The Casimir warp drive:Is the Casimir Effect a valid candidate to generate and sustain a Natario warp drive spacetime bubble??

Fernando Loup

The Casimir warp drive:Is the Casimir Effect a valid candidate to generate and sustain a Natario warp drive spacetime bubble??

Fernando Loup [*]
Residencia de Estudantes Universitas Lisboa Portugal

April 21, 2014

Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. The major drawback concerning warp drives is the huge amount of negative energy able to sustain the warp bubble.In order to perform an interstellar space travel to a "nearby" star at 20 light-years away in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!! Although the energy conditions of General Relativity forbids the existence of negative energy the Casimir Effect first predicted by Casimir in 1948 and verified experimentally by Lamoreaux in 1997 allows sub-microscopical amounts of it. Lamoreaux obtained experimentally negative energy densities of $10^{-4}\frac{Joules}{Meter^3}$.This is an extremely small value 10^{20} times lighter than the ones of a body of 1 kilogram in a cubic meter of space or better:100.000.000.000.000.000.000 times lighter than the ones of a body of 1 kilogram in a cubic meter of space. We introduce in this work a shape function that will low the negative energy density requirements in the Natario warp drive from $10^{48}\frac{Joules}{Meter^3}$ to $10^{-7}\frac{Joules}{Meter^3}$ a result 1000 times lighter than the ones obtained by Lamoreaux proving that the Casimir Effect can generate and sustain a Natario warp drive spacetime.We also discuss other warp drive drawbacks:collisions with hazardous interstellar matter(asteroids or comets) that may happen in a real interstellar travel and Horizons(causally disconnected portions of spacetime).

---

[*]spacetimeshortcut@yahoo.com

1 Introduction

The Warp Drive as a solution of the Einstein Field Equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2])(pg 34 in [5]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel.

The quest for negative energy densities in warp drive spacetimes is very important:The energy conditions of General Relativity allows the existence of positive energy densities only and the warp drive which requires negative energy densities is sustained by the equations of General Relativity.

So according to General Relativity the warp drive is impossible because it violates all the knows energy conditions.This was also stated by both Alcubierre and Natario (pg 8 in [1] pg 1 in [2]).

---

[1]do not violates Relativity

Although Classical Physics forbids the existence of negative energy densities there exists an effect in Quantum Physics that allows its existence however in very small sub-microscopical amounts.

The Casimir Effect first predicted theoretically by Casimir in 1948 [12] and experimentally demonstrated by Lamoreaux in 1997 [14] allows the existence of negative energy densities.Lamoreaux obtained experimentally negative energy densities of $10^{-4}\frac{Joules}{Meter^3}$.This is an extremely small value:In order to get an idea of how small this value is consider the fact that a body of 1 kilogram in a cubic meter of space possesses an energy density of $9\times 10^{16}\frac{Joules}{Meter^3}$. So the values obtained by Lamoreaux are 10^{20} times lighter than the ones of a body of 1 kilogram in a cubic meter of space or better:100.000.000.000.000.000.000 times lighter than the ones of a body of 1 kilogram in a cubic meter of space.

Alcubierre was the first to propose the Casimir Effect as a source of negative energy to sustain a warp drive(pg 9 in [1]) however the Casimir Effect can generate sub-microscopical amounts of it and warp drives needs astronomical quantities.

Some years ago Ridgely(pg 1 in [13]) proposed a very interesting idea:If the Casimir Effect produces sub-microscopical amounts of negative energy density and the warp drive needs enormous quantities then how about to modify the equations of the Casimir Effect in order to allow the production of macroscopical amounts of it???

Due to the sub-microscopical nature of the Planck Constant $h=6,626\times 10^{-34}\frac{Joules}{Seconds}$ which is the basis of all the Quantum Physics that supports the Casimir Effect we dont know if the Ridgely idea can be accomplished however it would be excellent to have macroscopical sources of negative energy density.

In this work we modified the Ridgely idea:Instead of a modification of the Casimir Effect we propose a modification of the geometry of the Natario warp drive in order to get negative energy density requirements compatible with the ones obtained by Lamoreaux for the Casimir Effect.We introduce a new Natario shape function that will low the negative energy density requirements from $10^{48}\frac{Joules}{Seconds}$ to $10^{-7}\frac{Joules}{Seconds}$ even for a spaceship moving at 200 times light speed.

This is a result 1000 times lighter than the ones obtained by Lamoreaux in 1997 and it proofs that the Casimir Effect can undoubtely generate and sustain a Natario warp drive spacetime.

Our result is only a demonstration of how far can the mathematics go in order to ameliorate the negative energy density requirements needed to sustain a warp drive.A real warp drive in interstellar space would need large and macroscopical amounts of negative energy density in order to deflect hazardous interstellar matter protecting the ship from impacts with asteroids comets or photons from gamma radiation so the Ridgely idea is still very attractive.

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.However we use the Alcubierre shape function to define its Natario counterpart.

In order to produce an independent and self-contained work we present here the entire Natario warp drive theory:The mathematics presented here is only a compilation of work already done by Natario without nothing new but perhaps with a more pedagogical presentation in order to captive the interest of non-familiarized readers.Anyone with basic knowledge in $3 + 1$ General Relativity should be able to read this work.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes and the International System of units for energetic purposes

This work is organized as follows:

- Section 2)-Introduces the Natario warp drive spacetime for people familiarized with basic $3 + 1$ General Relativity but without knowledge in Differential Forms in a pedagogical way.

- Section 3)-Outlines the problems of the energy conditions in General Relativity and introduces the Casimir Effect as the only known source of negative energy density already demonstrated experimentally although the amounts obtained are sub-microscopical.

- Section 4)-Outlines the problems of the immense magnitude in negative energy density when a ship travels with a speed of 200 times faster than light. The negative energy density for such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!..

- Section 5)-Covers the problem of interstellar navigation in the Natario warp drive spacetime and how collisions with hazardous objects in interstellar space(asteroids or comets) can be avoided

- Section 6)-We introduce a shape function that defines the Natario warp drive spacetime being this function an excellent candidate to lower the energy density requirements in the Natario warp drive to affordable levels completely obliterating the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!..We low the negative energy density to levels of 1000 times lighter than the ones obtained experimentally by Lamoreaux in 1997 for the Casimir Effect.

- Section 7)-Outlines the possibility of how to overcome the Horizon problem from an original point of view of General Relativity.

This work can be regarded as a companion of our work [11]

2 The Natario warp drive:a warp drive with zero expansion:

In 2001 Natario introduced a new warp drive spacetime defined using the canonical basis of the Hodge star in spherical coordinates defined as follows(pg 4 in [2])[2]:

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \tag{1}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \tag{2}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \tag{3}$$

We can introduce now the Natario definition for the warp drive:
(pg 4 in [2]):

- 1)-Any Natario vector nX generates a warp drive spacetime if $nX = 0$ for a small value of $|x|$ defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ for a large value of $|x|$ defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble and $X = vs$ or $X = -vs$ being the shift vector embedded inside the Natario vector.[3]

- 2)-The Natario vector nX is defined by the shift vector X as $nX = Xdx$ being $X = vs$ or $X = -vs$

Back again to the concept of the fish and the aquarium;

Imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed(a Natario vector $nX = vsdx$ and a shift vector $X = vs$ being vs the speed of the warp bubble) but inside the aquarium the fish is at the rest with respect to his local neighborhoods(a Natario vector $nX = 0$ and a shift vector $X = 0$ meaning a speed $vs = 0$ inside the aquarium because inside the aquarium the fish is stopped.The fish is at the rest with respect to its local neighborhoods which means to say the water inside the aquarium that is being carried out with the fish at the same time and the water is acting as a local coordinates frame).[4]

The spaceship is the fish and the aquarium is the warp bubble.

An observer in outer space at the rest with respect to a local coordinates frame outside the bubble of course(the vacuum of space surrounding him) sees a warp bubble passing by him at a large speed vs but inside the bubble the spaceship is at the rest with respect to a local coordinates frame inside the bubble(the vacuum of space surrounding the spaceship).

---

[2]See Appendix C on Hodge stars and differential forms
[3]The mathematical relations between the shift vector and the Natario vector are given in the Appendix B
[4]See Appendix G for an artistic presentation of a spaceship inside a Natario warp bubble

Applying the Natario equivalence between spherical and cartezian coordinates as shown below (pg 5 in [2])[5]:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \quad (4)$$

And for dx we have(pg 5 in [2]) :

$$dx = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \quad (5)$$

Inserting all this stuff in the Natario vector $nX = Xdx$ we would get the following expression for the corresponding shift vectors $X = vs$ or $X = -vs$(pg 5 in [2])

$$nX = Xdx = vs(t)d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \quad (6)$$

$$nX = Xdx = -vs(t)d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \quad (7)$$

Rewriting the Natario vector as(pg 5 in [2])
Note that we replaced $\frac{1}{2}$ by $f(r)$ in order to obtain(pg 5 in [2]):

$$nX = vs(t)d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (8)$$

$$nX = -vs(t)d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (9)$$

We get the final expressions for the Natario vector nX:(pg 5 in [2])

$$nX = v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim 2v_sf(r)\cos\theta\mathbf{e}_r - v_s(2f(r)+rf'(r))\sin\theta\mathbf{e}_\theta \quad (10)$$

$$nX = -v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_sf(r)\cos\theta\mathbf{e}_r + v_s(2f(r)+rf'(r))\sin\theta\mathbf{e}_\theta \quad (11)$$

From now on we will use this pedagogical approach that gives results practically similar the ones depicted in the original Natario vector shown above:

$$nX = v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim 2v_sf(r)\cos\theta dr - v_s(2f(r)+rf'(r))r\sin\theta d\theta \quad (12)$$

$$nX = -v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_sf(r)\cos\theta dr + v_s(2f(r)+rf'(r))r\sin\theta d\theta \quad (13)$$

In order to make the Natario warp drive holds true we need for the Natario vector nX a continuous Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble(pg 5 in [2])

---

[5]The mathematical demonstration of this expression will be given in the Appendix C on Hodge stars and differential forms

In order to avoid confusion with the Alcubierre shape function $f(rs)$ we will redefine the Natario shape function as $n(r)$ and the Natario vector as shown below

$$nX = v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s n(r)\cos\theta dr - v_s(2n(r) + rn'(r))r\sin\theta d\theta \tag{14}$$

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + rn'(r))r\sin\theta d\theta \tag{15}$$

Examining the following Natario vector $nX = Xdx$ or $nX = vsdx$ with a shift vector $X = vs$ using the Natario continuous shape function $n(r)$ being $n(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $n(r) = 0$ for small r(inside the warp bubble) while being $0 < n(r) < \frac{1}{2}$ in the walls of the warp bubble(Natario warped region)(pg 5 in [2]):

$$nX = v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s n(r)\cos\theta dr - v_s(2n(r) + rn'(r))r\sin\theta d\theta \tag{16}$$

We must study the behavior of the Natario vector nX for the Natario shape function $n(r)$ with a shift vector $X = vs(t)$ and a warp bubble speed $vs = 200$ with respect to a distant observer.Two hundred times faster than light.

According to Natario we have 3 possible values for the Natario shape function(pg 5 in [2])::

- 1)-Inside the warp bubble $n(r) = 0$

- 2)-Outside the warp bubble $n(r) = \frac{1}{2}$

- 3)-In the warp bubble walls(Natario warped region) $0 < n(r) < \frac{1}{2}$

Lets see what happens with the Natario vector nX for each one of these conditions:

- A)-Inside the warp bubble $n(r) = 0$

    Inside the warp bubble $n(r) = 0$ as a constant value.Then the derivatives of $n(r)$ vanishes and we can rewrite the Natario vector nX as follows:

$$nX = 2v_s n(r)\cos\theta dr - v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{17}$$

$$nX = 2v_s[0]\cos\theta dr - v_s(2[0])r\sin\theta d\theta \tag{18}$$

$$nX = 0!!! \tag{19}$$

    No motion at all!!!.The observer(the spaceship) inside the Natario warp bubble is completely at the rest with respect to its local spacetime neighborhoods and this observer dont feel any acceleration or any g-forces..Then for this observer $\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta = 0$.Remember that the Natario vector is still $nX = vsdx$.But the shift vector is still $X = vs$.The Natario warp bubble still moves with a speed $vs = 200$ with respect to a distant observer.Two hundred times faster

than light but the internal observer inside the warp bubble is completely at the rest and completely in safe from the g-forces that would kill him moving ar such hyper-fast velocities.The spaceship remains at the rest inside the warp bubble however the bubble moves with a speed 200 times faster than light with respect to distant external observers outside the bubble.

- B)-Outside the warp bubble $n(r) = \frac{1}{2}$

Outside the warp bubble $n(r) = \frac{1}{2}$ as a constant value.Then the derivatives of $n(r)$ vanishes and we can rewrite the Natario vector nX as follows:

$$nX \simeq 2v_s n(r) \cos\theta dr - v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{20}$$

$$nX \simeq 2v_s \frac{1}{2} \cos\theta dr - v_s(2\frac{1}{2})r\sin\theta d\theta \tag{21}$$

$$nX \simeq v_s \cos\theta dr - v_s r\sin\theta d\theta \tag{22}$$

Remember that in this case we have the Natario vector as still being $nX = vsdx$ with the shift vector defined as $X = vs$.But now we have $\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \neq 0$.The Natario vector do not vanishes and an external observer would see the Natario warp bubble passing by him at a $vs = 200$ two hundred times faster than light due to the shift vector $X = vs$

- C)-In the warp bubble walls(Natario warped region) $0 < n(r) < \frac{1}{2}$

This is the region where the walls of the Natario warp bubble resides.Following our example of the fish inside the aquarium then the Natario warped region would correspond to the glass of the aquarium walls.It is not a good idea to place an observer here because the energy needed to distort the spacetime generating the warp drive is placed in this region.The Natario vector nX would then be:

$$nX = v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s n(r)\cos\theta dr - v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{23}$$

Now the reader can understand the point of view of the idea outlined by Natario in 2001.The Natario vector $nX = vsdx = 0$ vanishes inside the warp bubble because inside the warp bubble there are no motion at all because $dx = 0$ while being $nX = vsdx \neq 0$ not vanishing outside the warp bubble because an external observer sees the warp bubble passing by him with a speed defined by the shift vector $X = vs$

According to pg 8 in [2] photons still outside the warp bubble but approaching the warp bubble from the front appears with the frequency highly Doppler Blueshifted when seen by the observer in the center of the bubble.Actually these photons are moving in the direction of the bubble towards it with a speed c but the bubble is moving in the direction of the photons and towards these with a speed $vs >> c$.Collisions between highly energetic photons and warp bubble walls are a major concern in warp drive science.We will address this issue later.

If the warp bubble moves with a speed vs then a source of photons(a star) is being seen by the observer in the center of the bubble with a relative approximation speed vs

Computing the Blueshift using the classical Doppler-Fizeau formula for a photon approaching the ship from the front we have:[6]

$$f = f_0 \frac{c + va}{c - vb} \tag{24}$$

The terms above are:

- 1)-f is the photon frequency seen by an observer

- 2)-f_0 is the original frequency of the emitted photon

- 3)-c is the light speed.in our case $c = 1$

- 4)-va is the relative speed of the light source approaching the observer.In our case is vs

- 5)-vb in the relative speed of the light source moving away from the observer. In our case because the photon is coming to the observer $vb = 0$

Rewriting the Doppler-Fizeau expression for an incoming photon approaching the warp bubble from the front we should expect for(see pg 8 in [2])[7]:

$$f = f_0(1 + vs) \tag{25}$$

$$f = f_0(1 + X) \tag{26}$$

Energy E is Planck Constsnt $\hbar$ multiplied by frequency so for the energy we would have:

$$E = E_0(1 + vs) \tag{27}$$

Note that as larger is $vs > 1$ as large is the energy of the photon and hence the Blueshift.The warp bubble walls would collide with these highly energetic photons.[8]This is a serious obstacle that compromises the physical feasibility of the warp drive.Again we will address this issue later.

---

[6]Remember that the warp drive do not obey Lorentz transformations so the classical formula can be applied to get the results of [2]

[7]Remember that $X = vs$ is the shift vector

[8]See again Appendix G.The spaceship is inside the bubble but the warp bubble is travelling in interstellar space filled with photons.Imagine a photon coming to the warp bubble walls:the blue region

Notice that outside the warp bubble the shift vector $X = vs$ with $vs > 1$ and we have (pg 6 in [2]).See also fig 1 pg 7 in [2].

$$\sin \alpha = \frac{1}{v_s} \tag{28}$$

According to Natario this is the familiar expression for the Mach cone angle[9].For superluminal(warp) speed scenario we prefer to redefine this as the Natario cone angle.See fig 3 pg 9 in [2]

But if $vs > 1$ then $\sin \alpha < 1$ and $\alpha < \frac{\pi}{2}$ and as higher are the values of vs then $\sin \alpha \simeq 0$ and also $\alpha \simeq 0$

As higher are the values of vs the Natario cone angle becomes zero so we can see that vs defines the inclination of the Natario cone angle. For "low" superluminal(warp) speeds the inclination is close to $\frac{\pi}{2}$ because although $vs > 1$ vs stands close to 1 while for "high" superluminal(warp) speed the Natario cone angle gets closer to zero because $vs >> 1$.Then according to pg 6 in [2]) we are left with 3 possible scenarios for the inclination of the Natario cone angle:

- 1)-$\|\mathbf{X}\| = 1$ and $vs = 1$.

- 2)- $X = vs$ and $vs > 1$.

- 3)-$X = vs$ and $vs >> 1$

We must examine each one of these possible scenarios:

- 1)-$\|\mathbf{X}\| = 1$ and $vs = 1$.

$$\sin \alpha = \frac{1}{\|\mathbf{X}\|} \tag{29}$$

$$\sin \alpha = 1 \dashrightarrow \alpha = \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| = 1 \tag{30}$$

This situation happens when the shift vector $X = vs$ and $vs = 1$.The warp drive reaches luminal speed.This means to say that the the warp drive reaches the Horizon[10].The Natario cone angle similar to the Mach cone angle appears(see pg 15 in [4]) with an inclination of $\frac{\pi}{2}$ because $sin\alpha = 1$ which means to say that the Natario cone angle appears perpendicular to the direction of motion.

This is a very important issue concerning warp drive science:we know that a jet when achieves the speed of the sound produces a sonic "boom" or a Mach "boom" that can break the glass of windows if the jet is flying at low altitude over a metropolitan area or can even damage the human hear if the jet is flying at a very low altitude over a populated area.

---

[9]See the graphical presentation of the Mach cone angle in the Appendix L.Compare this with fig 1 pg 7 in [2].Although we consider $\frac{1}{v_s}$ because we make $c = 1$ the expression really means $\frac{c}{v_s}$

[10]We will address the issue of Horizons later in this work

With the warp drive the same thing happens:although the interstellar space possesses a very low density when compared to the air it is not completely empty.The interstellar vacuum pure and simply do not exists:Interstellar space is filled with photons emitted by stars,space dust,asteroids comets or meteorites,ionized clouds of gas,supernova remnants etc.And remember that the light speed is much higher than the speed of the sound so in the end the result is the same.

When the jet reaches the speed of the sound the jet collides with the molecules of the gases of the atmosphere,water vapor,dust from the pollution etc and produces the sonic "boom" or the Mach "boom".

When the starship reaches the light speed it collides with the interstellar matter in front of the bubble producing the luminal "boom" or the Natario "boom".So to stay near a spaceship when it is close to reach the light speed is a very dangerous option.The ship is in safe inside the Natario warp bubble completely at the rest but the walls of the bubble are experiencing highly energetic collisions against the matter of the interstellar medium.Like the Doppler Blueshifts mentioned earlier this is a major concern to warp drive science.We will address this issue later.

This means to say that like a jet must take-off at subsonic speeds and move away from metropolitan or populated areas still at subsonic speeds gaining altitude in order to break the sound barrier at a safe distance or safe altitude a spaceship cannot break the light speed barrier near a planet.The collisions between the walls of the warp bubble and Doppler Blueshifted photons for a factor $vs = 200$ would release an immense amount of energy that could damage the surface of the planet[11].The ship must leave the planet at subluminal speeds and cross the entire planetary system still at subluminal speeds in order to reach its borders.Only outside the planetary system and faraway from any planet the ship can break the light speed barrier.[12] [13]

- 2)- $X = vs$ and $vs > 1$.

$$\sin \alpha = \frac{1}{v_s} \tag{31}$$

$$\sin \alpha < 1 \dashrightarrow \alpha < \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| > 1 \dashrightarrow \sin \alpha \simeq 1 \dashrightarrow \alpha \simeq \frac{\pi}{2} \tag{32}$$

This situation happens when $X = vs$ and $vs > 1$ and the warp drive reaches "low" superluminal(warp) Speed.The Natario cone angle similar to the Mach cone angle appears(see pg 15 in [4]) now with an inclination of less than $\frac{\pi}{2}$ but still close to it because vs is still close to 1.Example for a $vs = 2$(two times light speed) then $\sin \alpha = \frac{1}{2}$ and $\alpha = \frac{\pi}{6}$.[14]

---

[11]Think for a moment on hard X-rays from cosmic rays or gamma radiation also emitted by stars.And think on how many of these energetic photons exists per cubic centimeter of space.Now imagine a collision between each one of these photons with the energy multiplied by 200 with the walls of the warp bubble in the neighborhoods of a planet

[12]see the artistical presentations of this scenario in the Appendix I

[13]see also the artistical presentations of a Natario warp drive in real interstellar space Appendix M

[14]see the artistical presentations of this scenario in the Appendix J

- 3)-$X = vs$ and $vs >> 1$

$$\sin \alpha = \frac{1}{v_s} \tag{33}$$

$$\sin \alpha << 1 \dashrightarrow \alpha << \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| >> 1 \dashrightarrow \sin \alpha \simeq 0 \dashrightarrow \alpha \simeq 0 \tag{34}$$

This situation happens when $X = vs$ and $vs >> 1$ and the warp drive reaches "high" superluminal(warp) speed.The Natario cone angle similar to the Mach cone angle appears(see pg 15 in [4]) now with an inclination of much less than $\frac{\pi}{2}$ and highly inclined almost parallel to the direction of motion because $\sin \alpha \simeq 0$ so $\alpha \simeq 0$ too.Example for a $vs = 200$(two hundred times light speed) then $\sin \alpha = \frac{1}{200} \simeq 0$.[15]

Redefining the Natario vector nX as being the rate-of-strain tensor of Fluid Mechanics as shown below(pg 5 in [2]):

$$nX = X^r \mathbf{e}_r + X^\theta \mathbf{e}_\theta + X^\varphi \mathbf{e}_\varphi \tag{35}$$

$$nX = X^r dr + X^\theta r d\theta + X^\varphi r \sin \theta d\varphi \tag{36}$$

Remember that X^r,X^θ and X^φ are the corresponding shift vectors embedded in the Natario vector nX.[16]

Bur we already know that the Natario vector is given by:

$$nX = v_s(t)d\left[n(r)r^2 \sin^2 \theta d\varphi\right] \sim 2v_s n(r) \cos \theta dr - v_s(2n(r) + rn'(r))r \sin \theta d\theta \tag{37}$$

$$nX = -v_s(t)d\left[n(r)r^2 \sin^2 \theta d\varphi\right] \sim -2v_s n(r) \cos \theta dr + v_s(2n(r) + rn'(r))r \sin \theta d\theta \tag{38}$$

Note that in our expressions for the Natatio vector we do not have the term $X^\varphi r \sin \theta d\varphi$ and the Natario vector can be simply rewritten as:

$$nX = X^{rs} drs + X^\theta rs d\theta \tag{39}$$

Hence we should expect for the shift vectors X^r,X^θ the following expressions:

$$X^{rs} = -2v_s n(rs) \cos \theta \tag{40}$$

$$X^{rs} = 2v_s n(rs) \cos \theta \tag{41}$$

$$X^\theta = v_s(2n(rs) + (rs)n'(rs)) \sin \theta \tag{42}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs)) \sin \theta \tag{43}$$

---

[15]see the artistical presentations of this scenario in the Appendix K

[16]Again see the Appendices B and C for the relations between the shift vector,Natario vector and differential forms and Hodge stars

Applying the extrinsic curvature tensor for the shift vectors contained in the Natario vector nX we would get the following results(pg 5 in [2]):

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r) \cos\theta \tag{44}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r) \cos\theta; \tag{45}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} = v_s n'(r) \cos\theta \tag{46}$$

$$K_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\theta}{r}\right) + \frac{1}{r}\frac{\partial X^r}{\partial\theta}\right] = v_s \sin\theta\left(n'(r) + \frac{r}{2}n''(r)\right) \tag{47}$$

$$K_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\varphi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial X^r}{\partial\varphi}\right] = 0 \tag{48}$$

$$K_{\theta\varphi} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{X^\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial X^\theta}{\partial\varphi}\right] = 0 \tag{49}$$

Examining the first three results we can clearly see that(pg 5 in [2]):

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{50}$$

This is the trace of the extrinsic curvature tensor that defines the expansion of the normal volume elements and in the Natario warp drive this is zero.The Natario warp drive do not expands or contracts the spacetime.

A warp drive with zero expansion.

The observer(spaceship) is still immersed in the interior of the warp bubble and this bubble is carried out by the spacetime "stream" at faster than light velocities with the observer at the rest with respect to its local neighborhoods inside the bubble feeling no g-forces and no accelerations. Imagine an aquarium floating in the course of a river with a fish inside it...the walls of the aquarium are the walls of the warp bubble...Imagine that this river is a "rapid" and the aquarium is being carried out by the river stream...the aquarium walls do not expand or contract...an observer in the margin of the river would see the aquarium passing by him at an arbitrarily large speed but inside the aquarium the fish would be protected from g-forces or accelerations generated by the stream...because the fish would be at the rest with respect to its local spacetime neighborhoods inside the aquarium.The Natario warp drive is being carried out by the spacetime "stream" like a fish in the stream of a river due to the resemblances between the Natario vector nX and the rate-of-strain tensor of Fluid Mechanics

In the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the space-time expansion in the remaining direction(perpendicular)(pg 5 in [2]).[17]

---

[17]See Appendix H the artistic presentation of the Natario warp drive where the expansion in one direction is counter-balanced by a contraction in the opposite directions

The energy density in the Natario warp drive is given by the following expression(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2\sin^2\theta\right].$$

(51)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right].$$

(52)

This energy density is negative and violates all the known energy conditions of General Relativity making the warp drive almost impossible to be achieved.It also depends of the Natario shape function $n(r)$ and its derivatives.The Natario shape function must be analytical[18] in every point of the warp bubble.In order to generate the warp drive as a dynamical spacetime large outputs of energy are needed and this is a critical issue that will be solved perhaps by a real theory of Quantum Gravity but everything depends on the form or the behavior of the Natario shape function. We will address the question of the energy requirements later in this work and now we will introduce here the Natario warp drive continuous shape function $n(r)$.

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can define the Natario shape function $n(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[19].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]$$

(53)

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2}$$

(54)

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function can be defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)]$$

(55)

$$n(rs) = \frac{1}{2}[1 - [\frac{1}{2}[1 - tanh[@(rs - R)]]]]$$

(56)

---

[18]Must be continuous and differentiable in every point of its trajectory
[19]$tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

14

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

3 The Weak Energy Condition(WEC) in General Relativity and the Casimir Effect

The equation of the weak energy density condition (WEC) in General Relativity is given by:(see eq 4.72 pg 124 in [17],see eq 2.1.6 pg 41 in [16])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} \geq 0 \tag{57}$$

From above according to the WEC the term $T_{\mu\nu}$ represents a stress-emergy-momentum-tensor(SEMT) in a given local inertial frame that must be always positive-no negative masses are allowed in General Relativity(pg 124 in [17]) and the terms $u^{\mu}u^{\nu}$ are timelike vectors.

The WEC states that the energy density of any distribution of matter obeying a SEMT given by $T_{\mu\nu}$ as seen by an observer in a given spacetime moving with a four-velocity vector given by u^{μ} or u^{ν} must be non-negative for any future-directed timelike vectors u^{μ} or u^{ν}.(pg 41 in [16]).

However as stated by both Alcubierre and Natario the warp drive requires negative energy density and violates the WEC(and all the other energy conditions).(see pg 8 in [1],pg 1 in [2])

If negative masses are not allowed by General Relativity and the warp drive needs negative masses this means to say that the warp drive is impossible.

This is true from the point of view of the Classical Physics domain where General Relativity belongs:However the Quantum Physics domain allows sub-microscopical amounts of negative energy densities in a clear violation of the WEC.

The only known process to generate experimentally these sub-microscopical amounts of negative energy densities is the Casimir Effect.[20][21]

The Casimir Effect states that the vacuum energy density between two parallel conduction plates separated by a distance d is given by:(pg 42 in [16])

$$\rho = -\frac{\pi^2}{720}\frac{\hbar}{d^4} \tag{58}$$

The equation above was written in the Geometrized System of Units $c = G = 1$.In the International System of Units the same equation would be:

$$\rho = -\frac{\pi^2}{720}\frac{\hbar c}{d^4} \tag{59}$$

In the equation above $\pi = 3,1415926536$,$\hbar = \frac{h}{2\pi}$,h is the Planck Constant given by $h = 6,626 \times 10^{-34}\frac{Joules}{Seconds}$ and $c = 3 \times 10^8\frac{Meters}{Seconds}$ is the light speed.Except for the distance between the plates d all the terms above are constants.Then the equation of the Casimir Effect is better rewritten as:

---

[20]see Wikipedia the Free Encyclopedia on Energy Condition and Casimir Effect
[21]see also Appendix N

$$\rho = -\frac{C}{d^4} \tag{60}$$

With C being the Casimir Factor defined as:

$$C = \frac{\pi^2}{720}\hbar c \tag{61}$$

Computing the values of C we have:

$$C = \frac{\pi^2}{720}\hbar c = \frac{9,8696044012}{720} \times 1,0545606529 \times 10^{-34} \times 3 \times 10^8 = 4,3367068589 \times 10^{-28} \tag{62}$$

This is an extremely small value and the total negative energy density we can extract from the Casimir Effect is directly proportional to C and inversely proportional to the fourth power of the distance d.But since C is always a constant then the total negative energy density we can extract from the Casimir Effect depends only of the distance d of the separation of the plates.

The total negative energy density we can extract from the Casimir Effect is better written by the following equation:

$$\rho = -\frac{4,3367068589 \times 10^{-28}}{d^4} \tag{63}$$

The mass of the proton is $1.672621777 \times 10^{-27}$ kilograms[22] and C is a fixed value 10 times lighter than the magnitude of a proton mass.

So in order to obtain reasonable amounts of negative energy densities by the Casimir Effect the distance d between the plates must be sub-microscopically small or the effect will not be noticed.

For two plates separated by a distance $d = 1$ meter the total negative energy density would be:

$$\rho = -\frac{4,3367068589 \times 10^{-28}}{1^4} = -4,3367068589 \times 10^{-28} Joules/Meter^3 \tag{64}$$

For a distance $d = 10$ meters the total negative energy density would be:

$$\rho = -\frac{4,3367068589 \times 10^{-28}}{(10)^4} = -4,3367068589 \times 10^{-32} Joules/Meter^3 \tag{65}$$

These values are impossible to be measured experimentally at such distances.

The Casimir Effect was theorized in 1948 by Casimir (pg 3 in [12],pg 40 in [15]) but in 1948 Casimir did not possessed the needed technology to demonstrate it experimentally and the Casimir Effect remained a mathematical conjecture until 1997.

Lamoreaux in 1997 demonstrated experimentally the Casimir Effect for a distance between plates of about $d = 0,6\mu m$ to $d = 6\mu m$ (pg 1 in [14]).$1\mu m = 10^{-6} meters$.

---

[22]see Wikipedia the Free Encyclopedia

Computing the Casimir Effect for a distance $d = 1\mu m$ or $d = 10^{-6}meters$ we should expect for:

$$\rho = -\frac{4,3367068589 \times 10^{-28}}{(10^{-6})^4} = -\frac{4,3367068589 \times 10^{-28}}{(10^{-24})} = -4,3367068589 \times 10^{-4} Joules/Meter^3 \quad (66)$$

The value above is within the range of values obtained by Lamoreaux for the Casimir Effect and although microscopically small it can be measured.

The work of Lamoreaux is very important:He obtained microscopically small amounts of negative energy density and unfortunately the warp drive needs astronomical amounts of it but at least Lamoreaux demonstrated that the Casimir Effect is real and the negative energy exists.

Alcubierre was the first to propose the use of the Casimir Effect to generate the warp drive spacetime distortion.(pg 9 in [1]).

Ridgely proposed the following idea:If the Casimir Effect can only generate sub-microscopical amounts of negative energy density and the warp drive needs tremendous amounts of it then why not try to modify the Casimir Effect in order to generate large and macroscopical amounts of it in order to sustain a warp bubble?? (pg 1 in [13]).

Due to the microscopically small value of the Planck Constant $h = 6,626 \times 10^{-34} \frac{Joules}{Seconds}$ which is the basis of all the Quantum Mechanics thats supports the Casimir Effect we dont know if the Ridgely idea is achievable.Perhaps it is possible and we simply dont know how to enlarge the Casimir Effect in order to obtain the large amounts of exotic matter needed to sustain a warp drive perhaps with a different set of equations.

So in this work we propose a different idea:why not to modify the geometry of the warp drive spacetime in order to be satisfied by the small amounts of negative energy generated by the Casimir Effect??

A modification in the equations of the warp drive can reduce dramatically the needs of negative energy needed to sustain it making the requirements of the warp drive negative energy density as small as the experimental results obtained by Lamoreaux for the Casimir Effect.

In this work we demonstrate that this is perfectly possible for the Natario warp drive spacetime:As a matter of fact we are about to propose the so-called Casimir warp drive. Remember that the Einstein field equation of General Relativity allows curved or distorted spacetime geometries if we can generate the needed mass-energy configurations that will distort the spacetime geometry according to our needs.If we want a certain spacetime geometric configuration we need to generate a distribution of mass and energy specially tailored for the needed configuration. So warp drives can be classified by two conditions:geometry(the left side of the Einstein field equation) and energy(the right side of the Einstein field equation).

- 1)-From the point of view of geometry the warp drive can be classified as an Alcubierre or a Natario warp drive since there are no other known solutions.

- 2)-From the point of view of the energy that will distort the spacetime geometry generating the warp drive effect the only known solution is the Casimir warp drive.

4 The problem of the negative energy in the Natario warp drive space-time:The unphysical nature of warp drive

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{67}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression:

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{68}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for[23]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{69}$$

In the equatorial plane(1 + 1 dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{70}$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[24] of about $6 \times 10^{24}kg$

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths for both Alcubierre and Natario cases!!!!

And multiplying the mass of Earth by c^2 in order to get the total positive energy "stored" in the Earth according to the Einstein equation $E = mc^2$ we would find the value of $54 \times 10^{40} = 5,4 \times 10^{41}Joules$.

Earth have a positive energy of $10^{41}Joules$ and dividing this by the volume of the Earth(radius $R_{Earth} = 6300$ km approximately) we would find the positive energy density of the Earth.Taking the cube of the Earth radius $(6300000m = 6,3 \times 10^6)^3 = 2,5 \times 10^{20}$ and dividing $5,4 \times 10^{41}$ by $(4/3)\pi R_{Earth}^3$ we would find the value of $4,77 \times 10^{20}\frac{Joules}{m^3}$. So Earth have a positive energy density of $4,77 \times 10^{20}\frac{Joules}{m^3}$ and we are talking about negative energy densities with a factor of 10^{48} for the warp drive while the quantum theory allows only microscopical amounts of negative energy density.

---

[23]see Appendix A

[24]see Wikipedia:The free Encyclopedia

So we would need to generate in order to maintain a warp drive with 200 times light speed the negative energy density equivalent to the positive energy density of 10^{28} Earths!!!!

A number with 28 zeros!!!.Unfortunately we must agree with the major part of the scientific community that says:"Warp Drive is impossible and unphysical!!".

Remember that the Casimir Effect is the only physical known source of negative energy density and Lamoreaux obtained experimentally negative energy densities of about $10^{-4}\frac{Joules}{Meter^3}$ while a body of density of 1 kilogram per cubic meter possesses a density of about $9 \times 10^{16}\frac{Joules}{Meter^3}$

However looking better to the expression of the negative energy density in the equatorial plane of the Natario warp drive:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{71}$$

We can see that a very low derivative and hence its square can perhaps obliterate the huge factor of 10^{48} ameliorating the negative energy requirements to sustain the warp drive.

In section 6 we will introduce another Natario shape function that defines the Natario warp drive spacetime and this function allows the reduction of the negative energy density requirements to arbitrary low values completely obliterating the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!...

The values obtained with our new Natario shape function will make the Natario warp drive negative energy density requirements falls well within the Lamoreaux experimental values obtained for the Casimir Effect.

Why we cannot use the Natario shape function $n(rs)$ defined in the section 2 as:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{72}$$

With $f(rs)$ the Alcubierre shape function being:[25].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{73}$$

In order to get the total energy requirements needed to sustain the Natario warp drive??

The square of the derivative of the Alcubierre shape function is given by:

$$f'(rs)^2 = \frac{1}{4}\left[\frac{@^2}{cosh^4[@(rs - R)]}\right] \tag{74}$$

In the equatorial plane $y = 0$ and we can neglect the second order derivative of the Natario shape function and consequently its square.The square of the first order derivative is then given by:

$$n'(rs)^2 = \frac{1}{4}f'(rs)^2 \tag{75}$$

[25]$tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

$$n'(rs)^2 = \frac{1}{4}(\frac{1}{4}[\frac{@^2}{cosh^4[@(rs-R)]}]) \tag{76}$$

$$n'(rs)^2 = \frac{1}{16}[\frac{@^2}{cosh^4[@(rs-R)]}] \tag{77}$$

An interesting feature is the fact that the square of the derivative of the Natario shape function in the equatorial plane is 4 times lower than its Alcubierre counterpart.

Back again to the negative energy density in the Natario warp drive:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{78}$$

Inserting the results of the squares of the derivatives of the Natario shape function we get:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[\frac{3}{16}[\frac{@^2}{cosh^4[@(rs-R)]}]\right] \tag{79}$$

Now we must discuss a little bit of warp drive basics:

- 1)-According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

Then inside the bubble and outside the bubble the Natario shape function have always constant or fixed values(0 inside the bubble and $\frac{1}{2}$ outside the bubble) and its derivative is zero.Hence the region where the Natario shape function vary its values resulting in non-null derivatives is the Natario warped region($0 < n(rs) < \frac{1}{2}$) which means to say the walls of the warp bubble.

- 2)-Since the negative energy density depends on non-null derivatives of the Natario shape function this means to say that the negative energy density resides in the Natario warped region(warp bubble walls)[26]

Then the region where

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{80}$$

is the Natario warped region($0 < n(rs) < \frac{1}{2}$) with $n'(rs))^2 \neq 0$

Lets define the beginning of the Natario warped region where $n(rs)$ ceases to be zero as the point a and the end of the Natario warped region where $n(rs)$ is about to reach the value of $\frac{1}{2}$ as the point b.Remember that rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.So we have a certain value for rs in the beginning of the Natario warped region which is a and another value for rs in the end of the Natario warped region which is b.

The difference $b - a$ is the width of the Natario warped region.

---

[26]See Appendix G for a graphical presentation of the warp bubble walls

From section 2 we know that @ is the Alcubierre parameter related to the thickness of the bubble which can possesses arbitrary values and as large @ is as thicker or thinner the bubble becomes.Then for a very small thickness or width we must have a thickness parameter $@ >> |R|$ which means to say a very large value for @.

Back again to the negative energy density in the Natario warp drive:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[\frac{3}{16}[\frac{@^2}{cosh^4[@(rs - R)]}]\right] \tag{81}$$

Note that a large @ multiplied by $\frac{c^2}{G}\frac{v_s^2}{8\pi}$(which is 10^{48} for 200 times faster than light) will make the negative energy requirements even worst.The shape function introduced in the previous section is not suitable for a real Natario warp drive spacetime.In section 6 we will present a better function that will satisfy the Natario requirements well within the range of experimental values obtained by Lamoreaux for the Casimir Effect.

5 Horizons,infinite Doppler blueshifts and interstellar navigation in the Natario warp drive in a $1+1$ dimensional spacetime

The equation of the Natario warp drive spacetime is given by[27]:

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2]dt^2 + 2[X^{rs}drs + X^\theta rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{82}$$

The expressions for X^{rs} and X^θ in the equation above are given by:(see pg 5 in [2])

$$X^{rs} = -2v_s n(rs)\cos\theta \tag{83}$$

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{84}$$

$$X^\theta = v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{85}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{86}$$

We are interested in the two-dimensional $1+1$ version of the Natario warp drive in the dimensions rs and t (motion over the $x - axis$ only with $\theta = 0$ $cos\theta = 1$ and $\sin\theta = 0$.) given by:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{87}$$

With X^{rs} being given by:

$$X^{rs} = 2v_s n(rs) \tag{88}$$

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

A Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{89}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{90}$$

The shape function above gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor.For a while it is important to outline only that the warp factor $WF >> |R|$ is much greater then the modulus of the bubble radius.

---

[27]see Appendices B and C

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

An Horizon occurs every time an observer in the center of the warp bubble send a photon towards the front of the bubble.The photon will stop somewhere in the Natario warped region never reaching the outermost layers of the Natario warped region which are causally disconnected with respect to the observer in the center of the bubble.This means to say that the observer in the center of the bubble cannot signal or control the outermost regions of the warp bubble[28].

The motion of a photon being sent to the front of the bubble obeys the null-like geodesics $(ds^2 = 0)$

Inserting the condition of the null-like geodesics $(ds^2 = 0)$ in the equation of the Natario warp drive spacetime in $1 + 1$ dimensions we have:

$$ds^2 = 0 \dashrightarrow [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 = 0 \tag{91}$$

Solving the quadratic form above for $\frac{dx}{dx}$ being the speed of the photon being sent to the front(or the rear) of the warp bubble we can see that we have two solutions:one for the photon sent towards the front of the ship($-$ sign) and another for the photon sent to the rear of the ship($+$ sign). [29]

$$\frac{drs_{front}}{dt} = X^{rs} - 1 \tag{92}$$

$$\frac{drs_{rear}}{dt} = X^{rs} + 1 \tag{93}$$

$$X^{rs} = 2v_s n(rs) \tag{94}$$

We are interested in the result of the photon being sent to the front.

The Natario shape function $n(rs)$ is defined as being 0 inside the warp bubble and $\frac{1}{2}$ outside the warp bubble while being $\frac{1}{2} > n(rs) > 0$ in the Natario warped region according to pg 5 in [2].

Solving the Natario warp drive equation in a $1 + 1$ spacetime for a null-like interval $ds^2 = 0$ we will have the following equation for the motion of the photon sent to the front:

$$\frac{dx}{dt} = 2vsn(rs) - 1 \tag{95}$$

Inside the Natario warp bubble $n(rs) = 0$ and $2vsn(rs) = 0$.
Outside the Natario warp bubble $n(rs) = \frac{1}{2}$ and $2vsn(rs) = vs$.

Somewhere inside the Natario warped region when $n(rs)$ starts to increase from 0 to $\frac{1}{2}$ making the term $2vsn(rs)$ increases from 0 to vs and assuming a continuous behavior then in a given point $2vsn(rs) = 1$ and $\frac{dx}{dt} = 0$.The photon stops,A Horizon is established

---

[28]at least using photons.
[29]See Appendix D

The value of the Natario shape function when or where the photon stops is given by[30]:

$$n(rs) = \frac{1}{2vs} \tag{96}$$

Note that for superluminal speeds $(vs > 1)(c = 1)$ as higher the speed as close to zero $n(rs)$ becomes.So as higher the speed $(vs >>> 1)(vs-> +\infty)$ the point where the photon stops. approaches zero[31][32]

$$\lim_{vs->+\infty} (\frac{1}{2vs}) = 0 \tag{97}$$

The exact position rs of the so-called Eulerian observer where(or when) the photon stops in the Horizon of the $1+1$ Natario warp drive spacetime is given by:

$$rs = R + \frac{1}{@} arctanh(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}}) \tag{98}$$

As higher the thickness parameter $@$ becomes and also as higher the warp factor WF becomes as thicker or thinner the warp bubble becomes too.(see pgs 8,10 and 11 in [9])

The term $(\frac{1}{@} \simeq 0)$ as long as $(@ \simeq +\infty)$ and the terms $(\sqrt[W_F]{\frac{1}{vs}} \simeq 0)$ and $(\sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}} \simeq 1)$ giving $(2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}} \simeq 2)$ as long as $(vs \simeq +\infty)$

But $(\sqrt[W_F]{\frac{1}{vs}} > 0)$[33] and $(1 - \sqrt[W_F]{\frac{1}{vs}} \simeq 1)$ although we will always have the condition $(1 - \sqrt[W_F]{\frac{1}{vs}} < 1)$ implying in $(\sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}} < 1)$ and in consequence $(2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}} < 2)$ however we can clearly see that $(2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}} > 1)$ as long as $vs >> 1$ superluminal.

From the conclusions given above we can see that $(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}} < 0)$

So we can expect the following results concerning the position of the Eulerian observer in the Horizon of the $1+1$ Natario warp drive spacetime:

$$\frac{1}{@} \simeq 0 \tag{99}$$

$$arctanh(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}}) < 0 \tag{100}$$

$$\frac{1}{@} arctanh(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}}) << 0 \dashrightarrow \frac{1}{@} arctanh(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}}) \simeq 0 \tag{101}$$

---

[30]See Appendix E

[31]value of $n(rs)$ inside the bubble

[32]although $(vs-> +\infty)$ we will always have the condition $(vs < +\infty)$ then vs will never reach (∞) so $(\frac{1}{2vs}-> 0)$ but will never reaches zero and the condition $(\frac{1}{2} > \frac{1}{2vs} > 0)$ will always remains valid keeping the Horizon inside the Natario warped region

[33]again $(vs-> +\infty)$ and $(vs < +\infty)$ then vs will never reach (∞)

$$rs = R + \frac{1}{@}arctanh(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}}) < R \dashrightarrow rs = R + \frac{1}{@}arctanh(1 - 2 \sqrt[W_F]{1 - \sqrt[W_F]{\frac{1}{vs}}}) \simeq R \quad (102)$$

The photon never stops outside the bubble $(rs > R)$ and not over the bubble radius $(rs = R)$.It stops somewhere inside the Natario warped region where $(\frac{1}{2} > n(rs) > 0)$ because $(n(rs) = \frac{1}{2vs})$ being $(0 < \frac{1}{2vs} < \frac{1}{2})$ as long as $(vs >> 1)$.

-)-For a while we will neglect the geometrical distribution of negative energy density in the the $1 + 1$ dimensional Natario warp drive spacetime when a photon is sent to the front of the bubble:

We are ready now to examine the topic of the Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the warp bubble.(See pg 6 and 8 in [2]) [34]

The unit vector n defined below represents the direction of the corresponding light ray from the point of view of the Eulerian observer outside the warp bubble.(See pg 6 in [2])

$$n = \frac{dx}{dx} - 2vsn(rs) \quad (103)$$

In the Horizon $(\frac{dx}{dx} = 0)$ and $(2vsn(rs) = 1)$
Then we have:

$$n = \frac{dx}{dx} - 2vsn(rs) = 0 - 1 = -1 \quad (104)$$

The equation of the observed energy is given by(See pg 8 in [2]):

$$E_0 = E(1 + n.vsf(rs)) \quad (105)$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $2vsn(rs) = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.vsf(rs))} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \quad (106)$$

And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

-)-Considering now the geometrical distribution of negative energy density in the $1 + 1$ dimensional Natario warp drive spacetime when a photon is sent to the front of the bubble:

The infinite Doppler Blueshift do not happens in the Natario warp drive spacetime.This means to say that the Natario warp drive is stable and perfectly physically possible to be achieved.

---

[34]See Appendix F

Consider again the negative energy density in the Natario warp drive spacetime(see pg 5 in [2])[3536]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{107}$$

In pg 6 in [2] a warp drive with a x-axis only (a $1+1$ dimensional warp drive)is considered.In this case $[y^2 + z^2] = 0$ but the Natario energy density is not zero and given by:.

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2\right] \tag{108}$$

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{109}$$

Note that in front of the ship in the Natario warp drive spacetime even in the $1+1$ dimensional case there exists negative energy density in the front of the ship.This negative energy density deflects the photon before the point of the Horizon.

The amount of negative energy density in the Horizon can be computed inserting the value of the rs in the Horizon into the negative energy densxity equation

$$rs = R + \frac{1}{@}arctanh(1 - 2^{WF}\sqrt{1 - {}^{WF}\sqrt{\frac{1}{vs}}}) \tag{110}$$

And the negative energy density that exists in the Natario warped region before the Horizon point will deflect the photon before the Horizon point.

So Infinite Doppler Blueshifts can generate instabilities but the these instabilities never happens in the Natario warp drive spacetime because the photon never reaches the Horizon.The photon is deflected by the negative energy in front of the ship.

Still according with Natario in pg 7 before section 5.2 in [3] negative energy density means a negative mass density and a negative mass density generates a repulsive gravitational field that deflects the hazardous interstellar matter in the front of the warp bubble. This repulsive gravitational field in front of the ship in the Natario warp drive spacetime protects the ship from impacts with the interstellar matter and also protects the ship from photons highly Doppler Blueshifted approaching the ship from the front.So in the Natario warp drive the problem of the collisions with dangerous interstellar matter never happens.[37]

---

[35]$n(rs)$ is the Natario shape function.Equation written the Geometrized System of Units $c = G = 1$

[36]Equation written in Cartezian Coordinates.See Appendix A

[37]See Appendix M

Adapted from the negative mass in Wikipedia:The free Encyclopedia:

"if we have a small object with equal inertial and passive gravitational masses falling in the gravitational field of an object with negative active gravitational mass (a small mass dropped above a negative-mass planet, say), then the acceleration of the small object is proportional to the negative active gravitational mass creating a negative gravitational field and the small object would actually accelerate away from the negative-mass object rather than towards it."

6 Reducing the negative energy density requirements in the Natario warp drive in a $1 + 1$ dimensional spacetime to the experimental limits obtained by Lamoreaux for the Casimir Effect:The Casimir warp drive

From section 3 we know that Lamoreaux demonstrated experimentally the existence of exotic matter with negative energy densities however at sub-microscopical limits of about $10^{-4} \frac{Joules}{Meter^3}$.

This an incredible small amount of negative energy density:a body of mass 1 kilogram in a volume of one cubic meter possesses an energy density of $9 \times 10^{16} \frac{Joules}{Meter^3}$

So the negative energy density obtained by Lamoreaux is 10^{20} times lighter than the one of a 1 kilogram body in a cubic meter of space or better: 100.000.000.000.000.000.000 times lighter than the one of a 1 kilogram body in a cubic meter of space !!

From section 4 we know that the negative energy needed to sustain a warp drive wether in the Alcubierre or Natario geometries with a speed of 200 times faster than light in order to visit stars at 20 light-years away in a reasonable amount of time demands an amount of negative energy density of about $10^{48} \frac{Joules}{Meter^3}$.

This is an astronomical amount of negative energy density impossible to be generated by any laboratory with any present or future technology:Earth mass is nearly 10^{24} kilograms and this number is 10^{24} times bigger in magnitude than the mass of the Earth or better:1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the Earth!!

Remember also that warp drives can be classified according to their geometries as Alcubierre or Natario cases since there are no other known topologies.But the most important thing here is not the geometry:

The most important thing considering warp drives is the source of negative energy density that will generate the spacetime geometrical distortion.The Casimir Effect is until now the only known source available.

Now we are ready to demonstrate how the negative energy density requirements can be greatly reduced for the Natario warp drive in a $1 + 1$ dimensional spacetime in order to fill the Lamoreaux experimental limits.We will introduce here the so-called Casimir warp drive.

We already know the form of the equation of the Natario warp drive in a $1 + 1$ dimensional spacetime:[38]

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{111}$$

$$X^{rs} = 2v_s n(rs) \tag{112}$$

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

A Natario warp drive valid shape function can be given by:

---

[38]See Appendix B

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{113}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{114}$$

The shape function above gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that the Alcubierre shape function $f(rs)$ is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor that will squeeze the region where the derivatives of the Natario shape function are different than 0.The warp factor is always a fixed integer number directly proportional to the modulus of the bubble radius.$WF > |R|$.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

We must analyze the differences between this new Natario shape function with warp factors compared to the original Natario shape function presented in Section 2 and mainly the differences between their derivative squares essential to low the negative energy density requirements in the $1+1$ Natario warp drive spacetime.In order to do so we need to use the Alcubierre shape function.

- 1)-Alcubierre shape function and its derivative square:[39].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{115}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{116}$$

- 2)-original Natario shape function and its derivative square:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{117}$$

$$n'(rs)^2 = \frac{1}{16}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{118}$$

- 3)-Natario shape function with warp factors and its derivative square:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{119}$$

---

[39]$tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{120}$$

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}]] \tag{121}$$

$$n'(rs)^2 = [\frac{1}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs - R)]}] \tag{122}$$

- 4)-negative energy density in the $1 + 1$ Natario warp drive spacetime:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{123}$$

We already know that the region where the negative energy density is concentrated is the warped region in both Alcubierre $(1 > f(rs) > 0)$ and Natario $(0 < n(rs) < \frac{1}{2})$ cases.

And we also know that for a speed of 200 times light speed the negative energy density is directly proportional to 10^{48} resulting from the term $\frac{c^2}{G}\frac{v_s^2}{8\pi}$.

So in order to get a physically feasible Natario warp drive the derivative of the Natario shape function must obliterate the factor 10^{48}.

Examining first the negative energy density from the original Natario shape function:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{124}$$

$$n'(rs)^2 = \frac{1}{16}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{125}$$

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[\frac{3}{16}[\frac{@^2}{cosh^4[@(rs - R)]}]\right] \tag{126}$$

We already know from section 4 that @ is the Alcubierre parameter related to the thickness of the bubble and a large $@ > |R|$ means a bubble of very small thickness.On the other hand a small value of $@ < |R|$ means a bubble of large thickness.But @ cannot be zero and cannot be $@ << |R|$ so independently of the value of @ the factor $\frac{c^2}{G}\frac{v_s^2}{8\pi}$ still remains with the factor 10^{48} from 200 times light speed which is being multiplied by @'2 making the negative energy density requirements even worst!!

Examining now the negative energy density from the Natario shape function with warp factors:

$$n'(rs)^2 = [\frac{1}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs - R)]}] \tag{127}$$

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[[\frac{3}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs - R)]}]\right] \tag{128}$$

31

Comparing both negative energy densities we can clearly see that the differences between the equations is the term resulting from the warp factor which is:

$$WF^4[1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] \tag{129}$$

Inside the bubble $f(rs) = 1$ and $[1-f(rs)^{WF}]^{2(WF-1)} = 0$ resulting in a $n'(rs)^2 = 0$.This is the reason why the Natario shape function with warp factors do not have derivatives inside the bubble.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{2(WF-1)}] = 0$ resulting also in a $n'(rs)^2 = 0$.This is the reason why the Natario shape function with warp factors do not have derivatives outside the bubble.

Using the Alcubierre warped region we have:

In the Alcubierre warped region $1 > f(rs) > 0$.In this region the derivatives of the Natario shape function do not vanish because if $f(rs) < 1$ then $f(rs)^{WF} << 1$ resulting in an $[1-f(rs)^{WF}]^{2(WF-1)} << 1$ .Also if $f(rs) < 1$ then $[f(rs)^{2(WF-1)}] << 1$ too if we have a warp factor $WF > |R|$.

Note that if $[1-f(rs)^{WF}]^{2(WF-1)} << 1$ and $[f(rs)^{2(WF-1)}] << 1$ their product $[1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] <<<< 1$

Note that inside the Alcubierre warped region $1 > f(rs) > 0$ when $f(rs)$ approaches 1 $n'(rs)^2$ approaches 0 due to the factor $[1-f(rs)^{WF}]^{2(WF-1)}$ and when $f(rs)$ approaches 0 $n'(rs)^2$ approaches 0 again due to the factor $[f(rs)^{2(WF-1)}]$.

Back again to the negative energy density using the Natario shape function with warp factors:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[[\frac{3}{16}]WF^4[1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs-R)]}]\right] \tag{130}$$

Independently of the thickness parameter @ or the bubble radius R for a warp factor $WF = 600$ we have the following situations considering the Alcubierre warped region $1 > f(rs) > 0$:

- 1)-in the beginning of the Alcubierre warped region when $f(rs) = 0,9$ then $[f(rs)^{2(WF-1)}] = [(0,9)^{2(600-1)}] = (0,9)^{2(599)} = (0,9)^{1198} = 1,5223913522 \times 10^{-55}$

- 2)-in an intermediate region of the Alcubierre warped region when $f(rs) = 0,8$ then $[f(rs)^{2(WF-1)}] = [(0,8)^{2(600-1)}] = (0,8)^{2(599)} = (0,8)^{1198} = 7,9763539287 \times 10^{-117}$

- 3)-in the middle of the Alcubierre warped region when $f(rs) = 0,5$ then $[f(rs)^{2(WF-1)}] = [(0,5)^{2(600-1)}] = (0,5)^{2(599)} = (0,5)^{1198} = 0$

- 4)-in the end of the Alcubierre warped region when $f(rs) = 0,1$ then $[f(rs)^{2(WF-1)}] = [(0,1)^{2(600-1)}] = (0,1)^{2(599)} = (0,1)^{1198} = 0$

Note that when $f(rs) = 9,9999990000 \times 10^{-1}$ the term $[f(rs)^{2(WF-1)}] = 9,9988020717 \times 10^{-1}$ and this does not ameliorate the factor 10^{48} but the term $[1-f(rs)^{WF}]^{2(WF-1)} = 0$ because as stated before when $f(rs)$ approaches 1 $n'(rs)^2$ approaches 0 due to the factor $[1-f(rs)^{WF}]^{2(WF-1)}$

32

Note also that the Natario shape function with warp factors completely obliterated the term $\frac{c^2}{G}\frac{v_s^2}{8\pi}$ with the factor 10^{48} from 200 times light speed making the negative energy density requirements physically feasible!!

And remember that 10^{48} is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth which is 10^{24} kilograms !!!

Lamoreaux in 1997 obtained for the negative energy density from the Casimir Effect the experimental value of $10^{-4}\frac{Joules}{Meter^3}$.

Remember again that this is an incredible small amount of negative energy density:a body of mass 1 kilogram in a volume of one cubic meter possesses an energy density of $9 \times 10^{16}\frac{Joules}{Meter^3}$

So the negative energy density obtained by Lamoreaux is 10^{20} times lighter than the one of a 1 kilogram body in a cubic meter of space or better: 100.000.000.000.000.000.000 times lighter than the one of a 1 kilogram body in a cubic meter of space !!

The peak of energy for our modified Natario shape function occurs in the neighborhoods of the region where $f(rs) = 0,9$ and multiplying 10^{48} by 10^{-55} we get $10^{-7}\frac{Joules}{Meter^3}$

So our modified shape function can sustain a Natario warp bubble moving at a speed of 200 times faster than light with negative energy density requirements 1000 times lighter than the ones obtained by Lamoreaux.

The Casimir Effect can without shadows of doubt generate and sustain a Natario warp drive spacetime.

Writing together the equations of the Natario negative energy density with our modified shape function and the Casimir negative energy density

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[[\frac{3}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs-R)]}]\right] \qquad (131)$$

$$\rho = -\frac{\pi^2}{720}\frac{\hbar c}{d^4} \qquad (132)$$

We get finally:

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[[\frac{3}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs-R)]}]\right] = -\frac{\pi^2}{720}\frac{\hbar c}{d^4} \qquad (133)$$

Remember that the most important thing concerning warp drives is the negative energy density needed to generate and sustain the warp bubble.The formula given above is the equation of the so-called Casimir warp drive.

7 A causally connected superluminal Natario warp drive spacetime using micro warp bubbles

In 2002 Gauthier,Gravel and Melanson appeared with the idea of the micro warp bubbles.([7],[8])

According to them,microscopical particle-sized warp bubbles may have formed spontaneously immediately after the Big Bang and these warp bubbles could be used to transmit information at superluminal speeds.These micro warp bubbles may exist today.(see abs of [8])

A micro warp bubble with a radius of 10^{-10} meters could be used to transport an elementary particle like the electron whose Compton wavelength is 2.43×10^{-12} meters at a superluminal speed.These micro warp bubbles may have formed when the Universe had an age between the Planck time and the time we assume that Inflation started.(see pg 306 of [7])

Following the ideas of Gauthier,Gravel and Melanson ([7],[8]) a micro warp bubble can send information or particles at superluminal speeds.(abs of [8],pg 306 in [7]).

The idea of Gauthier,Gravel and Melanson ([7],[8]) to send information at superluminal speeds using micro warp bubbles is very interesting and as a matter of fact shows to us how to solve the Horizon problem. Imagine that we are inside a large superluminal warp bubble and we want to send information to the front.Photons sent from inside the bubble to the front would stop in the Horizon but we also know that incoming photons from outside would reach the bubble.[40] The external observer outside the bubble have all the bubble causally connected while the internal observer is causally connected to the point before the Horizon.Then the external observer can create the bubble while the internal observer cannot. This was also outlined by Everett-Roman in pg 3 in [6]. Unless we find a way to overcome the Horizon problem. We inside the large warp bubble could create and send one of these micro warp bubbles to the front of the large warp bubble but with a superluminal speed $vs2$ larger than the large bubble speed $X = 2vsn(rs)$.Then $vs2 >> X$ or $vs2 >> 2vsn(rs)$ and this would allow ourselves to keep all the warp bubble causally connected from inside overcoming the Horizon problem without the need of the "tachyonic" matter.

- 1)- Superluminal micro warp bubble sent towards the front of the large superluminal warp bubble $vs2 = \frac{dx}{dt} > X - 1 > vs - 1 \dashrightarrow X = 2vsn(rs)$

From above it easy to see that a micro warp bubble with a superluminal speed $vs2$ maintains a large superluminal warp bubble with speed vs causally connected from inside if $vs2 > vs$

---

[40]not true for the Natario warp drive because the negative energy density in the front with repulsive gravitational behavior would deflect all the photons sent from inside and outside the bubble effectively shielding the Horizon from the photon avoiding the catastrophical Infinite Doppler Blueshift

From the point of view of the astronaut inside the large warp bubble he is the internal observer with respect to the large warp bubble but he is the external observer from the point of view of the micro warp bubble so he keeps all the light-cone of the micro warp bubble causally connected to him so he can use it to send superluminal signals to the large warp bubble from inside.(Everett-Roman in pg 3 in [6]).

According with Natario in pg 7 before section 5.2 in [3] negative energy density means a negative mass density and a negative mass generates a repulsive gravitational field that deflect photons or positive mass-density particles from the interstellar medium or particles sent to the bubble walls by the astronaut inside the bubble.

However while the negative mass deflects the positive mass or photons[41] a negative mass always attracts another negative mass so the astronaut cannot send positive particles or photons to the large warp bubble but by sending micro warp bubbles these also possesses negative masses that will be attracted by the negative mass of the large warp bubble effectively being a useful way to send signals.

---

[41]by negative gravitational bending of light

8   Conclusion

In this work we demonstrated that the Casimir Effect can provide the negative energy density to generate and maintain a Natario warp drive.

This work was designed to be independent and self-contained so we introduced all the mathematical techniques needed to fully understand the Natario warp drive in order to transmit a solid, consistent and well-established theory.These techniques are not new and as a matter of fact this work is a compilation of work already done by Natario in order to provide here a clear and large coverage of this topic.

In order to illustrate how a spaceship inside a warp bubble is completely at the rest with respect to a frame in its local neighborhoods but it is seen by a remote distant observer faraway from the bubble with a speed of 200 times faster than light we used the metaphorical example of the fish and the aquarium.The warp bubble is the aquarium and the bubble walls are the glass of the aquarium floating in the stream of a river.An observer in the margin would see the aquarium passing by him at a large speed being carried out by the stream but inside the aquarium the fish is at the rest free from g-forces or accelerations.The warp drive distortion of the spacetime act like a river stream in the Natario model carrying the warp bubble with the spaceship inside at the rest but a distant observer sees the bubble passing by him at a large speed and in our case 200 times light speed.

We discussed the 3 major physical problems needed to be solved in order to make the Natario warp drive a suitable way for superluminal interstellar travel (1-Negative energy densities 2-Horizons and Doppler blueshifts 3-Collisions with hazardous interstellar matter) and we presented some theoretical solutions that may accomplish our goal.

We also presented a brief discussion of the problem of the energy conditions in General Relativity that requires always positive energy densities while the energy density for the warp drive is always negative.The Casimir Effect predicted theoretically in 1948 by Casimir [12] and experimentally demonstrated in 1997 by Lamoreaux [14] is the only known source of negative energy density.Lamoreaux obtained $10^{-4}\frac{Joules}{Meters^3}$ of negative energy density.This is a sub-microscopical quantity 10^{20} times lighter than the one of a 1 kilogram body in a cubic meter of space or better: 100.000.000.000.000.000.000 times lighter than the one of a 1 kilogram body in a cubic meter of space but at least we know that negative energy densities exists indeed.

For the negative energy density needed to travel at 200 times light speed we lowered the total amount from $10^{48}\frac{Joules}{Meters^3}$ which have a magnitude of 1.000.000.000.000.000.000.000.000 the magnitude of the mass of the Earth to $10^{-7}\frac{Joules}{Meters^3}$ using a Natario shape function with warp factors derived from the modulus of the bubble radius.Our result is 1000 times lighter than the negative energy density of $10^{-4}\frac{Joules}{Meters^3}$ obtained experimentally by Lamoreaux in 1997 for the Casimir Effect. Combined the equation for the negative energy density in the Natario warp drive with the equation of the negative energy density for the Casimir Effect we obtained the equation of the Casimir warp drive.

Also we demonstrated that the problems of interstellar navigation and impacts with hazardous objects(comets asteroids,photons) that exists in interstellar space can easily be circumvented in the Natario warp drive.

Horizons and causally disconnected portions of spacetime will need to wait for a theory that encompasses both General Relativity and the Non-Locality of Quantum Mechanics in order to be solved but perhaps General Relativity can by itself solve the problem if we really discover how to assemble warp bubbles from a negative energy density concentration.

But unfortunately although we already know now how to generate the negative energy density to sustain a warp drive by the Casimir Effect we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

9 Appendix A:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[42]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \qquad (134)$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \qquad (135)$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \qquad (136)$$

---

[42]$n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

10 Appendix B:Mathematical demonstration of the Natario warp drive equation using the Natario vector nX for a constant speed vs

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{137}$$

where X^i is the so-called shift vector.This shift vector is the responsible for the warp drive behavior defined as follows(pg 2 in [2]):

$$X^i = X, Y, Z \curvearrowright i = 1, 2, 3 \tag{138}$$

The warp drive spacetime is completely generated by the Natario vector nX(pg 2 in [2])

$$nX = X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}, \tag{139}$$

Defined using the canonical basis of the Hodge Star in spherical coordinates as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \tag{140}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \tag{141}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \tag{142}$$

Redefining the Natario vector nX as being the rate-of-strain tensor of fluid mechanics as shown below(pg 5 in [2]):

$$nX = X^r \mathbf{e}_r + X^\theta \mathbf{e}_\theta + X^\varphi \mathbf{e}_\varphi \tag{143}$$

$$nX = X^r dr + X^\theta rd\theta + X^\varphi r\sin\theta d\varphi \tag{144}$$

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{145}$$

$$X^i = r, \theta, \varphi \curvearrowright i = 1, 2, 3 \tag{146}$$

We are interested only in the coordinates r and θ according to pg 5 in [2])

$$ds^2 = dt^2 - (dr - X^r dt)^2 - (rd\theta - X^\theta dt)^2 \tag{147}$$

$$(dr - X^r dt)^2 = dr^2 - 2X^r dr dt + (X^r)^2 dt^2 \tag{148}$$

$$(rd\theta - X^\theta dt)^2 = r^2 d\theta^2 - 2X^\theta r d\theta dt + (X^\theta)^2 dt^2 \tag{149}$$

$$ds^2 = dt^2 - (X^r)^2 dt^2 - (X^\theta)^2 dt^2 + 2X^r drdt + 2X^\theta r d\theta dt - dr^2 - r^2 d\theta^2 \tag{150}$$

$$ds^2 = [1 - (X^r)^2 - (X^\theta)^2]dt^2 + 2[X^r dr + X^\theta r d\theta]dt - dr^2 - r^2 d\theta^2 \tag{151}$$

making $r = rs$ we have the Natario warp drive equation:

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2]dt^2 + 2[X^{rs}drs + X^\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{152}$$

According with the Natario definition for the warp drive using the following statement(pg 4 in [2]):any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble

The expressions for X^{rs} and X^θ are given by:(see pg 5 in [2])

$$nX \sim -2v_s n(rs) \cos\theta \mathbf{e}_{rs} + v_s(2n(rs) + (rs)n'(rs)) \sin\theta \mathbf{e}_\theta \tag{153}$$

$$nX \sim 2v_s n(rs) \cos\theta \mathbf{e}_{rs} - v_s(2n(rs) + (rs)n'(rs)) \sin\theta \mathbf{e}_\theta \tag{154}$$

$$nX \sim -2v_s n(rs) \cos\theta drs + v_s(2n(rs) + (rs)n'(rs)) \sin\theta rsd\theta \tag{155}$$

$$nX \sim 2v_s n(rs) \cos\theta drs - v_s(2n(rs) + (rs)n'(rs)) \sin\theta rsd\theta \tag{156}$$

But we already know that the Natario vector nX is defined by(pg 2 in [2]):

$$nX = X^{rs}drs + X^\theta rsd\theta \tag{157}$$

Hence we should expect for:

$$X^{rs} = -2v_s n(rs) \cos\theta \tag{158}$$

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{159}$$

$$X^\theta = v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{160}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{161}$$

We are interested in the two-dimensional $1+1$ version of the Natario warp drive in the dimensions rs and t (motion over the $x-axis$ only with $\theta = 0$ $cos\theta = 1$ and $\sin\theta = 0$) given by:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{162}$$

$$X^{rs} = 2v_s n(rs) \tag{163}$$

11 Appendix C:Differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{164}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{165}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{166}$$

From above we get the following results

$$dr \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{167}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{168}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{169}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [10]):

$$*dr = r^2\sin\theta(d\theta \wedge d\varphi) \tag{170}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{171}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{172}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{173}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{174}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta rd\theta \tag{175}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{176}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta \wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi \wedge dr)] \tag{177}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta \wedge d\varphi)] - [r\sin^2\theta(d\varphi \wedge dr)] \tag{178}$$

We know that the following expression holds true(see pg 9 in [9]):

$$d\varphi \wedge dr = -dr \wedge d\varphi \tag{179}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta \wedge d\varphi)] + [r\sin^2\theta(dr \wedge d\varphi)] \tag{180}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{181}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{182}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{183}$$

According to pg 10 in [9] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta \wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr \wedge d\varphi) \tag{184}$$

Because and according to pg 10 in [9]:

$$d(\alpha + \beta) = d\alpha + d\beta \tag{185}$$

$$d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{186}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{187}$$

42

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta) \wedge d\varphi + \sin^2\theta \wedge dd\varphi = 2sin\theta \cos\theta(d\theta \wedge d\varphi) \tag{188}$$

$$*[d(r^2)d\varphi] = 2rdr \wedge d\varphi + r^2 \wedge dd\varphi = 2r(dr \wedge d\varphi) \tag{189}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2 \sin\theta \cos\theta(d\theta \wedge d\varphi) + r \sin^2\theta(dr \wedge d\varphi) \tag{190}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2 \sin^2\theta d\varphi] \tag{191}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{192}$$

$$f(r)r^2 2sin\theta \cos\theta(d\theta \wedge d\varphi) + f(r)\sin^2\theta 2r(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{193}$$

$$2f(r)r^2 sin\theta \cos\theta(d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{194}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{195}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{196}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{197}$$

We can obtain the following result:

$$2f(r) \ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r) \ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{198}$$

$$2f(r) \ cos\theta e_r - 2f(r) \ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{199}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r) \ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{200}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{201}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{202}$$

43

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r) \ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \tag{203}$$

$$nX = -2vs(t)f(r) \ cos\theta e_r + vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \tag{204}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r) \ cos\theta dr - vs(t)[2f(r) + rf'(r)]r \sin\theta d\theta \tag{205}$$

$$nX = -2vs(t)f(r) \ cos\theta dr + vs(t)[2f(r) + rf'(r)]r \sin\theta d\theta \tag{206}$$

12 Appendix D:Solution of the quadratic form $ds^2 = 0$ for the Horizon problem in the Natario warp drive in a $1 + 1$ spacetime

The equation of the Natario warp drive spacetime in $1 + 1$ dimensions is given by:[43]

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{207}$$

Inserting the condition of the null-like geodesics $(ds^2 = 0)$ in the equation of the Natario warp drive spacetime in $1 + 1$ dimensions we have:

$$ds^2 = 0 \dashrightarrow [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 = 0 \tag{208}$$

$$[1 - (X^{rs})^2] + 2X^{rs}\frac{drs}{dt} - (\frac{drs}{dt})^2 = 0 \tag{209}$$

$$U = \frac{drs}{dt} \tag{210}$$

$$[1 - (X^{rs})^2] + 2X^{rs}U - U^2 = 0 \tag{211}$$

$$U^2 - 2X^{rs}U - [1 - (X^{rs})^2] = 0 \tag{212}$$

$$U = \frac{2X^{rs} \pm \sqrt{4(X^{rs})^2 + 4[1 - (X^{rs})^2]}}{2} \tag{213}$$

$$U = \frac{2X^{rs} \pm 2}{2} \tag{214}$$

$$U = \frac{drs}{dt} = X^{rs} \pm 1 \tag{215}$$

From above we can see that we have two solutions:one for the photon sent towards the front of the ship($-$ sign) and another for the photon sent to the rear of the ship($+$ sign).

$$U_{front} = \frac{drs_{front}}{dt} = X^{rs} - 1 \tag{216}$$

$$U_{rear} = \frac{drs_{rear}}{dt} = X^{rs} + 1 \tag{217}$$

$$X^{rs} = 2v_s n(rs) \tag{218}$$

---

[43]See Appendix B

13   Appendix E:Mathematical Demonstration of the horizon equation in the Natario warp drive for a $1 + 1$ spacetime

An Horizon occurs every time we get:[44][45]

$$U_{front} = \frac{drs_{front}}{dt} = X^{rs} - 1 = 2v_s n(rs) - 1 = 0 \tag{219}$$

The value of the $n(rs)$ in the Horizon is given by:

$$2v_s n(rs) - 1 = 0 \dashrightarrow 2v_s n(rs) = 1 \tag{220}$$

$$n(rs) = \frac{1}{2vs} = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{221}$$

$$\frac{1}{vs} = [1 - f(rs)^{WF}]^{WF} \tag{222}$$

$$\sqrt[WF]{\frac{1}{vs}} = 1 - f(rs)^{WF} \tag{223}$$

$$f(rs)^{WF} = 1 - \sqrt[WF]{\frac{1}{vs}} \tag{224}$$

$$f(rs) = \sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{225}$$

$$1 - tanh[@(rs - R)] = 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} \tag{226}$$

$$tanh[@(rs - R)] = 1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} \tag{227}$$

$$@(rs - R) = arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{228}$$

$$rs - R = \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{229}$$

$$rs = R + \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{230}$$

---

[44]See Appendix D

[45]When reading this appendix remember that $WF >> 1$ and $vs >> 1$

14 Appendix F:Infinite Doppler blueshift in the horizon for the Natario warp drive in a $1+1$ spacetime

Defining the following term X as being:

$$X = 2vsn(rs) \tag{231}$$

We know that in the Horizon the value of $n(rs)$ is given by:[46]

$$n(rs) = \frac{1}{2vs} \tag{232}$$

Then we have for X:

$$X = 2vsn(rs) = 1 \tag{233}$$

Also for the Horizon we have:

$$\frac{dx}{dt} = 0 \tag{234}$$

The photon stops!!![47]

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} = X - 1 = 2vsn(rs) - 1$

- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} = X + 1 = 2vsn(rs) + 1$

Or even better:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} - X = \frac{dx}{dt} - 2vsn(rs) = -1$

- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} + X = \frac{dx}{dt} + 2vsn(rs) = +1$

Then we can easily see that(pg 6 in [2]).:

$$\parallel \frac{dx}{dt} - X \parallel = 1 \tag{235}$$

$$\parallel \frac{dx}{dt} - 2vsn(rs) \parallel = 1 \tag{236}$$

Defining now the term n in function of X we get the following values for n:

- 1)-photon sent to the front of the bubble

$$n = \frac{dx}{dt} - X = \frac{dx}{dt} - 2vsn(rs) = -1 \dashrightarrow 2vsn(rs) = 1 \dashrightarrow \frac{dx}{dt} = 0 \tag{237}$$

- 1)-photon sent to the rear of the bubble

$$n = \frac{dx}{dt} + X = \frac{dx}{dt} + 2vsn(rs) = +1 \dashrightarrow 2vsn(rs) = 1 \dashrightarrow \frac{dx}{dt} = 0 \tag{238}$$

We are interested in the results for the photon sent to the front of the bubble.

---

[46]See Appendix E

[47]See Appendix D

We are ready now to examine the topic of the Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the Warp Bubble.See pg 6 and 8 in [2].

The unit vector n defined below represents the direction of the corresponding light ray from the point of view of the Eulerian observer outside the warp bubble.(See pg 6 in [2]).

$$n = \frac{dx}{dx} - X = \frac{dx}{dx} - 2vsn(rs) \tag{239}$$

In the Horizon $(\frac{dx}{dx} = 0)$ and $(X = 2vsn(rs) = 1)$
Then we have:

$$n = \frac{dx}{dx} - X = \frac{dx}{dx} - 2vsn(rs) = 0 - 1 = -1 \tag{240}$$

The equation of the observed energy is given by(See pg 8 in [2]):

$$E_0 = E(1 + n.X) \tag{241}$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $X = 2vsn(rs) = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.X)} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \tag{242}$$

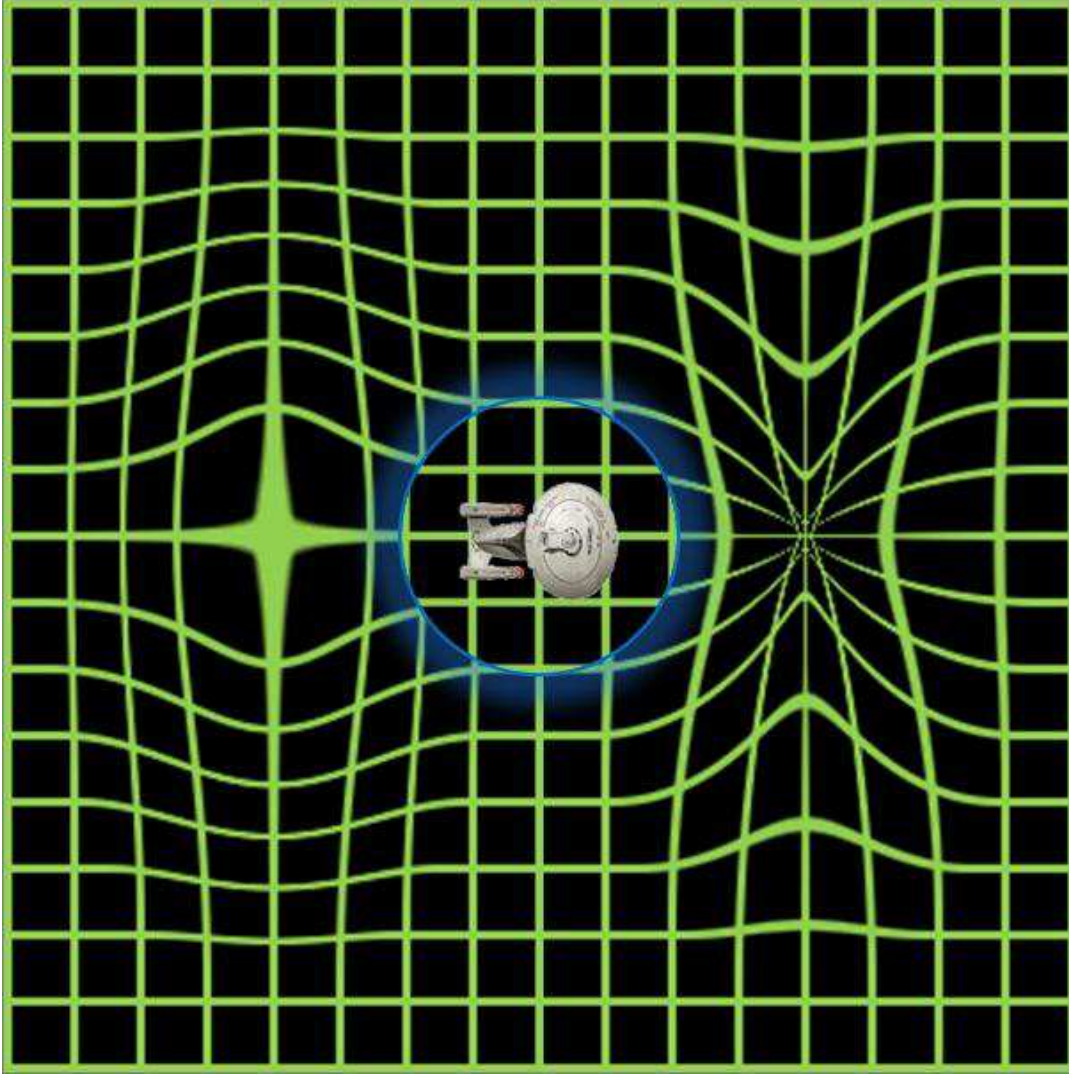And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

Figure 1: Artistic representation of the Natario Warp Bubble .(Source:Internet)

15 Appendix G:Artistic presentation of the Natario warp bubble

According to the Natario definition for the warp drive using the following statement(pg 4 in [2]):

- 1)-Any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ or $X = -vs$ for a large value of rs defined by Natario as the exterior of the bubble with $vs(t)$ being the speed of the bubble and X is the shift vector(pg 5 in [2]).

The blue region is the Natario warped region(bubble walls) where the negative energy is located.

Note that the negative energy surrounds entirely the "spaceship" inside the bubble deflecting incoming hazardous interstellar objects protecting the crew.[48]

A given Natario vector nX generates a Natario warp drive Spacetime if and only if satisfies these conditions stated below:

- 1)-A Natario vector nX being $nX = 0$ for a small value of rs(interior of the bubble)

- 2)-A Natario vector $nX = Xdx$ for a large value of rs(exterior of the bubble)

- 3)-A shift vector X depicting the speed of the bubble being $X = 0$(interior of the bubble) while $X = vs$ or $X = -vs$ being the shift vector seen by distant observers(exterior of the bubble).

The Natario vector nX is given by:

$$nX = v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim 2v_sn(rs)\cos\theta drs - v_s(2n(rs) + rsn'(rs))rs\sin\theta d\theta \qquad (243)$$

$$nX = -v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim -2v_sn(rs)\cos\theta drs + v_s(2n(rs) + rsn'(rs))rs\sin\theta d\theta \qquad (244)$$

This holds true if we set for the Natario vector nX a continuous Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the bubble) and $n(rs) = 0$ for small rs(inside the bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the bubble(pg 5 in [2])

The Natario vector $nX = 0$ vanishes inside the bubble because inside the bubble there are no motion at all because $dx = 0$ or $n(rs) = 0$ or $X = 0$ while being $nX = -vs(t)dx \neq 0$ or $nX = vs(t)dx \neq 0$ not vanishing outside the bubble because $n(rs)$ do not vanish.Then an external observer would see the bubble passing by him with a speed defined by the shift vector $X = -vs(t)$ or $X = vs(t)$.

The "spaceship" above lies in the interior of the bubble at the rest $X = vs = 0$ but the observer outside the bubble sees the "spaceship" passing by him with a speed $X = vs$.

See also pgs 14,15 and 16 in [4] and pgs 7,8 and 9 in [2] for more graphical presentations of the Natario warp bubble
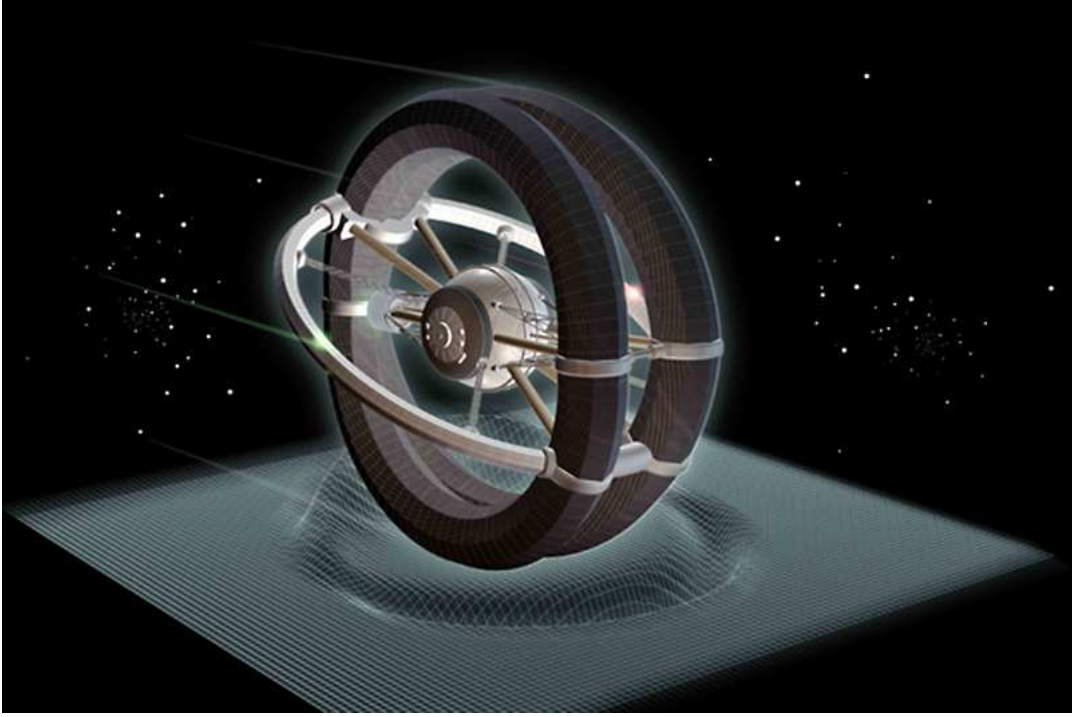
---

[48]See Appendix M

Figure 2: Artistic representation of the Natario warp drive .Note the Alcubierre Expansion of the Normal Volume Elements below.(Source:Internet)

16   Appendix H:Artistic presentation of the Natario warp drive

Note that according to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).

Remember also that the expansion of the normal volume elements in the Natario warp drive is given by the following expressions(pg 5 in [2]). :

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{245}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial \theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{246}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial \varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{247}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{248}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.

This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The "bars" in the figure were included to illustrate how the expansion in one direction can be counterbalanced by the contraction in the other directions.These "bars" keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [1] is also included

Note also that the energy density in the Natario Warp Drive being given by the following expressions(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2\sin^2\theta\right]. \tag{249}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{250}$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons so these will not reach the Horizon and the Natario warp drive will not suffer from Doppler blueshifts.The illustrated "bars" are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble.[49]

-)-Energy directly above the ship$(y - axis)$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{251}$$

-)-Energy directly in front of the ship$(x - axis)$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta\right]. \tag{252}$$

Note that as fast as the ship goes by,then the negative energy density requirements grows proportionally to the square of the bubble speed vs.

For a bubble speed 200 times faster than light then $(vs = 6 \times 10^{10})$ and this affects the negative energy requirements by the factor vs^2 being $(vs^2 = 3,6 \times 10^{21})$ [50]

This raises the amount of negative energy to enormous levels making this a major concern when studying warp drive spacetimes.

The derivatives of the Natario shape function $n(rs)$ must be very low in order to obliterate the factor vs^2.

---

[49]See Appendix M

[50]the mass of the Earth according to Wikipedia is $M_\oplus = 5,9722 \times 10^{24}$ kilograms.In tons this reaches 10^{21} tons,exactly the factor vs^2 for 200 times light speed

Note also that even in a $1 + 1$ dimensional spacetime the Natario warp drive retains the zero expansion behavior:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r) \cos \theta \tag{253}$$

$$K_{\theta\theta} = \frac{X^r}{r} = v_s n'(r) \cos \theta; \tag{254}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} = v_s n'(r) \cos \theta \tag{255}$$

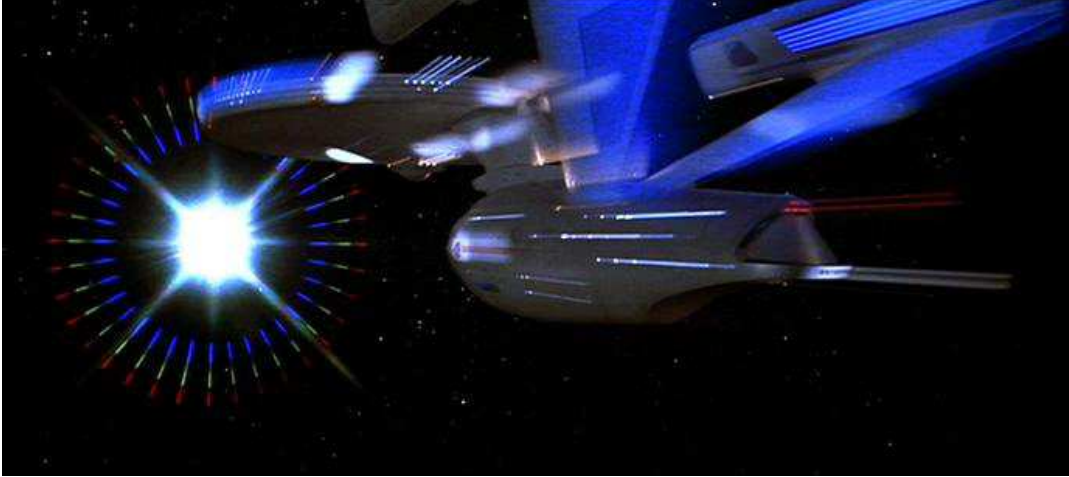$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{256}$$

Figure 3: Artistic representation of the Natario Warp Drive breaking the light speed barrier. (Source:Internet)

17 Appendix I:Artistic Presentation of the Natario warp drive entering in the luminal warp "shockwave" $X = 1$, Natario angle=$\alpha = \frac{\pi}{2}$

Although the figure above was "borrowed" from science fiction it depicts correctly what would happen for a Natario warp drive when it reaches luminal speeds($X = 1$). We already know that when $\|\mathbf{X}\| = 1$ and $vs = 1$ according to pg 6 in [2],(see also pg 15 in [4])

$$\sin \alpha = \frac{1}{\|\mathbf{X}\|} = 1 \tag{257}$$

$$\sin \alpha = 1 \dashrightarrow \alpha = \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| = 1 \tag{258}$$

From above we see that the white light "disk" appears in front of the ship perpendicular to the direction of motion because the Natario angle in this case is $\frac{\pi}{2}$.[51].

The white light "disk" represents the Natario "boom" that may harm the surface of a planet if the ship breaks the light speed barrier in the neighborhoods of it like the Mach "boom" harms glass of windows or human hears if a jet breaks the sound barrier at a low altitude and over populated areas .

---

[51]Compare this with the graphical presentation of the Mach cone angle Appendix L
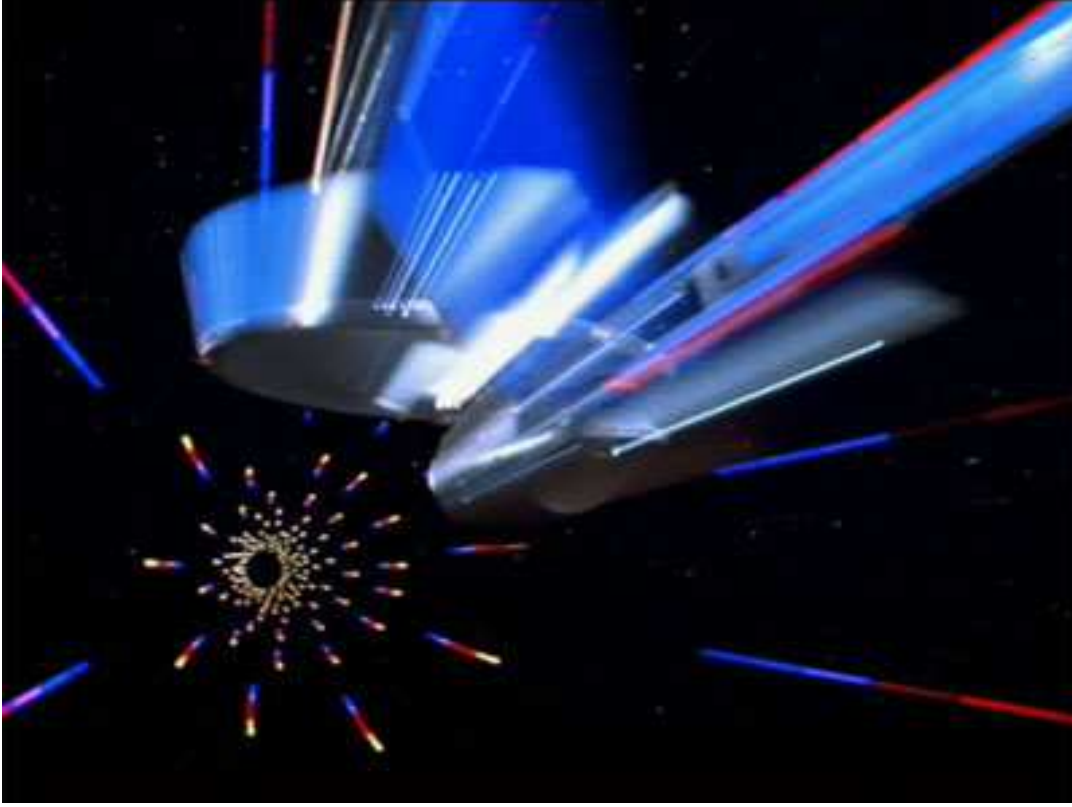
Figure 4: Artistic representation of the Natario Warp Drive at "low" superluminal(warp) speed. Note the Doppler lines . (Source:Internet)

18 Appendix J:Artistic presentation of the Natario warp drive at "low" superluminal(warp) speed $X = vs$ and $vs > 1$. Natario angle $\alpha < \frac{\pi}{2}$ $\alpha \simeq \frac{\pi}{2}$

Like in the previous artistic presentation the figure above was also "borrowed" from science fiction and also depicts correctly what would happen for a Natario warp drive when it reaches "low" superluminal(warp) speeds($X > 1$). We already know that when $X = vs$ and $vs > 1$ (pg 6 in [2]).

$$\sin \alpha = \frac{1}{v_s} \tag{259}$$

$$\sin \alpha < 1 \dashrightarrow \alpha < \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| > 1 \dashrightarrow \sin \alpha \simeq 1 \dashrightarrow \alpha \simeq \frac{\pi}{2} \tag{260}$$

From above we can see that $\sin \alpha < 1$ and $\alpha < \frac{\pi}{2}$.This means that the Natario angle is no longer perpendicular to the direction of motion and now have a small inclination given by the Natario angle itself.[52].

---

[52]Compare this with the graphical presentation of the Mach cone angle Appendix L

Figure 5: Artistic representation of the Natario Warp Drive at high superluminal(warp) speed. Note the Doppler lines . (Source:Internet)

19 Appendix K:Artistic presentation of the Natario warp drive at "high" superluminal(warp) speed $X = vs$ and $vs >> 1$. Natario angle $\alpha << \frac{\pi}{2}$ $\alpha \simeq 0$

Like in the two previous artistic presentation the figure above was also "borrowed" from science fiction and also depicts correctly what would happen for a Natario warp drive when it reaches "high" superluminal(warp) speeds($X >> 1$). We already know that when $X = vs$ and $vs >> 1$ (pg 6 in [2]).

$$\sin \alpha = \frac{1}{v_s} \tag{261}$$

$$\sin \alpha << 1 \dashrightarrow \alpha << \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| >> 1 \dashrightarrow \sin \alpha \simeq 0 \dashrightarrow \alpha \simeq 0 \tag{262}$$

In this case since $X >> 1$ $\sin \alpha << 1$ and $\alpha << \frac{\pi}{2}$.The Natario angle now appears almost parallel to the direction of motion and the inclination of the Natario angle is so hight that approaches zero.[53].

---

[53]Compare this with the graphical presentation of the Mach cone angle Appendix L
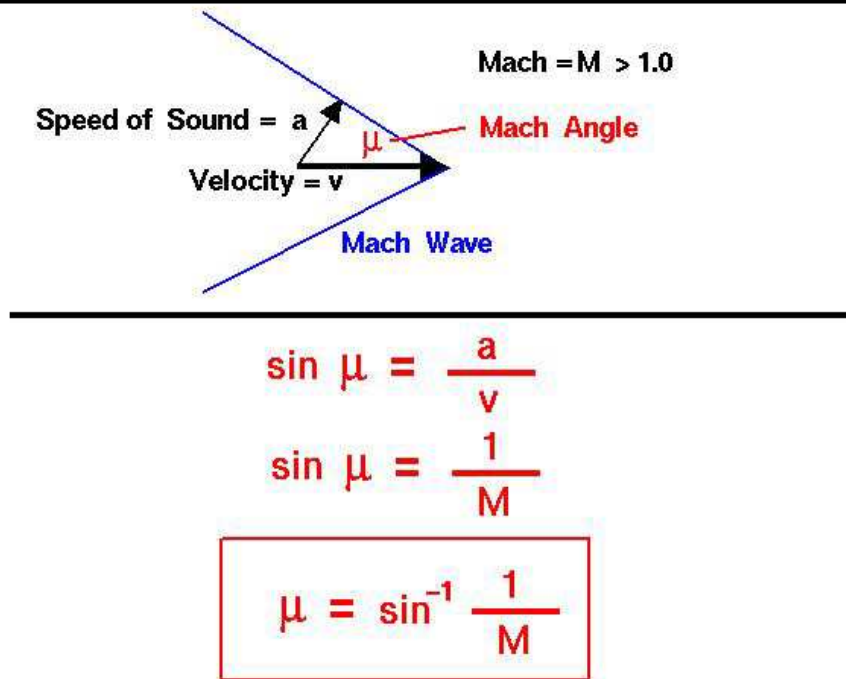
Figure 6: Graphical Presentation of the Mach cone angle . (Source:NASA)

20   Appendix L:Artistic presentation of the Mach cone angle

Above is being presented the graphical presentation of the Mach cone angle.We included this presentation here[54] to demonstrate the similarities that exists between the supersonic(Mach) speeds and the Mach cone angle with the superluminal(warp) speeds and the Natario cone angle.

Note that when a plane achieves the speed of the sound its velocity v equals the speed of the sound a and then

$$\sin \mu = \frac{a}{v} = 1 \tag{263}$$

In the above case the angle $\mu = \frac{\pi}{2}$ and the Mach "shockwave" is perpendicular to the direction of motion of the plane.

But when the plane accelerates to supersonic(Mach) speeds then $v >> a$ and $\sin \mu = \frac{a}{v} << 1$ with the angle $\mu \cong 0$.As fast the plane accelerates the Mach angle approaches zero and the Mach shockwave inclination tend to be parallel to the speed of the plane itself.

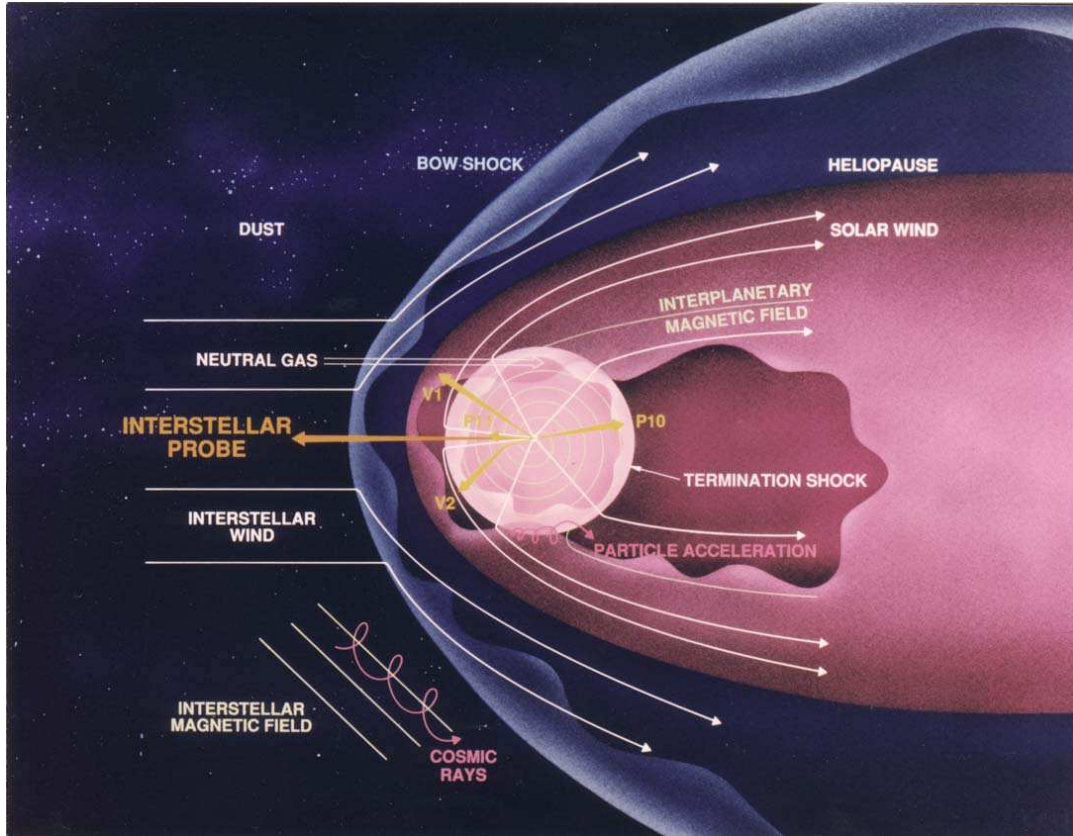[54]See the Remarks Section on Appendix L

Figure 7: Artistic representation of a Natario Warp Drive in a real superluminal space travel .Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter(brown arrows).(Source:Internet)

21 Appendix M:Artistic presentation of a Natario warp drive in a real faster than light interstellar spaceflight

Above is being presented the artistic presentation of a Natario warp drive in a real interstellar superluminal travel.The "ball" or the spherical shape is the Natario warp bubble with the negative energy surrounding the ship in all directions and mainly protecting the front of the bubble.[55]

The brown arrows in the front of the Natario bubble are a graphical presentation of the negative energy in front of the ship deflecting dust,neutral gases,interstellar wind and Doppler Blueshifted photons.

The angle between the blue bow shock and the direction of motion is the Natario angle defined by pg 6 in [2].[56]

The spaceship is at the rest and in complete safety inside the Natario bubble.

---

[55]See the Appendices G and H

[56]See the Appendices I,J,K and L

Lets illustrate the dangers of the collisions with interstellar matter in a real superluminal travel.

Consider the Natario "ball" of this presentation.The ship is at the rest inside the "ball" but the "ball" (warp bubble) is moving with a speed of 200 times faster than light. Imagine that our "ball" in interstellar space is moving towards two clouds of interstellar dust initially at the rest but in front of the ship :cloud A and cloud B.Cloud A lies in front of cloud B. So the ship will reach cloud A first. When the ship arrives at cloud A the repulsive gravitational field will deflect the particles but particles directly over the axis of the "ball" motion will be pushed away by the repulsive gravitational field from the negative energy of the warp bubble walls like a jet "stream" or a "dragging". Now the "ball" and "dragged" particles from cloud A are moving 200 times faster than light in the direction of cloud B.The "dragged" particles from the cloud A will reach the particles from cloud B initially at the rest and collisions will happens.These collisions will be highly energetic releasing enormous bursts of energy. [57] Then in front of the bubble a region of high energy will be formed.The ship at the rest inside the bubble is protected by the negative energy of the warp bubble walls but superluminal spaceships generating these huge amounts of energy in front of their bubbles must avoid to use faster than light speeds in the neighborhoods of planets because these have no protections against these energetic bursts.

A ship must leave a planet(Earth) at subluminal speeds and travel to the limits of the planetary system still at subluminal speeds and only in real interstellar space far away from any planet the ship can then trigger the faster than light motion.Once arriving at its destiny(Gliese $667c$) but still in interstellar space the ship must slow down from superluminal speeds entering in the planetary system to be visited at subluminal speeds in order to do not harm any planet.

According with Natario in pg 7 before section 5.2 in [3] negative energy density means a negative mass density and a negative mass density generates a repulsive gravitational field. This repulsive gravitational field in front of the ship in the Natario warp drive spacetime protects the ship from impacts with the hazardous interstellar matter and also protects the ship from photons highly Doppler Blueshifted approaching the ship from the front.So in the Natario warp drive the problem of the collisions with the interstellar matter never happens.

Adapted from the negative mass in Wikipedia:The free Encyclopedia:

"if we have a small object with equal inertial and passive gravitational masses falling in the gravitational field of an object with negative active gravitational mass (a small mass dropped above a negative-mass planet, say), then the acceleration of the small object is proportional to the negative active gravitational mass creating a negative gravitational field and the small object would actually accelerate away from the negative-mass object rather than towards it."

---

[57]Think on how many particles of space dust exists in each cubic centimeter of interstellar space and think on how many cubic centimeters of space exists in 20 light-years.
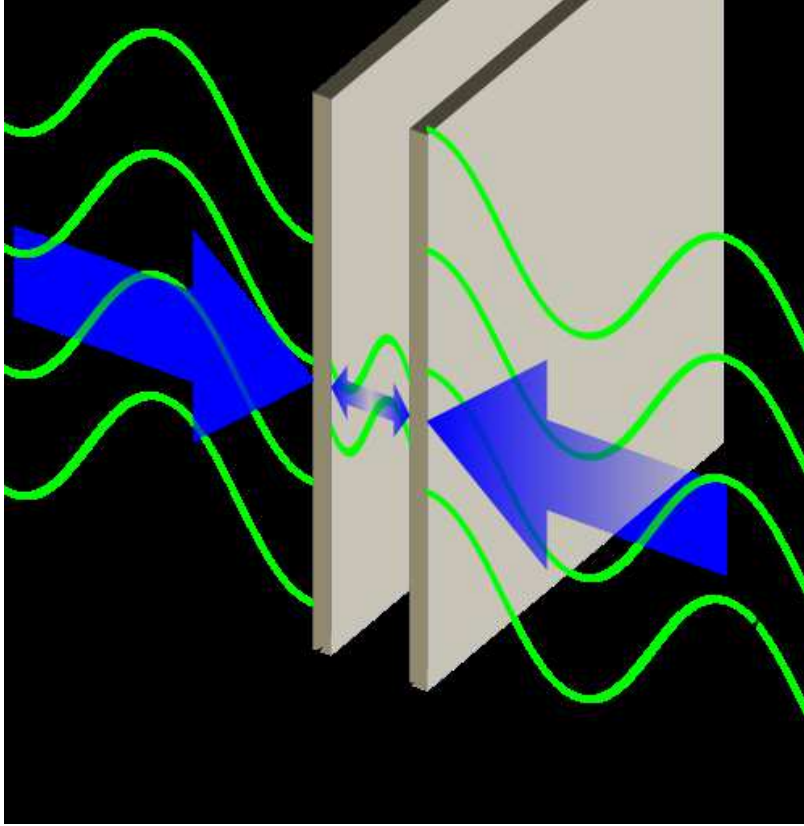
Figure 8: Artistic Presentation of the Casimir Effect

22　Appendix N:Artistic presentation of the Casimir Effect

The Casimir Effect states that the vacuum energy density between two parallel conduction plates separated by a distance d is given by:(pg 42 in [16])

$$\rho = -\frac{\pi^2}{720}\frac{\hbar}{d^4} \tag{264}$$

The equation above was written in the Geometrized System of Units $c = G = 1$.In the International System of Units the same equation would be:

$$\rho = -\frac{\pi^2}{720}\frac{\hbar c}{d^4} \tag{265}$$

As the Casimir plates are placed close together at a very small distance d a repulsive force(the small blue arrows in the inner region between the plates) appears.The repulsive behavior is due to the negative energy density that appears between the plates.

It was first theoretically predicted by Casimir in 1948 [12] but was experimentally demonstrated almost 50 years later by Lamoreaux in 1997 [14]

23   Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[58]

- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[59][60]

24   Remarks

Reference 4 can be obtained from the web pages of Professor Jose Natario at Instituto Superior Tecnico Lisboa Portugal as long as the site remains on-line. [61] [62]

Reference 16 can be obtained from the web pages of Professor Eric Poisson at University of Guelph Ontario Canada as long as the site remains on-line. [63] [64] [65]

Appendix L is available to the public from the NASA Glenn Research Center Ohio United States under the program destined for the education of high school students $(K-12)$.[66]

Although the main references of this work were taken from scientific sites available to the general public for consultation (eg:arXiv,HAL) we can provide a copy in PDF Acrobat reader of all our references for those interested.

---

[58]special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[59]"Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[60]appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

[61]http://www.math.ist.utl.pt/ jnatar/

[62]http://www.math.ist.utl.pt/ jnatar/seminario.pdf

[63]http://www.physics.uoguelph.ca/poisson/research/

[64]http://www.physics.uoguelph.ca/poisson/research/notes.html

[65]http://www.physics.uoguelph.ca/poisson/research/agr.pdf

[66]http://www.grc.nasa.gov/WWW/K-12/airplane/machang.html

25   Legacy

This work is dedicated as a Memorial to the Dutch physicist Hendrik Brught Gerhard Casimir(1909-2000) a man much ahead of his time.

In 1948 working in the laboratories of the industrial corporation Philips Groeilampenfabrieken Eidhoven The Netherlands he theorized for the first time the existence of matter with negative energy density also known as exotic matter by the Casimir Effect.The Casimir pioneer work is one of the most important works of all the history of physics(pg 41 in [15]) and the Casimir Effect is considered one of the most starling phenomenon of all the modern physics(pg 29 in [18]).

This work is also dedicated to the American physicist Steve Lamoreaux perhaps one of the best experimental physicists of all the history of physics.

In 1997 working at Los Alamos National Laboratory Los Alamos New Mexico United States of America he demonstrated experimentally for the first time the existence of the Casimir Effect and in consequence the existence of the exotic matter at sub-microscopical scales(pgs 1 and 8 in [14]).Inspired by the Lamoreaux breakthrough many other researchers developed new Casimir Effect measurements(pg 31 in [18]).

Much of the current research in modern physics is still theoretical(eg:Strings Theory,M-Theory,Quantum Gravity,TOE,Dark Matter,Dark Energy,Extra Dimensions etc) based over complex mathematical systems of equations but physics will always need good experimentalists like Lamoreaux in order to walk over solid ground

Without the work of Lamoreaux that confirmed the existence of exotic matter with negative energy density even at sub-microscopically small scales we would never be able to propose in this work the so-called Casimir warp drive.

References

[1] *Alcubierre M., (1994). Class.Quant.Grav. 11 L73-L77,gr-qc/0009013*

[2] *Natario J.,(2002). Class.Quant.Grav. 19 1157-1166,gr-qc/0110086*

[3] *Natario J.,(2006).Gen.Rel.Grav. 38 (2006) 475 − 484,gr-qc/0408085*

[4] *Natario J..(2003),Warp Drive Seminar Lectures.Instituto Superior Tecnico,Lisboa Portugal*

[5] *Lobo F.,(2007).,arXiv:0710.4474*

[6] *Everett A.E.,Roman T.A,(1997) ,Phys.Rev. D56 024023,gr-qc/9702049*

[7] *Gauthier C.,Gravel P.,Melanson J.,(2003). Gravitation and Cosmology. Vol 9 No.4(36) 301-306*

[8] *Gauthier C.,Gravel P.,Melanson J.,(2002). Int.J.Mod.Phys.A(IntJMPA). Vol 17 No.20 2761*

[9] *Introduction to Differential Forms,Arapura D.,2010*

[10] *Teaching Electromagnetic Field Theory Using Differential Forms,Warnick K.F. Selfridge R. H.,Arnold D. V.,IEEE Transactions On Education Vol 40 Num 1 Feb 1997*

[11] *Loup F.,(2014).,HAL-00937933*

[12] *Casimir H.B.G.,Proc.K.Ned.Akad.Wet 51-793,(1948)*

[13] *Ridgely C.T.,(2000).,PHYSICS/00100027*

[14] *Lamoreaux S.K.,(1997) ,Phys.Rev.Lett.V78-1-5*

[15] *Lamoreaux S.K.,(2007) ,Physics Today.40 issue of February 2007*

[16] *Advanced General Relativity Course - University of Guelph Ontario Canada,Poisson E.,(2002).*

[17] *Carroll S.(1997). gr-qc/9712019*

[18] *Lambrecth A.,(2002) ,Physics World.29 issue of September 2002*

.