Small first zeros of L-functions

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Abstract

From a family of L-functions with unitary symmetry, Hughes and Rudnick obtained results on the height of its lowest zero. We extend their results to other families of L-functions according to the type of symmetry coming from statistics for low-lying zeros.

Contents

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1 Introduction

1.1 Preview of results

The existence of a deep link between non-trivial zeros of natural families of L-functions and eigenvalues of random matrices has been speculated since Montgomery's work ([Mon73]) in the seventies. So, we are able to assign a classical compact group of matrices to many classical families of L-functions. We can refer to [ILS00], [FI03], [HR03], [RR11] or [Mil04]. Using the one-level density, Hughes and Rudnick obtained informations about the lowest zero of Dirichlet L-functions ([HR03], section 8). Our aim is to generalise these results.

Extreme low-lying zeros of a natural family of L-functions Let $\mathcal{F}(Q)$ be a finite set of L-functions with analytic conductor Q. We build the associate family $\mathcal{F} = \bigcup_{Q\geq 1} \mathcal{F}(Q)$. We assume Riemann hypothesis for any function in $\mathcal{F}$. We also assume the density theorem for $\mathcal{F}$ with test functions Φ satisfying supp $\widehat{\Phi} \subset [-\nu;\nu]$ with $\nu < \nu_{\max}(\mathcal{F})$. We write $W^*[\mathcal{F}]$ for the one-level density for non-trivial low-lying zeros of functions in $\mathcal{F}$. It turns out that the one-level densities that have been identified up to now have always been of the shape $W[G]$ whose Fourier transform is given by

$$\widehat{W[G]}(y) = \delta_0(y) + \frac{\delta}{2}\eta(y) + \varepsilon \tag{1.1}$$

where δ_0 is the Dirac function, η is defined on $\mathbb{R}$ by

$$\eta(y) = \begin{cases} 1 & \text{if} \quad |y| < 1 \\ \frac{1}{2} & \text{if} \quad |y| = 1 \\ 0 & \text{if} \quad |y| > 1 \end{cases}$$

and δ and ε are given in table 1.

G	U	O	Sp	SO^+	SO^-
δ	0	0	-1	1	-1
ε	0	1/2	0	0	1

Table 1: Value of (δ, ε).

Finally, for $G = Sp$, SO^+ or SO^-, let $n \geq 1$ be the only integer such that $n - 1 < \nu_{\max}(\mathcal{F}) \leq n$ and consider the equation in λ given by

$$\frac{\delta}{\lambda}\cos\theta_\lambda - \sum_{k=0}^{n-1} U_k(\lambda)\sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right)\left[\frac{\delta\alpha_R(k)}{2} - 1 + \varepsilon\beta_R(k)\right] + \frac{2\varepsilon}{\lambda}\sum_{k=0}^{n-1} U_k(\lambda)\cos\left(\theta_\lambda - k\delta\frac{\pi}{2}\right) = 0. \tag{1.2}$$

In this equation, let U_k be the k-th Chebyshev polynomial of the second kind, (δ, ε) is given in the table 1 and parameters θ_λ, α_R and β_R are defined in lemma 20 (page 35) and proposition 4 (page 38). We prove the following theorem on the smallest non negative imaginary part $\tilde{\gamma}_{f,1}$ of a non-trivial normalised zero of $L(f,.)$ in $\mathcal{F}$.

Theorem 1 *We have*

$$\limsup_{Q\to+\infty} \min_{L(f,.)\in\mathcal{F}(Q)} \tilde{\gamma}_{f,1} \leq \frac{1}{2\nu_{\max}(\mathcal{F})} \times$$

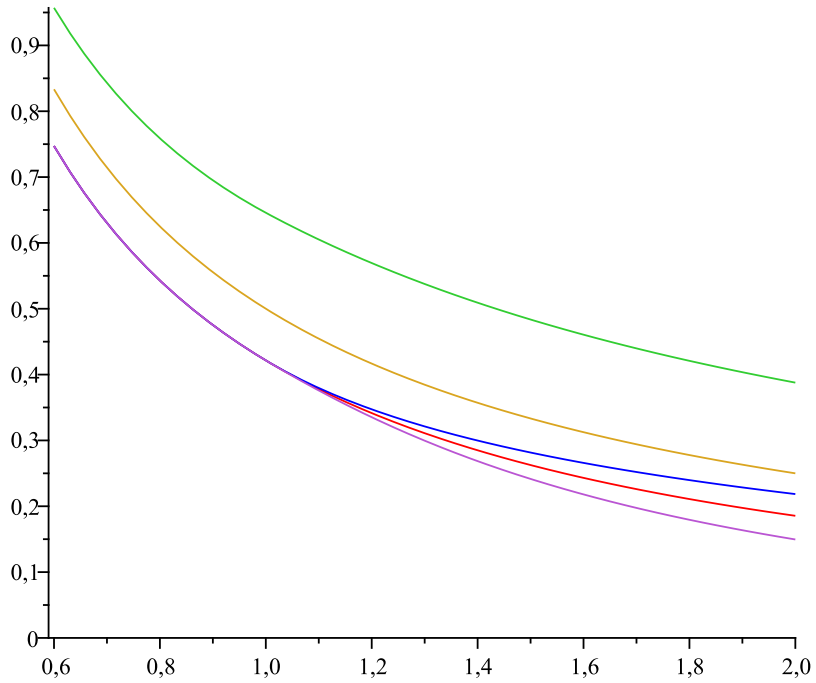
$$\begin{cases} 1 & \text{if } \ W^*[\mathcal{F}] = W[U] \\ 4V^{-1}\left(1+\frac{2}{\nu_{\max}(\mathcal{F})}\right) & \text{if } \ W^*[\mathcal{F}] = W[O] \\ 4V^{-1}\left(1+(\delta+2\varepsilon)\frac{2}{\nu_{\max}(\mathcal{F})}\right) & \text{if } \ W^*[\mathcal{F}] = W[Sp], W[SO^+] \text{ or } W[SO^-] \text{ and } \nu_{\max}(\mathcal{F}) \leq 1 \\ \frac{\nu_{\max}(\mathcal{F})}{\pi} \lim\limits_{R\to\nu_{\max}(\mathcal{F})/2^-} \lambda_R & \text{if } \ W^*[\mathcal{F}] = W[Sp], W[SO^+] \text{ or } W[SO^-] \text{ and } \nu_{\max}(\mathcal{F}) > 1 \end{cases}$$

where λ_R is the smallest positive root of equation (1.2) which is not a root of U_nU_{n-1} (with $n-1<\nu_{\max}(\mathcal{F})\leq n$) and where V is defined by

$$V : \left\{ \begin{array}{ccc} [0,\frac{1}{4}[\cup]\frac{1}{4},x_1[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \frac{\tan(2\pi x)}{2\pi x} \end{array} \right.$$

with $x_1 = \inf\left\{x>0,\ \frac{\tan(2\pi x)}{2\pi x}=1\right\} \approx 0,71.$

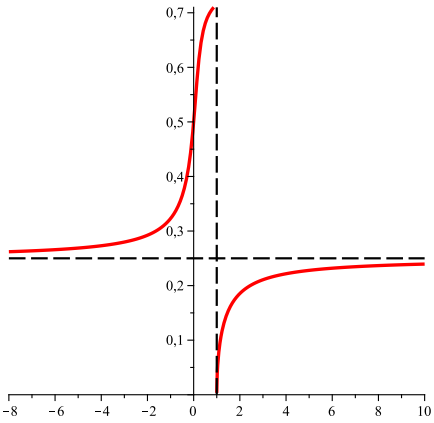
Remark 1 We plot the upper bound in theorem 1, denoted $M_{W^*[\mathcal{F}]}(\nu_{\max}(\mathcal{F}))$, for each $W^*[\mathcal{F}]$. From top to bottom, we have $W^*[\mathcal{F}] = W[Sp]$, $W^*[\mathcal{F}] = W[U]$, $W^*[\mathcal{F}] = W[SO^+]$, $W^*[\mathcal{F}] = W[O]$ and $W^*[\mathcal{F}] = W[SO^-]$.



Representative curves of $\nu_{\max}(\mathcal{F}) \longmapsto M_{W^*[\mathcal{F}]}(\nu_{\max}(\mathcal{F}))$

Remark 2 In the orthogonal case, when $\nu_{\max}(\mathcal{F})$ goes to infinity, we have

$$\frac{2}{\nu_{\max}(\mathcal{F})}V^{-1}\left(1+\frac{2}{\nu_{\max}(\mathcal{F})}\right)=\frac{\sqrt{6}}{\pi\nu_{\max}(\mathcal{F})^{3/2}}\left(1-\frac{6}{5\nu_{\max}(\mathcal{F})}+o\left(\frac{1}{\nu_{\max}(\mathcal{F})^{3/2}}\right)\right).$$



Representative curve of V^{-1}

Remark 3 In the case $1<\nu_{\max}(\mathcal{F})\leq 2$, we can simplify equation (1.2) as follows

$$(\delta+2\varepsilon)\frac{1-4\lambda^2}{\lambda}\left(\sin\lambda(1-R)-2\delta\lambda\cos\lambda R\right)$$
$$-\left[(\delta+2\varepsilon)(1-R)-1+4\varepsilon\right]\left[\cos\lambda(1-R)-2\delta\lambda\sin\lambda R-2\lambda\tan\Theta_R(\sin\lambda(1-R)-2\delta\lambda\cos\lambda R)\right]=0$$

where

$$\Theta_R=\frac{1}{2}\left(R-\frac{1}{2}\right)+\frac{\pi}{2}\left(1+\frac{\delta}{2}\right).$$

Moreover, this relation always vanishes in $\lambda=1/2$ which is a root of U_2.

In order to give several instances, let $H_k^*(q)$ be the set of primitive holomorphic cusp forms of prime level q end even weight $k\geq 2$. Let r be a positive integer. We define

$$\mathcal{H}_r(q)=\{L(Sym^r f,s),\ f\in H_k^*(q)\}$$

and if r is odd,

$$\mathcal{H}_r^{\pm}(q)=\{L(Sym^r f,s),\ f\in H_k^*(q) \text{ and } \varepsilon(Sym^r f)=\pm 1\}.$$

Then, we have several families of symmetric power L-functions

$$\mathcal{H}_r=\bigcup_{q \text{ prime}}\mathcal{H}_r(q) \quad \text{and} \quad \mathcal{H}_r^{\pm}=\bigcup_{q \text{ prime}}\mathcal{H}_r^{\pm}(q).$$

These families have been studied in [ILS00] (theorem 1.1, for the case $r=1$) and [RR11] (theorem B). We may sum up some properties of these natural families of L-functions in the following table where

$$\nu_{1,max}(1,k,\theta_0)=2 \text{ and } \nu_{1,max}(r,k,\theta_0)=\left(1-\frac{1}{2(k-2\theta_0)}\right)\frac{2}{r^2} \text{ if } r\geq 2 \text{ with } \theta_0=\frac{7}{64},$$

and for $\epsilon=\pm 1$,

$$\nu_{1,max}^{\epsilon}(1,k,\theta_0)=2 \text{ and } \nu_{1,max}^{\epsilon}(r,k,\theta_0)=\inf\left\{\nu_{1,max}(r,k,\theta_0),\frac{3}{r(r+1)}\right\} \text{ if } r\geq 2.$$

$\mathcal{F}$	$\mathcal{H}_r$ (r even)	$\mathcal{H}_r$ (r odd)	$\mathcal{H}_r^+$ (r odd)	$\mathcal{H}_r^-$ (r odd)
$\nu_{\max}(\mathcal{F})$	$\nu_{1,\max}(r,k,\theta_0)$	$\nu_{1,\max}(r,k,\theta_0)$	$\nu^{\epsilon}_{1,\max}(r,k,\theta_0)$	$\nu^{\epsilon}_{1,\max}(r,k,\theta_0)$
$\rho_{\max}(\mathcal{F})$	$\frac{1}{r^2}$	$\frac{1}{r^2}$	$\frac{1}{2r(r+2)}$	$\frac{1}{2r(r+2)}$
$W[\mathcal{F}]$	$W[Sp]$	$W[O]$	$W[SO^+]$	$W[SO^-]$
$W^*[\mathcal{F}]$	$W[Sp]$	$W[O]$	$W[SO^+]$	$W[Sp]$

Table 2: Several natural families of L-functions

Thus, thanks to theorem 1, we have

$$\begin{aligned}\limsup_{N\to+\infty} \min_{f\in H_k^*(N)} \tilde{\gamma}_{f,1} &\leq V^{-1}(2) < 0,19\\ \limsup_{N\to+\infty} \min_{f\in H_k^+(N)} \tilde{\gamma}_{f,1} &\leq 0,22\\ \limsup_{N\to+\infty} \min_{f\in H_k^-(N)} \tilde{\gamma}_{f,1} &\leq 0,39\end{aligned}$$

and

$$\limsup_{\substack{q\to+\infty\\ q \text{ prime}}} \min_{f\in H_k^*(q)} \tilde{\gamma}_{Sym^r f,1} \leq \frac{2}{\nu_{1,max}(r,k,\theta_0)} V^{-1}\left(1+(-1)^{r+1}\frac{2}{\nu_{1,max}(r,k,\theta_0)}\right)$$

and if r is odd, $\epsilon=\pm 1$, we have

$$\limsup_{\substack{q\to+\infty\\ q \text{ prime}}} \min_{\substack{f\in H_k^*(q)\\ \varepsilon(Sym^r f)=\epsilon}} \tilde{\gamma}_{Sym^r f,1} \leq \frac{2}{\nu^{\epsilon}_{1,max}(r,k,\theta_0)} V^{-1}\left(1+\epsilon\frac{2}{\nu^{\epsilon}_{1,max}(r,k,\theta_0)}\right).$$

Actually, since $\mathcal{H}^-(N)\neq\emptyset$ for large N, we get (see lemma 1 page 9) $\lim_{N\to+\infty}\min_{f\in H_k^*(N)} \tilde{\gamma}_{f,1}=0$.

Proportion of L-functions which have a small first zero Combining Bienaymé-Chebyshev inequality and statistics for low-lying zeros of symmetric power L-functions, we can obtain a positive proportion of L-functions in our family which have a small first zero. Hughes and Rudnick exposed this phenomenon in the case of Dirichlet L-functions which have a unitary symmetry. Symmetric power L-functions allows us to deal with all currently known symmetry group. $H_k^*(q)$ denotes the set of primitive holomorphic cusp forms of prime level q and even weight $k\geq 2$ and let $\omega_q(f)$ be the harmonic weight associated to f in $H_k^*(q)$. Let also $\varepsilon(Sym^r f)$ denotes the sign of the functional equation associated to the L-function $L(Sym^r f,s)$.

Theorem 2 *Let r be a fixed positive integer. Assuming the Riemann hypothesis for all symmetric power L-functions of order r.*

$$\text{If } \beta \geq \sqrt{\frac{\pi r^4}{4}\frac{6\pi^3 r^4 - 24(-1)^r \pi r^2 + 9\pi - \pi^3 + 2\sqrt{6}\sqrt{-\pi^4 + 6\pi^4 r^4 + 7\pi^2 + 12 - 24(-1)^r \pi^2 r^2}}{6\pi^4 r^4 + 48(-1)^{r+1}\pi^2 r^2 + 96 - 3\pi^2 - \pi^4}}$$

then

$$\liminf_{\substack{q\to+\infty \\ q \ prime}} \sum_{\substack{f\in H_k^*(q) \\ \tilde{\gamma}_{Sym^r f,1}\leq\beta}} \omega_q(f) \geq 1 - \frac{\pi^2}{6}\frac{16(\pi^2+3)\beta^4 + 8r^4(9-\pi^2)\beta^2 + (3+\pi^2)r^8}{[r^6\pi^2 - 4\beta^2 r^2\pi^2 + 16\beta^2(-1)^r]^2}.$$

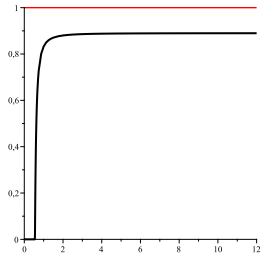
Moreover, if r is odd and $\sigma = \pm 1$. If $\frac{\beta}{r(r+2)\sqrt{\pi}} \geq$

$$\sqrt{\frac{\pi[24\pi^2 r^2(r+2)^2 + 48\sigma r(r+2) + 9 - \pi^2] + 2\sqrt{6}\sqrt{-\pi^4 + 24\pi^4 r^2(r+2)^2 + 7\pi^2 + 12 + 48\sigma\pi^2 r(r+2)}}{24\pi^4 r^2(r+2)^2 + 96\sigma\pi^2 r(r+2) + 96 - 3\pi^2 - \pi^4}}$$

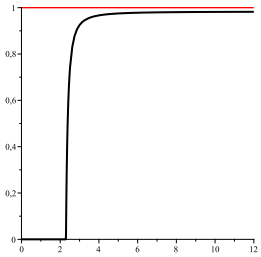
then

$$\liminf_{\substack{q\to+\infty \\ q \ prime}} 2 \sum_{\substack{f\in H_k^*(q) \\ \varepsilon(Sym^r f)=\sigma \\ \tilde{\gamma}_{Sym^r f,1}\leq\beta}} \omega_q(f) \geq 1 - \frac{\pi^2}{24}\frac{(\pi^2+3)\beta^4 + 2r^2(r+2)^2(9-\pi^2)\beta^2 + (\pi^2+3)r^4(r+2)^4}{[2\sigma\beta^2 + \pi^2\beta^2 r(r+2) - \pi^2 r^3(r+2)^3]^2}.$$

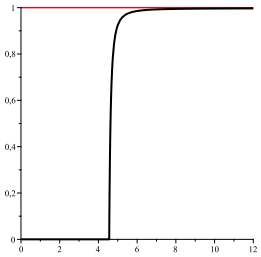
In order to give some examples for small values of r, we plot the graph of the function with parameter β which is associated to the lower bound in the first part of theorem 2. On these representative curves, the critical value of β corresponds to the minimal value of β which appears in theorem 2.



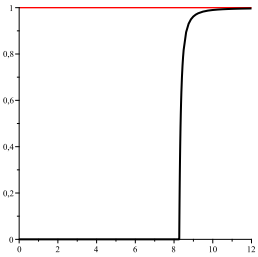
$r = 1$



$r = 2$



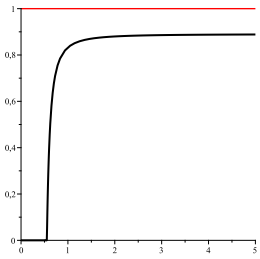
$r = 3$



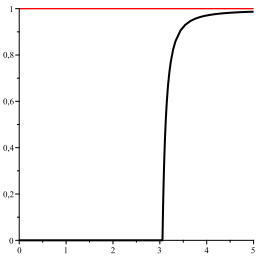
$r = 4$

Similarly, for a fixed small odd value of r, we plot the lower bounds of theorem 2 in different cases: when there is no restriction on the sign $\varepsilon(Sym^r f)$ of the functional equation, when $\varepsilon(Sym^r f) = +1$ and, finally, when $\varepsilon(Sym^r f) = -1$.

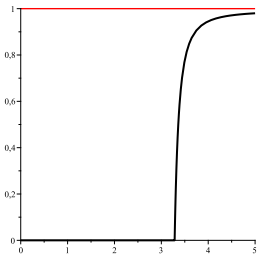
First, if $r = 1$:



$\varepsilon(Sym^r f) = \pm 1$

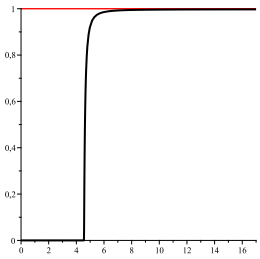


$\varepsilon(Sym^r f) = +1$

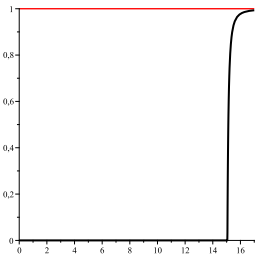


$\varepsilon(Sym^r f) = -1$

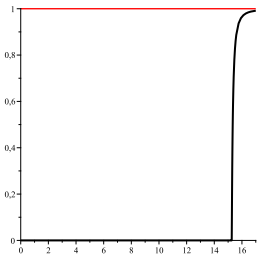
Second, if $r = 3$:



$\varepsilon(Sym^r f) = \pm 1$



$\varepsilon(Sym^r f) = +1$



$\varepsilon(Sym^r f) = -1$

1.2 Notations

The following notations will be used throughout this paper.

- $\lfloor x \rfloor$ denotes the floor of the real number x.
- $\sum_{n\geq 0}^{*}$ means the sum is running over odd non-negative integers.
- $\mathcal{C}_c^{\infty}(\mathbb{R})$ denotes the set of infinitely differentiable functions which are compactly supported.
- For $1 \leq p < +\infty$, $L^p(\mathbb{R})$ refers to the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|^p dx < +\infty$. In this case, we put $||f||_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}$. If f and g are in $L^2(\mathbb{R})$, let $(f,g)_{L^2} = \int_R f(x)g(x)dx$.
- If Φ is in $L^1(\mathbb{R})$, $\widehat{\Phi}(u) = \int_{\mathbb{R}} \Phi(x)e^{-2i\pi xu}dx$ is called the Fourier transform of Φ. When it is allowed, we can apply the inverse transform formula $\Phi(x) = \int_{\mathbb{R}} \widehat{\Phi}(u)e^{2i\pi xu}du$.
- If f and g are in $L^2(\mathbb{R})$, $f * g(u) = \int_{\mathbb{R}} f(t)g(u-t)dt$ is the convolution product of f and g.
- $\mathcal{S}_\nu(\mathbb{R})$ denotes the set of even Schwartz functions whose Fourier transforms are compactly supported in $[-R, R]$ with $0 < R < \nu$.

Acknowledgements I would like to thank Frederic Bayart for his availability and all their advices on Sobolev spaces.

1.3 Statistics for low-lying zeros

Let $\mathcal{F}$ be a natural family of L-functions[1] all of whose satisfies the Riemann hypothesis. Consider a L-function $L(f,s)$ in $\mathcal{F}$ with analytic conductor c_f. Let $\mathcal{F}(Q) = \{L(f,s) \in \mathcal{F}\ ,\ c_f = Q\}$. Each zero ρ_f of $L(f,s)$ which is on the critical line $\Re(s) = 1/2$ can be written $\rho_f = \frac{1}{2} + i\gamma_f$, and we denote $\tilde{\rho}_f = \frac{1}{2} + i\tilde{\gamma}_f$ with $\tilde{\gamma}_f = \gamma_f \frac{\ln c_f}{2\pi}$ the normalized zero.

In order to study the distribution of low-lying zeros of $L(f,s)$, for any test function Φ in $\mathcal{S}_\nu(\mathbb{R})$, we define the low-zeros sum

$$D[\Phi](f) = \sum_{\tilde{\gamma}_f} \Phi(\tilde{\gamma}_f)$$

where the sum is running over the imaginary parts of normalised zeros counted with multiplicity. $\mathcal{F}(Q)$ can be seen as a measurable space where measurable sets are all its subsets and which is equipped with the counting probability measure $\mu_{\mathcal{F}(Q)}$. $D[\Phi]$ is a measurable function on $\mathcal{F}(Q)$. However, we are unable to determine the asymptotic behaviour of $D[\Phi](f)$ for a single L-function. As a consequence, we must take into account a family of L-functions in order to obtain a significant result. That is why, we define the one-level density as the expectation of $D[\Phi]$:

$$\mathbb{E}_{\mathcal{F}(Q)}(D[\Phi]) = \frac{1}{|\mathcal{F}(Q)|} \sum_{L(f,s)\in\mathcal{F}(Q)} D[\Phi](f)$$

We also define the variance of the one-level density:

$$\mathbb{V}_{\mathcal{F}(Q)}(D[\Phi]) = \mathbb{E}([D[\Phi] - \mathbb{E}_{\mathcal{F}(Q)}(D[\Phi])]^2) = \frac{1}{|\mathcal{F}(Q)|} \sum_{L(f,s)\in\mathcal{F}(Q)} \left[D[\Phi](f) - \frac{1}{|\mathcal{F}(Q)|} \sum_{L(f,s)\in\mathcal{F}(Q)} D[\Phi](f) \right]^2$$

Our purpose is to find the asymptotic behaviour of $\mathbb{E}_{\mathcal{F}(Q)}(D[\Phi])$ and $\mathbb{V}_{\mathcal{F}(Q)}(D[\Phi])$ when Q goes to infinity. The density conjecture predicts

$$\lim_{Q\to+\infty} \frac{1}{|\mathcal{F}(Q)|} \sum_{L(f,s)\in\mathcal{F}(Q)} D[\Phi](f) = \int_{\mathbb{R}} \Phi(t) W[\mathcal{F}](t) dt$$

where $W[\mathcal{F}]$ is a density function characterised by $\mathcal{F}$. In order to estimate the one-level density, we convert sums over zeros to sums over primes. Unfortunately, we are able to evaluate these sums over primes only if the support of the test function is small. That's why, density theorems are proved only for test functions in $\mathcal{S}_{\nu_{\max}(\mathcal{F})}(\mathbb{R})$ with $\nu_{\max}(\mathcal{F})$ fixed. Currently, the maximal value for $\nu_{\max}(\mathcal{F})$ is 2 whereas the density conjecture does not predict any restriction on the support of test functions.

Sometimes, we will prefer using the harmonic measure for technical conveniences, rather than the Dirac one. For instance, in the case of symmetric power L-functions. Precisely, if A is a subset of $\mathcal{H}_r(q)$, let

$$\mu^h_{\mathcal{H}_r(q)}(A) = \sum_{L(Sym^r f,s)\in A} w_q(f)$$

[1] Our definition of L-function is the one of [IK04] chapter 5.

and

$$\mathbb{E}^h_{\mathcal{H}_r(q)}(D[\Phi]) = \sum_{f\in H_k^*(q)} \omega_q(f) D[\Phi](Sym^r f) \text{ where } \omega_q(f) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f,f\rangle_q}$$

$$\text{and} \quad \mathbb{V}^h_{\mathcal{H}_r(q)}(D[\Phi]) = \sum_{f\in H_k^*(q)} \omega_q(f) \left[D[\Phi](Sym^r f) - \mathbb{E}^h_{\mathcal{H}_r(q)}(D[\Phi])\right]^2.$$

Then, if r is odd and if A is a subset of $\mathcal{H}_r^{\pm}(q)$, let

$$\mu^h_{\mathcal{H}_r^{\pm}(q)}(A) = 2 \sum_{L(Sym^r f,s)\in A} w_q(f)$$

and

$$\mathbb{E}^h_{\mathcal{H}_r^{\pm}(q)}(D[\Phi]) = 2 \sum_{\substack{f\in H_k^*(q)\\ \varepsilon(Sym^r f)=\pm 1}} \omega_q(f) D[\Phi](Sym^r f)$$

$$\text{and} \quad \mathbb{V}^h_{\mathcal{H}_r^{\pm}(q)}(D[\Phi]) = 2 \sum_{\substack{f\in H_k^*(q)\\ \varepsilon(Sym^r f)=\pm 1}} \omega_q(f) \left[D[\Phi](Sym^r f) - \mathbb{E}^h_{\mathcal{H}_r^{\pm}(q)}(D[\Phi])\right]^2.$$

Actually, harmonic measures are asymptotic probability measures since we only have

$$\lim_{\substack{q\to+\infty\\ q \text{ prime}}} \sum_{f\in H_k^*(q)} \omega_q(f) = 1 \text{ and } \lim_{\substack{q\to+\infty\\ q \text{ prime}}} \sum_{\substack{f\in H_k^*(q)\\ \varepsilon(Sym^r f)=\pm 1}} \omega_q(f) = \frac{1}{2}. \tag{1.3}$$

The first relation comes from Petersson trace formula ([RR11], proposition 2.2) and the second one is subject to an assumption (Hypothesis $Nice(r,f)$ of [RR11]) we are assuming in order to get density theorems for these families.

1.4 What is the smallest zero of a L-function ?

If $L(f,s)$ is a self-dual L-function, the sign of the functional equation is equal to ± 1. Moreover, due to the following observation, we need to define $\tilde{\gamma}_{f,1}$ which appears in theorems 1 and 2 and to explain the consequences on density theorems.

Lemma 1 ([IK04], proposition 5.1) *Let $L(f,s)$ be a self-dual L-function with $\varepsilon(f) = -1$. Then $L\left(f, \frac{1}{2}\right) = 0$.*

If $\mathcal{F}$ is a natural family, all of whose L-functions $L(f,s)$ are self-duals and satisfy $\varepsilon(f) = -1$. Due to the previous lemma, we can denote non-trivial zeros of $L(f,s)$ by

$$\{\rho_{f,0}\} \cup \{\rho_{f,i}, i \in \mathbb{Z}^*\}$$

where $\rho_{f,0} = 1/2$ and $\rho_{f,i} = 1 - \rho_{f,-i}$ if $i \neq 0$. Moreover, we have:

$$... \leq Im(\rho_{f,-1}) \leq Im(\rho_{f,0}) = 0 \leq Im(\rho_{f,1}) \leq Im(\rho_{f,2}) \leq ...$$

In the other cases, we use the same notations without $\rho_{f,0}$.

We are going to expose its consequences on statistics of low-lying zeros. Let $\mathcal{F}$ a natural family of L-functions, we define

$$D^*[\Phi](f) = \sum_{i\in\mathbb{Z}^*} \Phi(\tilde{\gamma}_{f,i}).$$

The density function $W^*[\mathcal{F}]$ is defined to satisfy

$$\lim_{Q\to+\infty} \mathbb{E}_{\mathcal{F}(Q)}(D^*[\Phi]) = \int_{\mathbb{R}} \Phi(t)W^*[\mathcal{F}](t)dt. \tag{1.4}$$

If all L-functions $L(f,s)$ in $\mathcal{F}$ are self-duals and satisfy $\varepsilon(f) = -1$, we have

$$\mathbb{E}_{\mathcal{F}(Q)}(D[\Phi]) = \Phi(0) + \mathbb{E}_{\mathcal{F}(Q)}(D^*[\Phi]) \quad \text{and} \quad \mathbb{V}_{\mathcal{F}(Q)}(D[\Phi]) = \mathbb{V}_{\mathcal{F}(Q)}(D^*[\Phi]). \tag{1.5}$$

This phenomenon occurs for the family $\mathcal{H}^-$. It has been shown in [ILS00] (equation (1.18)) that the symmetry group associated to $\mathcal{H}^-$ is SO^-. In other words, we have

$$\lim_{\substack{N\to+\infty \\ N \text{ squarefree}}} \mathbb{E}_{\mathcal{H}^-(N)}(D[\Phi]) = \int_{\mathbb{R}} \Phi(t)W[SO^-](t)dt \quad \text{ie} \quad W[\mathcal{H}^-] = W[SO^-].$$

As a result, we deduce

$$\lim_{\substack{N\to+\infty \\ N \text{ squarefree}}} \mathbb{E}_{\mathcal{H}^-(N)}(D^*[\Phi]) = \int_{\mathbb{R}} \Phi(t)W[Sp](t)dt \quad \text{ie} \quad W^*[\mathcal{H}^-] = W[Sp]. \tag{1.6}$$

Similarly, the family $\mathcal{H}_r^-$ (r odd) has also SO^- as symmetry group ([RR11], theorem A). Nevertheless, for the harmonic measure, relation (1.5) becomes

$$\mathbb{E}^h_{\mathcal{H}_r^-(q)}(D[\Phi]) = 2\Phi(0)\mu^h_{\mathcal{H}_r(q)}\left(\mathcal{H}_r^-(q)\right) + \mathbb{E}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi])$$

and

$$\mathbb{V}^h_{\mathcal{H}_r^-(q)}(D[\Phi]) = \mathbb{V}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi]) + 2\Phi(0)\left[\Phi(0)\mu^h_{\mathcal{H}_r(q)}\left(\mathcal{H}_r^-(q)\right) + \mathbb{E}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi])\right]\left[1 - 2\mu^h_{\mathcal{H}_r(q)}\left(\mathcal{H}_r^-(q)\right)\right]^2.$$

Thanks to relation (1.3), we have

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{E}^h_{\mathcal{H}_r^-(q)}(D[\Phi]) = \Phi(0) + \lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{E}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi]) \quad \text{and} \quad \lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{V}^h_{\mathcal{H}_r^-(q)}(D[\Phi]) = \lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{V}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi]).$$

As a result, with theorems A and D of [RR11], we get

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{E}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi]) = \int_{\mathbb{R}} \Phi(t)W[Sp](t)dt \quad \text{and} \quad \lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{V}^h_{\mathcal{H}_r^-(q)}(D^*[\Phi]) = 2\int_{\mathbb{R}} |u|\widehat{\Phi}(u)^2 du. \tag{1.7}$$

2 Proportion of L-functions with a small smallest zero

In this part, we prove theorem 2. The starting point is the following proposition. We do not give a proof of this result since it is essentially the same than the proof theorem 8.3 from [HR03].

Proposition 1 *Let g be in $\mathcal{S}_R(\mathbb{R})$ and $\Phi(x) = (x^2 - \beta^2)g^2(x)$. Let*

$$B(g) = \sqrt{\frac{\int_{\mathbb{R}} x^2 g^2(x) W^*[\mathcal{F}](x)dx}{\int_{\mathbb{R}} g^2(x) W^*[\mathcal{F}](x)dx}}.$$

We assume

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{E}^h_{\mathcal{F}(q)}(D^*[\Phi]) = \int_{\mathbb{R}} \Phi(t)W^*[\mathcal{F}](t)dt \quad \text{and} \quad \lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{V}^h_{\mathcal{F}(q)}(D^*[\Phi]) = \mathbb{V}_{\mathcal{F}}(\Phi).$$

Then, if $\beta > B(g)$, we have

$$\liminf_{\substack{q\to+\infty \\ q \text{ prime}}} \mu^h_{\mathcal{F}(q)}\left(\{L(f,s) \in \mathcal{F};\, \tilde{\gamma}_{f,1} < \beta\}\right) \geq 1 - \frac{\mathbb{V}_{\mathcal{F}}(\Phi)}{\left[\int_{\mathbb{R}} \Phi(t)W^*[\mathcal{F}](t)dt\right]^2} =: Borne_{\mathcal{F}}(\beta^2). \tag{2.1}$$

Remark 4 The same result holds for all natural family of L-functions with the counting probability measure instead of the harmonic measure.

This proposition gives a result only if the right member term is positive. We denote $\beta_{\min}(g)$ the smallest value of $\beta > B(g)$ such that this property is satisfied. Thanks to $\beta_{\min}(g) > B(g)$, we may detect a zero. We have:

$$\liminf_{\substack{q\to+\infty \\ q \text{ prime}}} \min_{L(f,s)\in\mathcal{F}(q)} \tilde{\gamma}_{f,1} \leq \beta_{\min}(g)$$

Actually, we prove a better upper bound in theorem 1.

Ricotta and Royer proved in [RR11] (theorems A, B and D), that if Φ is in $\mathcal{S}_{\rho_{\max}(\mathcal{F})}(\mathbb{R})$, then

$$\lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{E}^h_{\mathcal{F}(q)}(D[\Phi]) = \int_{\mathbb{R}} \Phi(t)W[\mathcal{F}](t)dt \quad \text{and} \quad \lim_{\substack{q\to+\infty \\ q \text{ prime}}} \mathbb{V}^h_{\mathcal{F}(q)}(D[\Phi]) = 2\int_{\mathbb{R}} |u|\widehat{\Phi}(u)^2 du$$

where $\rho_{\max}(\mathcal{F})$ and $W[\mathcal{F}]$ are given in table 2. Since $\rho_{\max}(\mathcal{F}) < 1$, we have

$$\int_{\mathbb{R}} \Phi(t)W^*[\mathcal{F}](t)dt = \widehat{\Phi}(0) + \frac{\sigma_{\mathcal{F}}}{2}\Phi(0)$$

with:

$\mathcal{F}$	$\mathcal{H}_r$ (r even)	$\mathcal{H}_r$ (r odd)	$\mathcal{H}_r^+$ (r odd)	$\mathcal{H}_r^-$ (r odd)
$\sigma_{\mathcal{F}}$	$(-1)^{r+1} = -1$	$(-1)^{r+1} = 1$	1	-1

In order to obtain explicit lower bound in the previous proposition, we specialise relation (2.1) in a fixed test function.

Lemma 2 *If* $0 < R < 1/2$, $\widehat{g}_0(u) = \cos\left(\frac{\pi u}{2R}\right)\mathbb{1}_{[-R,R]}(u)$ *and* $\Phi(x) = (x^2-\beta^2)g_0^2(x)$, *then*

$$Borne_{\mathcal{F}}(\beta^2) = 1 - \frac{2\pi^2R^2}{3}\frac{256(3+\pi^2)R^4\beta^4 + 32(9-\pi^2)R^2\beta^2 + \pi^2 + 3}{\left(128\sigma_{\mathcal{F}}R^3\beta^2 + 16\pi^2R^2\beta^2 - \pi^2\right)^2} \quad \textit{and} \quad B(g_0) = \frac{1}{4R\sqrt{1+\sigma_{\mathcal{F}}\frac{8R}{\pi^2}}}.$$

Proof: We have

$$\widehat{\Phi}(u) = \left[\frac{2R-|u|}{2}\left(\frac{1}{16R^2} - \beta^2\right)\cos\left(\frac{\pi u}{2R}\right) - \frac{1}{\pi}\left(\frac{1}{16R} + \beta^2R\right)\sin\left(\frac{\pi|u|}{2R}\right)\right].\mathbb{1}_{[-2R,2R]}(u)$$
$$\widehat{\Phi}(0) = \frac{1}{16R} - \beta^2R \qquad \text{and} \qquad \Phi(0) = -\frac{16\beta^2R^2}{\pi^2}.$$

Thanks to several integration by parts, we obtain

$$\int_{\mathbb{R}} |u|\widehat{\Phi}(u)^2du = \frac{768R^4\beta^4 + 3 + 288\beta^2R^2 + \pi^2 - 32\beta^2R^2\pi^2 + 256R^4\beta^4\pi^2}{768\pi^2}.$$

□

Lemma 3 *Let* $Borne^+_{\mathcal{F}}(\beta^2) = \max\left\{0; Borne_{\mathcal{F}}(\beta^2)\right\}$. *The following table sums up variations of* $\beta \longmapsto Borne^+_{\mathcal{F}}(\beta^2)$ *where*

$$\beta_{\min}(g_0) = \frac{1}{4R}\sqrt{\frac{\pi^2(3\pi^2 + 24\sigma_{\mathcal{F}}R - 2(\pi^2-9)R^2) + 4\pi R\sqrt{9\pi^4 + 72\sigma_{\mathcal{F}}R\pi^2 - 6(\pi^4 - 7\pi^2 - 12)R^2}}{3\pi^4 + 48\sigma_{\mathcal{F}}R\pi^2 + (192 - 6\pi^2 - 2\pi^4)R^2}}.$$

β	$B(g_0)$ $\quad$ $\beta_{\min}(g_0)$ $\quad$ $+\infty$
$Borne^+_{\mathcal{F}}(\beta^2)$	$0 \longrightarrow 0$ ↗ $1 - \frac{2\pi^2R^2}{3}\frac{\pi^2+3}{(\pi^2+8R\sigma_{\mathcal{F}})^2}$

Proof: First, we study the sign of $Borne_{\mathcal{F}}(X)$. It is the same as the quadratic polynomial's

$$Y(X) = 3\pi^4 - 6\pi^2R^2 - 2\pi^4R^2 + X\left(-96R^2\pi^4 - 768\sigma_{\mathcal{F}}\pi^2R^3 - 576R^4\pi^2 + 64R^4\pi^4\right)$$
$$+X^2\left(768R^4\pi^4 + 12288\sigma_{\mathcal{F}}R^5\pi^2 + 49152R^6 - 1536R^6\pi^2 - 512R^6\pi^4\right).$$

Let $L(R)$ be its leading coefficient and $\Delta(R)$ its discriminant. We get

$$\Delta(R) \quad = \quad \underbrace{-98304(\pi^4 - 7\pi^2 - 12)\pi^2}_{<0}R^6(R-R_1)(R-R_2)$$

with

$$R_1 \quad = \quad \min\left\{\pi^2\frac{12\sigma_{\mathcal{F}} + \sqrt{6(\pi^2-3)(\pi^2-4)}}{2(\pi^4-7\pi^2-12)}; \pi^2\frac{12\sigma_{\mathcal{F}} - \sqrt{6(\pi^2-3)(\pi^2-4)}}{2(\pi^4-7\pi^2-12)}\right\}$$
$$R_2 \quad = \quad \max\left\{\pi^2\frac{12\sigma_{\mathcal{F}} + \sqrt{6(\pi^2-3)(\pi^2-4)}}{2(\pi^4-7\pi^2-12)}; \pi^2\frac{12\sigma_{\mathcal{F}} - \sqrt{6(\pi^2-3)(\pi^2-4)}}{2(\pi^4-7\pi^2-12)}\right\}.$$

Numerical values *If* $\sigma_{\mathcal{F}} = -1$, *then* $R_1 \approx -8,330$ *and* $R_2 \approx 1,074$.
If $\sigma_{\mathcal{F}} = 1$, *then* $R_1 \approx -1,074$ *and* $R_2 \approx 8,330$.

In addition,

$$L(R) \quad = \quad \underbrace{256(192-6\pi^2-2\pi^4)}_{<0} R^4(R-R_3)(R-R_4)$$

with

$$\begin{aligned} R_3 &= \min\left\{\pi^2\frac{-24\sigma_{\mathcal{F}}+\sqrt{6\pi^2(\pi^2+3)}}{192-6\pi^2-2\pi^4}, \pi^2\frac{-24\sigma_{\mathcal{F}}-\sqrt{6\pi^2(\pi^2+3)}}{192-6\pi^2-2\pi^4}\right\} \\ R_4 &= \max\left\{\pi^2\frac{-24\sigma_{\mathcal{F}}+\sqrt{6\pi^2(\pi^2+3)}}{192-6\pi^2-2\pi^4}, \pi^2\frac{-24\sigma_{\mathcal{F}}-\sqrt{6\pi^2(\pi^2+3)}}{192-6\pi^2-2\pi^4}\right\}. \end{aligned}$$

Numerical values *If $\sigma_{\mathcal{F}}=-1$, then $R_3\approx -8,210$ and $R_4\approx 0,573$.*
If $\sigma_{\mathcal{F}}=1$, then $R_3\approx -0,573$ and $R_4\approx 8,210$.

Therefore, we always have

$$R_1 < R_3 < 0 < \frac{1}{2} < R_4 < R_2.$$

Let $W(R)$ be degree 1 monomial coefficient in Y, then

$$W(R) \quad = \quad 32\pi^2R^2(-3\pi^2-24\sigma_{\mathcal{F}}R+2(\pi^2-9)R^2).$$

Since $0<R<1/2$, we have $\Delta(R)\geq 0$ and $L(R)\geq 0$. The polynomial Y has two real roots,

$$X_1(R)=\frac{-W(R)-\sqrt{\Delta(R)}}{2L(R)} \text{ and } X_2(R)=\frac{-W(R)+\sqrt{\Delta(R)}}{2L(R)}.$$

Now, we want compare to $X_1(R)$, $X_2(R)$ and $B(g_0)^2$. Since $L(R)$ is positive , $X_1(R)\leq X_2(R)$. Moreover,

$$\begin{aligned} & [B(g_0)^2-X_1(R)][B(g_0)^2-X_2(R)] \\ = \quad & \underbrace{\frac{\pi^2(\pi^4+3\pi^2-96)(8\pi^2R^2+24R^2+24\sigma_{\mathcal{F}}\pi^2R+3\pi^4)}{8R^2[\pi^2+8R\sigma_{\mathcal{F}}]^2[-3\pi^4-48\sigma_{\mathcal{F}}R\pi^2+2(\pi^4+3\pi^2-96)R^2]^2}}_{>0}(R-R_3)(R-R_4). \end{aligned}$$

As a result, we have $X_1(R)\leq B(g_0)^2\leq X_2(R)$ and $\beta_{\min}(g_0)^2=X_2(R)$.
Then, we prove that $Borne_{\mathcal{F}}(X)$ is increasing on $[X_2(R);+\infty[$. We compute

$$\frac{dBorne_{\mathcal{F}}}{dX}(X)=-\frac{256R^4B(g_0)^6}{3\pi^4(X-B(g_0)^2)^3}\left[16R^2(2R\sigma_{\mathcal{F}}(\pi^2-9)-3\pi^2)X-2R\sigma_{\mathcal{F}}(\pi^2+3)-3\pi^2\right].$$

If $0<R<1/2$, then $2R\sigma_{\mathcal{F}}(\pi^2-9)-3\pi^2<0$. As a consequence:

$$\frac{dBorne_{\mathcal{F}}}{dX}(X)\geq 0 \quad \Longleftrightarrow \quad X\geq \frac{1}{16R^2}\frac{2R\sigma_{\mathcal{F}}(\pi^2+3)+3\pi^2}{2R\sigma_{\mathcal{F}}(\pi^2-9)-3\pi^2}$$

Since the right member of this inequality is always negative (when $0<R<1/2$), then $Borne^+_{\mathcal{F}}$ is an increasing function.

□

Proof of theorem 2 We specialise the inequality in proposition 1 in the test function g_0 with $0 < R < \rho_{\max}(\mathcal{F})$. Using lemma 2, if $\beta > B(g_0)$, we have

$$\liminf_{\substack{q\to+\infty \\ q \text{ prime}}} \mu^h_{\mathcal{F}(q)}\left(\{L(f,s)\in\mathcal{F};\, \tilde{\gamma}_{f,1} < \beta\}\right) \geq Borne^+_{\mathcal{F}}(\beta^2).$$

Thanks to lemma 3, if $\beta > \beta_{\min}(g_0)$, we get

$$\liminf_{\substack{q\to+\infty \\ q \text{ prime}}} \mu^h_{\mathcal{F}(q)}\left(\{L(f,s)\in\mathcal{F};\, \tilde{\gamma}_{f,1} < \beta\}\right) \geq Borne_{\mathcal{F}}(\beta^2).$$

To obtain theorem 2 , we evaluate the right member in $R = \rho_{\max}(\mathcal{F})/2$.

3 The smallest zero

In this part, we prove theorem 1. Our starting point is the following proposition. We do not write the proof that is essentially given in [HR03] (theorem 8.1).

Proposition 2 *The infimum being taken over functions g in $\mathcal{S}_{\nu_{\max}(\mathcal{F})/2}(\mathbb{R})$, we have*

$$\limsup_{Q\to+\infty} \min_{L(f,s)\in\mathcal{F}(Q)} \tilde{\gamma}_{f,1} \leq \inf\left\{\sqrt{\frac{\int_{\mathbb{R}} x^2 g^2(x) W^*[\mathcal{F}](x)dx}{\int_{\mathbb{R}} g^2(x) W^*[\mathcal{F}](x)dx}}\right\}. \tag{3.1}$$

We give a sketch of the proof of theorem 1.

Sketch of the proof We have to determine an explicit expression of the right member term in (3.1). Let

$$\mathfrak{m} := \inf_{g\in\mathcal{S}_\nu(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} x^2 g^2(x) W[G](x)dx}{\int_{\mathbb{R}} g^2(x) W[G](x)dx}$$

with $\nu = \nu_{\max}(\mathcal{F})/2$ and $W[G] = W^*[\mathcal{F}]$.

- In subsection 3.1 (lemma 5), we prove the existence of a function $\tilde{B}$ and a set H^∞_ν such that $\mathfrak{m} = \inf_{\widehat{g}\in H^\infty_\nu\setminus\{0\}} \tilde{B}(\widehat{g})$. Precisely, we define $\tilde{B}$ by

$$\tilde{B}(h) = \frac{1}{4\pi^2} \frac{\int_{\mathbb{R}} h'(u)^2 du - \frac{\delta}{2}\int_{-1}^1 h'*h'(u)du}{\int_{\mathbb{R}} h(u)^2 du + \frac{\delta}{2}\int_{-1}^1 h*h(u)du + \varepsilon\left(\int_{\mathbb{R}} h(u)du\right)^2} \tag{3.2}$$

  where (δ,ε) is given in table 1.

- In subsection 3.1 (lemma 6), by a density argument, if

$$H_R = \left\{h:\mathbb{R}\to\mathbb{R} \text{ even, continuous, } \mathcal{C}^1 \text{ on }]-R,R[, \text{ with supp } h\subset[-R,R]\right\}$$

  and

$$\tilde{\mathfrak{m}}_R = \inf_{h\in H_R\setminus\{0\}} \tilde{B}(h),$$

  we prove $\mathfrak{m} = \lim_{R\to\nu^-} \tilde{\mathfrak{m}}_R$.

- In subsection 3.1 (lemma 7), by topological arguments, there exists $\mathfrak{h}_R$ in the Sobolev space H_0^1 such that $\tilde{B}(\mathfrak{h}_R) = \tilde{\mathfrak{m}}_R$.
- In subsection 3.3 (lemma 10), thanks to Fourier theory, $\mathfrak{h}_R$ satisfies a Volterra equation with temporal shifts

$$\mathfrak{h}_R(u) = \varphi(u) + \frac{\delta}{2}\int_u^R \mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)dt \tag{3.3}$$

  where φ is an explicit function which is defined with the unknown parameter $\tilde{\mathfrak{m}}_R$. Then, $\mathfrak{h}_R$ is in H_R.
- Then, we solve the previous Volterra equation in subsection 3.4. As a result, we obtain an explicit expression of $\mathfrak{h}_R$ which also depends on the unknown parameter $\tilde{\mathfrak{m}}_R$.
- We determine $\sqrt{\tilde{\mathfrak{m}}_R}$ in subsection 3.5 by solving the equation $\tilde{\mathfrak{m}}_R = \tilde{B}(\mathfrak{h}_R)$.
- To conclude, we use $\sqrt{\mathfrak{m}} = \lim_{R\to\nu^-}\sqrt{\tilde{\mathfrak{m}}_R}$.

3.1 Density and topological arguments

This section is devoted to the proof of the following lemma which sums up the three first steps of the preceding sketch of the proof. Let H_0^1 be the Sobolev space defined by

$$H_0^1 = \Big\{u \in L^2(]-R,R[) \text{ with } u(-R) = u(R) = 0, \\ \exists v \in L^2(\mathbb{R}) \text{ such that } \forall \phi \in \mathcal{C}_c^\infty(]-R,R[),\ \int_{\mathbb{R}} u\phi' = -\int_{\mathbb{R}} v\phi\Big\}.$$

If u is in H_0^1 then v is called the weak derivative of u and is denoted by u'. The space H_0^1 is equipped with the inner product

$$\langle u, v\rangle_{H_0^1} = \langle u', v'\rangle_{L^2} = \int_{-R}^R u'v'.$$

H_0^1 is a reflexive separable Hilbert space ([Bre11], paragraphe VIII.3).

Lemma 4 *We have*

$$\mathfrak{m} = \lim_{R\to\nu^-} \tilde{\mathfrak{m}}_R.$$

In addition, for each $R > 0$, there exists $\mathfrak{h}_R$ in H_0^1 such that $\tilde{m}_R = \tilde{B}(\mathfrak{h}_R)$.

Before proving this result, we need to prove some technical lemmas.

A new expression

Lemma 5 *Let $H_\nu^\infty = \{h \in \mathcal{C}_c^\infty(\mathbb{R}),$ even and with supp $h \subset [-R,R]$ such that $0 < R < \nu\}$. Then,*

$$\mathfrak{m} = \inf_{h\in H_\nu^\infty\setminus\{0\}} \tilde{B}(h).$$

Proof: Thanks to Plancherel theorem, inversion formula, Parseval formula and relation (1.1), we have:

$$\begin{aligned}\int_{\mathbb{R}} g^2(x)W[G](x)dx &= \int_{\mathbb{R}} \widehat{g^2}(y)\widehat{W[G]}(y)dy = \widehat{g^2}(0) + \frac{\delta}{2}\int_{-1}^{1}\widehat{g^2}(y)dy + \varepsilon\int_{\mathbb{R}}\widehat{g^2}(y)dy \\ &= \int_{\mathbb{R}} g^2(x)dx + \frac{\delta}{2}\int_{-1}^{1}\widehat{g}(y)*\widehat{g}(y)dy + \varepsilon g^2(0) \\ &= \int_{\mathbb{R}}\widehat{g}^2(y)dy + \frac{\delta}{2}\int_{-1}^{1}\widehat{g}(y)*\widehat{g}(y)dy + \varepsilon\left(\int_{\mathbb{R}}\widehat{g}(y)dy\right)^2\end{aligned}$$

Similarly, since $\widehat{xg(x)} = \frac{-1}{2i\pi}\widehat{g}'$, we prove

$$\int_{\mathbb{R}} x^2 g^2(x)W[G](x)dx = \frac{1}{4\pi^2}\int_{\mathbb{R}}\widehat{g}'(u)^2du - \frac{\delta}{2}\int_{-1}^{1}\widehat{g}'*\widehat{g}'(u)du.$$

In addition, since the set of Schwartz functions is invariant by Fourier transformation, we have:

$$g \in \mathcal{S}_\nu(\mathbb{R}) \Leftrightarrow \widehat{g} \in H_\nu^\infty$$

Therefore,

$$\mathfrak{m} = \inf_{\widehat{g}\in H_\nu^\infty\setminus\{0\}} \tilde{B}(\widehat{g}).$$

□

A density argument

Lemma 6 *We have*

$$\inf_{h\in H_\nu^\infty\setminus\{0\}} \tilde{B}(h) = \lim_{R\to\nu^-}\tilde{m}_R.$$

Proof: We define mollifiers (ρ_n) by

$$\rho_n : \left\{\begin{array}{l}\mathbb{R}\longrightarrow\mathbb{R} \\ x\longmapsto \frac{n}{\int_{\mathbb{R}}\rho(t)dt}\rho(nx)\end{array}\right. \quad \text{where} \quad \rho(x) = \left\{\begin{array}{ll} e^{\frac{1}{|x|^2-1}} & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1\end{array}\right. .$$

The function ρ_n is non-negative, smooth with supp $\rho_n \subset [-1/n; 1/n]$ and such that $\int_{\mathbb{R}}\rho_n(u)du = 1$. We recall two properties (see e.g. [Bre11], Theorem 4.22 and Theorem 4.15).

P1 Let $1 \leq p < \infty$, if $g \in L^p(\mathbb{R})$ then $\rho_n * g$ tends to g in $L^p(\mathbb{R})$.

P2 Let $1 \leq p \leq \infty$, if $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ then $f*g \in L^p(\mathbb{R})$ and $||f*g||_p \leq ||f||_1||g||_p$.

Let $0 < R < \nu$ and $\eta > 0$.
There exists h in H_R such that $\tilde{m}_R \leq \tilde{B}(h) \leq \tilde{m}_R + \eta$. Let $h_n = \rho_n * h$. For large n, h_n is in H_ν^∞. We also have $h_n' = \rho_n * h'$. Thanks to property **P1**, we have

$$\lim_{n\to+\infty}||h_n||_2^2 = ||h||_2^2, \quad \lim_{n\to+\infty}||h_n'||_2^2 = ||h'||_2^2 \quad \text{and} \quad \lim_{n\to+\infty}\int_{\mathbb{R}}h_n(u)du = \int_{\mathbb{R}}h(u)du.$$

Thanks to property **P2**, we may write

$$\begin{aligned} & \left|\int_{-1}^{1} h_n * h_n(u) - h * h(u) du\right| \leq \int_{\mathbb{R}} |h_n * h_n(u) - h * h(u)|\, du \\ \leq \quad & ||(h_n - h) * h_n||_1 + ||h * (h_n - h)||_1 \leq (||h_n||_1 + ||h||_1)||h - h_n||_1. \end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} \int_{-1}^{1} h_n * h_n(u) du = \int_{-1}^{1} h * h(u) du \text{ and similarly } \lim_{n\to+\infty} \int_{-1}^{1} h_n' * h_n'(u) du = \int_{-1}^{1} h' * h'(u) du.$$

Then, $\lim_{n\to+\infty} \tilde{B}(h_n) = \tilde{B}(h)$. There exists h_n in H_ν^∞ such that $|\tilde{B}(h) - \tilde{B}(h_n)| \leq \eta$. As a result, for all $\eta > 0$, there exists h_n in H_ν^∞ such that $|\tilde{m}_R - \tilde{B}(h_n)| \leq 2\eta$. Then, for all $R < \nu$, we have $\inf_{h\in H_\nu^\infty \setminus \{0\}} \tilde{B}(h) \leq \tilde{m}_R$. As a consequence, we have

$$\inf_{h\in H_\nu^\infty \setminus \{0\}} \tilde{B}(h) \leq \lim_{R\to\nu^-} \tilde{m}_R.$$

On the other side, let h be in H_ν^∞. There exists $R_0 < \nu$ such that supp $h \subset [-R_0; R_0]$. Since h is in H_{R_0} and $R \mapsto \tilde{m}_R$ is decreasing, we get

$$\lim_{R\to\nu^-} \tilde{m}_R \leq \tilde{m}_{R_0} \leq \tilde{B}(h).$$

Therefore,

$$\lim_{R\to\nu^-} \tilde{m}_R \leq \inf_{h\in H_\nu^\infty \setminus \{0\}} \tilde{B}(h).$$

□

Some compact operators Relation (3.2) allows us to extend $\tilde{B}$ to $H_0^1 \setminus \{0\}$.

Lemma 7 *If $R > 0$, there exists $\mathfrak{h}_R$ in $H_0^1 \setminus \{0\}$ such that $\tilde{m}_R = \tilde{B}(\mathfrak{h}_R)$.*

Proof: Let K be the operator of $L^2(]-R, R[)$ defined by

$$K[h](u) = \frac{\delta}{2} \int_{u-1}^{u+1} h(t) dt + \varepsilon \int_{\mathbb{R}} h(t) dt.$$

We may write $\tilde{B}$ on the shape

$$\tilde{B}(h) = \frac{\langle h', h' \rangle_{L^2} + \langle Kh', h' \rangle_{L^2}}{\langle h, h \rangle_{L^2} + \langle Kh, h \rangle_{L^2}}.$$

Since K is a Hilbert-Schmidt operator, it is a compact operator of $L^2(]-R, R[)$. Denote by I the identity function of $L^2(]-R, R[)$, the transformations that have been done in lemma 5 show that, for all h in $L^2(]-R, R[)$, $\langle (I+K)[h], h \rangle_{L^2} \geq 0$ with equality if and only if $h = 0$.

Since smooth function compactly supported in $]-R, R[$ are dense in H_0^1, the infimum of $\tilde{B}$ over H_0^1 is also equal to $\tilde{m}_R$.

Let (g_n) be a sequence of non zero functions in H_R such that $\lim_{n\to+\infty}\tilde{B}(g_n)=\tilde{m}_R$. We consider $h_n = g_n/\langle g_n', g_n'\rangle_{L^2}$. Since for all real number $t\neq 0$ and all h in H_R, $\tilde{B}$ satisfies $\tilde{B}(t.h)=\tilde{B}(h)$, we have

$$\lim_{n\to+\infty}\tilde{B}(h_n)=\tilde{m}_R \quad \text{and} \quad \|h_n'\|_{L^2}=\|h_n\|_{H_0^1}=1.$$

Sequences (h_n) and (h_n') are bounded in the Hilbert space $L^2(]-R,R[)$. Since the unit ball of $L^2(]-R,R[)$ is compact for the weak topology, up to consider sub-sequences, there exists h and k in $L^2(]-R,R[)$ such that (h_n) (respectively (h_n')) tends to h (respectively k) weakly. Moreover, for all function ϕ in $\mathcal{C}_c^\infty(]-R,R[)$, we may write

$$\int g\phi' = \lim_{n\to+\infty}\int g_n\phi' = -\lim_{n\to+\infty}\int g_n'\phi = -\int k\phi.$$

Then g belongs to H^1, $g'=k$ and g_n tends to g weakly in H^1. In addition, there exists a compact embedding of H^1 into the set of continuous function on $[-R,R]$ equipped with the norm of uniform convergence ([Bre11], theorem 8.2 et 8.8), we may conclude ([Bre11], 6.1 remark 2) h_n converges uniformly to h (precisely to its continuous representative which will be always identified with h). As a result, h is an even functions belonging to H_0^1 and $\lim_{n\to+\infty}\|h_n\|_{L^2}=\|h\|_{L^2}$.

Since (h_n) and (h_n') are weakly convergent in $L^2(]-R,R[)$ and since K is a compact operator, the sequence (Kh_n) and (Kh_n') converge respectively to Kh and Kh' strongly in $L^2(]-R,R[)$. Then, we get ([Bre11], proposition 3.5 (iv))

$$\lim_{n\to+\infty}\langle Kh_n,h_n\rangle_{L^2}=\langle Kh,h\rangle_{L^2} \text{ and } \lim_{n\to+\infty}\langle Kh_n',h_n'\rangle_{L^2}=\langle Kh',h'\rangle_{L^2}.$$

To sum up, we have

$$\lim_{n\to+\infty}\langle (I+K)h_n',h_n'\rangle_{L^2}=1+\langle Kh',h'\rangle_{L^2}, \quad \lim_{n\to+\infty}\langle (I+K)h_n,h_n\rangle_{L^2}=\langle (I+K)h,h\rangle_{L^2}$$

and, since (h_n') tends weakly to h' in L^2, we may deduce $\|h'\|_{L^2}\leq \liminf\|h_n'\|_{L^2}=1$.

Furthermore, $1+\langle Kh',h'\rangle_{L^2}$ is non zero. Indeed, if it was zero, we would have

$$0=1+\langle Kh',h'\rangle_{L^2}\geq \|h'\|_{L^2}+\langle Kh',h'\rangle_{L^2}=\langle (I+K)h',h'\rangle_{L^2}\geq 0.$$

So, $\|h'\|_{L^2}=1$ and $\langle (I+K)h',h'\rangle_{L^2}=0$, thus $\|h'\|_{L^2}=1$ and h' should be zero !

As a result $\langle (I+K)h,h\rangle_{L^2}\neq 0$ because $\tilde{m}_R$ is finite and $1+\langle Kh',h'\rangle_{L^2}$ is non zero.

Then

$$\frac{\|h'\|_{L^2}+\langle Kh',h'\rangle_{L^2}}{\langle (I+K)h,h\rangle_{L^2}}=\tilde{B}(h)\geq \tilde{m}_R=\lim_{n\to+\infty}\tilde{B}(h_n)=\frac{1+\langle Kh',h'\rangle_{L^2}}{\langle (I+K)h,h\rangle_{L^2}},$$

and $\|h'\|_{L^2}\geq 1$. Since we already have $\|h'\|_{L^2}\leq 1$, it comes $\|h'\|_{L^2}=1$. To conclude, the function h is non zero and satisfies $\tilde{B}(h)=\tilde{m}_R$.

□

Remark 5 $\tilde{B}$ is not continuous when $B_{||\cdot||_{H^1}}(0,1)\to\mathbb{R}$ is equipped with the weak topology. Indeed, since the unit ball is weakly compact, $\tilde{B}$ should be bounded. However, by considering $h_n(u)=\cos\left(\frac{(2n+1)\pi u}{2R}\right)\mathbb{1}_{[-R,R]}(u)$, we remark that $\tilde{B}$ cannot be bounded.

The proof of lemma 4 comes from lemmas 5, 6 and 7.

3.2 Fourier analysis

For technical conveniences, let $B(h) = 16R^2\tilde{B}(h)$ and $\mathfrak{m}_R = \inf_{h\in H_R} B(h)$. Then

$$\mathfrak{m}_R = B(\mathfrak{h}_R) = 16R^2\tilde{\mathfrak{m}}_R.$$

Let

$$\Omega_R = \left\{ (c_n) \in \mathbb{R}^{\mathbb{N}}, \text{ such that } c_{2n} = 0 \text{ and } x \mapsto \sum_{n\geq 0} c_n \cos\left(\frac{\pi n x}{2R}\right) \text{ is continuous and in} H_0^1 \right\}.$$

In this section, we prove some technical lemmas.

Lemma 8 *Let*

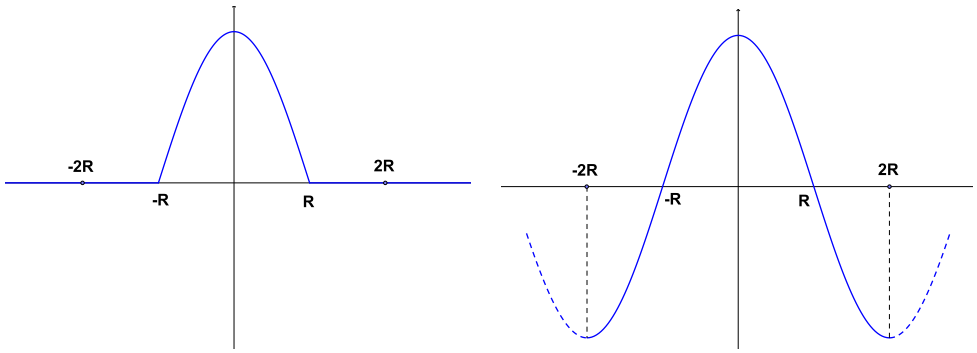
$$\Psi : \left\{ \begin{array}{lll} H_0^1 \cap \{continuous, even\} & \longrightarrow & \Omega_R \\ h & \longmapsto & \left(\frac{1-(-1)^n}{2R}\int_0^R h(t)\cos\left(\frac{\pi n t}{2R}\right)dt\right)_{n\geq 0} \end{array} \right. .$$

Then, Ψ is a bijective function.

Proof: For each h in H_0^1, we associate an even $4R$-periodic function $\tilde{h}$ which is defined by:

$$\text{for all } x \in [0,R], \ \ \tilde{h}(x) = h(x) \text{ and } \tilde{h}(R+x) = -h(R-x).$$

We also have $h = \tilde{h}\cdot \mathbb{1}_{[-R,R]}$.



representative curve of h representative curve of $\tilde{h}$

As a result, the mean value of $\tilde{h}$ is equal to zero. Actually, if c_n denotes the n-th Fourier coefficient of $\tilde{h}$, we have

$$c_n = \frac{1}{4R}\int_{-2R}^{2R} \tilde{h}(t)e^{-\frac{i\pi nt}{2R}}dt = \frac{1-(-1)^n}{2R}\int_0^R h(t)\cos\left(\frac{\pi n t}{2R}\right)dt.$$

Since the Fourier series of a function in H_0^1 is normally convergent, we get

$$h(u) = \tilde{h}(u) \cdot \mathbb{1}_{[-R,R]}(u) = 2\left[\sum_{n\geq 0} c_n \cos\left(\frac{\pi n x}{2R}\right)\right] \mathbb{1}_{[-R,R]}(u).$$

Then Ψ is well defined and the last relation gives us an explicit expression of Ψ^{-1}.

□

As a consequence, since B is defined on the set of continuous and even functions in H_0^1, we may define B on Ω_R by $B(c) := B(\Psi^{-1}(c))$.

Lemma 9 *Let $c = (c_n)$ be in Ω_R. If $R \leq 1/2$ then*

$$B(c) = \frac{\sum_{n\geq 0}^{*} n^2 c_n^2}{\sum_{n\geq 0}^{*} c_n^2 + (\delta + 2\varepsilon)\frac{8R}{\pi^2}\left(\sum_{n\geq 0}^{*} \frac{(-1)^{\frac{n-1}{2}}}{n} c_n\right)^2}$$

and, if $R > 1/2$, we have

$$B(c) = \frac{\sum_{n\geq 0}^{*} n^2 c_n^2 - \frac{\delta}{2R}\sum\sum_{m,n\geq 0} mn\mu_{m,n} c_m c_n}{\sum_{n\geq 0}^{*} c_n^2 + \frac{\delta}{2R}\sum\sum_{m,n\geq 0} \lambda_{m,n} c_m c_n + \frac{16R\varepsilon}{\pi^2}\left(\sum_{n\geq 0}^{*} \frac{(-1)^{\frac{n-1}{2}} c_n}{n}\right)^2}$$

where, for n and m odd numbers:

$$\lambda_{m,n} = \begin{cases} \frac{8R^2 mn(-1)^{\frac{m+n}{2}}}{\pi^2(m^2-n^2)}\left[\frac{1}{n^2}\cos\frac{\pi n}{2R} - \frac{1}{m^2}\cos\frac{\pi m}{2R}\right] - \frac{8R^2(-1)^{\frac{m+n}{2}}}{mn\pi^2} & si\ m \neq n \\ \frac{2R(2R-1)}{n\pi}\sin\frac{\pi n}{2R} - \frac{8R^2}{\pi^2 n^2}\cos\frac{n\pi}{2R} + \frac{8R^2}{\pi^2 n^2} & si\ m = n \end{cases}$$

$$\text{and} \qquad \mu_{m,n} = \begin{cases} \frac{8R^2(-1)^{\frac{m+n}{2}}}{\pi^2(m^2-n^2)}\left[\cos\frac{\pi m}{2R} - \cos\frac{\pi n}{2R}\right] & si\ m \neq n \\ -\frac{2R(2R-1)}{n\pi}\sin\frac{\pi n}{2R} & si\ m = n \end{cases}$$

Proof: We are giving an explicit expression of B on Ω_R. Let h be an even and continuous function in H_0^1 such that $c = (c_n) = \Psi(h)$. Thanks to Parseval formula, we have:

$$\begin{aligned} \frac{1}{2R}\int_{\mathbb{R}} h(u)^2 du &= \frac{1}{4R}\int_{-2R}^{2R} \tilde{h}(u)^2 du = \sum_{n\in\mathbb{Z}} |c_n|^2 = 2\sum_{n\geq 0} c_n^2 \\ \frac{1}{2R}\int_{\mathbb{R}} h'(u)^2 du &= \frac{1}{4R}\int_{-2R}^{2R} \tilde{h}'(u)^2 du = \frac{\pi^2}{4R^2}\sum_{n\in\mathbb{Z}} n^2|c_n|^2 = \frac{\pi^2}{2R^2}\sum_{n\geq 0} n^2 c_n^2 \end{aligned}$$

In addition, we may write

$$\int_{\mathbb{R}} h(u)du = \int_{-R}^{R} \tilde{h}(u)du = \frac{4R}{\pi}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(2n+1)} c_{2n+1} = \frac{8R}{\pi}\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)} c_{2n+1}.$$

Remember $B(h) = 16R^2\tilde{B}(h)$ where $\tilde{B}(h)$ is given in (3.2). We obtain an explicit expression of $B(c)$ using

$$\int_{-1}^{1} h * h(x)dx = \begin{cases} \left(\int_{\mathbb{R}} h(x)dx\right)^2 & \text{if } R \leq \frac{1}{2} \\ 4\sum_{m\geq 0}\sum_{n\geq 0} c_m c_n \lambda_{m,n} & \text{if } R > \frac{1}{2} \end{cases}$$

and

$$\int_{-1}^{1} h' * h'(x)dx = \begin{cases} 0 & \text{if } R \leq \frac{1}{2} \\ \frac{\pi^2}{R^2}\sum_{m\geq 0}\sum_{n\geq 0} mnc_m c_n \mu_{m,n} & \text{if } R > \frac{1}{2} \end{cases}.$$

□

Hughes and Rudnick proved that $\mathfrak{m} = \frac{1}{16\nu^2}$ if $G = U$. We extend their result.

Corollary 1 *If $G = O$ or if $G = SO^{\pm}$ and $0 < R \leq 1/2$ then*

$$\frac{1}{1+R} < \mathfrak{m}_R \leq \frac{1}{1+\frac{8}{\pi^2}R} < 1.$$

If $G = SO^{\pm}$ with $R > 1/2$, then $\mathfrak{m}_R < 1$. If $G = Sp$ and $0 < R < 1/2$, then

$$1 < \mathfrak{m}_R < \frac{1}{1-\frac{8}{\pi^2}R} < 2.$$

Proof: Thanks to the Cauchy-Schwarz inequality, if $G = O$ or if $G = SO^{\pm}$ with $0 < R \leq 1/2$, we have

$$B(c) = \frac{\sum_{n\geq 0}^{*} n^2 c_n^2}{\sum_{n\geq 0}^{*} c_n^2 + (\delta + 2\varepsilon)\frac{8R}{\pi^2}\left(\sum_{n\geq 0}^{*} \frac{(-1)^{\frac{n-1}{2}}}{n} c_n\right)^2} \geq \frac{\sum_{n\geq 0}^{*} n^2 c_n^2}{(1+R)\sum_{n\geq 0}^{*} c_n^2} \geq \frac{1}{1+R}$$

and one of both inequalities is strict. In order to obtain an upper bound for $\mathfrak{m}_R$, we specialise B in $e = (e_n)$ which is defined by $e_n = 0$ for all n except $e_1 = 1$. The upper bound comes easily if $G = O$, if $G = SO^{\pm}$ or if $G = Sp$ and $0 < R \leq 1/2$. If $G = SO^{\pm}$ and $R > 1/2$, we have

$$B(e) = \frac{1 - \frac{\delta}{2R}\mu_{1,1}}{1 + \frac{\delta}{2R}\lambda_{1,1} + \frac{16R\varepsilon}{\pi^2}} = \frac{1 + \frac{\delta(2R-1)}{\pi}\sin\frac{\pi}{2R}}{1 + \frac{\delta(2R-1)}{\pi}\sin\frac{\pi}{2R} + \frac{4R\delta}{\pi^2}\left(1 - \cos\frac{\pi}{2R}\right) + \frac{16R\varepsilon}{\pi^2}} < 1.$$

Finally, if $G = Sp$ and $R < 1/2$, for all c in Ω_R, we have

$$B(c) = \frac{\sum_{n\geq 0}^{*} n^2 c_n^2}{\sum_{n\geq 0}^{*} c_n^2 - \frac{8R}{\pi^2}\left(\sum_{n\geq 0}^{*} \frac{(-1)^{\frac{n-1}{2}}}{n} c_n\right)^2} \geq \frac{\sum_{n\geq 0}^{*} n^2 c_n^2}{\sum_{n\geq 0}^{*} c_n^2} \geq 1$$

and one of both inequalities is strict. Therefore, thanks to lemma 7, we have $\mathfrak{m}_R > 1$.

□

Corollary 1 implies, except if $G = Sp$ and $R > 1/2$, that $\mathfrak{m}_R$ is not the square of an integer. We make the following hypothesis which will be justified page 43.

"Sp hypothesis": If $G = Sp$ and $R > 1/2$ then $\mathfrak{m}_R$ is not the square of an integer.

Remark 6 In the symplectic case with $R > 1/2$, by evaluating B in e, we get:

$$\mathfrak{m}_R \leq B(e) = \frac{1 - \frac{2R-1}{\pi}\sin\frac{\pi}{2R}}{1 - \frac{2R-1}{\pi}\sin\frac{\pi}{2R} - \frac{4R}{\pi^2}\left(1 - \cos\frac{\pi}{2R}\right)} = \frac{12R}{\pi^2} + O(1)$$

Therefore, since $\mathfrak{m}_R = 16R^2\tilde{\mathfrak{m}}_R$, corollary 1 and proposition 2 show:

$$\limsup_{Q\to+\infty} \min_{L(f,s)\in\mathcal{F}(Q)} \tilde{\gamma}_{f,1} \leq \lim_{R\to\nu_{\max}(\mathcal{F})/2} \sqrt{\tilde{\mathfrak{m}}_R} \ll \begin{cases} \frac{1}{\nu_{\max}(\mathcal{F})^{1/2}} & \text{if } W^*[\mathcal{F}] = W[Sp] \\ \frac{1}{\nu_{\max}(\mathcal{F})} & \text{if } W^*[\mathcal{F}] = W[SO^+] \\ \frac{1}{\nu_{\max}(\mathcal{F})^{3/2}} & \text{if } W^*[\mathcal{F}] = W[O] \\ \frac{1}{\nu_{\max}(\mathcal{F})^2} & \text{if } W^*[\mathcal{F}] = W[SO^-] \end{cases}$$

As a result, density conjecture agrees with random matrix prediction.

3.3 A Volterra equation with temporal shifts

Since the unitary case has been solved, we assume G is one of the following compact groups O, SO^-, SO^+ or Sp. In this section, we prove the following lemma.

Lemma 10 *The optimal test function $\mathfrak{h}_R$ satisfies, for all $0 \leq u \leq R$,*

$$\mathfrak{h}_R(u) = \varphi(u) + \frac{\delta}{2}\int_u^R \mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)dt \tag{3.4}$$

where (δ, ε) is given in the table 1 and with

$$\varphi(u) = \frac{\pi k_{\delta,\varepsilon}}{2\mathfrak{m}_R \cos\frac{\sqrt{\mathfrak{m}_R}\pi}{2}}\left[\cos\frac{\sqrt{\mathfrak{m}_R}\pi u}{2R} - \cos\frac{\sqrt{\mathfrak{m}_R}\pi}{2}\right]\cdot \mathbb{1}_{[-R,R]}(u)$$

where

$$k_{\delta,\varepsilon} = \frac{2\mathfrak{m}_R}{\pi}\left[\frac{\delta}{2}\int_{R-1}^R \mathfrak{h}_R(x)dx + \varepsilon\int_{-R}^R \mathfrak{h}_R(x)dx\right]. \tag{3.5}$$

Remark 7 We identify $\mathfrak{h}_R$ to its continuous representative. As a result, equation (3.4) proves that $\mathfrak{h}_R$ is of class $\mathcal{C}^1$ on $]-R, R[$. In other words, the function $\mathfrak{h}_R$ belongs to H_R.

Before proving this result, we need to prove some technical lemmas. However, we may immediately deduce the following corollary.

Corollary 2 *If $G = O$ or if $G = SO^{\pm}$ or Sp with $R \leq 1/2$, then*

$$\sqrt{\mathfrak{m}} = \frac{1}{\nu}V^{-1}\left(1 + (\delta + 2\varepsilon)\frac{1}{\nu}\right)$$

where V has been defined in section "Preview of results".

Proof: If $G = O$, then $\delta = 0$ and lemma 10 gives us immediately $\mathfrak{h}_R = \varphi$. Similarly, if $G = SO^+$, SO^- or Sp, $R \leq 1/2$ and if $0 < t < R$ then $t + 1 \notin [-R, R]$ and $t - 1 \notin [-R, R]$. Therefore, in this case, we also have $\mathfrak{h}_R = \varphi$. We may sum up these remarks by

$$\mathfrak{h}_R(u) = \varphi(u) = \frac{(\delta + 2\varepsilon)\int_{-R}^{R} \mathfrak{h}_R(x)dx}{2\cos\left(\frac{\pi\sqrt{\mathfrak{m}_R}}{2}\right)}\left[\cos\frac{\sqrt{\mathfrak{m}_R}\pi u}{2R} - \cos\frac{\sqrt{\mathfrak{m}_R}\pi}{2}\right] \cdot \mathbb{1}_{[-R,R]}(u).$$

As a result, we have

$$\int_{-R}^{R} \mathfrak{h}_R(u)du \cdot \left[1 - (\delta + 2\varepsilon)R\left(\frac{2}{\pi\sqrt{\mathfrak{m}_R}}\tan\left(\frac{\pi\sqrt{\mathfrak{m}_R}}{2}\right) - 1\right)\right] = 0.$$

Since $\int_{-R}^{R} \mathfrak{h}_R(u)du \neq 0$ (otherwise $\mathfrak{h}_R = 0$), we may conclude

$$\frac{2}{\pi\sqrt{\mathfrak{m}_R}}\tan\left(\frac{\pi\sqrt{\mathfrak{m}_R}}{2}\right) = 1 + \frac{\delta + 2\varepsilon}{R}.$$

The result comes easily from lemmas 5 and 6.

□

Remark 8 We will use this phenomenon in the next section in order to determine $\mathfrak{h}_R$ in full generality.

Let

$$S_n(t) = \sin\left(\frac{\pi nt}{2R}\right)\mathbb{1}_{[-R,R]}(t) \quad \text{and} \quad C_n(t) = \cos\left(\frac{\pi nt}{2R}\right)\mathbb{1}_{[-R,R]}(t).$$

Lemma 11 *n denotes an odd positive integer. Let h be in H_R with $c = (c_m) = \Psi(h)$, then:*

$$\sum_{m\geq 0}\left[mn\mu_{m,n} + \mathfrak{m}_R\lambda_{m,n}\right]c_m$$
$$= -\frac{2R}{n\pi}(n^2 - \mathfrak{m}_R)(S_n * h)(1) + \frac{8R^2(-1)^{\frac{n-1}{2}}}{n\pi^2}\mathfrak{m}_R\sum_{m\geq 0}\frac{(-1)^{\frac{m-1}{2}}}{m}\left[1 - \cos\frac{m\pi}{2R}\right]c_m$$

Proof: We have:

$$\sum_{m\geq 0}\left[mn\mu_{m,n} + \mathfrak{m}_R\lambda_{m,n}\right]c_m$$
$$= -(n^2 - \mathfrak{m}_R)\left(\frac{2R(2R-1)}{n\pi}\sin\left(\frac{\pi n}{2R}\right)c_n + \frac{8R^2}{n\pi^2}\sum_{m\neq n}\frac{m(-1)^{\frac{m+n}{2}}}{m^2 - n^2}\cos\left(\frac{\pi n}{2R}\right)c_m\right)$$
$$-\frac{8R^2\mathfrak{m}_R}{n^2\pi^2}\cos\left(\frac{\pi n}{2R}\right)c_n - \mathfrak{m}_R\sum_{m\geq 0}\frac{8R^2(-1)^{\frac{m+n}{2}}}{nm\pi^2}c_m + \frac{8R^2}{\pi^2}\sum_{m\neq n}\frac{n(-1)^{\frac{m+n}{2}}}{m(m^2 - n^2)}(m^2 - \mathfrak{m}_R)\cos\left(\frac{\pi m}{2R}\right)c_m$$

Since

$$\frac{n}{m(m^2 - n^2)} = \frac{1}{n}\left(\frac{m}{m^2 - n^2} - \frac{1}{m}\right),$$

we may conclude

$$\sum_{m\geq 0}\left[mn\mu_{m,n}+\mathfrak{m}_R\lambda_{m,n}\right]c_m$$
$$=-(n^2-\mathfrak{m}_R)\left(\frac{2R(2R-1)}{n\pi}\sin\left(\frac{\pi n}{2R}\right)c_n+\frac{8R^2}{n\pi^2}\sum_{m\neq n}{}^{*}\frac{(-1)^{\frac{m+n}{2}}m}{m^2-n^2}\left[\cos\left(\frac{\pi n}{2R}\right)-\cos\left(\frac{\pi m}{2R}\right)\right]c_m\right)$$
$$+\frac{8R^2\mathfrak{m}_R(-1)^{\frac{n-1}{2}}}{n\pi^2}\sum_{m\geq 0}{}^{*}\frac{(-1)^{\frac{m-1}{2}}}{m}\left[1-\cos\left(\frac{\pi m}{2R}\right)\right]c_m.$$

In addition, we have

$$(S_n*h)(1)=\int_{1-R}^{R}h(u)\sin\left(\frac{\pi n}{2R}(1-u)\right)du$$
$$=\sum_{m\geq 0}{}^{*}c_m\int_{1-R}^{R}\sin\left(\frac{\pi(m-n)u}{2R}+\frac{\pi n}{2R}\right)-\sin\left(\frac{\pi(m+n)u}{2R}-\frac{\pi n}{2R}\right)du$$
$$=(2R-1)\sin\left(\frac{\pi n}{2R}\right)c_n+\frac{4R}{\pi}\sum_{m\neq n}{}^{*}\frac{(-1)^{\frac{m+n}{2}}m}{m^2-n^2}\left[\cos\left(\frac{\pi n}{2R}\right)-\cos\left(\frac{\pi m}{2R}\right)\right]c_m.$$

□

Lemma 12 *Let $\mathfrak{c}=(\mathfrak{c}_n)=\Psi(\mathfrak{h}_R)$. Then, for all odd positive integer n, we have*

$$\mathfrak{c}_n=\frac{(-1)^{\frac{n-1}{2}}k_{\delta,\varepsilon}}{n[n^2-\mathfrak{m}_R]}-\frac{\delta}{n\pi}(S_n*\mathfrak{h}_R)(1)=\frac{(-1)^{\frac{n-1}{2}}k_{\delta,\varepsilon}}{n[n^2-\mathfrak{m}_R]}+\frac{2R\delta}{n^2\pi^2}(C_n*\mathfrak{h}_R')(1).$$

Proof: We define a norm on Ω_R by

$$||c||=\sum_{n=0}^{+\infty}n|c_n|+\left(\sum_{n=0}^{+\infty}n^2c_n^2\right)^{1/2}$$

and we consider five differentiable functions H, T, S, A and G on $(\Omega_R,||\cdot||)$ defined by:

$$H(c)=\sum_{n\geq 0}n^2c_n^2,\qquad T(c)=\sum_{n\geq 0}c_n^2,\qquad S(c)=\sum_{n\geq 0}{}^{*}\frac{(-1)^{\frac{n-1}{2}}}{n}c_n,$$
$$A(c)=\sum_{m\geq 0}\sum_{n\geq 0}mn\mu_{m,n}c_mc_n\quad\text{and}\qquad G(c)=\sum_{m\geq 0}\sum_{n\geq 0}\lambda_{m,n}c_mc_n.$$

With lemma 9, we may write

$$B=\frac{H-\frac{\delta}{2R}A}{T+\frac{\delta}{2R}G+\frac{16R\varepsilon}{\pi^2}S^2}\text{ if }R>\frac{1}{2}\quad\text{or}\quad B=\frac{H}{T+(\delta+2\varepsilon)\frac{8R}{\pi^2}S^2}\text{ if }R\leq\frac{1}{2}.$$

Since $B(\mathfrak{c})=\mathfrak{m}_R$, $\mathfrak{c}$ is a critical point. Therefore, if $R>1/2$, we have $dB_{|_{\mathfrak{c}}}=0$ hence:

$$\forall n\text{ odd},\quad(n^2-\mathfrak{m}_R)\mathfrak{c}_n=\frac{\delta}{2R}\sum_{m\geq 0}\left[mn\mu_{m,n}+\mathfrak{m}_R\lambda_{m,n}\right]\mathfrak{c}_m+\varepsilon\frac{16RS(\mathfrak{c})\mathfrak{m}_R}{\pi^2}\frac{(-1)^{\frac{n-1}{2}}}{n}$$

Thanks to lemma 11, "Sp hypothesis" and corollary 1, we may conclude

$$\mathfrak{c}_n = \frac{(-1)^{\frac{n-1}{2}} k_{\delta,\varepsilon}}{n[n^2-\mathfrak{m}_R]} - \frac{\delta}{n\pi}(S_n * \mathfrak{h}_R)(1) = \frac{(-1)^{\frac{n-1}{2}} k_{\delta,\varepsilon}}{n[n^2-\mathfrak{m}_R]} + \frac{2R\delta}{n^2\pi^2}(C_n * \mathfrak{h}'_R)(1) \tag{3.6}$$

where

$$\begin{aligned} k_{\delta,\varepsilon} &= \frac{4R\mathfrak{m}_R}{\pi^2}\left[\delta {\sum_{m\geq 0}}^* \frac{(-1)^{\frac{m-1}{2}}}{m}\left(1-\cos\frac{m\pi}{2R}\right)\mathfrak{c}_m + 4\varepsilon {\sum_{m\geq 1}}^* \frac{(-1)^{\frac{m-1}{2}}}{m}\mathfrak{c}_m\right] \\ &= \frac{2\mathfrak{m}_R}{\pi}\left[\frac{\delta}{2}\int_{R-1}^{R}\mathfrak{h}_R(x)dx + \varepsilon\int_{-R}^{R}\mathfrak{h}_R(x)dx\right]. \end{aligned} \tag{3.7}$$

Similarly, $\mathfrak{c}$ satisfies (3.6) and (3.7) in the case $R \leq 1/2$ due to the fact that, if $R \leq 1/2$, we have $(S_n * \mathfrak{h}_R)(1) = 0$ and $\int_{R-1}^{R}\mathfrak{h}_R(x)dx = \int_{-R}^{R}\mathfrak{h}_R(x)dx$.

□

Proof of lemma 10 Assume $0 \leq u \leq R$. Since $\mathfrak{h}_R(u) = 2\sum_{n\geq 0}\mathfrak{c}_n\cos\left(\frac{\pi nu}{2R}\right)$, lemma 12 gives

$$\mathfrak{h}_R(u) = 2k_{\delta,\varepsilon}{\sum_{n\geq 0}}^* \frac{(-1)^{\frac{n-1}{2}}\cos\left(\frac{\pi nu}{2R}\right)}{n(n^2-\mathfrak{m}_R)} + \frac{4R\delta}{\pi^2}{\sum_{n\geq 0}}^* \frac{(C_n * \mathfrak{h}'_R)(1)}{n^2}\cos\left(\frac{\pi nu}{2R}\right).$$

First, using relation 1.444.6 of [GR07], we have

$$\begin{aligned} {\sum_{n\geq 0}}^* \frac{(C_n * \mathfrak{h}'_R)(1)}{n^2}\cos\left(\frac{\pi nu}{2R}\right) &= \frac{\pi}{8}\int_{-R}^{R}\left(\pi - \frac{\pi|u+t|}{2R} - \frac{\pi|u-t|}{2R}\right)\mathfrak{h}'_R(1-t)dt \\ &= \frac{\pi^2}{8R}\int_{u}^{R}\mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)dt. \end{aligned}$$

Then, using relation 1.445.6 of [GR07], we compute

$$2k_{\delta,\varepsilon}{\sum_{n\geq 0}}^* \frac{(-1)^{\frac{n-1}{2}}\cos\left(\frac{\pi nu}{2R}\right)}{n(n^2-\mathfrak{m}_R)} = \frac{\pi k_{\delta,\varepsilon}}{2\mathfrak{m}_R\cos\frac{\sqrt{\mathfrak{m}_R}\pi}{2}}\left[\cos\frac{\sqrt{\mathfrak{m}_R}\pi u}{2R} - \cos\frac{\sqrt{\mathfrak{m}_R}\pi}{2}\right] = \varphi(u).$$

3.4 Optimal test Function

In this section, we solve the Volterra equation with temporal shifts which appears in lemma 10. Precisely, we prove that, except for a countable or finite number of value of R, the previous Volterra equation admits one and only one solution in H_R. We are giving an explicit expression of $\mathfrak{h}_R$.

An appropriate partition

For technical convenience, since R will tend to ν (thanks to lemma 6), we may assume that $2R$ is not an integer. Similarly, since $n \geq 1$ is the only integer such that $n-1 < 2\nu \leq n$, we may

assume $\frac{n-1}{2} < R < \frac{n}{2}$, then $n = \lfloor 2R \rfloor + 1$.
Let D be the derivative operator and T, T^{-1} the shift operators defined by:

$$D[f] = f', \quad T[f](u) = f(u+1) \quad \text{and} \quad T^{-1}[f](u) = f(u-1).$$

We build a partition of $[-R, R]$ which is invariant by $T[id]$, $T^{-1}[id]$ and by symmetry. We define

$$\begin{cases} a_{n-2i} = R - i & \text{if} \quad 0 \leq i \leq \frac{n-1}{2} \\ a_{n-2i-1} = \lfloor 2R \rfloor - R - i & \text{if} \quad 0 \leq i \leq \frac{n-2}{2} \end{cases}$$

and $I_0 =]-a_1, a_1[$, $I_k =]a_k, a_{k+1}[$, $I_{-k} =]-a_{k+1}, -a_k[$ for $1 \leq k \leq n-1$. We have

$$[-R, R] = \bigcup_{k=-\lfloor 2R \rfloor}^{\lfloor 2R \rfloor} \overline{I_k}.$$

Moreover, if $|k| < n-2$ then $T^{\pm 1}(I_k) = I_{k \pm 2}$ and, if $k \in \{n-2, n-1\}$ then $T(I_k) \cap \text{ supp } \mathfrak{h}_R = \emptyset$.

A differential equation with temporal shifts

By derivation and integration, we remark that $\mathfrak{h}_R$ satisfies the previous Volterra equation is equivalent to $\mathfrak{h}_R$ satisfies the following differential equation with temporal shifts:
for all u in $]-R, R[$,

$$\mathfrak{h}_R'(u) = \varphi'(u) - \frac{\delta}{2} \left[\mathfrak{h}_R(u+1) - \mathfrak{h}_R(u-1) \right]. \tag{3.8}$$

Let

$$\lambda = \frac{\sqrt{\mathfrak{m}_R}\pi}{2R} \qquad \text{and} \qquad w = \frac{-\pi^2 k_{\delta,\varepsilon}}{4R\sqrt{\mathfrak{m}_R} \cos \frac{\pi\sqrt{\mathfrak{m}_R}}{2}}.$$

We consider polynomial sequences (T_n) and (U_n) which are defined by induction:

$$\begin{cases} T_0 = 1 \\ T_1 = X \\ T_{n+1} = 2XT_n - T_{n-1} \end{cases} \qquad \begin{cases} U_0 = 1 \\ U_1 = 2X \\ U_{n+1} = 2XU_n - U_{n-1} \end{cases}$$

Polynomials T_n (resp. U_n) are called Chebychev polynomials of first kind (resp. second kind). They satisfy, for all real number θ and all non-negative integer n,

$$T_n(\cos\theta) = \cos n\theta \quad \text{and} \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

As a result, we have

$$T_n(X) = \prod_{k=0}^{n-1} \left[X - \cos\left(\frac{\pi}{2n} + \frac{k\pi}{n} \right) \right], \quad U_n(X) = \prod_{k=1}^{n} \left[X - \theta_n(k) \right] \text{ where } \theta_n(k) = \cos\left(\frac{k\pi}{n+1} \right).$$

In this subsection, we prove the following proposition:

Proposition 3 *Even functions h which are compactly supported on $[-R,R]$, $\mathcal{C}^1$ on each I_k and which satisfy the differential equation (3.8) on $\cup I_k$ are those that satisfy:*

- *If $0 \leq k \leq n-1$, then:*

$$h|_{I_{n-(2k+1)}}(u) = \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} r_j(I_{n-1}) U_k\left(\theta_n(j)\right) \sin\left(\left[u - \frac{n-2k-1}{2}\right]\theta_n(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-1}{2}\right]\right) + r_n^\lambda(k)\sin\left(\lambda\left[u - \frac{n-2k-1}{2}\right] + \theta_n^\lambda(k)\right)$$

- *If $0 \leq k \leq n-2$, then:*

$$h|_{I_{n-2(k+1)}}(u) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} r_j(I_{n-2}) U_k\left(\theta_{n-1}(j)\right) \sin\left(\left[u - \frac{n-2(k+1)}{2}\right]\theta_{n-1}(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2(k+1)}{2}\right]\right) + r_{n-1}^\lambda(k)\sin\left(\lambda\left[u - \frac{n-2k-2}{2}\right] + \theta_{n-1}^\lambda(k)\right)$$

where $r_j(I_{n-1})$ with $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$ and $r_j(I_{n-2})$ with $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ are arbitrary real numbers and where $r_n^\lambda(k)$ and $\theta_n^\lambda(k)$ refer respectively to the modulus and the argument of the complex number

$$\frac{iw}{\lambda + \delta\sin\lambda}\left[\frac{U_k(\lambda)}{U_n(\lambda)}e^{-i\left(\lambda+\frac{\pi\delta}{2}\right)\frac{n+1}{2}} - e^{i\left(\lambda+\frac{\pi\delta}{2}\right)\frac{n-2k-1}{2}} + \frac{U_{n-k-1}(\lambda)}{U_n(\lambda)}e^{i\left(\lambda+\frac{\pi\delta}{2}\right)\frac{n+1}{2}}\right]e^{-i\frac{\pi\delta}{2}\frac{n-2k-1}{2}}.$$

Before proving this result, we need to prove some technical lemmas.

Transformation in a linear differential equation With corollary 2, we may assume $\delta \neq 0$ (hence $\delta^2 = 1$) and $n \geq 2$. Moreover, throughout this section h refers to an even function which is compactly supported on $[-R,R]$, $\mathcal{C}^1$ on each I_k and which satisfies the differential equation (3.8) on $\cup I_k$.
Let Q_k the operator which has the following recursive definition:

$$\begin{cases} Q_1 = I \\ Q_2 = D + \frac{\delta}{2}(T^{-1} - T) \\ Q_{k+1} = DQ_k + \frac{1}{4}Q_{k-1} + \left(\frac{\delta}{2}\right)^k\left(T^{-k} + (-1)^k T^k\right) \end{cases}$$

Lemma 13 *h satisfies the following linear differential equations:*

- *On I_{n-1}, if $1 \leq k \leq n$, h satisfies:*

$$\left(\frac{i}{2}\right)^k U_k\left(\frac{1}{i}D\right)[h] = Q_k[\varphi'] + \left(\frac{\delta}{2}\right)^k T^{-k}[h]$$

- *On I_{n-2}, if $1 \leq k \leq n-1$, h satisfies:*

$$\left(\frac{i}{2}\right)^k U_k\left(\frac{1}{i}D\right)[h] = Q_k[\varphi'] + \left(\frac{\delta}{2}\right)^k T^{-k}[h]$$

- *On I_0, if $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, h satisfies:*

$$2\left(\frac{i}{2}\right)^k T_k\left(\frac{1}{i}D\right)[h] = Q_k[\varphi'] + \left(\frac{\delta}{2}\right)^k \left(T^{-k}[h] + (-1)^k T^k[h]\right)$$

Proof: We prove the first relation by induction for $k = 1, ..., n$. If $k = 1$,the differential equation (3.8) gives:

$$\text{if } u \in I_{n-1}, \;\; h'(u) = \varphi'(u) + \frac{\delta}{2}h(u-1). \tag{3.9}$$

The result comes easily from the definition of Q_1 and U_1. If $k = 2$, we derive relation (3.9) and we apply the differential equation (3.8).
We assume the result holds for k and $k-1$ with $2 \leq k \leq n-1$. Then, with recursive relation on U_k, we get:

$$\begin{aligned}
&\left(\frac{i}{2}\right)^{k+1} U_{k+1}\left(\frac{1}{i}D\right)[h] = \frac{2}{i}\left(\frac{i}{2}\right)^{k+1} DU_k\left(\frac{1}{i}D\right)[h] - \left(\frac{i}{2}\right)^{k+1} U_{k-1}\left(\frac{1}{i}D\right)[h] \\
= \;& D\left(\left(\frac{i}{2}\right)^k U_k\left(\frac{1}{i}D\right)[h]\right) + \frac{1}{4}\left(\frac{i}{2}\right)^{k-1} U_{k-1}\left(\frac{1}{i}D\right)[h] \\
= \;& DQ_k[\varphi'] + \frac{1}{4}Q_{k-1}[\varphi'] + \left(\frac{\delta}{2}\right)^k T^{-k}[h'] + \left(\frac{\delta}{2}\right)^{k+1} T^{-(k-1)}[h] \\
& \qquad \text{since } k \leq n-1 \text{ we may apply differential equation (3.8).} \\
= \;& DQ_k[\varphi'] + \frac{1}{4}Q_{k-1}[\varphi'] + \left(\frac{\delta}{2}\right)^k T^{-k}[\varphi'] + \left(\frac{\delta}{2}\right)^{k+1} T^{-(k+1)}[h]
\end{aligned}$$

Since we apply this relation with u in I_{n-1}, we have $T^{k+1}[h](u) = 0$. Therefore, we have

$$\left(\frac{i}{2}\right)^{k+1} U_{k+1}\left(\frac{1}{i}D\right)[h] = Q_{k+1}[\varphi'] + \left(\frac{\delta}{2}\right)^{k+1} T^{-(k+1)}[h] \quad \text{on } I_{n-1}.$$

We prove the other relations of this lemma in the same way.

□

Lemma 14 *Let f be a function which is $\mathcal{C}^1$ on $]-R, R[$ and with supp $f \subset [-R, R]$. Then, for all $1 \leq k \leq n$ we have*

$$Q_k[f|_{I_{n-1}}] = \left(\frac{i}{2}\right)^{k-1} \sum_{j=0}^{k-1} \left(\frac{\delta}{i}\right)^j U_{k-j-1}\left(\frac{1}{i}D\right) T^{-j}[f|_{I_{n-1}}].$$

Similarly, for all $1 \leq k \leq n-1$ we have

$$Q_k[f|_{I_{n-2}}] = \left(\frac{i}{2}\right)^{k-1} \sum_{j=0}^{k-1} \left(\frac{\delta}{i}\right)^j U_{k-j-1}\left(\frac{1}{i}D\right) T^{-j}[f|_{I_{n-2}}].$$

Proof: By induction on $k = 1, ..., n$. The result comes easily from the definition of Q_k if $k = 1$ or $k = 2$. We assume the result holds for k and $k-1$ with $2 \leq k \leq n-1$. Since $T^k[f|_{I_{n-1}}] = 0$, we have

$$\begin{aligned}
Q_{k+1} &= DQ_k + \frac{1}{4}Q_{k-1} + \left(\frac{\delta}{2}\right)^k T^{-k} \\
&= \left(\frac{i}{2}\right)^k \left(\sum_{j=0}^{k-1} \left(\frac{\delta}{i}\right)^j \frac{2D}{i} U_{k-j-1}\left(\frac{1}{i}D\right) T^{-j} - \sum_{j=0}^{k-2} \left(\frac{\delta}{i}\right)^j U_{k-j-2}\left(\frac{1}{i}D\right) T^{-j} \right) + \left(\frac{\delta}{2}\right)^k T^{-k} \\
&= \left(\frac{i}{2}\right)^k \sum_{j=0}^{k-2} \left(\frac{\delta}{i}\right)^j \left[\frac{2D}{i} U_{k-j-1}\left(\frac{1}{i}D\right) - U_{k-j-2}\left(\frac{1}{i}D\right) \right] T^{-j} + \left(\frac{i}{2}\right)^k \left(\frac{\delta}{i}\right)^{k-1} U_1\left(\frac{1}{i}D\right) T^{-(k-1)} \\
&\qquad + \left(\frac{\delta}{2}\right)^k T^{-k} \\
&= \left(\frac{i}{2}\right)^k \sum_{j=0}^{k} \left(\frac{\delta}{i}\right)^j U_{k-j}\left(\frac{1}{i}D\right) T^{-j}.
\end{aligned}$$

Similarly, we prove the other relation of this lemma.

□

Using lemmas 13 and 14, we prove that $h|_{I_{n-1}}$ and $h|_{I_{n-2}}$ satisfy linear non-homogeneous differential equations with constant coefficients. Precisely,

Corollary 3 *We have*

$$i^n U_n\left(\frac{1}{i}D\right)[h|_{I_{n-1}}] = \text{Im}\ \left[iw\delta^n e^{i\lambda(u-n)} \frac{1 - (i\delta e^{i\lambda})^n U_n(\lambda) + (i\delta e^{i\lambda})^{n+1} U_{n-1}(\lambda)}{\lambda + \delta \sin\lambda} \right]$$

and

$$i^{n-1} U_{n-1}\left(\frac{1}{i}D\right)[h|_{I_{n-2}}] = \text{Im}\ \left[iw\delta^{n-1} e^{i\lambda(u-n+1)} \frac{1 - (i\delta e^{i\lambda})^{n-1} U_{n-1}(\lambda) + (i\delta e^{i\lambda})^{n} U_{n-2}(\lambda)}{\lambda + \delta \sin\lambda} \right].$$

Proof: We have $T^{-n}[h|_{I_{n-1}}] = 0$ and $T^{-(n-1)}[h|_{I_{n-2}}] = 0$. Thus, with lemma 13, we get

$$i^n U_n\left(\frac{1}{i}D\right)[h|_{I_{n-1}}] = 2^n Q_n[\varphi'|_{I_{n-1}}] \quad \text{and} \quad i^{n-1} U_{n-1}\left(\frac{1}{i}D\right)[h|_{I_{n-2}}] = 2^{n-1} Q_{n-1}[\varphi'|_{I_{n-2}}].$$

Let $e_\lambda(u) = we^{i\lambda u}$. We have $\varphi'(u) = \text{Im}\,[e_\lambda(u)]$. Since $D^2(e_\lambda) = (i\lambda)^2 e_\lambda$, thanks to lemma 14, we write

$$Q_n(e_\lambda|_{I_{n-1}}) = \left(\frac{i}{2}\right)^{n-1} \sum_{j=0}^{n-1} \left(\frac{\delta}{i}\right)^j U_{n-j-1}(\lambda) T^{-j}(e_\lambda|_{I_{n-1}})$$

and

$$Q_{n-1}(e_\lambda|_{I_{n-2}}) = \left(\frac{i}{2}\right)^{n-2} \sum_{j=0}^{n-2} \left(\frac{\delta}{i}\right)^j U_{n-j-2}(\lambda) T^{-j}(e_\lambda|_{I_{n-2}}).$$

Due to the fact that

$$\sum_{j=0}^{n-1} U_j(X)z^j = \frac{1 - z^n U_n(X) + z^{n+1}U_{n-1}(X)}{1 - 2zX + z^2}, \tag{3.10}$$

we have

$$Q_n(e_\lambda|_{I_{n-1}}) = \frac{\delta^{n-1} i e^{-i\lambda}}{2^n} \frac{1 - (i\delta e^{i\lambda})^n U_n(\lambda) + (i\delta e^{i\lambda})^{n+1} U_{n-1}(\lambda)}{\lambda\delta + \sin\lambda} e_\lambda(u - n + 1)$$

and

$$Q_n(e_\lambda|_{I_{n-2}}) = \frac{\delta^{n-2} i e^{-i\lambda}}{2^{n-1}} \frac{1 - (i\delta e^{i\lambda})^{n-1} U_{n-1}(\lambda) + (i\delta e^{i\lambda})^n U_{n-2}(\lambda)}{\lambda\delta + \sin\lambda} e_\lambda(u - n + 2).$$

We may conclude since $Q_n(\varphi') = \mathrm{Im}(Q_n(e_\lambda))$.

□

Lemma 15 *$R \mapsto \tilde{\mathfrak{m}}_R$ is a strictly decreasing function.*

Proof: We are assuming that there exist $R_1 < R_2$ such that $\tilde{\mathfrak{m}}_{R_1} = \tilde{\mathfrak{m}}_{R_2}$. Since $R \mapsto \tilde{\mathfrak{m}}_R$ is a decreasing function, we may assume there exists an integer n_0 such that

$$\frac{n_0 - 1}{2} < R_1 < R_2 < \frac{n_0}{2}.$$

Let $\lambda = \lambda_{R_1} = \lambda_{R_2} = 2\pi\sqrt{\tilde{\mathfrak{m}}_{R_1}}$. There exists $\mathfrak{h}_{R_1}$ in H_{R_1} such that $\tilde{B}(\mathfrak{h}_{R_1}) = \tilde{m}_{R_1} = \tilde{m}_{R_2}$. Thanks to the Volterra equation (3.4), $\mathfrak{h}_{R_1}|_{I_{n_0-1}(R_2)}$ is smooth. Using corollary 3, we have:

$$i^{n_0} U_{n_0}\left(\frac{1}{i}D\right)[\mathfrak{h}_{R_1}|_{I_{n_0-1}(R_2)}](u) = \mathrm{Im}\left[i w_{R_2} \delta^{n_0} e^{i\lambda(u-n_0)} \frac{1 - (i\delta e^{i\lambda})^{n_0} U_{n_0}(\lambda) + (i\delta e^{i\lambda})^{n_0+1} U_{n_0-1}(\lambda)}{\lambda + \delta\sin\lambda}\right]$$

Since supp $\mathfrak{h}_{R_1} \subset [-R_1, R_1]$, we have $\mathfrak{h}_{R_1}|_{I_{n_0-1}(R_2)\setminus I_{n_0-1}(R_1)} = 0$. Thus, for all u in $I_{n_0-1}(R_2)$, we have

$$\mathrm{Im}\left[i w_{R_2} \delta^{n} e^{i\lambda(u-n_0)} \frac{1 - (i\delta e^{i\lambda})^{n_0} U_{n_0}(\lambda) + (i\delta e^{i\lambda})^{n_0+1} U_{n_0-1}(\lambda)}{\lambda + \delta\sin\lambda}\right] = 0.$$

Then, on $I_{n_0-1}(R_2)$, $\mathfrak{h}_{R_1}|_{I_{n_0-1}(R_2)}$ satisfies $U_{n_0}\left(\frac{1}{i}D\right)[\mathfrak{h}_{R_1}|_{I_{n_0-1}(R_2)}](u) = 0$ and $\mathfrak{h}_{R_1}(u) = 0$ on $I_{n_0-1}(R_2)\setminus I_{n_0-1}(R_1) \neq \emptyset$. Using Picard-Lindelöf theorem, we may conclude $\mathfrak{h}_{R_1}|_{I_{n_0-1}(R_2)} = 0$. Thus, supp $\mathfrak{h}_{R_1} \subset [-a_{n_0-1}(R_2), a_{n_0-1}(R_2)]$ and $\tilde{\mathfrak{m}}_{a_{n_0-1}(R_2)} = \tilde{\mathfrak{m}}_{R_2}$ with $a_{n_0-1}(R_2) = \lfloor 2R_2 \rfloor - R_2$. As a result, for all R such that $a_{n_0-1}(R_2) < R < R_2$, we have $\tilde{m}_R = \tilde{m}_{R_2}$. By induction, there exists r_k (with $r_1 = a_{n-1}(R_2)$) in $]\frac{n_0-k-1}{2}, \frac{n_0-k}{2}[$ such that for all R in $[r_k, R_2[$, we have $\tilde{m}_R = \tilde{m}_{R_2}$. As a result, $R \mapsto \tilde{m}_R$ is a constant function on $]r_{n_0-1}, 1/2[$ which contradicts corollary 2.

□

Corollary 4 *Except for at most $n-1$ values of R in $]\frac{n-1}{2}, \frac{n}{2}[$, we have $U_n(\lambda)U_{n-1}(\lambda) \neq 0$.*

Proof: $U_n(\lambda)U_{n-1}(\lambda) = 0$ if λ is one of the $n-1$ positive roots of U_nU_{n-1}. Since $R \mapsto \lambda = 2\pi\sqrt{\tilde{m}_R}$ is a strictly decreasing function, $U_n(\lambda)U_{n-1}(\lambda) \neq 0$ excepted for at most $n-1$ values of R in $]\frac{n-1}{2}, \frac{n}{2}[$.

□

Since R will tends to ν, we may assume $U_n(\lambda)U_{n-1}(\lambda) \neq 0$.

Lemma 16 *There exists* $2n-1$ *complex numbers* $z_1(I_{n-1}), ..., z_n(I_{n-1})$ *and* $z_1(I_{n-2}), ..., z_{n-1}(I_{n-2})$ *such that*

$$h|_{I_{n-1}}(u) = \text{Im}\left[\sum_{j=1}^{n} z_j(I_{n-1})e^{iu\theta_n(j)} + z_n^\lambda(0)e^{i\lambda\left(u-\frac{n-1}{2}\right)}\right]$$

$$\textit{and} \quad h|_{I_{n-2}}(u) = \text{Im}\left[\sum_{j=1}^{n-1} z_j(I_{n-2})e^{iu\theta_{n-1}(j)} + z_{n-1}^\lambda(0)e^{i\lambda\left(u-\frac{n-2}{2}\right)}\right]$$

where

$$z_n^\lambda(0) = r_n^\lambda(0)e^{i\theta_n^\lambda(0)} = \frac{iw}{(i\delta)^n U_n(\lambda)} e^{-i\lambda\frac{n+1}{2}} \frac{1-(i\delta e^{i\lambda})^n U_n(\lambda) + (i\delta e^{i\lambda})^{n+1}U_{n-1}(\lambda)}{\lambda + \delta\sin\lambda}.$$

Proof: We solve differential equations of corollary 3. The general solution of the homogeneous equation is any functions which may be written

$$u \longmapsto \text{Im}\left[\sum_{j=1}^{n} z_j(I_{n-1})e^{iu\theta_n(j)}\right].$$

Finally, with corollary 4, it is easy to check that $u \longmapsto \text{Im}\left[z_n^\lambda(0)e^{i\lambda\left(u-\frac{n-1}{2}\right)}\right]$ is a particular solution of our differential equation. Similarly, we success in obtaining an explicit expression of $h|_{I_{n-2}}$.

□

Extension of the optimal test function In lemma 16, we have an explicit expression of h only on $I_{n-2} \cup I_{n-1}$. Thanks to the differential equation with temporal shifts (3.8), we extend this explicit expression to $[-R, R]$.

Lemma 17 *With* $U_{-1} = 0$, *if* $k = 0, ..., n-1$ *then:*

$$h|_{I_{n-(2k+1)}} = \left(\frac{i}{\delta}\right)^k T^k U_k\left(\frac{1}{i}D\right)[h|_{I_{n-1}}] - \frac{2}{\delta}\sum_{j=0}^{k}\left(\frac{i}{\delta}\right)^{j-1} U_{j-1}\left(\frac{1}{i}D\right)T^j[\varphi']$$

Similarly, if $k = 0, ..., n-2$, *then:*

$$h|_{I_{n-2(k+1)}} = \left(\frac{i}{\delta}\right)^k T^k U_k\left(\frac{1}{i}D\right)[h|_{I_{n-2}}] - \frac{2}{\delta}\sum_{j=0}^{k}\left(\frac{i}{\delta}\right)^{j-1} U_{j-1}\left(\frac{1}{i}D\right)T^j[\varphi']$$

Proof: We prove the first relation by induction on $k = 0, .., n-1$. If $k = 0$, there is nothing to prove. If $k = 1$, the result comes easily from relation (3.8). Assume the result holds for $k-1$ and k with $1 \leq k \leq n-2$. Using relation (3.8), we get

$$h'|_{I_{n-(2k+1)}}(u+1) = \varphi'(u+1) + \frac{\delta}{2}\left(h|_{I_{n-(2k+3)}}(u) - h|_{I_{n-(2k-1)}}(u+2)\right).$$

Therefore, we have

$$h|_{I_{n-(2k+3)}} = T^2[h|_{I_{n-(2k-1)}}] + \frac{2}{\delta}\left(TD[h|_{I_{n-(2k+1)}}] - T[\varphi']\right).$$

The result follows easily from the induction hypothesis.

□

Lemma 18 *If* $0 \leq k \leq n-1$, *then*

$$h|_{I_{n-(2k+1)}}(u) = \mathrm{Im}\left[(i\delta)^k \sum_{j=1}^{n} z_j(I_{n-1}) U_k(\theta_n(j)) e^{i(u+k)\theta_n(j)} + z_n^\lambda(k) e^{i\lambda\left(u - \frac{n-2k-1}{2}\right)}\right]$$

and if $0 \leq k \leq n-2$, *then*

$$h|_{I_{n-2(k+1)}}(u) = \mathrm{Im}\left[(i\delta)^k \sum_{j=1}^{n-1} z_j(I_{n-2}) U_k(\theta_{n-1}(j)) e^{i(u+k)\theta_{n-1}(j)} + z_{n-1}^\lambda(k) e^{i\lambda\left(u - \frac{n-2k-2}{2}\right)}\right]$$

where

$$\begin{aligned} z_n^\lambda(k) &= r_n^\lambda(k) e^{i\theta_n^\lambda(k)} \\ &= \frac{iw}{\lambda + \delta \sin\lambda}\left[(i\delta)^{k-n} e^{-i\lambda\frac{n+1}{2}} \frac{U_k(\lambda)}{U_n(\lambda)} - e^{i\lambda\frac{n-2k-1}{2}} + (i\delta)^{k+1} e^{i\lambda\frac{n+1}{2}} \frac{U_{n-k-1}(\lambda)}{U_n(\lambda)}\right]. \end{aligned} \tag{3.11}$$

Proof: With lemmas 16 and 17, we may write

$$\begin{aligned} h|_{I_{n-(2k+1)}}(u) = \mathrm{Im}\ \Bigg[&\sum_{j=1}^{n} z_j(I_{n-1}) \left(\frac{i}{\delta}\right)^k U_k(\theta_n(j)) e^{i(u+k)\theta_n(j)} + z_n^\lambda(0) \left(\frac{i}{\delta}\right)^k U_k(\lambda) e^{i\lambda\left(u - \frac{n-2k-1}{2}\right)} \\ &- \frac{2w}{\delta} \sum_{j=0}^{k} \left(\frac{i}{\delta}\right)^{j-1} U_{j-1}\left(\frac{1}{i}D\right) e^{i\lambda(u+j)}\Bigg]. \end{aligned}$$

Thanks to relation (3.10), we get

$$\begin{aligned} \frac{2w}{\delta} \sum_{j=0}^{k} \left(\frac{i}{\delta}\right)^{j-1} U_{j-1}\left(\frac{1}{i}D\right) e^{i\lambda(u+j)} &= 2w\delta e^{i\lambda(u+1)} \sum_{j=0}^{k-1} \left(\frac{i}{\delta}\right)^{j} U_j(\lambda) e^{i\lambda j} \\ &= iwe^{i\lambda u} \frac{1 - (i\delta e^{i\lambda})^k U_k(\lambda) + (i\delta e^{i\lambda})^{k+1} U_{k-1}(\lambda)}{\lambda + \delta \sin\lambda}. \end{aligned}$$

Thus we have:

$$z_n^\lambda(0)\left(\frac{i}{\delta}\right)^k U_k(\lambda)e^{i\lambda\left(u-\frac{n-2k-1}{2}\right)} - \frac{2w}{\delta}\sum_{j=0}^{k}\left(\frac{i}{\delta}\right)^{j-1} U_{j-1}\left(\frac{1}{i}D\right)e^{i\lambda(u+j)}$$
$$= \frac{iw}{\lambda+\delta\sin\lambda}e^{i\lambda\left(u-\frac{n-2k-1}{2}\right)}\left((i\delta)^{k-n}e^{-i\lambda\frac{n+1}{2}}\frac{U_k(\lambda)}{U_n(\lambda)} + (i\delta)^{k+1}e^{i\lambda\frac{n+1}{2}}\frac{U_k(\lambda)U_{n-1}(\lambda)}{U_n(\lambda)} - e^{i\lambda\frac{n-2k-1}{2}}\right.$$
$$\left. -(i\delta)^{k+1}e^{i\lambda\frac{n+1}{2}}U_{k-1}(\lambda)\right)$$

We may conclude since, if $n \geq 2$ and $1 \leq j \leq n-1$, then

$$U_{n-1}U_j - U_nU_{j-1} = U_{n-1-j}. \tag{3.12}$$

□

Even conditions h is assumed to be even. We are exploiting this fact in order to obtain some restrictions on complex numbers z_j.

Lemma 19 *h is given lemma 18. Then h is an even function if and only if we have:*
For $1 \leq j \leq \left\lfloor\frac{n+1}{2}\right\rfloor$,

$$\overline{z_j(I_{n-1})} - z_{n+1-j}(I_{n-1}) + (i\delta)^{n-1}U_{n-1}(\theta_n(j))e^{i(n-1)\theta_n(j)}\left(z_j(I_{n-1}) - \overline{z_{n+1-j}(I_{n-1})}\right) = 0,$$

and, for $1 \leq j \leq \left\lfloor\frac{n}{2}\right\rfloor$,

$$\overline{z_j(I_{n-2})} - z_{n-j}(I_{n-2}) + (i\delta)^{n-2}U_{n-2}(\theta_{n-1}(j))e^{i(n-2)\theta_{n-1}(j)}\left(z_j(I_{n-2}) - \overline{z_{n-j}(I_{n-2})}\right) = 0.$$

Proof: First, using relation (3.8), we prove that h is even if and only if $h(u) = h(-u)$ for all $u \in I_{n-2} \cup I_{n-1}$. Then, due to the fact that $\overline{z_n^\lambda(n-1)} = -z_n^\lambda(0)$, if $u \in I_{n-1}$, we have:

$$h|_{I_{n-1}}(u) = h_{I_{-(n-1)}}(-u)$$
$$\iff \text{Im}\left[\sum_{j=1}^{n} z_j(I_{n-1})e^{iu\theta_n(j)}\right] = \text{Im}\left[(i\delta)^{n-1}\sum_{j=1}^{n} z_j(I_{n-1})U_{n-1}(\theta_n(j))e^{i(n-1)\theta_n(j)}e^{-iu\theta_n(j)}\right]$$

Since $\theta_n(n+1-k) = -\theta_n(k)$, we get

$$\text{Im}\left[\sum_{j=1}^{n} z_j(I_{n-1})e^{iu\theta_n(j)}\right] = \text{Im}\left[\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\left(z_j(I_{n-1}) - \overline{z_{n+1-j}(I_{n-1})}\right)e^{iu\theta_n(j)}\right]$$

and

$$\text{Im}\left[(i\delta)^{n-1}\sum_{j=1}^{n} z_j(I_{n-1})U_{n-1}(\theta_n(j))e^{i(n-1)\theta_n(j)}e^{-iu\theta_n(j)}\right]$$
$$= -\text{Im}\left[(-i\delta)^{n-1}\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\left(\overline{z_j(I_{n-1})} - z_{n+1-j}(I_{n-1})\right)U_{n-1}(\theta_n(j))e^{-i(n-1)\theta_n(j)}e^{iu\theta_n(j)}\right].$$

Therefore, for all $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$, we get

$$\overline{z_j(I_{n-1})} - z_{n+1-j}(I_{n-1}) + (i\delta)^{n-1} U_{n-1}(\theta_n(j)) e^{i(n-1)\theta_n(j)} \left(z_j(I_{n-1}) - \overline{z_{n+1-j}(I_{n-1})} \right) = 0.$$

□

Corollary 5 *For all $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$, there exists a real number $r_j(I_{n-1})$ (which may be negative) such that*

$$\overline{z_j(I_{n-1})} - z_{n+1-j}(I_{n-1}) = r_j(I_{n-1}) e^{i\left[\frac{n-1}{2}\left(\theta_n(j)+\frac{\pi\delta}{2}\right)+\frac{j\pi}{2}\right]}.$$

For all $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, there exists a real number $r_j(I_{n-2})$ (which may be negative) such that

$$\overline{z_j(I_{n-2})} - z_{n-j}(I_{n-2}) = r_j(I_{n-2}) e^{i\left[\frac{n-2}{2}\left(\theta_{n-1}(j)+\frac{\pi\delta}{2}\right)+\frac{j\pi}{2}\right]}.$$

Proof: We have

$$U_{n-1}(\theta_n(j)) = U_{n-1}\left(\cos\left(\frac{j\pi}{n+1}\right)\right) = (-1)^{j+1}.$$

Therefore, if $\overline{z_j(I_{n-1})} - z_{n+1-j}(I_{n-1}) = r_j e^{i\theta_j}$, the previous lemma gives

$$r_j \left(e^{i\theta_j} + (i\delta)^{n-1} (-1)^{j-1} e^{i(n-1)\theta_j} e^{-i\theta_j} \right) = 0.$$

Then $r_j = 0$ or $e^{i\theta_j} = \pm e^{i\left[\frac{n-1}{2}\left(\theta_n(j)+\frac{\pi\delta}{2}\right)+\frac{j\pi}{2}\right]}$.

□

Proof of proposition 3 From lemma 18, we have for $0 \leq k \leq n-1$:

$$h|_{I_{n-(2k+1)}}(u) = \mathrm{Im}\left[(i\delta)^k \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(z_j(I_{n-1}) - \overline{z_{n+1-j}(I_{n-1})} \right) U_k(\theta_n(j)) e^{i(u+k)\theta_n(j)} + z_n^\lambda(k) e^{i\lambda\left(u-\frac{n-2k-1}{2}\right)} \right]$$

Then, corollary 5 gives:

$$\begin{aligned} h|_{I_{n-(2k+1)}}(u) \quad = \quad & \mathrm{Im}\left[\sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} r_j(I_{n-1}) U_k(\theta_n(j)) e^{i\left[\left(u-\frac{n-2k-1}{2}\right)\theta_n(j) - \frac{\pi}{2}\left(j+\delta\frac{n-2k-1}{2}\right)\right]} \right. \\ & \left. + r_n^\lambda(k) e^{i\left[\lambda\left(u-\frac{n-2k-1}{2}\right)+\theta_n^\lambda(k)\right]} \right] \end{aligned}$$

We obtain $h|_{I_{n-(2k+2)}}$ in the same way. To conclude, one can easily check that such a function satisfies the differential equation with temporal shifts (3.8).

□

Continuity conditions

Since $\mathfrak{h}_R$ satisfies the differential equation (3.8), we have an explicit expression of $\mathfrak{h}_R$ thanks to proposition 3. However, due to the fact that $\mathfrak{h}_R$ is a continuous function, there are some restrictions on coefficients r_j which are associated to the explicit expression of $\mathfrak{h}_R$. In this subsection, we prove that, except for at most a finite number of R in $]\frac{n-1}{2}, \frac{n}{2}[$, there is one and only one continuous function which satisfies the differential equation (3.8) and we give the exact values of the corresponding coefficients r_j.

Lemma 20 *$\mathfrak{h}_R$ is a continuous function if and only if coefficients $r_j(I_{n-1})$ and $r_j(I_{n-2})$ are satisfying the following linear system: for $0 \leq k \leq n-1$,*

$$\begin{aligned}
&\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} r_j(I_{n-2}) U_k\left(\theta_{n-1}(j)\right) \sin\left(\left[a_{n-1} - \frac{n-2}{2}\right]\theta_{n-1}(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-2}{2}\right]\right) \\
&\quad - \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} r_j(I_{n-1}) U_k\left(\theta_n(j)\right) \sin\left(\left[a_{n-1} - \frac{n-1}{2}\right]\theta_n(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-1}{2}\right]\right) \\
&\qquad = r_n^\lambda(k) \sin\left(\lambda\left[a_{n-1} - \frac{n-1}{2}\right] + \theta_n^\lambda(k)\right) - r_{n-1}^\lambda(k) \sin\left(\lambda\left[a_{n-1} - \frac{n-2}{2}\right] + \theta_{n-1}^\lambda(k)\right) \\
&\qquad\qquad = r_\lambda U_k(\lambda) \sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right)
\end{aligned}$$

where r_λ and θ_λ refer to the modulus and the argument of the complex number:

$$z_\lambda = \frac{-2iwe^{-i\lambda a_{n-1}}}{U_n(\lambda)U_{n-1}(\lambda)} \frac{1 - (i\delta e^{i\lambda})^n U_n(\lambda) + (i\delta e^{i\lambda})^{n+1} U_{n-1}(\lambda)}{-2i\delta e^{i\lambda}(\lambda + \delta \sin\lambda)} = \frac{-2iwe^{-i\lambda a_{n-1}}}{U_n(\lambda)U_{n-1}(\lambda)} \sum_{k=0}^{n-1} (i\delta e^{i\lambda})^k U_k(\lambda)$$

Proof: We use the fact that $\mathfrak{h}_R$ is a continuous function on $\inf I_j$ for $j = -(n-1)...n-1$. Precisely, h_R is continuous on $a_{n-(2k+1)} > 0$ (ie $a_{n-1}, a_{n-3}...$) then

$$\text{for } 0 \leq k \leq \left\lfloor \frac{n-2}{2} \right\rfloor, \qquad \lim_{u \to a^-_{n-(2k+1)}} \mathfrak{h}_R|_{I_{n-2(k+1)}}(u) = \lim_{u \to a^+_{n-(2k+1)}} \mathfrak{h}_R|_{I_{n-(2k+1)}}(u).$$

Moreover, h_R is continuous on $a_{n-2(k+1)} < 0$ (ie $a_{-n}, a_{-(n-2)}, ...$) then

$$\text{for } \left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n-1, \qquad \lim_{u \to a^-_{n-2(k+1)}} \mathfrak{h}_R|_{I_{n-2(k+1)}}(u) = \lim_{u \to a^+_{n-2(k+1)}} \mathfrak{h}_R|_{I_{n-(2k+1)}}(u).$$

Therefore, for $0 \leq k \leq n-1$, we have

$$\begin{aligned}
&\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} r_j(I_{n-2}) U_k\left(\theta_{n-1}(j)\right) \sin\left(\left[a_{n-1} - \frac{n-2}{2}\right]\theta_{n-1}(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-2}{2}\right]\right) \\
&\quad - \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} r_j(I_{n-1}) U_k\left(\theta_n(j)\right) \sin\left(\left[a_{n-1} - \frac{n-1}{2}\right]\theta_n(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-1}{2}\right]\right) \\
&\qquad = r_n^\lambda(k) \sin\left(\lambda\left[a_{n-1} - \frac{n-1}{2}\right] + \theta_n^\lambda(k)\right) - r_{n-1}^\lambda(k) \sin\left(\lambda\left[a_{n-1} - \frac{n-2}{2}\right] + \theta_{n-1}^\lambda(k)\right).
\end{aligned}$$

Reciprocally, if $\mathfrak{h}_R$ is continuous on $\inf I_j$, since $\mathfrak{h}_R$ is even and $-\inf I_j = \sup I_{-j}$, then $\mathfrak{h}_R$ is a continuous function.

□

We consider the real matrix M_R which is associated to this linear system and $\det M_R$ refers to the determinant of M_R. Precisely, let $M_R = (m_{k,j})_{\substack{k=0,\dots,n-1 \\ j=1,\dots,n}}$ defined by:

$$\begin{cases} m_{k,j} = U_k\left(\theta_{n-1}(j)\right)\sin\left(\left[a_{n-1} - \frac{n-2}{2}\right]\theta_{n-1}(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-2}{2}\right]\right) & \text{if } 1 \le j \le \lfloor\frac{n}{2}\rfloor \\ m_{k,j+\lfloor\frac{n}{2}\rfloor} = U_k\left(\theta_n(j)\right)\sin\left(\left[a_{n-1} - \frac{n-1}{2}\right]\theta_n(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-1}{2}\right]\right) & \text{if } 1 \le j \le \lfloor\frac{n+1}{2}\rfloor \end{cases}$$

Therefore, the linear system of lemma 20 is

$$M_R \cdot \begin{pmatrix} r_1(I_{n-2}) \\ \vdots \\ r_{\lfloor\frac{n}{2}\rfloor}(I_{n-2}) \\ -r_1(I_{n-1}) \\ \vdots \\ -r_{\lfloor\frac{n+1}{2}\rfloor}(I_{n-1)}) \end{pmatrix} = r_\lambda \begin{pmatrix} U_0\left(\lambda\right)\sin\left(\theta_\lambda\right) \\ \vdots \\ U_k\left(\lambda\right)\sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right) \\ \vdots \\ U_{n-1}\left(\lambda\right)\sin\left(\theta_\lambda - (n-1)\delta\frac{\pi}{2}\right) \end{pmatrix}.$$

Lemma 21 *Except for a finite number of values of R in $]\frac{n-1}{2}; \frac{n}{2}[$, the matrix M_R is invertible.*

Proof: We may decompose Chebychev polynomials as

$$U_{2k}(X) = \sum_{j=0}^{k} a_{2k,j}X^{2j},\ U_{2k+1}(X) = \sum_{j=0}^{k} a_{2k+1,j}X^{2j+1} \text{ with } a_{2k,k} = 2^{2k} \text{ and } a_{2k+1,k} = 2^{2k+1}. \quad (3.13)$$

We consider N_R the real matrix which is defined by $N_R = (n_{k,j})$ such that, for $0 \le k \le n-1$:

$$\begin{cases} n_{k,j} = \theta_{n-1}(j)^k\sin\left(\left[a_{n-1} - \frac{n-2}{2}\right]\theta_{n-1}(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-2}{2}\right]\right) & \text{if } 1 \le j \le \lfloor\frac{n}{2}\rfloor \\ n_{k,j+\lfloor\frac{n}{2}\rfloor} = \theta_n(j)^k\sin\left(\left[a_{n-1} - \frac{n-1}{2}\right]\theta_n(j) - \frac{\pi}{2}\left[j + \delta\frac{n-2k-1}{2}\right]\right) & \text{if } 1 \le j \le \lfloor\frac{n+1}{2}\rfloor \end{cases}$$

We consider rows of M_R and N_R. Precisely, let

$$M_R = \begin{pmatrix} L_0 \\ \vdots \\ L_{n-1} \end{pmatrix} \quad \text{and} \quad N_R = \begin{pmatrix} \tilde{L}_0 \\ \vdots \\ \tilde{L}_{n-1} \end{pmatrix}.$$

Thanks to relation (3.13), for $0 \le k \le \lfloor\frac{n-1}{2}\rfloor$, we have

$$L_{2k} = \sum_{j=0}^{k} a_{2k,j}(-1)^{k-i}\tilde{L}_{2j}, \text{ and for } 0 \le k \le \lfloor\frac{n-2}{2}\rfloor,\ L_{2k+1} = \sum_{j=0}^{k} a_{2k+1,j}(-1)^{k-i}\tilde{L}_{2j+1}.$$

As a result, we have

$$\det M_R = \left(\prod_{j=0}^{n-1} 2^k\right)\det N_R = 2^{\frac{n(n-1)}{2}}\det N_R.$$

In addition, we may write $N_R = (n_{k,j})$ with:

$$\begin{cases} n_{k,j} = \frac{1}{2i}\left([i\delta\theta_{n-1}(j)]^k e^{i(R\theta_{n-1}(j)+\Phi_j)} - [-i\delta\theta_{n-1}(j)]^k e^{-i(R\theta_{n-1}(j)+\Phi_j)}\right) & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \\ n_{k,j+\lfloor \frac{n}{2} \rfloor} = \frac{1}{2i}\left([i\delta\theta_n(j)]^k e^{i(R\theta_n(j)+\tilde{\Phi}_j)} - [-i\delta\theta_n(j)]^k e^{-i(R\theta_n(j)+\tilde{\Phi}_j)}\right) & \text{if } 1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor \end{cases}$$

Due to the multilinearity of determinant, we have

$$\det N_R = \frac{(i\delta)^{\frac{n(n-1)}{2}}}{(2i)^n} \sum_{\epsilon=(\epsilon',\epsilon'')} V[\epsilon',\epsilon''](-1)^{\sigma(\epsilon)} e^{i\left[\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \epsilon'(j)(R\theta_{n-1}(j)+\Phi_j)+\sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \epsilon''(j)\left(R\theta_n(j)-\tilde{\Phi}\right)\right]}$$

where the sum is running over functions $\epsilon : \{1,..,n\} \longmapsto \{-1,+1\}$ which may split in $\epsilon = (\epsilon',\epsilon'')$ where $\epsilon(j) = \epsilon'(j)$ if $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $\epsilon(j + \lfloor \frac{n}{2} \rfloor) = \epsilon''(j)$ if $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$. In addition, $\sigma(\epsilon) = |\{1 \leq j \leq n;\, \epsilon(j) = -1\}|$. Finally, $V[\epsilon',\epsilon'']$ refers to the determinant of the Vandermonde matrix associated to real numbers $(\epsilon'(1)\theta_{n-1}(1),..,\epsilon'(\lfloor \frac{n}{2} \rfloor)\theta_{n-1}(\lfloor \frac{n}{2} \rfloor),\epsilon''(1)\theta_n(1),..,\epsilon''(\lfloor \frac{n+1}{2} \rfloor)\theta_n(\lfloor \frac{n+1}{2} \rfloor))$. Since $V[\epsilon',\epsilon''] \neq 0$ for all ϵ, we may conclude $R \longmapsto \det N_R$ vanishes at most a finite number of times on $]\frac{n-1}{2};\frac{n}{2}[$. The same result holds for $R \longmapsto \det M_R$ since $\det M_R = 2^{\frac{n(n-1)}{2}} \det N_R$.

□

Corollary 6 *For all R in $]\frac{n-1}{2};\frac{n}{2}[$ such that $\det M_R \neq 0$, there exists one and only one continuous function satisfying the differential equation with temporal shifts* (3.8)*. Moreover:*

- *If $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$,*

$$r_j(I_{n-2}) = r_\lambda \sum_{k=0}^{n-1} U_k(\lambda) \sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right) \Delta_{k+1,j}. \tag{3.14}$$

- *If $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$,*

$$r_j(I_{n-1}) = -r_\lambda \sum_{k=0}^{n-1} U_k(\lambda) \sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right) \Delta_{k+1,j+\lfloor \frac{n}{2} \rfloor}. \tag{3.15}$$

with $\Delta_{k,j} = \frac{(-1)^{k+j}}{\det M_R} M_{k,j}$ where $M_{k,j}$ denotes the minor of M_R obtained by removing from M_R its k-th row and j-th column.

3.5 Exact value of the minimum

In this section, we finish proving theorem 1. Remember that some cases have been solved in corollary 2.

Thanks to the previous section, we have an explicit expression of the optimal test function $\mathfrak{h}_R$. Nevertheless, this explicit expression depends on the unknown parameter λ which is related to $\tilde{\mathfrak{m}}_R$ by the following relation:

$$\lambda^2 = 4\pi^2 \tilde{\mathfrak{m}}_R$$

In order to conclude, we solve the equation $\tilde{\mathfrak{m}}_R = \tilde{B}(\mathfrak{h}_R)$ where $\tilde{\mathfrak{m}}_R$ is the only unknown parameter.

Two equations, one unknown parameter It is technically easier to express λ instead of $\tilde{\mathfrak{m}}_R$. Thanks to relation (3.2), the relation $\tilde{\mathfrak{m}}_R = \tilde{B}(\mathfrak{h}_R)$ may be written:

$$\lambda^2 = \frac{\int_{\mathbb{R}} \mathfrak{h}'_R(u)^2 du - \frac{\delta}{2}\int_{-1}^{1} \mathfrak{h}'_R * \mathfrak{h}'_R(u) du}{\int_{\mathbb{R}} \mathfrak{h}_R(u)^2 du + \frac{\delta}{2}\int_{-1}^{1} \mathfrak{h}_R * \mathfrak{h}_R(u) du + \varepsilon \left(\int_{\mathbb{R}} \mathfrak{h}_R(u) du\right)^2} \tag{3.16}$$

Furthermore, the relation (3.5) may be written:

$$\frac{w}{\lambda}\cos\lambda R + \frac{\delta}{2}\int_{R-1}^{R} \mathfrak{h}_R(x)dx + \varepsilon \int_{-R}^{R} \mathfrak{h}_R(x)dx = 0 \tag{3.17}$$

As a result, λ is satisfying two equations.

Lemma 22 *Relation (3.17) implies relation (3.16).*

Proof: The differential equation with temporal shifts give

$$\int_{\mathbb{R}} \mathfrak{h}'_R(u)^2 du - \frac{\delta}{2}\int_{-1}^{1} \mathfrak{h}'_R * \mathfrak{h}'_R(u) du = \int_{\mathbb{R}} \mathfrak{h}'_R(u)\left[\mathfrak{h}'_R(u) + \frac{\delta}{2}\int_{u-1}^{u+1} \mathfrak{h}'_R(t)dt\right] du = \int_{\mathbb{R}} \mathfrak{h}'_R(u)\varphi'(u)du.$$

Thanks to the Volterra equation we have

$$\int_{\mathbb{R}} \mathfrak{h}_R(u)^2 du + \frac{\delta}{2}\int_{-1}^{1} \mathfrak{h}_R * \mathfrak{h}_R(u) du = \int_{\mathbb{R}} \mathfrak{h}_R(u)\varphi(u)du + \frac{\delta}{2}\left(\int_{\mathbb{R}} \mathfrak{h}_R(u)du\right)\left(\int_{R-1}^{R} \mathfrak{h}_R(u)du\right).$$

Finally, using an integration by parts, we get

$$\int_{\mathbb{R}} \mathfrak{h}'_R(u)\varphi'(u)du - \lambda^2 \int_{\mathbb{R}} \mathfrak{h}_R(u)\varphi(u)du = -w\lambda\cos\lambda R \int_{\mathbb{R}} \mathfrak{h}_R(u).$$

Therefore, we may write equation (3.16) as

$$-\lambda^2\left(\frac{w}{\lambda}\cos\lambda R + \frac{\delta}{2}\int_{R-1}^{R} \mathfrak{h}_R(x)dx + \varepsilon\int_{-R}^{R} \mathfrak{h}_R(x)dx\right)\int_{\mathbb{R}} \mathfrak{h}_R(u) = 0.$$

□

As a result, we use relation (3.17) in order to determine λ.

End of the proof of theorem 1 In this subsection, we finish proving theorem 1. Precisely,

Proposition 4 *If $G = SO^+$, SO^- or Sp with $n \geq 2$, then $\lambda_R := \lambda$ is the smallest positive root of*

$$\frac{\delta}{\lambda}\cos\theta_\lambda - \sum_{k=0}^{n-1} U_k(\lambda)\sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right)\left[\frac{\delta\alpha_R(k)}{2} - 1 + \varepsilon\beta_R(k)\right] + \frac{2\varepsilon}{\lambda}\sum_{k=0}^{n-1} U_k(\lambda)\cos\left(\theta_\lambda - k\delta\frac{\pi}{2}\right) = 0 \tag{3.18}$$

which is not a root of U_nU_{n-1} *and where* $\alpha_R(k)$ *and* $\beta_R(k)$ *are defined by:*

$$\begin{aligned}\alpha_R(k) &= 2\sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \frac{\sin\left[\left(R-\frac{n}{2}\right)\theta_{n-1}(j)\right]\sin\left[\frac{\pi}{2}\left(j+\delta\frac{n-2}{2}\right)\right]}{\theta_{n-1}(j)}\Delta_{k+1,j} \\ &\qquad +2\sum_{j=1}^{\lfloor \frac{n+1}{2}\rfloor} \frac{\sin\left[\left(R-\frac{n-1}{2}\right)\theta_{n}(j)\right]\sin\left[\frac{\pi}{2}\left(j+\delta\frac{n-1}{2}\right)\right]}{\theta_{n}(j)}\Delta_{k+1,j+\lfloor \frac{n}{2}\rfloor}\end{aligned}$$

and

$$\begin{aligned}\beta_R(k) &= 2\sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \frac{\sin\left[\left(R-\frac{n}{2}\right)\theta_{n-1}(j)\right]}{\theta_{n-1}(j)}\Delta_{k+1,j}\sum_{l=0}^{n-2} U_l(\theta_{n-1}(j)\sin\left[\frac{\pi}{2}\left(j+\delta\frac{n-2l-2}{2}\right)\right] \\ &\qquad +2\sum_{j=1}^{\lfloor \frac{n+1}{2}\rfloor} \frac{\sin\left[\left(R-\frac{n-1}{2}\right)\theta_{n}(j)\right]}{\theta_{n}(j)}\Delta_{k+1,j+\lfloor \frac{n}{2}\rfloor}\sum_{l=0}^{n-1} U_l(\theta_n(j))\sin\left[\frac{\pi}{2}\left(j+\delta\frac{n-2l-1}{2}\right)\right].\end{aligned}$$

Before proving this result, we need to prove some technical lemmas. We may write $\mathfrak{h}_R$ as a sum of two non-continuous even functions. Let φ_λ and ψ which are defined on $\mathbb{R}\backslash\{a_{-n},...,a_n\}$ by:

- If $0\leq k\leq n-1$, let

$$\varphi_\lambda|_{I_{n-(2k+1)}}(u)=r_n^\lambda(k)\sin\left(\lambda\left[u-\frac{n-2k-1}{2}\right]+\theta_n^\lambda(k)\right).$$

- If $0\leq k\leq n-2$, let

$$\varphi_\lambda|_{I_{n-2(k+1)}}(u)=r_{n-1}^\lambda(k)\sin\left(\lambda\left[u-\frac{n-2k-2}{2}\right]+\theta_{n-1}^\lambda(k)\right).$$

- supp $\varphi_\lambda\subset[-R,R]$.

- If $u\in\mathbb{R}\backslash\{a_{-n},...,a_n\}$, let

$$\psi(u)=\mathfrak{h}_R(u)-\varphi_\lambda(u).$$

Even though φ_λ and ψ are not continuous functions, they are smooths on each I_k. Furthermore, they satisfy

$$\mathfrak{h}_R(u)=\psi(u)+\varphi_\lambda(u)\qquad\text{and}\qquad\varphi_\lambda''=-\lambda^2\varphi_\lambda.$$

Lemma 23 *We have*

$$\begin{aligned}\int_{R-1}^{R}\psi(u)du &= r_\lambda\sum_{k=0}^{n-1}U_k(\lambda)\sin\left(\theta_\lambda-k\delta\frac{\pi}{2}\right)\alpha_R(k) \\ \text{and}\quad \int_{-R}^{R}\psi(u)du &= r_\lambda\sum_{k=0}^{n-1}U_k(\lambda)\sin\left(\theta_\lambda-k\delta\frac{\pi}{2}\right)\beta_R(k).\end{aligned}$$

Proof: We may write:

$$\begin{aligned}\int_{R-1}^{R}\psi(u)du &= \int_{I_{n-1}}\psi|_{I_{n-1}}(u)du+\int_{I_{n-2}}\psi|_{I_{n-2}}(u)du\\ &= 2\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}r_j(I_{n-2})\frac{\sin\left[\left(R-\frac{n}{2}\right)\theta_{n-1}(j)\right]}{\theta_{n-1}(j)}\sin\left[\frac{\pi}{2}\left(j+\delta\frac{n-2}{2}\right)\right]\\ &\qquad -2\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}r_j(I_{n-1})\frac{\sin\left[\left(R-\frac{n-1}{2}\right)\theta_n(j)\right]}{\theta_n(j)}\sin\left[\frac{\pi}{2}\left(j+\delta\frac{n-1}{2}\right)\right]\end{aligned}$$

The result comes easily from relations (3.14) and (3.15) of corollary 6. Similarly, since

$$\int_{-R}^{R}\psi(u)du=\sum_{k=0}^{n-1}\int_{I_{n-1}}\psi|_{I_{n-2k-1}}(u-k)du+\sum_{k=0}^{n-2}\int_{I_{n-2}}\psi|_{I_{n-2k-2}}(u-k)du$$

we obtain the second part of this lemma.

□

Lemma 24 *For $0\leq k\leq n-2$, we have*

$$\lim_{u\to\sup I_{n-2k-2}}\varphi'_\lambda|_{I_{n-2k-2}}(u)-\lim_{u\to\inf I_{n-2k-1}}\varphi'_\lambda|_{I_{n-2k-1}}(u)=\lambda U_k(\lambda)\mathrm{Re}\ \left((i\delta)^{-k}z_\lambda\right)$$

and for $1\leq k\leq n-1$,

$$\lim_{u\to\sup I_{n-2k-1}}\varphi'_\lambda|_{I_{n-2k-1}}(u)-\lim_{u\to\inf I_{n-2k}}\varphi'_\lambda|_{I_{n-2k}}(u)=\lambda U_{n-k-1}(\lambda)\mathrm{Re}\ \left((i\delta)^{-(n-k-1)}z_\lambda\right)$$

where the complex number z_λ is defined in lemma 20.

Proof: Since $\mathfrak{h}_R$ satisfies the differential equation with temporal shifts (3.8), we may conclude $\mathfrak{h}'_R$ is continuous on $]-R,R[$. Therefore, for $0\leq k\leq n-2$, we get

$$\begin{aligned}&\lim_{u\to\sup I_{n-2k-2}}\varphi'_\lambda|_{I_{n-2k-2}}(u)-\lim_{u\to\inf I_{n-2k-1}}\varphi'_\lambda|_{I_{n-2k-1}}(u)\\ &= \lambda\mathrm{Re}\ \left[e^{i\lambda\left(a_{n-1}-\frac{n-2}{2}\right)}z^\lambda_{n-1}(k)-e^{i\lambda\left(a_{n-1}-\frac{n-1}{2}\right)}z^\lambda_n(k)\right]\end{aligned}$$

and, for $1\leq k\leq n-1$,

$$\lim_{u\to\sup I_{n-2k-1}}\varphi'_\lambda|_{I_{n-2k-1}}(u)-\lim_{u\to\inf I_{n-2k}}\varphi'_\lambda|_{I_{n-2k}}(u)=\lambda\mathrm{Re}\ \left[e^{i\lambda\left(a_n-\frac{n-1}{2}\right)}z^\lambda_n(k)-e^{i\lambda\left(a_n-\frac{n}{2}\right)}z^\lambda_{n-1}(k-1)\right].$$

The result comes easily from relations (3.11) and (3.12).

□

Lemma 25 *We have*

$$\begin{aligned}\int_{\mathbb{R}}\varphi_\lambda(u)du &= -\frac{2r_\lambda}{\lambda}\sum_{k=0}^{n-1}U_k(\lambda)\cos\left(\theta_\lambda-k\delta\frac{\pi}{2}\right)\\ \textit{and}\quad \int_{R-1}^{R}\varphi_\lambda(u)du &= -\frac{2w}{\delta\lambda}\cos\lambda R-\frac{2}{\lambda}r_\lambda\cos\theta_\lambda+2\delta\mathrm{Re}\ \left[iz_\lambda\sum_{k=0}^{n-1}(-i\delta)^kU_k(\lambda)\right].\end{aligned}$$

Proof: We may write:

$$\begin{aligned}\int_{\mathbb{R}} \varphi_\lambda(u)du &= -\frac{1}{\lambda^2}\int_{\mathbb{R}} \varphi_\lambda''(u)du = -\frac{1}{\lambda^2}\sum_{k=-(n-1)}^{n-1} [\varphi_\lambda'|_{I_k}]_{\inf I_k}^{\sup I_k} \\ &= -\frac{1}{\lambda^2}\left[2\lim_{u\to\sup I_{n-1}} \varphi_\lambda'|_{I_{n-1}}(u) + \sum_{k=-(n-1)}^{n-2} \lim_{u\to\sup I_k} \varphi_\lambda'|_{I_k}(u) - \lim_{u\to\inf I_{k+1}} \varphi_\lambda'|_{I_{k+1}}(u)\right]\end{aligned}$$

Since $z_n^\lambda(0) = (i\delta)^{-(n-1)}e^{-i\lambda\left(a_n-\frac{n-1}{2}\right)}U_{n-1}(\lambda)z_\lambda$, we get

$$\lim_{u\to\sup I_{n-1}} \varphi_\lambda'|_{I_{n-1}}(u) = -\lambda \mathrm{Re}\ \left(z_\lambda(i\delta)^{-(n-1)}U_{n-1}(\lambda)\right)$$

and the previous lemma gives

$$\int_{\mathbb{R}} \varphi_\lambda(u)du = \frac{-2}{\lambda}\sum_{k=0}^{n-1} U_k(\lambda)\mathrm{Re}\ \left((i\delta)^{-k}z_\lambda\right).$$

Similarly, we have:

$$\begin{aligned}&\int_{R-1}^{R} \varphi_\lambda(u)du \\ &= \frac{-1}{\lambda^2}\left[\lim_{u\to\sup I_{n-1}} \varphi'|_{I_{n-1}}(u) + \lim_{u\to\sup I_{n-2}} \varphi'|_{I_{n-2}}(u) - \lim_{u\to\inf I_{n-1}} \varphi'|_{I_{n-1}}(u) - \lim_{u\to\inf I_{n-2}} \varphi'|_{I_{n-2}}(u)\right]\end{aligned}$$

The previous lemma gives

$$\lim_{u\to\sup I_{n-2}} \varphi'|_{I_{n-2}}(u) - \lim_{u\to\inf I_{n-1}} \varphi'|_{I_{n-1}}(u) = \lambda \mathrm{Re}\ (z_\lambda).$$

In addition, since $z_{n-1}^\lambda(0) = i\delta\frac{U_n(\lambda)}{U_{n-1}(\lambda)}e^{i\lambda/2}z_n^\lambda(0) - 2w\delta\lambda\mathrm{Re}\ (e^{i\lambda R})$, we get

$$\lim_{u\to\inf I_{n-2}} \varphi_\lambda'|_{I_{n-2}}(u) = -2w\delta\lambda\cos(\lambda R) - \lambda\mathrm{Re}\ \left[U_n(\lambda)z_\lambda(i\delta)^{-n}\right].$$

Therefore, we may write

$$\int_{R-1}^{R} \varphi_\lambda(u)du = -\frac{2w}{\delta\lambda}\cos\lambda R - \frac{2}{\lambda}r_\lambda\cos\theta_\lambda + \frac{1}{\lambda}\mathrm{Re}\ \left[z_\lambda(1-(i\delta)^{-n}U_n(\lambda) + (i\delta)^{-(n+1)}U_{n-1}(\lambda))\right].$$

The result comes easily from relation (3.10).

□

To conclude, since $\mathfrak{h}_R = \psi + \varphi_\lambda$, we may easily transform equation (3.17) thanks to both lemmas 23 and 25. Proposition 4 follows immediately due to the fact that $r_\lambda \neq 0$ (otherwise $\mathfrak{h}_R = 0$).

Particular case In this subsection, we are assuming $n = 2$ and we give a simpler expression of equation (3.18) than in proposition 4.

Corollary 7 *If $G = SO^+$, SO^- or Sp and $\frac{1}{2} < R < 1$ then λ_R is the smallest positive root of*

$$(\delta + 2\varepsilon)\frac{1-4\lambda^2}{\lambda}\left(\sin\lambda(1-R) - 2\delta\lambda\cos\lambda R\right)$$
$$-\left[(\delta+2\varepsilon)(1-R) - 1 + 4\varepsilon\right]\left[\cos\lambda(1-R) - 2\delta\lambda\sin\lambda R - 2\lambda\tan\Theta_R(\sin\lambda(1-R) - 2\delta\lambda\cos\lambda R)\right] = 0$$

which is not a root of U_1U_2 and where

$$\Theta_R = \frac{1}{2}\left(R - \frac{1}{2}\right) + \frac{\pi}{2}\left(1 + \frac{\delta}{2}\right).$$

Proof: The matrix M_R which appears in corollary 6 can be written

$$M_R = \begin{pmatrix} -1 & \sin\left[\frac{1}{2}\left(\frac{1}{2} - R\right) - \frac{\pi}{2}\left(1 + \frac{\delta}{2}\right)\right] \\ 0 & \sin\left[\frac{1}{2}\left(\frac{1}{2} - R\right) - \frac{\pi}{2}\left(1 - \frac{\delta}{2}\right)\right] \end{pmatrix}.$$

Therefore,

$$\det M_R = -\delta\cos\Theta_R \neq 0.$$

We get $\Delta_{1,1} = -1$, $\Delta_{1,2} = 0$, $\Delta_{2,1} = -\delta\tan\Theta_R$ and $\Delta_{2,2} = \frac{\delta}{\cos\Theta_R}$. In addition, if $k \in \{1,2\}$, we have:

$$\begin{aligned} \alpha_R(k) &= 2(R-1)\Delta_{k+1,1} - 2\left(\cos\Theta_R + \delta\sin\Theta_R\right)\Delta_{k+1,2} \\ \beta_R(k) &= \alpha_R(k) + 2\left(\cos\Theta_R + \delta\sin\Theta_R\right)\Delta_{k+1,2} \end{aligned}$$

Therefore:

$$\begin{cases} \alpha_R(0) = \beta_R(0) = 2(1-R) \\ \alpha_R(1) = 2\delta(1-R) - 2(\delta + \tan\Theta_R) \\ \beta_R(1) = 2\delta(1-R) - 4(\delta + \tan\Theta_R) \end{cases}$$

As a result, since $\delta\varepsilon = -\varepsilon$, we get

$$\begin{aligned} & \frac{\delta}{\lambda}\cos\theta_\lambda - \sum_{k=0}^{n-1} U_k(\lambda)\sin\left(\theta_\lambda - k\delta\frac{\pi}{2}\right)\left[\frac{\delta\alpha_R(k)}{2} - 1 + \varepsilon\beta_R(k)\right] + \frac{2\varepsilon}{\lambda}\sum_{k=0}^{n-1} U_k(\lambda)\cos\left(\theta_\lambda - k\delta\frac{\pi}{2}\right) \\ = \;& \frac{\delta+2\varepsilon}{\lambda}\cos\theta_\lambda - \left(\frac{\delta\alpha_R(0)}{2} + \varepsilon\beta_R(0) - 1 + 4\varepsilon\right)\sin\theta_\lambda + 2\lambda\delta\left(\frac{\delta\alpha_R(1)}{2} + \varepsilon\beta_R(1) - 1\right)\cos\theta_\lambda \\ = \;& (\delta+2\varepsilon)\frac{1-4\lambda^2}{\lambda}\cos\theta_\lambda - \left[(\delta+2\varepsilon)(1-R) - 1 + 4\varepsilon\right]\left[\sin\theta_\lambda - 2\lambda\tan(\Theta_R)\cos\theta_\lambda\right]. \end{aligned}$$

In addition, we have

$$\begin{aligned} z_\lambda &= \frac{-2iwe^{-i\lambda(1-R)}}{U_1(\lambda)U_2(\lambda)}\left(1 + i\delta U_1(\lambda)e^{i\lambda}\right) \\ &= \frac{-2w}{U_1(\lambda)U_2(\lambda)}\left(\sin\lambda(1-R) - 2\delta\lambda\cos\lambda R + i\left[\cos\lambda(1-R) - 2\delta\lambda\sin\lambda R\right]\right). \end{aligned}$$

The result comes easily from these relations.

□

Comments on the "Sp hypothesis" In the symplectic case with $R > 1/2$, our proof of theorem 1 is submitted to the "Sp hypothesis". However, we prove that this theorem is still true even though $\mathfrak{m}_R^2$ is an odd positive integer. Therefore, throughout this paragraph, we are assuming $G = Sp$, $R > 1/2$ and $m_R = N^2$ where N is an odd integer.

Lemma 26 *$\mathfrak{h}_R$ satisfies $\int_{R-1}^{R} \mathfrak{h}_R(u)du = 0$ and, for all $0 \leq u \leq R$,*

$$\mathfrak{h}_R(u) = \varphi(u) + \frac{\delta}{2}\int_u^R \mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)dt$$

with

$$\varphi(u) = \left(\frac{-4}{N\pi}\int_0^R \left[\mathfrak{h}_R'(t) - \frac{\delta}{2}\left(\mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)\right)\right]\sin\left(\frac{\pi Nt}{2R}\right)dt\right)\left(\cos\left(\lambda u\right) - \cos(\lambda R)\right)\cdot \mathbb{1}_{[-R,R]}(u).$$

Proof: In lemma 12, we prove that for all odd positive integer n,

$$(n^2 - \mathfrak{m}_R)\mathfrak{c}_n = \frac{(-1)^{\frac{n-1}{2}}k_{\delta,\varepsilon}}{n} - (n^2 - \mathfrak{m}_R)\frac{\delta}{n\pi}(S_n * \mathfrak{h}_R)(1).$$

First, with $n = N$, we may deduce $k_{\delta,\varepsilon} = \int_{R-1}^{R} \mathfrak{h}_R(u)du = 0$. Second, for all $n \neq N$, we get

$$\mathfrak{c}_n = -\frac{\delta}{n\pi}(S_n * \mathfrak{h}_R)(1).$$

Since $\mathfrak{h}_R(u) = 2\sum_{n\geq 0}\mathfrak{c}_n\cos\left(\frac{\pi nu}{2R}\right)\cdot\mathbb{1}_{[-R,R]}(u)$, for all u in $[0,R]$, we write

$$\mathfrak{h}_R(u) = 2\left[\mathfrak{c}_N + \frac{\delta}{N\pi}S_N * \mathfrak{h}_R(1)\right]\cos\left(\frac{\pi Nu}{2R}\right) - \frac{2\delta}{\pi}{\sum_{n\geq 0}}^{*}\frac{S_n * \mathfrak{h}_R(1)}{n}\cos\left(\frac{\pi nu}{2R}\right).$$

The sum in the right member of this equality has been computed in the proof of lemma 10 and we have

$$\mathfrak{c}_N + \frac{\delta}{N\pi}S_N * \mathfrak{h}_R(1) = \frac{-2}{N\pi}\int_0^R \left[\mathfrak{h}_R'(t) - \frac{\delta}{2}\left(\mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)\right)\right]\sin\left(\frac{\pi Nt}{2R}\right)dt.$$

□

Therefore, changing w with

$$w = \frac{4\lambda}{N\pi}\int_0^R \left[\mathfrak{h}_R'(t) - \frac{\delta}{2}\left(\mathfrak{h}_R(t+1) - \mathfrak{h}_R(t-1)\right)\right]\sin\left(\frac{\pi Nt}{2R}\right)dt,$$

this Volterra equation with temporal shift has been solved in section 3.4. Thus, we get an explicit expression of $\mathfrak{h}_R$. Now, several cases may occur. First, this explicit expression of $\mathfrak{h}_R$ doesn't satisfy the compatibility equation $\int_{R-1}^{R}\mathfrak{h}_R(u)du = 0$, then the "$Sp$ hypothesis" is true. Second, if h_R satisfies the compatibility equation, then on account of the fact that the argument of the complex number z_λ is independent of w, λ is a root of equation (3.18). Since $\mathfrak{m}_R$ is the smallest critical value of B, λ is still the smallest root of equation (3.18).

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