and  $(n-i)$  times at  $b$  and is positive on  $]a, b[$ . We say that such a basis is the *Bernstein basis relative to  $(a, b)$*  if it additionally satisfies  $\sum_{i=0}^n B_i = \mathbb{I}$ , where  $\mathbb{I}$  is the constant function  $\mathbb{I}(x) = 1$ ,  $x \in [a, b]$ . Let us recall that  $\mathbb{E}$  possesses a Bernstein basis relative to  $(a, b)$  if and only if, firstly it contains constants, and secondly the  $n$ -dimensional space  $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$  is an EC-space on  $[a, b]$ . Note that the second property is not an automatic consequence of the first one, see [8] and other references therein.

As an instance, given any pairwise distinct  $\lambda_0, \dots, \lambda_k$ , the so-called *Müntz space*  $M(\lambda_0, \dots, \lambda_k)$ , spanned over a given positive interval  $[a, b]$  (i.e.,  $a > 0$ ) by the power functions  $x^{\lambda_i}$ ,  $0 \leq i \leq k$ , is a  $(k+1)$ -dimensional EC-space on  $[a, b]$ . If  $\lambda_0 = 0$ , since  $D(M(\lambda_0, \dots, \lambda_k)) = M(\lambda_1 - 1, \dots, \lambda_k - 1)$ , the space  $M(\lambda_0, \dots, \lambda_k)$  possesses a Bernstein basis relative to  $(a, b)$ .

For the rest of the section we assume that  $\mathbb{E} \subset C^n([a, b])$  contains constants and that  $D\mathbb{E}$  is an  $(n$ -dimensional) EC-space on  $[a, b]$ . We denote by  $(B_0, \dots, B_n)$  the Bernstein basis relative to  $(a, b)$  in  $\mathbb{E}$ .

**Definition 2.1** A linear operator  $\mathbb{B} : C^0([a, b]) \rightarrow \mathbb{E}$  is said to be a *Bernstein operator based on  $\mathbb{E}$*  when, firstly it is of the form

$$\mathbb{B}F := \sum_{i=0}^k F(\zeta_i) B_i, \quad \text{for some } a = \zeta_0 < \zeta_1 < \dots < \zeta_n = b, \quad (2)$$

and secondly it reproduces a two-dimensional EC-space  $\mathbb{U}$  on  $[a, b]$ , in the sense that  $\mathbb{B}V = V$  for all  $V \in \mathbb{U}$ .

Any Bernstein operator  $\mathbb{B}$  is positive (i.e.,  $F \geq 0$  implies  $\mathbb{B}F \geq 0$ ) and shape preserving due to the properties of Bernstein bases in EC-spaces, see [8]. Everything concerning Bernstein-type operators in EC-spaces with no Bernstein bases can be deduced from Bernstein operators as defined above [8], [9].

**Theorem 2.2** Given  $n \geq 2$ , let  $\mathbb{E} \subset C^n([a, b])$  contain constants. Assume that  $D\mathbb{E}$  is an  $n$ -dimensional EC-space on  $[a, b]$ . For a function  $U \in \mathbb{E}$ , expanded in the Bernstein basis relative to  $(a, b)$  as  $U := \sum_{i=0}^n u_i B_i$ , the following properties are equivalent:

- (i)  $u_0, \dots, u_n$  form a strictly monotonic sequence;
- (ii) there exists a nested sequence  $\mathbb{E}_1 \subset \mathbb{E}_2 \subset \dots \subset \mathbb{E}_{n-1} \subset \mathbb{E}_n := \mathbb{E}$ , where  $\mathbb{E}_1 := \text{span}(\mathbb{I}, U)$  and where, for  $i = 1, \dots, n-1$ ,  $\mathbb{E}_i$  is an  $(i+1)$ -dimensional EC-space on  $[a, b]$ ;
- (iii) there exists a Bernstein operator based on  $\mathbb{E}$  which reproduces  $U$ .

In [8] it was proved that there exists a one-to-one correspondence between the set of all Bernstein operator based on  $\mathbb{E}$  and the set of all two-dimensional EC-spaces  $\mathbb{U}$  they reproduce. In particular, if (i) holds, then the unique Bernstein operator based on  $\mathbb{E}$  reproducing  $U$  is defined by (2) with

$$\zeta_i := U^{-1}(u_i), \quad 0 \leq i \leq n. \quad (3)$$

Note that this is meaningful since (i) implies the strict monotonicity of  $U$  on  $[a, b]$ . Condition (ii) of Th. 2.2 yields the following corollary.

**Corollary 2.3** Given an integer  $n \geq 1$ , consider a nested sequence

$$\mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \dots \subset \mathbb{E}_p \subset \mathbb{E}_{p+1} \subset \dots, \quad (4)$$

where  $\mathbb{E}_n$  contains constants and for any  $p \geq n$ ,  $D\mathbb{E}_p$  is a  $p$ -dimensional EC-space on  $[a, b]$ . Let  $U \in \mathbb{E}_n$  be a non-constant function reproduced by a Bernstein operator  $\mathbb{B}_n$  based on  $\mathbb{E}_n$ . Then,  $U$  is also reproduced by a Bernstein operator  $\mathbb{B}_p$  based on  $\mathbb{E}_p$  for any  $p > n$ .

**Remark 2.4** In the situation described in Corollary 2.3, a natural question arises: given  $F \in C^0([a, b])$ , does the sequence  $\mathbb{B}_k F$ ,  $k \geq n$ , converges to  $F$  in  $C^0([a, b])$  equipped with the infinite norm? Obviously,

for this to be true for any  $F \in C^0([a, b])$ , it is necessary that  $\cup_{k \geq n} \mathbb{E}_k$  be dense in  $C^0([a, b])$ . The example of Müntz spaces proves that this is not always satisfied.

### 3. Müntz spaces over positive intervals

Throughout this section we consider a fixed positive interval  $[a, b]$ , a fixed infinite sequence of real numbers  $\lambda_k$ ,  $k \geq 0$ , assumed to satisfy

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_k < \lambda_{k+1} < \dots, \quad \lim_{k \rightarrow +\infty} \lambda_k = +\infty. \quad (5)$$

We are interested with the corresponding nested sequence of Müntz spaces

$$M(\lambda_0) \subset M(\lambda_0, \lambda_1) \subset \dots \subset M(\lambda_0, \dots, \lambda_k) \subset M(\lambda_0, \dots, \lambda_k, \lambda_{k+1}) \subset \dots \quad (6)$$

Given any  $n \geq 1$ , for each  $k \geq n$ , we can select a Bernstein operator  $\mathbb{B}_k$  based on  $M(\lambda_0, \dots, \lambda_k)$ . Assume the sequence  $\mathbb{B}_k$ ,  $k \geq n$ , to satisfy

$$\lim_{k \rightarrow +\infty} \|F - \mathbb{B}_k F\|_\infty = 0 \quad \text{for any } F \in C^0([a, b]). \quad (7)$$

Then, the union of all spaces  $M(\lambda_0, \dots, \lambda_k)$ ,  $k \geq 0$ , is dense in  $C^0([a, b])$  equipped with the infinite norm. As is well-known, this holds if and only if the sequence (5) fulfils the so-called *Müntz density condition* below [4], [6],

$$\sum_{i \geq 1} \frac{1}{\lambda_i} = +\infty. \quad (8)$$

As an instance, the Müntz condition (8) is satisfied when  $\lambda_k = \ell + 1$  for all  $k \geq 1$ . This case was addressed in [8]. Convergence – in the sense of (7) – was proved there under the assumption that each  $\mathbb{B}_k$  reproduced the function  $x^{\lambda_1}$ . This convergence result includes the classical Bernstein operators [5] obtained with  $\ell = 0$ . Below we extend it to the general interesting situation of sequences of Müntz Bernstein operators  $\mathbb{B}_k$  all reproducing the same two-dimensional EC-space (see Remark 2.4).

**Theorem 3.1** *Given  $n \geq 1$ , let  $\mathbb{E}_1 \subset M(\lambda_0, \dots, \lambda_n)$  be a two-dimensional EC-space reproduced by a Bernstein operator  $\mathbb{B}_k$  based on  $M(\lambda_0, \dots, \lambda_k)$  for any  $k \geq n$ . Then, if the Müntz density condition (8) holds, the sequence  $\mathbb{B}_k$ ,  $k \geq n$ , converges in the sense of (7).*

Before starting the proof, let us introduce some notations. For  $k \geq 1$ , denote by  $(B_{k,0}, \dots, B_{k,k})$  the Bernstein basis relative to  $(a, b)$  in the Müntz space  $M(\lambda_0, \dots, \lambda_k)$ . We consider the functions

$$U^*(x) = x^{\lambda_1}, \quad V_p(x) := x^{\lambda_p}, \quad p \geq 2, \quad x \in [a, b],$$

expanded in the successive Bernstein bases as

$$U^* = \sum_{i=0}^k u_{k,i}^* B_{k,i} \quad \text{for all } k \geq 1, \quad V_p = \sum_{i=0}^k v_{p,k,i} B_{k,i} \quad \text{for all } k \geq p. \quad (9)$$

With these notations, the key-point to prove Theorem 3.1 is the following lemma, for the proof of which we refer to [2], see also [1].

**Lemma 3.2** *Assume that the Müntz density condition (8) holds. Then, we have*

$$\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} \left| (u_{k,i}^*)^{\frac{\lambda_p}{\lambda_1}} - v_{p,k,i} \right| = 0 \quad \text{for all } p \geq 2. \quad (10)$$

*Proof of Theorem 3.1:* • Let us start with the simplest example  $n = 1$ . Then,  $\mathbb{E}_1 = \text{span}(\mathbb{1}, U^*)$ . For each  $k \geq 1$ , the unique operator  $\mathbb{B}_k^*$  which reproduces  $\mathbb{E}_1$  is given by

$$\mathbb{B}_k^* F := \sum_{i=0}^k F(\zeta_{k,i}^*) B_{k,i}, \quad \text{with, for } i = 0, \dots, k, \quad \zeta_{k,i}^* := (u_{k,i}^*)^{\frac{1}{\lambda_1}} \quad (11)$$

According to Korovkin's theorem for positive linear operators [7], we just have to select a function  $F$  so that  $\mathbb{I}, U^*, F$  span a three-dimensional EC-space on  $[a, b]$  and prove that  $\lim_{k \rightarrow +\infty} \|F - \mathbb{B}_k^* F\|_\infty = 0$  for this specific  $F$ . We can thus choose for instance  $F := V_2$ . Actually we will more generally prove the result with  $F = V_p$ , for any  $p \geq 2$ . Using (9) and (11), we obtain, for any  $k \geq p$ ,

$$\|\mathbb{B}_k^* V_p - V_p\|_\infty = \left\| \sum_{i=0}^k (V_p(\zeta_{k,i}^*) - v_{p,k,i}) B_{k,i} \right\|_\infty \leq \max_{0 \leq i \leq k} |V_p(\zeta_{k,i}^*) - v_{p,k,i}|. \quad (12)$$

On account of (11), Lemma 3.2 yields the expected result

$$\lim_{k \rightarrow +\infty} \|\mathbb{B}_k^* V_p - V_p\|_\infty = 0 \quad \text{for each } p \geq 2.$$

• We now assume that  $n > 1$ . Select a strictly increasing function  $U \in \mathbb{E}_1$ . Condition (ii) of Theorem 2.2 enables us to select a function  $V \in M(\lambda_0, \dots, \lambda_n)$  so that the functions  $\mathbb{I}, U, V$  span a three-dimensional EC-space on  $[a, b]$ . For any  $k \geq n$ , expand  $U, V$  as

$$U = \sum_{i=0}^k u_{k,i} B_{k,i}, \quad V = \sum_{i=0}^k v_{k,i} B_{k,i}.$$

We know that, for each  $k \geq n$ , the sequence  $(u_{k,0}, \dots, u_{k,k})$  is strictly increasing, and that the Bernstein operator  $\mathbb{B}_k$  is defined by formula (2) with  $\zeta_{k,i} := U^{-1}(u_{k,i})$  for  $i = 0, \dots, k$ . Via expansions of  $U$  and  $V$  in the basis  $(\mathbb{I}, U^*, V_2, \dots, V_n)$  of the Müntz space  $M(\lambda_0, \dots, \lambda_n)$ , Lemma 3.2 readily proves that

$$\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |U(\zeta_{k,i}^*) - u_{k,i}| = 0 = \lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |V(\zeta_{k,i}^*) - v_{k,i}| \quad (13)$$

The left part in (13) can be written as  $\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |U(\zeta_{k,i}^*) - U(\zeta_{k,i})| = 0$ . On this account, the uniform continuity of the function  $V \circ U^{-1}$  and the right part in (13) prove that  $\lim_{k \rightarrow +\infty} \max_{0 \leq i \leq k} |V(\zeta_{k,i}) - v_{k,i}| = 0$ , thus implying that  $\lim_{k \rightarrow +\infty} \|\mathbb{B}_k V - V\|_\infty = 0$ . By Korovkin's theorem, (7) is satisfied.  $\square$

**Remark 3.3** Given  $n \geq 2$ , one can apply Theorem 3.1 with  $\mathbb{E}_1 := \text{span}(\mathbb{I}, V_n) = M(\lambda_0, \lambda_n)$ , due to the nested sequence of Müntz spaces  $M(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_n)$  for  $1 \leq i \leq n$ . Note that Theorem 3.1 contains in particular the *Bernstein-type result* expected in [3].

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