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MÖBIUS FUNCTION OF SEMIGROUP POSETS THROUGH HILBERT SERIES

JONATHAN CHAPPELON*, IGNACIO GARCÍA-MARCO, LUIS PEDRO MONTEJANO,
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ABSTRACT. In this paper, we investigate the Möbius function $\mu_{\mathcal{S}}$ associated to a (locally finite) poset arising from a semigroup \mathcal{S} of \mathbb{Z}^m . In order to do this, we introduce and develop a new approach to study $\mu_{\mathcal{S}}$ by using the Hilbert series associated to \mathcal{S} . The latter allows us to provide formulas for $\mu_{\mathcal{S}}$ when \mathcal{S} is a semigroup with unique Betti element, and when \mathcal{S} is a complete intersection numerical semigroup with three generators. We also give a characterization for a locally finite poset to be isomorphic to a semigroup poset. We are thus able to calculate the Möbius function of certain posets (for instance the classical arithmetic Möbius function) by computing the Möbius function of the corresponding semigroup poset.

1. INTRODUCTION

The *Möbius function* is an important concept associated to (*locally finite*) posets¹. It can be considered as a generalization of the classical Möbius arithmetic function on the integers (given by the Möbius function of the poset obtained from the positive integers partially ordered by divisibility). The Möbius function has been extremely useful in tackling many different problems. For instance, the inclusion-exclusion principle can be retrieved by considering the set of all subsets of a finite set partially ordered by inclusion. We refer the reader to [18] for a large number of applications of the Möbius function. In this paper, we investigate the Möbius function associated to posets arising naturally from subsemigroups of \mathbb{Z}^m as follows. Let a_1, \dots, a_n be nonzero vectors in \mathbb{Z}^m and let $\mathcal{S} = \langle a_1, \dots, a_n \rangle$ denote the semigroup generated by a_1, \dots, a_n , that is,

$$\mathcal{S} = \langle a_1, \dots, a_n \rangle = \{x_1 a_1 + \dots + x_n a_n \mid x_1, \dots, x_n \in \mathbb{N}\}.$$

We say that \mathcal{S} is *pointed* if $\mathcal{S} \cap (-\mathcal{S}) = \{0\}$, where $-\mathcal{S} := \{-x \mid x \in \mathcal{S}\}$. Whenever \mathcal{S} is pointed, \mathcal{S} induces on \mathbb{Z}^m a poset structure whose partial order $\leq_{\mathcal{S}}$ is defined by $x \leq_{\mathcal{S}} y$ if and only if $y - x \in \mathcal{S}$ for all x and y in \mathbb{Z}^m . This (locally finite) poset will be denoted by $(\mathbb{Z}^m, \leq_{\mathcal{S}})$. We denote by $\mu_{\mathcal{S}}$ the Möbius function associated to $(\mathbb{Z}^m, \leq_{\mathcal{S}})$. As far as we are aware, $\mu_{\mathcal{S}}$ has only been investigated when \mathcal{S} is a *numerical semigroup*, i.e., when $\mathcal{S} \subset \mathbb{N}$ and $\gcd\{a_1, \dots, a_n\} = 1$. Moreover, the only known results concerning $\mu_{\mathcal{S}}$ are an old theorem due to Deddens [5] which determines the value of $\mu_{\mathcal{S}}$ when \mathcal{S} has exactly two generators, and a recent paper due to Chappelton and Ramírez Alfonsín [4] where the authors investigate $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, a + d, \dots, a + kd \rangle$ with $a, k, d \in \mathbb{Z}^+$. In both papers the authors approach the problem by a thorough study of the intrinsic properties of each semigroup. It is worth pointing out that the results and techniques developed in this paper are deeply inspired by those of [4].

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¹The Möbius function was first introduced by Gian-Carlo Rota exactly 50 years ago, see [18].

In this work we introduce and develop a new and more general method to study $\mu_{\mathcal{S}}$ by means of the *Hilbert series of the semigroup* \mathcal{S} . This enables us to provide formulas for $\mu_{\mathcal{S}}$ when \mathcal{S} belongs to some families of semigroups. Finally, we study when a locally finite poset is isomorphic to a semigroup poset.

This paper is organized as follows. In the next section, we review some classic notions of the Möbius function and interpret and use them for semigroup posets. In Section 3, we recall some definitions and results about Hilbert series and prove two general results relating the Möbius function of $(\mathbb{Z}^m, \leq_{\mathcal{S}})$ and the Hilbert series of \mathcal{S} for every pointed semigroup \mathcal{S} . The latter are the key results that we use in Section 4 to both provide explicit formulas for $\mu_{\mathcal{S}}$ when $\mathcal{S} = \mathbb{N}^m$ or \mathcal{S} is a *semigroup with a unique Betti element* or $\mathcal{S} = \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$ is a *complete intersection numerical semigroup*. Finally, in Section 5, we determine when a locally finite poset \mathcal{P} is isomorphic to the poset associated to a semigroup \mathcal{S} and, thus, $\mu_{\mathcal{P}}$ can be computed by means of $\mu_{\mathcal{S}}$. In particular, we are able to recover the classical arithmetic Möbius function and the Möbius function of the set of all subsets of a finite set partially ordered by inclusion.

2. MÖBIUS FUNCTION ASSOCIATED TO A SEMIGROUP POSET

Let $(\mathcal{P}, \leq_{\mathcal{P}})$ be a partially ordered set, or *poset* for short. The *strict partial order* $<_{\mathcal{P}}$ is the reduction of $\leq_{\mathcal{P}}$ given by $a <_{\mathcal{P}} b$ if and only if $a \leq_{\mathcal{P}} b$ and $a \neq b$. For any a and b in the poset \mathcal{P} , the *segment* between a and b is defined by

$$[a, b]_{\mathcal{P}} := \{c \in \mathcal{P} \mid a \leq_{\mathcal{P}} c \leq_{\mathcal{P}} b\}.$$

A poset is said to be *locally finite* if every segment has finite cardinality. In this paper, we only consider locally finite posets.

Let a and b be elements of the poset \mathcal{P} . A *chain* of length $l \geq 0$ between a and b is a subset of $[a, b]_{\mathcal{P}}$ containing a and b , with cardinality $l + 1$ and totally ordered by $<_{\mathcal{P}}$, that is $\{a_0, a_1, \dots, a_l\} \subset [a, b]_{\mathcal{P}}$ such that

$$a = a_0 <_{\mathcal{P}} a_1 <_{\mathcal{P}} a_2 <_{\mathcal{P}} \dots <_{\mathcal{P}} a_{l-1} <_{\mathcal{P}} a_l = b.$$

For any nonnegative integer l , we denote by $C_l(a, b)$ the set of all chains of length l between a and b . The cardinality of $C_l(a, b)$ is denoted by $c_l(a, b)$. This number always exists because the poset \mathcal{P} is supposed to be locally finite. For instance, the number of chains $c_2(2, 12)$, where the poset is the set \mathbb{N} partially ordered by divisibility, is equal to 2. Indeed, there are exactly 2 chains of length 2 between 2 and 12 in $[2, 12]_{\mathbb{N}} = \{2, 4, 6, 12\}$, which are $\{2, 4, 12\}$ and $\{2, 6, 12\}$.

For any locally finite poset \mathcal{P} , the *Möbius function* $\mu_{\mathcal{P}}$ is the integer-valued function on $\mathcal{P} \times \mathcal{P}$ defined by

$$(1) \quad \mu_{\mathcal{P}}(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b),$$

for all elements a and b of the poset \mathcal{P} . Note that this sum is always finite because, for a and b given, there exists a maximal length of a possible chain between a and b since the segment $[a, b]_{\mathcal{P}}$ has finite cardinality.

The concept of Möbius function for a locally finite poset (\mathcal{P}, \leq) was introduced by Rota in [18]. In this paper, Rota proves the following property of the Möbius function: for all $(a, b) \in \mathcal{P} \times \mathcal{P}$,

$$(2) \quad \mu_{\mathcal{P}}(a, a) = 1 \quad \text{and} \quad \sum_{c \in [a, b]_{\mathcal{P}}} \mu_{\mathcal{P}}(a, c) = 0.$$

In this work we will consider posets associated to semigroups of \mathbb{Z}^m . We will begin by summarizing some generalities on semigroups that are useful for the understanding of this work. We refer the readers to [3] or, more generally, to [14] for further details.

Let $\mathcal{S} := \langle a_1, \dots, a_n \rangle \subset \mathbb{Z}^m$ denote the subsemigroup of \mathbb{Z}^m generated by $a_1, \dots, a_n \in \mathbb{Z}^m$, i.e.,

$$\mathcal{S} := \langle a_1, \dots, a_n \rangle = \{x_1 a_1 + \dots + x_n a_n \mid x_1, \dots, x_n \in \mathbb{N}\}.$$

The semigroup \mathcal{S} induces the binary relation $\leq_{\mathcal{S}}$ on \mathbb{Z}^m given by

$$x \leq_{\mathcal{S}} y \iff y - x \in \mathcal{S}.$$

It turns out that $\leq_{\mathcal{S}}$ is an order if and only if \mathcal{S} is pointed (see [3]). Moreover, whenever \mathcal{S} is pointed the poset $(\mathbb{Z}^m, \leq_{\mathcal{S}})$ is locally finite.

We denote by $\mu_{\mathcal{S}}$ the Möbius function associated to $(\mathbb{Z}^m, \leq_{\mathcal{S}})$. It is easy to see that $\mu_{\mathcal{S}}$ can be considered as a univariate function of \mathbb{Z}^m . Indeed, for all $x, y \in \mathbb{Z}^m$ and for all $l \geq 0$, we have that

$$(3) \quad c_l(x, y) = c_l(0, y - x).$$

The above follows since the set $C_l(x, y)$ is in bijection with $C_l(0, y - x)$. Indeed the map that assigns the chain $\{x_0, x_1, \dots, x_l\} \in C_l(x, y)$ to the chain $\{0, x_1 - x_0, \dots, x_l - x_0\} \in C_l(0, y - x)$ is clearly a bijection. Thus, by definition of $\mu_{\mathcal{S}}$ and equality (3) we obtain

$$\mu_{\mathcal{S}}(x, y) = \mu_{\mathcal{S}}(0, y - x)$$

for all $x, y \in \mathbb{Z}$.

In the sequel of this paper we shall only consider the reduced Möbius function $\mu_{\mathcal{S}} : \mathbb{Z}^m \rightarrow \mathbb{Z}$ defined by

$$\mu_{\mathcal{S}}(x) := \mu_{\mathcal{S}}(0, x), \quad \text{for all } x \in \mathbb{Z}^m.$$

Thus, the formula given by (2) can be more easily presented when the locally finite poset is $(\mathbb{Z}^m, \leq_{\mathcal{S}})$.

Proposition 2.1. *Let \mathcal{S} be a pointed semigroup and let $x \in \mathbb{Z}^m$. Then,*

$$\sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(x - b) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Firstly, from (1) we observe that $\mu_{\mathcal{S}}(b) = 0$ for all $b \notin \mathcal{S}$. Since \mathcal{S} is pointed, if $b \in \mathcal{S}$ then $-b \notin \mathcal{S}$ and, hence, $\sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(0 - b) = \mu_{\mathcal{S}}(0) = 1$. Take $x \neq 0$, then from (2)

$$0 = \sum_{b \in [0, x]_{\mathbb{Z}^m}} \mu_{\mathcal{S}}(b) = \sum_{\substack{b \in \mathcal{S} \\ x - b \in \mathcal{S}}} \mu_{\mathcal{S}}(b) = \sum_{\substack{b \in \mathcal{S} \\ x - b \in \mathcal{S}}} \mu_{\mathcal{S}}(x - b) = \sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(x - b).$$

□

The formula presented in Proposition 2.1 will be very useful to get most of our results.

3. THE HILBERT AND MÖBIUS SERIES

In this section we present two results (Theorem 3.2 and Theorem 3.3), both relating the Hilbert series of the semigroup \mathcal{S} with the Möbius function of the poset $(\mathbb{Z}^m, \leq_{\mathcal{S}})$. Before proving these theorems we recall some basic notions on multivariate Hilbert series. For a thorough study of multivariate Hilbert series we refer the reader to [12].

Let k be any field and let $\mathcal{S} = \langle a_1, \dots, a_n \rangle$ be a subsemigroup of \mathbb{Z}^m . The semigroup \mathcal{S} induces a grading in the ring of polynomials $k[x_1, \dots, x_n]$ by assigning $\deg_{\mathcal{S}}(x_i) := a_i$ for all $i \in \{1, \dots, n\}$. Then the \mathcal{S} -degree of the monomial $m := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is $\deg_{\mathcal{S}}(m) := \sum \alpha_i a_i$; we say that a polynomial is \mathcal{S} -homogeneous if all its monomials have the same \mathcal{S} -degree.

An ideal is \mathcal{S} -homogeneous if it is generated by \mathcal{S} -homogeneous polynomials. For all $b \in \mathbb{Z}^m$, we denote by $k[x_1, \dots, x_n]_b$ the k -vector space formed by all \mathcal{S} -homogeneous polynomials of \mathcal{S} -degree b .

Consider $I \subset k[x_1, \dots, x_n]$ an \mathcal{S} -homogeneous ideal, then for all $b \in \mathbb{Z}^m$ we denote by I_b the k -vector space formed by the \mathcal{S} -homogeneous polynomials in I of \mathcal{S} -degree b . Note that I_b is a k -vector subspace of $k[x_1, \dots, x_n]_b$. The quotient ring $k[x_1, \dots, x_n]/I$ is also graded by taking $(k[x_1, \dots, x_n]/I)_b := k[x_1, \dots, x_n]_b/I_b$ for all $b \in \mathbb{Z}^m$.

Whenever \mathcal{S} is pointed the k -vector space $k[x_1, \dots, x_n]_b$ has finite dimension for all $b \in \mathbb{Z}^m$ by [12, Proposition 4.1.19]. Hence, one can define the *multigraded Hilbert function* of $M := k[x_1, \dots, x_n]/I$ as

$$HF_M : \mathbb{Z}^m \longrightarrow \mathbb{N},$$

where $HF_M(b) := \dim_k(M_b) = \dim_k(k[x_1, \dots, x_n]_b) - \dim_k(I_b)$ for all $b \in \mathbb{Z}^m$.

For every $b = (b_1, \dots, b_m) \in \mathbb{Z}^m$, we denote by \mathbf{t}^b the monomial $t_1^{b_1} \cdots t_m^{b_m}$ in the Laurent polynomial ring $\mathbb{Z}[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$. We define the *multivariate Hilbert series* of M as the following formal power series in $\mathbb{Z}[[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]]$:

$$\mathcal{H}_M(\mathbf{t}) := \sum_{b \in \mathbb{Z}^m} HF_M(b) \mathbf{t}^b.$$

For every \mathcal{S} -homogeneous ideal I , the Hilbert series of $M = k[x_1, \dots, x_n]/I$ can be expressed as a quotient of polynomials in the Laurent polynomial ring in the following way (see [12, Theorem 5.8.19]):

$$\mathcal{H}_M(\mathbf{t}) = \frac{\mathbf{t}^\alpha h(t_1, \dots, t_m)}{(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n})},$$

where $\alpha \in \mathbb{Z}^m$ and $h(t_1, \dots, t_m) \in \mathbb{Z}[t_1, \dots, t_m]$.

Denote by $I_{\mathcal{S}}$ the *toric ideal* of \mathcal{S} , i.e., the kernel of the homomorphism of k -algebras

$$\varphi : k[x_1, \dots, x_n] \longrightarrow k[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$$

induced by $\varphi(x_i) = \mathbf{t}^{a_i}$ for all $i \in \{1, \dots, n\}$.

It is well known that $I_{\mathcal{S}}$ is \mathcal{S} -homogeneous (see [19, Corollary 4.3]). Therefore, it makes sense to study the multivariate Hilbert series of $k[x_1, \dots, x_n]/I_{\mathcal{S}}$ with respect to the grading induced by \mathcal{S} .

Proposition 3.1. *Let \mathcal{S} be a pointed semigroup and $M := k[x_1, \dots, x_n]/I_{\mathcal{S}}$. Then,*

$$\mathcal{H}_M(\mathbf{t}) = \sum_{b \in \mathcal{S}} \mathbf{t}^b.$$

Proof. Take $b \in \mathbb{Z}^m$, then $k[x_1, \dots, x_n]_b = \{0\}$ and $HF_M(b) = 0$ whenever $b \notin \mathcal{S}$. Let us prove that $HF_M(b) = 1$ for all $b \in \mathcal{S}$. Indeed, φ induces an isomorphism of k -vector spaces between M_b and $\{\alpha \mathbf{t}^b \mid \alpha \in k\}$, for all $b \in \mathcal{S}$. Hence, $HF_M(b) = \dim_k(M_b) = 1$. \square

From now on, we denote by $\mathcal{H}_{\mathcal{S}}(\mathbf{t})$ the multivariate Hilbert series of $k[x_1, \dots, x_n]/I_{\mathcal{S}}$ and we call it the *Hilbert series* of \mathcal{S} . Let us now relate $\mu_{\mathcal{S}}$ with the Hilbert series of \mathcal{S} .

Theorem 3.2. *Let c_1, \dots, c_k be nonzero vectors in \mathbb{Z}^m and denote*

$$(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k}) \mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b \mathbf{t}^b \in \mathbb{Z}[[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]].$$

Then,

$$\sum_{b \in \mathbb{Z}^m} f_b \mu_{\mathcal{S}}(x - b) = 0 \text{ for all } x \notin \{\sum_{i \in A} c_i \mid A \subset \{1, \dots, k\}\}.$$

Proof. Firstly, from Proposition 3.1, we obtain that

$$f_b = \sum_{\substack{A \subset \{1, \dots, k\} \\ b = \sum_{i \in A} c_i \in \mathcal{S}}} (-1)^{|A|},$$

for all $b \in \mathbb{Z}^m$. Set $\Delta := \{\sum_{i \in A} c_i \mid A \subset \{1, \dots, k\}\}$. By Proposition 2.1, for all $x \notin \Delta$ and $A \subset \{1, \dots, k\}$ we have that

$$\sum_{b \in \mathcal{S}} \mu_{\mathcal{S}} \left(x - \sum_{i \in A} a_i - b \right) = 0.$$

Hence, for all $x \notin \Delta$ it follows that

$$\sum_{b \in \mathbb{Z}^m} \alpha_b \mu_{\mathcal{S}}(x - b) = \sum_{b \in \mathcal{S}} \sum_{A \subset \{1, \dots, k\}} (-1)^{|A|} \mu_{\mathcal{S}} \left(x - \sum_{i \in A} c_i - b \right) = 0,$$

where $\alpha_b = \sum_{\substack{A \subset \{1, \dots, k\} \\ b = \sum_{i \in A} c_i \in \mathcal{S}}} (-1)^{|A|} = f_b$, which completes the proof. \square

We notice that the formula $(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k}) \mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b \mathbf{t}^b$ might have an infinite number of terms. Nevertheless, for every $x \in \mathbb{Z}^m$ the formula $\sum_{b \in \mathbb{Z}^m} f_b \mu_{\mathcal{S}}(x - b) = 0$ only involves a finite number of nonzero summands since \mathcal{S} is pointed.

As a consequence of this result, whenever we know an explicit expression of $\mathcal{H}_{\mathcal{S}}(\mathbf{t})$ as

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \frac{f(\mathbf{t})}{(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k})},$$

where $f(\mathbf{t}) \in \mathbb{Z}[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$ for some $c_1, \dots, c_k \in \mathbb{Z}^m$, we can derive a recursive formula for the Möbius function. This is the case of the so called almost arithmetic semigroups, i.e., subsemigroups $\mathcal{S} = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}$ such that a_1, \dots, a_{n-1} form an arithmetic progression and $a_n \in \mathbb{Z}^+$. In [16, Theorem 3] the authors find an expression of the Hilbert series of these semigroups as a quotient of two polynomials needing some previously computed parameters. In this setting, Theorem 3.2 provides a recursive formula for $\mu_{\mathcal{S}}$ which also depends on these parameters. Unfortunately, this recursive formula does not allow us to give an explicit expression of $\mu_{\mathcal{S}}$.

Now, we consider the *Möbius series* $\mathcal{G}_{\mathcal{S}}$, i.e., the generating function of the Möbius function

$$\mathcal{G}_{\mathcal{S}}(\mathbf{t}) := \sum_{b \in \mathbb{Z}^m} \mu_{\mathcal{S}}(b) \mathbf{t}^b \in \mathbb{Z}[[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]].$$

Theorem 3.3. *Let \mathcal{S} be a pointed semigroup. Then,*

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) \cdot \mathcal{G}_{\mathcal{S}}(\mathbf{t}) = 1.$$

Proof. From the definitions of $\mathcal{H}_{\mathcal{S}}(\mathbf{t})$ and $\mathcal{G}_{\mathcal{S}}(\mathbf{t})$, we obtain that

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) \cdot \mathcal{G}_{\mathcal{S}}(\mathbf{t}) = \left(\sum_{b \in \mathcal{S}} \mathbf{t}^b \right) \left(\sum_{b \in \mathbb{Z}^m} \mu_{\mathcal{S}}(b) \mathbf{t}^b \right) = \sum_{b \in \mathbb{Z}^m} \left(\sum_{c \in \mathcal{S}} \mu_{\mathcal{S}}(b - c) \right) \mathbf{t}^b.$$

The result follows by Proposition 2.1. \square

Therefore, Theorem 3.3 states that whenever we can explicitly compute the inverse of $\mathcal{H}_{\mathcal{S}}(\mathbf{t})$ we will be able to obtain $\mu_{\mathcal{S}}$. Corollary 3.4 (below) explains how to obtain $\mu_{\mathcal{S}}$ when $\mathcal{H}_{\mathcal{S}}(\mathbf{t})$ has a certain form.

Let $B = (b_1, b_2, \dots, b_k)$ be a k -tuple of nonzero vectors in \mathbb{Z}^m such that the semigroup $\mathcal{T} := \langle b_1, \dots, b_k \rangle$ is pointed and take $b \in \mathbb{Z}^m$. The *denumerant* $d_B(b)$ is the number of

non-negative integer representations of b by b_1, \dots, b_k , that is, the number of solutions of $b = \sum_{i=1}^k x_i b_i$, where x_i is a nonnegative integer for all i . As \mathcal{T} is pointed, then $d_B(b)$ is finite for all $b \in \mathbb{Z}^m$ and $d_B(0) = 1$. It is well known (see, e.g., [12, Theorem 5.8.15]) that its generating function is given by

$$\sum_{b \in \mathbb{Z}^m} d_B(b) \mathbf{t}^b = \frac{1}{(1 - \mathbf{t}^{b_1})(1 - \mathbf{t}^{b_2}) \cdots (1 - \mathbf{t}^{b_k})}.$$

Corollary 3.4. *Let \mathcal{S} be a pointed semigroup such that its Hilbert series is of the form*

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \frac{(1 - \mathbf{t}^{b_1})(1 - \mathbf{t}^{b_2}) \cdots (1 - \mathbf{t}^{b_k})}{\sum_{i=1}^r c_i \mathbf{t}^{d_i}},$$

where $b_1, \dots, b_k \in \mathbb{Z}^m$ are nonzero, $d_1, \dots, d_r \in \mathbb{Z}^m$ and c_1, \dots, c_r are integers. Then,

$$\mu_{\mathcal{S}}(b) = \sum_{i=1}^r c_i \cdot d_B(b - d_i),$$

for all $b \in \mathbb{Z}^m$, where $B = (b_1, \dots, b_k)$.

Proof. From Theorem 3.3 and the definition of the denominator, we have

$$\begin{aligned} \mathcal{G}_{\mathcal{S}}(\mathbf{t}) &= \frac{1}{\mathcal{H}_{\mathcal{S}}(\mathbf{t})} = \frac{\sum_{i=1}^r c_i \mathbf{t}^{d_i}}{(1 - \mathbf{t}^{b_1})(1 - \mathbf{t}^{b_2}) \cdots (1 - \mathbf{t}^{b_k})} = \left(\sum_{i=1}^r c_i \mathbf{t}^{d_i} \right) \sum_{b \in \mathbb{Z}^m} d_A(b) \mathbf{t}^b \\ &= c_1 \sum_{b \in \mathbb{Z}^m} d_A(b) \mathbf{t}^{m+d_1} + \cdots + c_r \sum_{b \in \mathbb{Z}^m} d_A(b) \mathbf{t}^{b+d_r} \\ &= c_1 \sum_{b \in \mathbb{Z}^m} d_A(b - d_1) \mathbf{t}^b + \cdots + c_r \sum_{b \in \mathbb{Z}^m} d_A(b - d_r) \mathbf{t}^b \\ &= \sum_{b \in \mathbb{Z}^m} \left(\sum_{i=1}^r c_i \cdot d_A(b - d_i) \right) \mathbf{t}^b. \end{aligned}$$

This concludes the proof. □

The above result is particularly important in the following section in order to obtain explicit formulas of the Möbius function $\mu_{\mathcal{S}}$ when \mathcal{S} is a complete intersection semigroup. Recall that a pointed semigroup $\mathcal{S} = \langle a_1, \dots, a_n \rangle$ is a complete intersection semigroup if its corresponding toric ideal $I_{\mathcal{S}}$ is a complete intersection. Moreover, $I_{\mathcal{S}}$ is a complete intersection if it is generated by $n - d$ \mathcal{S} -homogeneous polynomials, where d is the dimension of the \mathbb{Q} -vector space spanned by a_1, \dots, a_n . Whenever $I_{\mathcal{S}}$ is generated by $n - d$ \mathcal{S} -homogeneous polynomials of \mathcal{S} -degrees $b_1, \dots, b_{n-d} \in \mathbb{Z}^m$, by [12, Page 341], we have that

$$(4) \quad \mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \frac{(1 - \mathbf{t}^{b_1}) \cdots (1 - \mathbf{t}^{b_{n-d}})}{(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n})}.$$

So, we can use Corollary 3.4.

For characterizations of complete intersection toric ideals we refer the reader to [2, 7].

4. EXPLICIT FORMULAS FOR THE MÖBIUS FUNCTION

This section is devoted to obtain explicit formulas for the Möbius function $\mu_{\mathcal{S}}$ when \mathcal{S} belongs to some families of subsemigroups of \mathbb{Z}^m . As far as we are aware, the only known results concerning $\mu_{\mathcal{S}}$ are an old theorem due to Deddens [5] that determines the values of $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, b \rangle \subset \mathbb{N}$, and a recent paper due to Chappelton and Ramírez Alfonsín [4] where the authors investigate $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, a + d, \dots, a + kd \rangle \subset \mathbb{N}$ with $a, k, d \in \mathbb{Z}^+$ and obtain a semi-explicit formula when a is even and $k = 2$. In both papers the authors approach the problem by a thorough study of the intrinsic properties of each semigroup. Here we first provide in Theorem 4.1 an explicit formula for $\mu_{\mathcal{S}}$ when $\mathcal{S} = \mathbb{N}^m$; this formula will be used in the last section. We then provide formulas for $\mu_{\mathcal{S}}$ when \mathcal{S} is a semigroup with a unique Betti element (Theorem 4.2) and when \mathcal{S} is a complete intersection numerical semigroup generated by three elements (Theorem 4.5).

These families generalize the two mentioned above, since $\mathcal{S} = \langle a, b \rangle \subset \mathbb{N}$ is a semigroup with a unique Betti element and $\mathcal{S} = \langle a, a + d, a + 2d \rangle \subset \mathbb{N}$ with a even and $\gcd\{a, d\} = 1$ is a complete intersection (see [14, Theorem 3.5] or [1, Corollary 3.7]).

The results included in this section are consequences of Theorem 3.3 (for completeness, we also include a second proof of Theorem 4.5 by using Theorem 3.2 as an appendix).

4.1. The semigroup \mathbb{N}^m .

Let $\{e_1, \dots, e_m\}$ denote the canonical basis of \mathbb{N}^m . For $\mathcal{S} = \langle e_1, \dots, e_m \rangle = \mathbb{N}^m$ the following result holds.

Theorem 4.1. *For $\mathcal{S} = \mathbb{N}^m$, the Möbius function is given by*

$$\mu_{\mathbb{N}^m}(x) = \begin{cases} (-1)^{|A|} & \text{if } x = \sum_{i \in A} e_i \text{ for some } A \subset \{1, \dots, m\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We observe that

$$\mathcal{H}_{\mathbb{N}^m}(\mathbf{t}) = \sum_{b \in \mathbb{N}^m} \mathbf{t}^b = \frac{1}{(1 - t_1) \cdots (1 - t_m)}.$$

Therefore, by Theorem 3.3 we derive that

$$\mathcal{G}_{\mathbb{N}^m}(\mathbf{t}) = (1 - t_1) \cdots (1 - t_m) = \sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} \prod_{i \in A} t_i = \sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} \mathbf{t}^{\sum_{i \in A} e_i},$$

and the result follows. \square

4.2. Semigroups with a unique Betti element.

A semigroup $\mathcal{S} \subset \mathbb{N}^m$ is said to have a *unique Betti element* $b \in \mathbb{N}^m$ if its corresponding toric ideal is generated by a set of \mathcal{S} -homogeneous polynomials of common \mathcal{S} -degree b . These semigroups were studied in detail by García-Sánchez, Ojeda and Rosales in [8]. In particular they prove in [8, Corollary 10] that they are complete intersection semigroups.

Theorem 4.2. *Let $\mathcal{S} = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}^m$ be a semigroup with a unique Betti element $b \in \mathbb{N}^m$ and denote by d the dimension of the \mathbb{Q} -vector space generated by a_1, \dots, a_n . Then,*

$$\mu_{\mathcal{S}}(x) = \sum_{j=1}^t (-1)^{|A_j|} \binom{k_{A_j} + n - d - 1}{k_{A_j}},$$

where $\{A_1, \dots, A_t\} = \{A \subset \{1, \dots, n\} \mid \exists k_A \in \mathbb{N} \text{ such that } x - \sum_{i \in A} a_i = k_A b\}$.

Proof. By (4) we have that

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \frac{(1 - \mathbf{t}^b)^{n-d}}{\prod_{i=1}^n (1 - \mathbf{t}^{a_i})} = \frac{(1 - \mathbf{t}^b)^{n-d}}{\sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} \mathbf{t}^{\sum_{i \in A} a_i}}.$$

Thus, applying Corollary 3.4, we have that for all $x \in \mathbb{Z}^m$

$$\mu_{\mathcal{S}}(x) = \sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} d_B \left(x - \sum_{i \in A} a_i \right),$$

where B is the $(n-d)$ -uple (b, \dots, b) . The equality

$$d_B(y) = \begin{cases} \binom{k+n-d-1}{k} & \text{if } y = kb \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in \mathbb{Z}^m$, completes the proof. \square

In the particular case when $\mathcal{S} = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}$ is a numerical semigroup with a unique Betti element $b \in \mathbb{N}$, it is proved in [8] that there exist pairwise relatively prime integers $b_1, \dots, b_n \geq 2$ such that $a_i := \prod_{j \neq i} b_j$, for all $i \in \{1, \dots, n\}$, and $b = \prod_{i=1}^n b_i$. In this case Theorem 4.2 can be refined as follows.

Corollary 4.3. *Let $\mathcal{S} = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}$ be a numerical semigroup with a unique Betti element $b \in \mathbb{N}$. Then,*

$$\mu_{\mathcal{S}}(x) = \begin{cases} (-1)^{|A|} \binom{k+n-2}{k} & \text{if } x = \sum_{i \in A} a_i + kb \text{ for some } A \subset \{1, \dots, n\}, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As $d = 1$, by Theorem 4.2 it only suffices to prove that for every $A_1, A_2 \subset \{1, \dots, n\}$, if b divides $\sum_{i \in A_1} a_i - \sum_{i \in A_2} a_i$, then $A_1 = A_2$. We take $b_1, \dots, b_n \geq 2$ such that $a_i = \prod_{j \neq i} b_j$, then by [8, Example 12] (see also [11, Theorem 2.9 and Remark 2.10]) we have that $I_{\mathcal{S}} = (f_2, \dots, f_n)$, where $f_i := x_1^{b_1} - x_i^{b_i}$ for all $i \in \{2, \dots, n\}$. Assume that there exist $A_1, A_2 \subset \{1, \dots, n\}$, $A_1 \neq A_2$ and that $\sum_{i \in A_1} a_i - \sum_{i \in A_2} a_i = kb$ for some $k \in \mathbb{N}$. Thus, the binomial $g := \prod_{i \in A_1} x_i - x_1^{b_1 k} \prod_{i \in A_2} x_i \neq 0$ belongs to $I_{\mathcal{S}}$ and so it can be written as a combination of f_2, \dots, f_n . However, this is a contradiction because $x_i^{b_i}$ does not divide $\prod_{j \in A_1} x_j$ for all $i \in \{1, \dots, n\}$. \square

As a direct consequence of this result we recover Dedden's classical result.

Corollary 4.4. *Let $a, b \in \mathbb{Z}^+$ be relatively prime integers and consider $\mathcal{S} := \langle a, b \rangle$. Then,*

$$\mu_{\mathcal{S}}(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } a + b \pmod{ab}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

4.3. Three generated complete intersection numerical semigroups.

We provide a semi-explicit formula for $\mu_{\mathcal{S}}$ when \mathcal{S} is a complete intersection numerical semigroup minimally generated by the set $\{a_1, a_2, a_3\}$.

Many authors have studied three generated numerical semigroups and one can find several (equivalent) characterizations of the complete intersection property for them. In particular, Herzog proves in [9] that \mathcal{S} is a complete intersection if and only if $\gcd\{a_i, a_j\} a_k \in \langle a_i, a_j \rangle$ with $\{i, j, k\} = \{1, 2, 3\}$. We assume that $\gcd\{a_2, a_3\} a_1 \in \langle a_2, a_3 \rangle$ and denote

$d := \gcd\{a_2, a_3\}$. Note that $d \geq 2$, otherwise $a_1 \in \langle a_2, a_3 \rangle$, which contradicts the minimality of $\{a_1, a_2, a_3\}$. In [9], the author also proves that if we take $\gamma_2, \gamma_3 \in \mathbb{N}$ such that $da_1 = \gamma_2 a_2 + \gamma_3 a_3$, then

$$(5) \quad I_{\mathcal{S}} = (x_1^d - x_2^{\gamma_2} x_3^{\gamma_3}, x_2^{a_3/d} - x_3^{a_2/d}),$$

whose \mathcal{S} -degrees are da_1 and $a_2 a_3/d$, respectively.

For every $x \in \mathbb{Z}$ there exists a unique $\alpha(x) \in \{0, \dots, d-1\}$ such that $\alpha(x)a_1 \equiv x \pmod{d}$. It is easy to check that for every $x, y \in \mathbb{Z}$,

$$(6) \quad \alpha(x-y) = \begin{cases} \alpha(x) - \alpha(y) & \text{if } \alpha(x) \geq \alpha(y), \\ d + \alpha(x) - \alpha(y) & \text{otherwise.} \end{cases}$$

Theorem 4.5. *Let $\mathcal{S} = \langle a_1, a_2, a_3 \rangle$ be a numerical semigroup such that $da_1 \in \langle a_2, a_3 \rangle$ where $d := \gcd\{a_2, a_3\}$. For all $x \in \mathbb{Z}$, we have*

$$\mu_{\mathcal{S}}(x) = 0$$

if $\alpha(x) \geq 2$, or

$$\mu_{\mathcal{S}}(x) = (-1)^\alpha (d_B(x') - d_B(x' - a_2) - d_B(x' - a_3) + d_B(x' - a_2 - a_3))$$

otherwise, where $x' := x - \alpha(x)a_1$ and $B := (da_1, a_2 a_3/d)$.

First proof. By (5) $I_{\mathcal{S}}$ is generated by two \mathcal{S} -homogeneous polynomials whose \mathcal{S} -degrees are da_1 and $a_2 a_3/d$. Thus, applying (4) we obtain

$$\begin{aligned} \mathcal{H}_{\mathcal{S}}(t) &= \frac{(1 - t^{da_1})(1 - t^{a_2 a_3/d})}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})} \\ &= \frac{(1 - t^{da_1})(1 - t^{a_2 a_3/d})}{1 - t^{a_1} - t^{a_2} - t^{a_3} + t^{a_1+a_2} + t^{a_1+a_3} + t^{a_2+a_3} - t^{a_1+a_2+a_3}}. \end{aligned}$$

Hence, from Corollary 3.4, we have

$$(7) \quad \mu_{\mathcal{S}}(x) = d_B(x) - d_B(x - a_1) - d_B(x - a_2) - d_B(x - a_3) + d_B(x - (a_1 + a_2)) + d_B(x - (a_1 + a_3)) + d_B(x - (a_2 + a_3)) - d_B(x - (a_1 + a_2 + a_3)),$$

for all integers x , where $B := (da_1, a_2 a_3/d)$. Since $\alpha(da_1) = \alpha(a_2 a_3/d) = 0$, it follows that $\alpha(y) = 0$ if $y \in \langle da_1, a_2 a_3/d \rangle$. As a consequence of this, $d_B(y) = 0$ whenever $\alpha(y) \neq 0$.

We denote $C := \{0, a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_3 + a_1, a_1 + a_2 + a_3\}$ and observe that $\alpha(y) \in \{0, 1\}$ for all $y \in C$.

We distinguish three different cases upon the value of $\alpha := \alpha(x)$, for $x \in \mathbb{Z}$.

Case 1. $\alpha \geq 2$.

We deduce that $\alpha(x-y) = \alpha(x) - \alpha(y) \neq 0$ and $d_B(x-y) = 0$, for all $y \in C$. Therefore, using (7), we obtain that $\mu_{\mathcal{S}}(x) = 0$.

Case 2. $\alpha = 1$.

We deduce that $\alpha(x-y) \neq 0$ and $d_B(x-y) = 0$ for all $y \in \{0, a_2, a_3, a_2 + a_3\}$. Therefore, using (7), we obtain that

$$\mu_{\mathcal{S}}(x) = -d_B(x - a_1) + d_B(x - a_1 - a_2) + d_B(x - a_1 - a_3) - d_B(x - a_1 - a_2 - a_3).$$

Case 3. $\alpha = 0$.

Since $d \geq 2$, we deduce that $\alpha(x-y) \neq 0$ and $d_B(x-y) = 0$ for all $y \in \{a_1, a_1 + a_2, a_1 + a_3, a_1 + a_2 + a_3\}$. Therefore, using (7), we obtain that

$$\mu_{\mathcal{S}}(x) = d_B(x) - d_B(x - a_2) - d_B(x - a_3) + d_B(x - a_2 - a_3).$$

This completes the proof. \square

Theorem 4.5 yields an algorithm for computing $\mu_{\mathcal{S}}(x)$ for all $x \in \mathbb{Z}$ which relies on the computation of four denumerants of the form $d_B(y)$, where $B = (da_1, a_2a_3/d)$. It is worth to mention that in [15, Section 4.4] there are several results and methods to compute this type of denumerants.

Also note that Theorem 4.5 generalizes [4, Theorem 3], where the authors provide a semi-explicit formula for $\mathcal{S} = \langle 2q, 2q + e, 2q + 2e \rangle$ where $q, e \in \mathbb{Z}^+$ and $\gcd\{2q, 2q + e, 2q + 2e\} = 1$. It is easy to see that these semigroups are complete intersection, it suffices to set $a_1 := 2q + e$, $a_2 := 2q$, $a_3 := 2q + 2e$ and observe that $\gcd\{a_2, a_3\}a_1 = 2a_1 = a_2 + a_3 \in \langle a_2, a_3 \rangle$. For some other families of complete intersection numerical semigroups we refer the reader to [1, 17].

5. WHEN IS A POSET EQUIVALENT TO A SEMIGROUP POSET?

Let $D = \{d_1, \dots, d_m\}$ be a finite set and let us consider the (locally finite) poset \mathcal{P} of multisets of D ordered by inclusion. For every $S, T \in \mathcal{P}$ such that $T \subset S$, it is well known that

$$(8) \quad \mu_{\mathcal{P}}(T, S) = \begin{cases} (-1)^{|S \setminus T|} & \text{if } T \subset S \text{ and } S \setminus T \text{ is a set,} \\ 0 & \text{otherwise.} \end{cases}$$

We consider the semigroup $\mathbb{N}^m = \langle e_1, \dots, e_m \rangle$ and the map $\psi : \mathcal{P} \rightarrow \mathbb{N}^m$ defined as $\psi(S) = (s_1, \dots, s_m)$ where s_i denotes the multiplicity of d_i in S for all $S \in \mathcal{P}$. We consider the order in \mathbb{N}^m induced by the semigroup \mathbb{N}^m , i.e., $\alpha \leq_{\mathbb{N}^m} \beta$ if and only if $\beta - \alpha \in \mathbb{N}^m$ for all $\alpha, \beta \in \mathbb{N}^m$. We have that ψ is an *order isomorphism*, i.e., an order preserving and order reflecting bijection. Thus, we can say that the poset of multisets of a finite set is a particular case of semigroup poset. This implies that for all $S, T \in \mathcal{P}$ such that $T \subset S$, then $\mu_{\mathcal{P}}(T, S) = \mu_{\mathbb{N}^m}(\psi(T), \psi(S)) = \mu_{\mathbb{N}^m}(\psi(S) - \psi(T))$ and by Theorem 4.1 we retrieve the formula (8).

It is worth pointing out that we can also use the tools of previous sections to recover the classical arithmetic Möbius function. Recall that for all $a, b \in \mathbb{N}$ such that $a \mid b$, we have that

$$(9) \quad \mu(a, b) = \begin{cases} (-1)^r & \text{if } b/a \text{ is a product of } r \text{ different prime numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for every $m \in \mathbb{Z}^+$, we denote by p_1, \dots, p_m the first m prime numbers and by \mathbb{N}_m the set of integers that can be written as a product of powers of p_1, \dots, p_m . Then, for all $m \geq 1$ the map $\psi : \mathbb{N}_m \rightarrow \mathbb{N}^m$ defined as $\psi(p_1^{\alpha_1} \cdots p_m^{\alpha_m}) = (\alpha_1, \dots, \alpha_m)$ is an order isomorphism between \mathbb{N}_m ordered by divisibility and the poset $(\mathbb{N}^m, \leq_{\mathbb{N}^m})$. Hence, for every $a, b \in \mathbb{N}$ we consider $m \in \mathbb{Z}^+$ such that $a, b \in \mathbb{N}_m$ and we recover the formula (9) by means of the Möbius function of \mathbb{N}^m given in Theorem 4.1.

In these two examples we easily find ad-hoc order isomorphisms between the poset \mathcal{P} and a semigroup poset which allows us to compute $\mu_{\mathcal{P}}$. In this section, we provide tools to do this systematically.

Let $(\mathcal{P}, \leq_{\mathcal{P}})$ be a locally finite poset. For every $x \in \mathcal{P}$ we denote $\mathcal{P}_x := \{y \in \mathcal{P} \mid x \leq_{\mathcal{P}} y\}$ and we consider the restricted Möbius function $\mu_{\mathcal{P}}(-, x) : \mathcal{P}_x \rightarrow \mathbb{Z}$. It is clear that if there exists a pointed semigroup \mathcal{S} and an order isomorphism $\psi : (\mathcal{P}_x, \leq_{\mathcal{P}}) \rightarrow (\mathcal{S}, \leq_{\mathcal{S}})$, then $\mu_{\mathcal{P}}(-, x)$ can be computed by means of the Möbius function of $(\mathcal{S}, \leq_{\mathcal{S}})$ since $\mu_{\mathcal{P}}(y, x) = \mu_{\mathcal{S}}(\psi(y))$ for all $y \in \mathcal{P}_x$. In this section we characterize in Theorem 5.3 when there exists such a pointed semigroup and such an isomorphism in terms of \mathcal{P}_x .

The poset \mathcal{P}_x is said to be *autoequivalent* if and only if for all $y \in \mathcal{P}_x$ there exists an order isomorphism $g_y : \mathcal{P}_x \rightarrow \mathcal{P}_y$ such that $g_y \circ g_z = g_z \circ g_y$ for all $y, z \in \mathcal{P}_x$ and g_x is the identity. For all $y \in \mathcal{P}_x$ we denote by $l_1(y) := \{z \in \mathcal{P} \mid \nexists u \in \mathcal{P} \text{ such that } y \leq u \leq z\}$. Whenever \mathcal{P}_x is autoequivalent with isomorphisms $\{g_y\}_{y \leq x}$ and $l_1(x)$ is a finite set of n elements, we associate to \mathcal{P} a subgroup $L_{\mathcal{P}} \subset \mathbb{Z}^n$ in the following way.

Let $l_1(x) = \{x_1, \dots, x_n\} \subset \mathcal{P}$ and consider the map

$$f : \mathbb{N}^n \rightarrow \mathcal{P}$$

defined as $f(0, \dots, 0) = x$, and for all $\alpha \in \mathbb{N}^n$ and all $i \in \{1, \dots, n\}$, $f(\alpha + e_i) = g_{x_i}(f(\alpha))$, where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{N}^n . In particular, $f(e_i) = g_{x_i}(f(0)) = g_{x_i}(x) = x_i$ for all $i \in \{1, \dots, n\}$.

Lemma 5.1. *f is well defined and is surjective.*

Proof. Suppose that $\alpha + e_i = \beta + e_j$, then we set $\gamma := \alpha - e_j = \beta - e_i \in \mathbb{N}^n$. Thus, $f(\alpha + e_i) = g_{x_i}(f(\alpha)) = g_{x_i}(g_{x_j}(f(\gamma))) = g_{x_j}(g_{x_i}(f(\gamma))) = g_{x_j}(f(\beta)) = f(\beta + e_j)$ and f is well defined.

Take $y \in \mathcal{P}_x$. If $y = x$, then $y = f(0)$. If $y \neq x$, then there exists $z \in \mathcal{P}_x$ such that $y \in l_1(z)$; therefore $y = g_z(x_j)$ for some $j \in \{1, \dots, n\}$. We claim that if $z = f(\alpha)$, then $y = f(\alpha + e_j)$. Indeed, $f(\alpha + e_j) = g_{x_j}(f(\alpha)) = g_{x_j}(z) = g_{x_j}(g_z(x)) = g_z(g_{x_j}(x)) = g_z(x_j) = y$. \square

Now we set $L_{\mathcal{P}} := \{\alpha - \beta \in \mathbb{Z}^n \mid f(\alpha) = f(\beta)\}$.

Lemma 5.2. *$L_{\mathcal{P}}$ is a subgroup of \mathbb{Z}^n .*

Proof. If $\gamma \in L_{\mathcal{P}}$, then $-\gamma \in L_{\mathcal{P}}$ clearly. Moreover, if $\gamma_1, \gamma_2 \in L_{\mathcal{P}}$, then $\gamma_1 + \gamma_2 \in L_{\mathcal{P}}$. Indeed, take $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n$ such that $f(\alpha) = f(\alpha')$, $\gamma_1 = \alpha - \alpha'$, $f(\beta) = f(\beta')$ and $\gamma_2 = \beta - \beta'$. Then $f(\alpha + \beta) = f(\alpha' + \beta) = f(\alpha' + \beta')$ and the lemma is proved. \square

For every subgroup $L \subset \mathbb{Z}^n$ the *saturation* of L is the group

$$\text{Sat}(L) := \{\gamma \in \mathbb{Z}^n \mid \exists d \in \mathbb{Z}^+ \text{ such that } d\gamma \in L\}.$$

Theorem 5.3. *Let \mathcal{P} be a locally finite poset and let $x \in \mathcal{P}$. Then, (\mathcal{P}_x, \leq) is isomorphic to $(\mathcal{S}, \leq_{\mathcal{S}})$ for some (pointed) semigroup $\mathcal{S} \subset \mathbb{Z}^n$ if and only if \mathcal{P}_x is autoequivalent, $l_1(x)$ is finite and $L_{\mathcal{P}} = \text{Sat}(L_{\mathcal{P}})$.*

Proof. (\Rightarrow) Let $\mathcal{S} \subset \mathbb{Z}^n$ be a (pointed) semigroup and denote by $\{a_1, \dots, a_n\}$ its unique minimal set of generators. Assume that $\psi : \mathcal{P}_x \rightarrow \mathcal{S}$ is an order isomorphism; let us prove that \mathcal{P}_x is autoequivalent, that $l_1(x) = n$ and that $L_{\mathcal{P}} = \text{Sat}(L_{\mathcal{P}})$. Firstly, we observe that setting $x_i := \psi^{-1}(a_i)$, then $l_1(x) = \{x_1, \dots, x_n\}$. Now, for every $y \in \mathcal{P}_x$ we set

$$g_y : \mathcal{P}_x \rightarrow \mathcal{P}_y \\ z \mapsto \psi^{-1}(\psi(z) + \psi(y)),$$

then it is straightforward to check that g_y is an order isomorphism. Moreover, g_x is the identity map in \mathcal{P}_x and $g_y \circ g_z = g_z \circ g_y$ for all $y, z \in \mathcal{P}_x$.

Now we take $f : \mathbb{N}^n \rightarrow \mathcal{P}_x$ the map associated to $\{g_y\}_{y \leq x}$, i.e., $f(0) = x$ and if $f(\alpha) = y$, then $f(\alpha + e_j) = g_{x_j}(f(\alpha))$. We claim that $\psi(f(\alpha)) = \sum \alpha_i a_i \in \mathcal{S}$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Indeed, $\psi(f(0)) = \psi(x) = 0$ and if we assume that $\psi(f(\alpha)) = \sum \alpha_i a_i$ for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then $\psi(f(\alpha + e_j)) = \psi(g_{x_j}(f(\alpha))) = \psi(\psi^{-1}(\sum \alpha_i a_i + a_j)) = \sum \alpha_i a_i + a_j$, as desired.

Since $L_{\mathcal{P}} \subset \text{Sat}(L_{\mathcal{P}})$ by definition, let us prove that $\text{Sat}(L_{\mathcal{P}}) \subset L_{\mathcal{P}}$. We take $\gamma \in \text{Sat}(L_{\mathcal{P}})$, then $d\gamma \in L_{\mathcal{P}}$ for some $d \in \mathbb{Z}^+$. This means that there exist $\alpha, \beta \in \mathbb{N}^n$ such that $f(\alpha) = f(\beta)$ and $d\gamma = \alpha - \beta$. Hence we have that $\sum \alpha_i a_i = \psi(f(\alpha)) = \psi(f(\beta)) = \sum \beta_i a_i$;

which implies that $\sum \gamma_i a_i = 1/d (\sum (\alpha_i - \beta_i) a_i) = 0$. Thus, if we take $\alpha', \beta' \in \mathbb{N}^m$ such that $\gamma = \alpha' - \beta'$, then $\psi(f(\alpha')) = \psi(f(\beta'))$ and, whence, $f(\alpha') = f(\beta')$ and $\gamma \in L_{\mathcal{P}}$.

(\Leftarrow) Since $L_{\mathcal{P}} = \text{Sat}(L_{\mathcal{P}})$, we have that $\mathbb{Z}^n/L_{\mathcal{P}}$ is a torsion free group; hence there exists a group isomorphism $\rho : \mathbb{Z}^n/L_{\mathcal{P}} \rightarrow \mathbb{Z}^m$, where $m = n - \text{rk}(L_{\mathcal{P}})$. We denote $a_i := \rho(e_i + L_{\mathcal{P}})$ for all $i \in \{1, \dots, n\}$ and set $\mathcal{S} := \langle a_1, \dots, a_n \rangle \subset \mathbb{Z}^m$. We claim that (\mathcal{P}_x, \leq) and $(\mathcal{S}, \leq_{\mathcal{S}})$ are isomorphic. More precisely, it is straightforward to check that the map

$$\begin{aligned} \psi : \mathcal{P}_x &\longrightarrow \mathcal{S} \\ y &\longmapsto \sum \alpha_i a_i, \text{ if } f(\alpha) = y \end{aligned}$$

is an order isomorphism. \square

In algebraic terms, the idea under (\Leftarrow) in Theorem 5.3 is that whenever \mathcal{P}_x is autoequivalent and $l_1(x)$ is finite, the subgroup $L_{\mathcal{P}}$ defines a lattice ideal $I := (\{\mathbf{x}^\alpha - \mathbf{x}^\beta \mid \alpha - \beta \in L_{\mathcal{P}}\})$. Moreover, \mathcal{P}_x is isomorphic to a semigroup poset $(\mathcal{S}, \leq_{\mathcal{S}})$ if and only if the ideal I itself is the toric ideal of a semigroup \mathcal{S} , but this happens if and only if I is prime or, equivalently, if $L_{\mathcal{P}} = \text{Sat}(L_{\mathcal{P}})$ (see, e.g., [6] or [10]).

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APPENDIX: SECOND PROOF OF THEOREM 4.5

In this appendix we give a second proof of Theorem 4.5 by using Theorem 3.2.

Proposition 5.4. *Let $\mathcal{S} = \langle a_1, a_2, a_3 \rangle$ be a numerical semigroup such that $da_1 \in \langle a_2, a_3 \rangle$ where $d := \gcd\{a_2, a_3\}$. Then,*

(a) $\mu_{\mathcal{S}}(x) = \mu_{\mathcal{S}}(x - da_1) + \mu_{\mathcal{S}}(x - a_2a_3/d) - \mu_{\mathcal{S}}(x - da_1 - a_2a_3/d)$ for all $x \in \mathbb{Z} \setminus C$, and

(b) $\sum_{i=0}^{d-1} \mu_{\mathcal{S}}(x - ia_1) = \sum_{i=0}^{d-1} \mu_{\mathcal{S}}(x - ia_1 - a_2a_3/d)$ for all $x \in \mathbb{Z} \setminus C'$,

where $C' = \{0, a_2, a_3, a_2 + a_3\}$ and $C = C' \cup \{0, a_1, a_1 + a_2, a_1 + a_3, a_1 + a_2 + a_3\}$.

Proof. By (5), $I_{\mathcal{S}}$ is generated by two \mathcal{S} -homogeneous polynomials whose \mathcal{S} -degrees are da_1 and a_2a_3/d . Applying (4) we get that

$$\mathcal{H}_{\mathcal{S}}(t) = \frac{(1 - t^{da_1})(1 - t^{a_2a_3/d})}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})} = \frac{1 - t^{da_1} - t^{a_2a_3/d} + t^{da_1+a_2a_3/d}}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})}.$$

A direct application of Theorem 3.2 proves (a). To prove (b) it suffices to observe that $(1 - t^{da_1})/(1 - t^{a_1}) = 1 + t + \dots + t^{d-1}$, giving

$$\mathcal{H}_{\mathcal{S}}(t) = \frac{(1 - t^{da_1})(1 - t^{a_2a_3/d})}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})} = \frac{\sum_{i=0}^{d-1} (t^i - t^{i+(a_2a_3/d)})}{(1 - t^{a_2})(1 - t^{a_3})},$$

and to apply again Theorem 3.2. \square

Second proof of Theorem 4.5. We set $C' := \{0, a_2, a_3, a_2 + a_3\}$ and $C := C' \cup \{a_1, a_1 + a_2, a_1 + a_3, a_1 + a_2 + a_3\}$. We observe that for all $y \in C$, $\alpha(y) \in \{0, 1\}$. Moreover $\alpha(y) = 0$ if $y \in C'$ and $\alpha(y) = 1$ if $y \in C \setminus C'$. Let x be an integer, we distinguish three different cases upon the value of $\alpha := \alpha(x)$.

Case 1. $\alpha \geq 2$.

We observe that $\alpha(x - \lambda(da_1) - \delta(a_2a_3/d)) = \alpha(x) \geq 2$ for all $\lambda, \delta \in \mathbb{N}$, which implies that $x - \lambda(da_1) - \delta(a_2a_3/d) \notin C$. By iteratively applying Proposition 5.4 (a) we get that $\mu_{\mathcal{S}}(x) = 0$.

Case 2. $\alpha = 1$.

By Proposition 5.4 (b), we have that

$$\sum_{i=0}^{d-1} \mu_{\mathcal{S}}(y - ia_1) = \sum_{i=0}^{d-1} \mu_{\mathcal{S}}(y - ia_1 - a_2a_3/d)$$

for all $y \notin C'$. Moreover, for $i \geq 2$, by *Case 1* we have that $\mu_{\mathcal{S}}(x - ia_1) = \mu_{\mathcal{S}}(x - ia_1 - a_2a_3/d) = 0$. This gives that

$$\mu_{\mathcal{S}}(y) + \mu_{\mathcal{S}}(y - a_1) = \mu_{\mathcal{S}}(y - a_2a_3/d) + \mu_{\mathcal{S}}(y - a_1 - a_2a_3/d)$$

for all $y \notin C'$. We set $\sigma(y) := \mu_{\mathcal{S}}(y) + \mu_{\mathcal{S}}(y - a_1)$ and have that

$$(10) \quad \sigma(y) = \sigma(y - a_2a_3/d)$$

for all $y \notin C'$. Moreover, we observe that $\mu_{\mathcal{S}}(x - \lambda a_2a_3/d) = 1$ for every $\lambda \in \mathbb{N}$; hence, $x - \lambda a_2a_3/d \notin C'$. Applying (10) iteratively, we get that $\sigma(x) = 0$ and $\mu_{\mathcal{S}}(x) = -\mu_{\mathcal{S}}(x - a_1)$.

Case 3. $\alpha = 0$.

We denote $\tau(y) := \mu_{\mathcal{S}}(y) - \mu_{\mathcal{S}}(y - da_1)$, then by Proposition 5.4 (a) we have that

$$(11) \quad \tau(y) = \tau(y - a_2a_3/d) \text{ if } y \notin C.$$

We take $\lambda \in \mathbb{N}$ the minimum integer such that $x - \lambda a_2 a_3 / d < 0$ or $x - \lambda a_2 a_3 / d \in C$; an iterative application of (11) yields $\tau(x) = \tau(x - \lambda da_1)$, hence $\mu_S(x) = \mu_S(x - da_1) + \tau(x - \lambda da_1)$. Let us compute $\tau(x - \lambda da_1)$. If $x' := x - \lambda da_1 < 0$, then $\tau(x') = 0$. Since $\alpha(x') = \alpha(x) = 0$, we have that $x' \in C'$. A direct computation yields $\tau(0) = 1$, $\tau(a_2) = \tau(a_3) = -1$, $\tau(a_2 + a_3) = \mu_S(da_1) - \mu_S(0) = 2 - 1 = 1$ if $a_2 + a_3 = da_1$ or $\tau(a_2 + a_3) = \mu_S(a_2 + a_3) - \mu_S(a_2 + a_3 - da_1) = 1 - 0 = 1$ if $a_2 + a_3 \neq da_1$. Hence,

$$\mu_S(x) = \mu_S(x - da_1) + \begin{cases} 1 & \text{if } x \equiv 0 \pmod{a_2 a_3 / d}, \\ -1 & \text{if } x \equiv a_2 \pmod{a_2 a_3 / d}, \\ -1 & \text{if } x \equiv a_3 \pmod{a_2 a_3 / d}, \\ 1 & \text{if } x \equiv a_2 + a_3 \pmod{a_2 a_3 / d}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that when $B = (da_1, a_2 a_3 / d)$, then $d_B(y)$ is the number of $i \in \mathbb{N}$ such that $y - i(da_1) \geq 0$ and $y - i(da_1) \equiv 0 \pmod{a_2 a_3 / d}$ and thus the result follows by applying the above recursive formula. \square

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