Automorphisms of open surfaces with irreducible boundary

Adrien Dubouloz, Stéphane Lamy

AUTOMORPHISMS OF OPEN SURFACES WITH IRREDUCIBLE BOUNDARY

ADRIEN DUBOULOZ AND STÉPHANE LAMY

ABSTRACT. Let (S, B_S) be the log pair associated with a projective completion of a smooth quasi-projective surface V. Under the assumption that the boundary B_S is irreducible, we obtain an algorithm to factorize any automorphism of V into a sequence of simple links. This factorization lies in the framework of the log Mori theory, with the property that all the blow-ups and contractions involved in the process occur on the boundary. When the completion S is smooth, we obtain a description of the automorphisms of V which is reminiscent of a presentation by generators and relations except that the "generators" are no longer automorphisms. They are instead isomorphisms between different models of V preserving certain rational fibrations. This description enables one to define normal forms of automorphisms and leads in particular to a natural generalization of the usual notions of affine and Jonquières automorphisms of the affine plane. When V is affine, we show however that except for a finite family of surfaces including the affine plane, the group generated by these affine and Jonquières automorphisms, which we call the tame group of V, is a proper subgroup of $\mathrm{Aut}(V)$.

INTRODUCTION

Smooth affine surfaces with a rich group of algebraic automorphisms have been intensively studied after the pioneering work of M.H. Gizatullin and V.I. Danilov in the seventies. Affine surfaces whose automorphism group acts with a dense orbit with finite complement were first characterized by M.H. Gizatullin [12] in terms of the structure of their boundary divisors in smooth minimal projective completions. Namely, except for finitely many exceptional cases, these surfaces are precisely those which admit completions by chains of proper nonsingular rational curves. Their automorphism groups have been studied by V.I. Danilov and M.H. Gizatullin in a series of papers [13, 14]. They established in particular that their automorphism groups can be described as fundamental groups of graphs of groups attached to well-chosen families of projective completions. The vertices of these graphs correspond to classes up to isomorphism of suitable projective models of the affine surfaces under consideration while the arrows are determined by certain birational relations between these. It is however difficult to extract from them more concrete geometric properties of automorphisms or the existence of interesting subgroups due to the fact that they have in general uncountably many vertices and uncountably many edges between any pairs of vertices.

Affine surfaces V as above have the nice geometric property that they come equipped with families of $\mathbb{A}^1$-fibrations $\pi\colon V \to \mathbb{A}^1$, that is, surjective morphisms with general fibers isomorphic to the affine line. The original approach of M.H. Gizatullin and V.I. Danilov has been recently reworked by J. Blanc and the first author [5] with a particular focus on the interactions between automorphisms and these fibrations. This led to introduce simpler graphs encoding equivalence classes of rational fibrations from which it is possible to decide for instance if the automorphism group of V is generated by

Date: March 2014.

The second author was partially supported by a Marie Curie Intra European Fellowship to visit the Mathematics Institute of Warwick, on leave from the Institut Camille Jordan, Université Lyon 1.

Both authors were partially supported by ANR Grant "BirPol" ANR-11-JS01-004-01.

automorphisms preserving these fibrations. However the methods used in *loc. cit.* remain close to the ones introduced by M.H. Gizatullin and V.I. Danilov, depending in particular on properties of birational maps that are a priori specific to the 2-dimensional case.

As a step towards a hypothetical theory to study automorphisms of higher dimensional affine varieties by methods of birational geometry, it is natural to try to reformulate these existing results in the framework of log Mori theory. Since every smooth affine surface admitting a completion by a chain of smooth rational curves admits in fact such a completion by a particular chain $C_0, C_1, \dots, C_r$, $r \geq 1$, whose self-intersections are respectively $0, a_1, \dots, a_r$, where $a_1 \leq -1$ and $a_i \leq -2$ for all $i = 2, \dots, r$, we see that after contracting the curves $C_1, \dots, C_r$, we obtain a completion by a possibly singular projective surface S with an irreducible boundary $B_S = C_0$. So given a smooth quasi-projective surface V, we would like more generally to describe the automorphism group of V say when there exists a completion $S \supset V$ where S is a possibly singular projective surface with $S \smallsetminus V$ equal to an irreducible curve. More precisely, we look for a factorization in the framework of the log Mori theory for automorphisms of V that do not extend as biregular automorphisms on S.

When V admits a completion into a (log) Mori fiber space S, and $f\colon S \dashrightarrow S$ is the birational self-map associated to an automorphism of V, the (log) Sarkisov program gives a factorization of f into so-called elementary links between intermediate (log) Mori fiber spaces. As already expressed in [6], the hope is that a refinement of such an algorithm could allow to understand the structure of polynomial automorphisms of the affine 3-space $\mathbb{A}^3$. Here we have in mind to complete $\mathbb{A}^3$ by the projective space $\mathbb{P}^3$, and to apply the algorithm to the birational map from $\mathbb{P}^3$ to $\mathbb{P}^3$ induced by an automorphism of $\mathbb{A}^3$. It seems natural to expect an algorithm which is *proper*, that is where all the blow-ups and contractions occur on the boundary divisor.

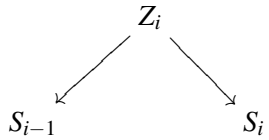
A natural first step is to check if at least in the 2-dimensional case, the log Sarkisov program satisfies this property, and so could be used to give a good description of the automorphism groups of quasi-projective surfaces V admitting completions into log Mori fiber spaces. But maybe surprisingly it turns out that applying the log Sarkisov program to such a completion S does not provide a satisfactory description: In general the links occurring in a factorization of a birational transformation of S induced by an automorphism of V do not preserve the inner quasi-projective surface V (see Proposition 1). This is not the case for $\mathbb{A}^2$, but it is worth noting that the phenomenon occurs for the 3-dimensional affine space: There exist some automorphisms of $\mathbb{A}^3$ for which the usual Sarkisov factorization is *not* proper (see [22, §1.2.3]).

This motivated the search for an alternative algorithm for which all the blow-ups and contractions would occur on the boundary divisor. It is such an algorithm, together with applications and examples, that we propose in this paper, the main point being a shift in focus from the existence of completions with a log Mori fiber space structure to the existence of completions by one irreducible divisor. This last property might turn out to be the right one for studying automorphisms of $\mathbb{A}^3$.

Before stating our main result, let us introduce the class of *dlt completions* of a smooth quasi-projective surface V: These are divisorially log terminal pairs (S, B_S) consisting of a projective completion S of V and a reduced boundary divisor $B_S = \sum E_i$, such that the support of B_S is exactly $S \smallsetminus V$. Also, by a *strictly birational map of dlt completions* we mean a birational map $f : (S, B_S) \dashrightarrow (S', B_{S'})$ which induces an isomorphism $S \smallsetminus B_S \to S' \smallsetminus B_{S'}$ and which is not a biregular isomorphism. With these definitions, our factorization result reads as follows.

Theorem 1. *Let $f\colon V \xrightarrow{\sim} V'$ be an isomorphism of smooth quasi-projective surfaces, and let S, S' be dlt completions of V and V' with irreducible boundary divisors $B_S, B_{S'}$. Then if the induced map $f\colon S \dashrightarrow S'$*

is strictly birational, we can factorize f into a finite sequence of n links of the following form



where $S_0 = S, S_1, \ldots, S_n = S'$ are dlt completions of V with an irreducible boundary, Z_i is for all $i = 1, \ldots, n$ a dlt completion of V with two boundary components, and $Z_i \to S_{i-1}$, $Z_i \to S_i$ are the divisorial contractions associated with each one of the two $K+B$ extremal rays with support in the boundary of Z_i.

The existence of the above decomposition, which was already considered by the authors in [9] (unpublished), is in fact a particular case of more general factorization results developed later on by Y.M. Polyakova: see [24] where she reformulates the problem in terms of relations induced by certain classes of birational maps in suitable categories of 2-dimensional log-terminal pairs. This approach certainly provides a nice theoretical framework for studying automorphisms of quasi-projective surfaces in general: for instance, one can recover from it the description of M.H Gizatullin and V.I. Danilov in terms of fundamental groups of graphs of groups. However, it remains too abstract to give precise handle on the properties of these automorphism groups and their subgroups. In our view, such a factorization result is only a preliminary step for the understanding of these groups, and a second crucial step consists in extracting from it some particular classes of birational maps which are relevant for the study of precise properties of these groups. For instance, in [5] the question was to decide whether the automorphism group of an affine surface admitting a completion by a chain of smooth rational curves is generated by automorphisms preserving $\mathbb{A}^1$-fibrations. The problem was solved by introducing two classes of birational maps called *fibered modifications* and *reversions*, roughly characterized by the respective properties that they preserve an $\mathbb{A}^1$-fibration or exchange it to another one, and then by using an appropriate factorization result to deduce that any automorphism can be decomposed in a finite sequence of such maps.

Here, as an application of our factorization result, we follow a similar strategy to describe the structure of the automorphism group of a quasi-projective surface V admitting a smooth completion (S, B_S) with irreducible boundary $B_S \simeq \mathbb{P}^1$, a case which is essentially complementary to the situations in which the combinatorial methods developed in [5] give a satisfactory description. Affine surfaces of this type have been first studied by Gizatullin and Danilov [14]: They established in particular that their isomorphy types as abstract affine surfaces depend only on the self-intersection B_S^2 of the boundary B_S in a smooth completion (S, B_S) and not on the choice of a particular smooth completion S or boundary divisor B_S (except in the case $B_S^2 = 4$ where there are two models). They described their automorphism groups in terms of the action of certain groups on a "space of tails" which essentially encodes the isomorphy types of smooth completions (S, B_S) of a fixed affine surface V. Here we follow a different approach based on a natural generalizations of the classical notions of *Affine* and *Jonquières* automorphisms for the affine plane. Roughly, for a given affine surface V, affine automorphisms in our sense are characterized by the property that they come as restrictions of biregular automorphisms of various smooth completions (S, B_S) while Jonquières automorphisms are automorphisms which preserve certain $\mathbb{A}^1$-fibration on V. With these notions, we obtain a kind of presentation by generators and relations closely related to the one considered by Gizatullin and Danilov in *loc. cit.* and reminiscent of the usual description given by Jung's Theorem for automorphisms of the affine plane.

It is classical that $\mathrm{Aut}(V)$ is generated by these two classes of automorphisms when V is $\mathbb{A}^2$ or a smooth affine quadric surface and we are able to prove that this holds more generally for every affine surface V admitting a smooth completion (S,B_S) with rational irreducible boundary of self-intersection $B_S^2 \leq 4$. On the other hand, we show that this property fails for those admitting smooth completions (S,B_S) with $B_S^2 \geq 5$. We also derive from our description that if $B_S^2 \geq 5$ then $\mathrm{Aut}(V)$ is "much bigger" than the automorphism group of $\mathbb{A}^2$, in the sense that the proper normal subgroup generated by affine and Jonquières automorphisms of V cannot be generated by a countable family of algebraic subgroups (see Proposition 22).

The article is organized as follows. In Section 1, we briefly review the log Sarkisov program and we illustrate the reason why it does not provide a satisfactory algorithm to obtain informations about automorphism groups of quasi-projective surfaces. In Section 2, we review the geometry of dlt completions, establish our factorization Theorem 1 and discuss some of its properties. Then in Section 3 we apply our algorithm to the case of quasi-projective surfaces V admitting smooth completions with irreducible boundaries. We observe that our algorithm yields a kind of presentation by generators and relations for the automorphisms of V (Proposition 16) and enables to define a notion of normal forms for automorphisms. We then consider the situation where V is affine and discuss the structure of the automorphism group (Proposition 22). Finally, section 4 is devoted to the explicit study of various examples of affine surfaces V admitting smooth completions with irreducible boundaries which illustrate the increasing complexity of the groups $\mathrm{Aut}(V)$ in terms of the self-intersection of their boundary divisors.

1. Quasi-projective surfaces with log Mori fiber space completions and the log Sarkisov program

Many interesting quasi-projective surfaces with a rich automorphism group admit completions into dlt pairs (S,B_S) which are log Mori fiber spaces $g\colon (S,B_S) \to Y$, i.e., g has connected fibers, Y is a normal curve or a point, and all the curves contracted by g are numerically proportional and of negative intersection with the divisor $K_S + B_S$. Examples of such situations include the affine plane $\mathbb{A}^2$ or quasi-projective surfaces obtained as complements of either a section or a fiber in a $\mathbb{P}^1$-bundle over a smooth projective curve. In this context, the log Sarkisov program established by Bruno and Matsuki [6] gives an effective algorithm to factorize a birational map $f\colon S \dashrightarrow S'$ between log Mori fiber spaces into a sequence of elementary links for which we control the complexity of the intermediate varieties in the sense that at any step they differ from a log Mori fiber space by at most one divisorial contraction. As it was established by Takahashi [23, p.401] for the case of $\mathbb{A}^2$, it seems natural to expect in general that given a quasi-projective surface V and a log Mori fiber space S completing V, applying this algorithm to birational maps $f\colon S \dashrightarrow S$ corresponding to automorphisms of V would lead to a good description of the automorphism group of V. Unfortunately, this is not the case as it turns out in general that the birational transformations involved in the algorithm do not preserve the inner quasi-projective surface V. In this section we briefly review the mechanism of the log Sarkisov program of Bruno and Matsuki and illustrate this phenomenon.

1.1. Overview of the log Sarkisov program for projective surfaces.

Let $f\colon S \dashrightarrow S'$ be a birational map between 2-dimensional log Mori fiber spaces (S,B_S) and $(S',B_{S'})$. We assume further that the latter are log MMP related, i.e. that they can be both obtained from a same pair (X,B_X) consisting of a smooth surface X and a simple normal crossing divisor B_X by running the log Minimal Model Program. We denote by $\pi\colon X \to S$ the corresponding morphism and by $C_i \subset X$ the irreducible components of its exceptional locus.

The algorithm depends on two main discrete invariants of the birational map f which are defined as follows. First, we choose an ample divisor H' on S'. We denote by $H_S \subset S$ (resp. $H_X \subset X$, etc...) the strict transform of a general member of the linear system $|H'|$. The *degree* μ of f is then defined as the positive rational number $\frac{H_S.C}{-(K_S+B_S).C}$ where C is any curve contained in a fiber of the log Mori fiber structure on S. For the second invariant, the fact that π is obtained by running the log MMP implies that in the ramification formulas

$$K_X + B_X = \pi^*(K_S + B_S) + \sum a_i C_i, \quad H_X = \pi^* H_S - \sum m_i C_i$$

we have $a_i > 0$ for every i, which enables to define the *maximal multiplicity* λ of f as the maximum of the positive rational numbers $\lambda_i = \frac{m_i}{a_i}$.

If $\lambda > \mu$, then the algorithm predicts the existence of a maximal extraction, that is, an extremal divisorial contraction $Z \to S$ whose exceptional divisor realizes the maximal multiplicity λ. Then either Z is itself a log Mori fiber space, or there exists another extremal divisorial contraction from Z that brings us back to a log Mori fiber space. These operations done, one shows that we have simplified f in the sense that: either μ went down; or μ remained constant but λ went down; or μ and λ remained constant but the number of exceptional divisors in X realizing the maximal multiplicity λ went down. Otherwise, if $\lambda \leq \mu$, the algorithm predicts that either S is equipped with a second structure of log Mori fiber space for which the associated degree μ is strictly smaller, or there exists an extremal divisorial contraction from S to another log Mori fiber space for which μ is again strictly smaller.

The four types of elementary links occurring in the factorization procedure can be summarized by the following diagrams:

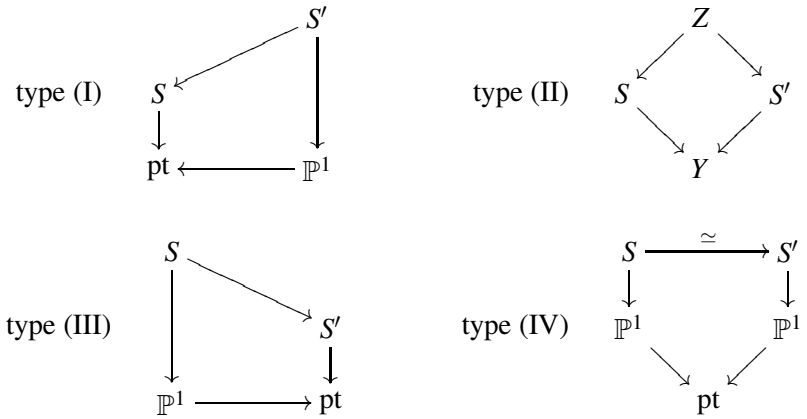


FIGURE 1. The four types of links of the log Sarkisov program.

The above program works for 2-dimensional dlt pairs (S, B_S). Bruno and Matsuki [6] also established the existence of the analogue program in dimension 3 for Kawamata log terminal (klt) pairs (Y, B_Y) generalizing the original 3-dimensional version previously written down by Corti [7]. For klt pairs in any dimension, Hacon and McKernan [15] recently gave a proof of the existence of a factorization of birational maps between log Mori fiber spaces into sequences of links of types (I), ..., (IV) (the definition of these links is slightly more complicated in higher dimension because of the presence of isomorphisms in codimension 1). However, their description, based on the results in [4], is much less effective and does not take the form of an explicit algorithm. In any case, we shall see in the next

subsection that anyone of these factorization results is in general inadequate to study the automorphism group of an open surface V.

1.2. **Inadequacy of the log Sarkisov program.** The following criterion shows that for a large class of quasi-projective surfaces V admitting completions into log Mori fiber spaces (S, B_S), any procedure which factors a birational map $S \dashrightarrow S$ into sequences of links of types (I), ..., (IV) between log Mori fiber spaces will affect in a nontrivial way the inner surface V.

Proposition 1. *Let V be a quasi-projective surface admitting a completion into a log Mori fiber space $\rho\colon S \to C$ over a smooth projective curve C. Suppose further that each irreducible component of the boundary $S \smallsetminus V$ has nonnegative self-intersection, and is not contained in a fiber of any log Mori fiber space structure on S. Then a strictly birational map $\phi\colon S \dashrightarrow S$, cannot admit a factorization into a sequence of Sarkisov links of type (I), ..., (IV), each restricting to an isomorphism on V.*

Proof. Since $\rho\colon S \to C$ is a log Mori fiber space over a curve, an elementary link starting from S is necessarily of type (II), (III) or (IV). Links of type (IV) only change the considered log Mori fiber space structure on S to another structure of the same type. Since ϕ is strictly birational, it cannot be factored into a sequence of links of type (IV). Therefore, after a sequence of links of type (IV), one has necessarily to perform a link of type (II) or (III) with respect to the log Mori fiber space structure $\rho'\colon S \to C$ at this step. Since by assumption the components of the boundary have non-negative self-intersection hence cannot be contracted, we see that a link of type (III) never restricts to an isomorphism on V. Consider now the possibility of a link of type (II). After performing the extraction $Z \to S$ with center at a point $q \in S$, the morphism $Z \to S'$ is the contraction of the strict transform of the unique fiber F of the log Mori fiber space $\rho' : S \to C$ passing though q. Our hypothesis implies that F is not an irreducible component of the boundary $S \smallsetminus V$, and so, the link does not restrict to an isomorphism on V. □

1.3. **Example.** As an illustration of Proposition 1, let us consider the case of the smooth affine surface V defined as the complement of the diagonal D in $\mathbb{P}^1 \times \mathbb{P}^1$. The birational map

$$f\colon (x,y) \in \mathbb{A}^2 \dashrightarrow \left(x + \frac{1}{x-y}, y + \frac{1}{x-y}\right) \in \mathbb{A}^2$$

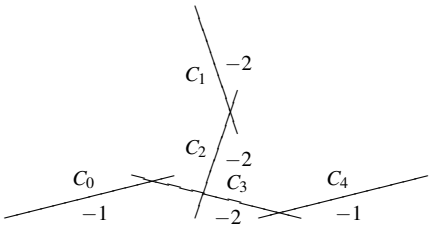
preserves the levels $x - y = constant$, and extends via the embedding $(x,y) \in \mathbb{A}^2 \hookrightarrow ([x:1],[y:1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ to a birational map from $S = \mathbb{P}^1 \times \mathbb{P}^1$ to $S' = \mathbb{P}^1 \times \mathbb{P}^1$ inducing an isomorphism on $V = \mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus D$, where D is the closure of the diagonal $x - y = 0$ in $\mathbb{A}^2$. The unique proper[1] base point of f is the point $p = ([1:0],[1:0])$, and the unique contracted curve is the diagonal D. Straightforward calculations in local charts show that we can resolve f by performing 4 blow-ups that give rise to divisors $C_1, \ldots, C_4$ arranged as on Figure 2. We denote by C_0 the strict transform of the diagonal D. Note that C_4 is the strict transform of the diagonal in S'.

Choosing $H' = D$ as an ample divisor on S', the coefficients a_i in the ramification formulas

$$K_X + B_X = \pi^*(K_S + B_S) + \sum a_i C_i, \text{ and } H_X = \pi^* H_S - \sum m_i C_i,$$

are easy to compute. For the m_i, one exploits for instance the fact that the strict transform H_S of a general member of $|D|$ is a rational curve of bidegree $(3,3)$ with a double point at p and at each of the infinitely near base points of f. The results are tabulated in Figure 2.

[1] By *proper* we mean a base point which is not an infinitely near point.



	a_i	m_i	λ_i
C_1	1	2	2
C_2	1	4	4
C_3	1	6	6
C_4	2	8	4

FIGURE 2. Resolution of f and coefficients in the ramification formulas.

The maximal multiplicity is thus realized by the divisor C_3 and a maximal extraction $Z \to S$ is obtained by first blowing-up three times to produce C_1, C_2 and C_3 and then contracting C_1 and C_2 creating a cyclic quotient singularity. The boundary $Z \smallsetminus V$ consists of two irreducible curves C_0 and C_3, the latter supporting the unique singular point on the surface. Furthermore, there exist 4 irreducible curves on Z that correspond to $K+B$ extremal rays:

- C_3, which is the exceptional divisor associated with the maximal multiplicity;
- C_0, which is the strict transform of the diagonal on S;
- The strict transforms of the 2 rules D_+ and D_- of $\mathbb{P}^1 \times \mathbb{P}^1$ crossing at p.

Now the log Sarkisov program imposes to contract one of the two curves D_+ or D_- above (precisely: the one that was a fiber for the chosen structure of log Mori fiber space on $\mathbb{P}^1 \times \mathbb{P}^1$) to reach a new log Mori fiber space. But this birational contraction does not restrict to an isomorphism on the affine surface V.

However, the above computation shows that we are left with a third option which consists in contracting the strict transform C_0 of D. This is precisely the curve that our alternative algorithm will impose to contract to get a new projective surface S_1 supporting a cyclic quotient singularity along the new boundary $B_{S_1} = C_3$. By construction, the corresponding birational map $S \dashrightarrow S_1$ induces an isomorphism on the inner affine surface V but it turns out that S_1 is no longer a log Mori fiber space. Indeed, its divisor class group is isomorphic to $\mathbb{Z}^2$, generated by the strict transforms of D_+ and D_-. On the other hand, one checks that these curves generate the only $K+B$ extremal rays on S_1, each of these giving rise to a divisorial contraction $S_1 \to \mathbb{P}^2$. Note in particular that even though it consists of a maximal extraction $Z \to S$ followed by a divisorial contraction $Z \to S_1$, the birational map $S \dashrightarrow S_1$ just constructed is not a Sarkisov link of type (II).

Summing up, Proposition 1 and the above example show that for quasi-projective surfaces V admitting completions into log Mori fiber spaces, there does not exist any factorization process for which each elementary step is simultaneously a link of type (I), ..., (IV) between log Mori fiber spaces and a birational map restricting to an isomorphism on V. So we cannot escape the dilemma that inevitably we have to abandon one of these properties.

2. THE FACTORIZATION ALGORITHM

Here we first review basic facts on 2-dimensional dlt pairs and discuss the geometry of the boundaries of dlt completions involved in our main statement. Then we prove Main Theorem 1 and discuss some additional properties of the factorization.

2.1. **Singularities and geometry of boundaries.** The fact that an automorphism of a normal quasi-projective surface V extends to an automorphism of its minimal desingularisation enables to restrict without loss of generality to the case of a smooth quasi-projective surface. On the other hand since an extremal contraction starting from a smooth log surface may yield a singular one, it is necessary to allow some kind of singularities on the projective completions S of V. Following recent work of Fujino [11], the widest framework where the log Mori Program is established in dimension 2 is the one of pairs (S, B_S) with log canonical singularities. However, it is enough for our purpose to work with the subclass of dlt pairs (S, B_S).

2.1.1. *Hirzebruch-Jung singularities.* Before giving the characterization of these pairs that will be used in the sequel, let us first recall that an isolated singular point p of a surface S is called a Hirzebruch-Jung cyclic quotient singularity of type $A_{n,q}$, $n \geq 2$, $1 \leq q \leq n-1$, $\gcd(n,q) = 1$ if it is analytic locally isomorphic to the quotient of $\mathbb{A}^2$ by the action of the group $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ of complex n-th roots of unity defined by $(x,y) \mapsto (\varepsilon x, \varepsilon^q y)$. As it is well-known (see e.g. [2, page 99]), the exceptional locus of the minimal resolution $\pi\colon \overline{S} \to S$ of p consists of a chain of rational curves $E_1, \ldots, E_s$ with self-intersections $E_i^2 = -a_i \leq -2$ determined by the expression

$$\frac{n}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ldots}}$$

as a continued fraction. Recall that cyclic quotient singularities are log terminal, i.e., in the ramification formula $K_{\overline{S}} = \pi^* K_S + \sum c_i E_i$ one has $-1 < c_i$ for every i. For such singularities, one has in fact $-1 < c_i \leq 0$. Indeed, otherwise, we can write $K_{\overline{S}} = \pi^* K_S + A - B$ where A and B are effective $\mathbb{Q}$-divisors supported on the exceptional locus of π and without common components. Since $A^2 < 0$, it follows that $K_{\overline{S}} \cdot A = (A-B)\cdot A < 0$ and hence, there would exist an index i such that $K_{\overline{S}} \cdot E_i < 0$. But then E_i would be a (-1)-curve which is absurd.

2.1.2. *Dlt pairs.* For a definition of such pairs in general, we refer the reader to [18, Definition 2.8]. In our situation, [19, Proposition 2.42] combined with the local description of log terminal singularities of surfaces which can be found in [17, see in particular page 57, case (3)] leads to the following equivalent definition:

Definition 2. A pair (S, B_S) consisting of a projective surface S and a nonempty reduced divisor $B_S = \sum B_i$ such that $S \smallsetminus B_S$ is smooth is called divisorially log terminal (dlt) if the following conditions are satisfied:

- The B_i are smooth irreducible curves with normal crossings, that is each common point of two components is a normal crossing at a smooth point of S;
- A singular point p of S is a Hirzebruch-Jung singularity $A_{n,q}$ and the strict transform of B_S in the minimal resolution $\pi\colon \bar{S} \to S$ of p meets the exceptional chain of rational curves $E_1, \ldots, E_s$ transversally at a unique point of the initial or final curve E_1 or E_s.

In particular, a dlt pair (S, B_S) with irreducible boundary divisor B_S is a purely log terminal (plt) pair.

Note that the above conditions guarantee in particular that the total transform of B_S in the minimal resolution $\pi\colon \tilde{S} \to S$ of the singularities of S is a simple normal crossing divisor.

2.1.3. *Geometry of the boundary.* Let us first introduce notations and terminology that will be used in the sequel. Given a strictly birational map of dlt completions $f\colon (S, B_S) \dashrightarrow (S', B_{S'})$ with irreducible boundaries, we denote by $\pi\colon \tilde{S} \to S$ and $\pi'\colon \tilde{S}' \to S'$ the minimal resolutions of the singularities of S and S' respectively. We denote by $\tilde{S} \xleftarrow{\sigma} X \xrightarrow{\sigma'} \tilde{S}'$ the minimal resolution of the base points of the

birational map $\tilde{f}\colon \tilde{S} \dashrightarrow \tilde{S}'$ induced by f. Recall [1, Theorem 1.3.7] that X and the birational morphisms σ, σ' are uniquely determined up to isomorphism by the following universal property: given another resolution $\tilde{S} \leftarrow X' \to \tilde{S}'$, there exists a unique birational morphism $X' \to X$ such that the obvious diagram commutes. In particular, X does not contain (-1)-curves that are exceptional for both $\pi\circ\sigma : X \to S$ and $\pi'\circ\sigma' : X \to S'$. This implies that if the sequence of blow-ups $\sigma' : X \to \tilde{S}'$ is not empty, the (-1)-curve produced as the last exceptional divisor of the sequence is the strict transform of B_S. Note also that by construction the boundary of $\tilde{S}$ and X are simple normal crossing divisors, with each irreducible component a smooth rational curve.
The following result shows that the existence of strictly birational maps of dlt completions $f : (S, B_S) \dashrightarrow (S', B_{S'})$ imposes strong constraints on the boundaries:

Proposition 3. *Let $f\colon (S, B_S) \dashrightarrow (S', B_{S'})$ be a strictly birational map of dlt completions with irreducible boundaries. Then the following holds:*

(1) The boundaries B_S and $B_{S'}$ are both isomorphic to $\mathbb{P}^1$;
(2) S admits at most two singularities;
(3) f admits a unique proper base point $\mathcal{B}(f)$, and if S has exactly two singularities then $\mathcal{B}(f)$ coincides with one of these singularities.

Proof. Recall (see e.g. [16, Theorem 5.2 page 410]) that if $h : M \dashrightarrow M'$ is a birational map between normal surfaces, and $p \in M$ is a proper base point of h, then there exists a curve $C \subset M'$ such that $h^{-1}(C) = p$. In our situation, since $B_{S'}$ is the only curve that can be transformed to a point by f^{-1}, it follows that f has a unique proper base point $\mathcal{B}(f) = f^{-1}(B_{S'}) \in B_S$. This implies in turn that $f(B_S)$ cannot be equal to $B_{S'}$ and so must be equal to a point $p' = \mathcal{B}(f^{-1}) \in B_{S'}$. In particular, with the notation above, the strict transforms on the minimal resolution X of B_S and $B_{S'}$ are smooth rational curves (they come either from the resolution of a $A_{n,q}$ singularity, or from the blow-up of a smooth point), and they are not equal. This gives (1).

Now suppose that the union of the singularities of S and of $\mathcal{B}(f)$ consists of at least three distinct points supported on B_S. Then on X, the strict transform of B_S is a boundary component with at least three neighbors. If $\sigma' \neq \mathrm{id}$ then the first contraction must be the one of the strict transform of B_S, which is impossible since the boundary divisor is simply normal crossing for all surfaces between X and $\tilde{S}'$. Hence $\sigma' = \mathrm{id}$, but again this gives a contradiction, since on $\tilde{S}'$ all divisors except maybe the strict transform of $B_{S'}$ which is distinct from that of B_S must have at most two neighbors. This proves (2) and (3). □

2.2. **Proof of the factorization Theorem 1.** The proof relies on the following lemma which characterizes the possible extremal rays supported on the boundaries of dlt completions (S, B_S).

Lemma 4. *Let (S, B_S) be a dlt completion of a smooth quasi-projective surface V.*

(1) A (smooth) rational curve $C \subset B_S$ with at least two neighboring components in B_S is not a $K_S + B_S$ extremal ray.
(2) If $C \subset B_S$ is a smooth rational curve with only one neighboring component in B_S and supporting at most one singularity of S, then $(K_S + B_S).C < 0$.
(3) Let $C \subset B_S$ be a curve supporting exactly one singularity p of S, and denote by $\overline{C}$ the strict transform of C in the minimal resolution of p. If $\overline{C}^2 < 0$ then $C^2 < 0$.

Proof. Let n be the number of neighbors of C in B_S and let $p_1,\dots,p_r$ the singular points of S supported along C. By the adjunction formula (see e.g. 2.2.4 in [25]), we have

$$(K_S+B_S)\cdot C = (K_S+C)\cdot C + n = \deg(K_C+\mathrm{Diff}_C(0)) + n = -2+\sum_{i=1}^{r}(1-\frac{1}{m_i})+n,$$

where $m_i \geq 2$ is the index of the singular point p_i, $i=1,\dots,r$. This implies (1) and (2). For (3), let $\pi : \overline{S} \to S$ be a minimal resolution of p and let $E = E_1$ be the unique π-exceptional curve that intersects the strict transform $\overline{C}$ of C. We write $\overline{C} = \pi^*C - bE - R$, $K_{\overline{S}} = \pi^*(K_S) + cE + R'$ where $b > 0$, $0 \geq c > -1$ (see §2.1.1) and where R, R' are π-exceptional divisors whose supports do not meet $\overline{C}$. The fact that (S, B_S) is a dlt pair implies that $c - b > -1$ whence that $1 > b$. The assertion follows since $C^2 = \pi^*C\cdot\overline{C} = (\overline{C}+bE+R)\cdot\overline{C} = \overline{C}^2 + b < \overline{C}^2 + 1$. □

Proof of Theorem 1. Recall that we have a strictly birational map $f\colon (S,B_S) \dashrightarrow (S',B_{S'})$ restricting to an isomorphism $V = S\setminus B_S \simeq V' = S'\setminus B_{S'}$. As in subsection 2.1.3, we let $\pi\colon \tilde{S}\to S$ and $\pi'\colon \tilde{S}'\to S'$ be the minimal resolutions of singularities and we let $\tilde{S} \xleftarrow{\sigma} X \xrightarrow{\sigma'} \tilde{S}'$ be the minimal resolution of the base points of the induced birational map $\tilde{f}$. By Proposition 3(1) and the description of Hirzebruch-Jung singularities given in §2.1.1, the divisor B_X is then a tree of rational curves. The irreducible components of B_X are exceptional for at least one of the two morphisms $\pi\circ\sigma$ or $\pi'\circ\sigma'$, thus they all have a strictly negative self-intersection. Since B_X is a tree, there exists a unique sub-chain $E_0, E_1, \dots, E_n = E_0{}'$ of B_X joining the strict transforms E_0 and $E_0{}'$ of B_S and $B_{S'}$ respectively. We proceed by induction on the number $n+1$ of components in this chain. The integer $n \geq 1$ will also be the number of links needed to factorize f. We use the same notation for the curves E_i, $i = 0,\dots,n$ and their images or strict transforms in the different surfaces that will come into play.

To construct the first link $S = S_0 \dashrightarrow S_1$, we consider the minimal partial resolution $\tilde{S} \leftarrow Y \dashrightarrow \tilde{S}'$ of $\tilde{f}$ dominated by X and containing the divisor E_1 defined as follows:

- If $\tilde{f} : \tilde{S} \dashrightarrow \tilde{S}'$ is either a morphism or has a proper base point supported outside from E_0, then E_1 is one of the exceptional divisor of π, and the boundary $B_{\tilde{S}}$ is a chain of rational curves with E_0, E_1 intersecting in one point. In this case we put $Y = \tilde{S}$.

- Otherwise, if $\tilde{f} : \tilde{S} \dashrightarrow \tilde{S}'$ has a proper base point on E_0 then by definition of the resolution X, the divisor E_1 is produced by blowing-up successively the base points of $\tilde{f}$ as long as they lie on E_0, E_1 being the last divisor produced by this process. We let $Y \to \tilde{S}$ be the intermediate surface thus obtained. By construction, the image of the curves contracted by the induced birational morphism $X \to Y$ are all located outside E_0 and the self-intersections of E_0 in X and Y are equal. The divisor B_Y is a chain that looks as in Figure 3. The wavy curves labeled "Sing" correspond to the (possible) chains

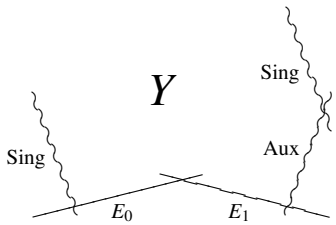
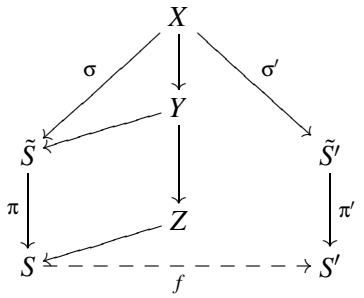


FIGURE 3. The boundary divisor of Y.

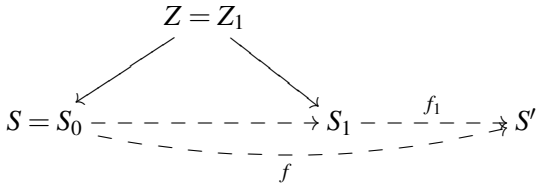
of rational curves obtained by desingularisation of S, and the wavy curve labeled "Aux" corresponds to

the (possible) chain of auxiliary rational curves, each with self-intersection -2, obtained by resolving the base points of $\tilde{f}$ before getting E_1.

In both cases, we have $E_0^2 < 0$ on Y, since this self-intersection is the same as the one on X. So all irreducible components of B_Y have a strictly negative self-intersection. By running the $K+B$ MMP on Y we can successively contract all the components of the boundary B_Y except E_0 and E_1. Indeed at each step Y is a minimal resolution of the intermediate surface, and each extremity component C of the boundary chain supports at most one singularity: Lemma 4 ensures that C is $K+B$ negative and has negative self-intersection whence generates a $K+B$ extremal ray giving rise to a divisorial extremal contraction. We note (Z, E_0+E_1) the dlt pair obtained from the pair (Y, B_Y) by this sequence of contractions.



By construction, Z dominates S via the divisorial contraction of the $K+B$ extremal curve E_1. Again by Lemma 4, E_0 generates a $K+B$ extremal ray in Z, and $E_0^2 < 0$ on Z. So there exists a $K+B$ divisorial extremal contraction $Z \to S_1$ contracting E_0 and yielding a new dlt pair (S_1, B_{S_1}) with reduced boundary B_{S_1} consisting of the strict transform of E_1. We obtain the first expected link and the map $f\colon S \dashrightarrow S'$ factorizes via a birational map $f_1\colon S_1 \dashrightarrow S'$.



Furthermore, the minimal resolution X_1 of the induced birational map $\tilde{f}_1 : \tilde{S}_1 \dashrightarrow \tilde{S}'$ between the minimal desingularisations of $\tilde{S}_1$ and $\tilde{S}'$ induced by f_1 is dominated by X. More precisely, since E_0 is the only possible (-1)-curve on X which is exceptional for both induced morphism $X \to \tilde{S}_1$ and $X \to \tilde{S}'$, X_1 is either equal to X if $E_0^2 \leq -2$ or is obtained from X by first contracting E_0 and then all successive (-1)-curves occurring in the minimal resolution of a singular point of S supported on B_S and distinct from the proper base point of f (see Figure 3 above). It follows that the chain associated to $f_1\colon S_1 \dashrightarrow S'$ as defined at the beginning of the proof consists of the curves $E_1, \ldots, E_n = E_0{}'$ hence has length n. We conclude by induction that we can factorize f into exactly n links. □

Remark 5. A by-product of the proof above is the following property of the intermediate surfaces Z_i with two boundary components that appear in the Theorem: each one of the boundary component supports at most one singularity. Note also that neither Lemma 4 nor the above proof tell something about the possible $K_{Z_i}+B_{Z_i}$ extremal curves on these intermediate surfaces that do not belong to the boundary: in the example given in §1.2 above, we have four $K+B$ extremal rays but only two of them were supported on the boundary.

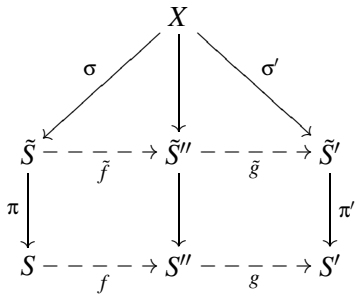
We introduce a concept that will prove useful in the next section.

Definition 6. If $f\colon (S,B_S) \dashrightarrow (S'',B_{S''})$ and $g\colon (S'',B_{S''}) \dashrightarrow (S',B_{S'})$ are strictly birational maps of dlt completions, we will say that f and g are in *special position* if $\mathcal{B}(f^{-1}) = \mathcal{B}(g)$ and in *general position* otherwise.

It follows in particular from the construction of the factorization $f = f_n \dots f_1\colon (S,B_S) \dashrightarrow (S',B_{S'})$ given in the proof above that for every $i = 1,\dots,n-1$, f_i and f_{i+1} are in general position. In general, see Remark 8 below, the factorization into elementary links of a composition of two strictly birational maps of dlt completions with irreducible boundaries does not coincide with the concatenation of the factorizations of these maps. The following corollary provides however a sufficient condition for this property to hold. In particular the condition is satisfied when all the surfaces into play are smooth.

Corollary 7. *Let $f : (S,B_S) \dashrightarrow (S'',B_{S''})$ and $g : (S'',B_{S''}) \dashrightarrow (S',B_{S'})$ be birational maps of dlt completions with irreducible boundaries. If f and g are in general position and at least one of the two points $\mathcal{B}(g)$ or $\mathcal{B}(f^{-1})$ is a smooth point of S'' then the factorization of $g\circ f$ into elementary links given by Theorem 1 is equal to the concatenation of the factorizations of f and g. Furthermore, one has then $\mathcal{B}(g\circ f) = \mathcal{B}(f)$ and $\mathcal{B}((g\circ f)^{-1}) = \mathcal{B}(g^{-1})$.*

Proof. Up to replacing f and g by their inverses, we may assume that $\mathcal{B}(g)$ is a smooth point of S''. As before we denote by $\tilde{S}$ the minimal desingularisation of S (same with S', S'') and by $\tilde{f}$, $\tilde{g}$ the induced birational maps. The hypothesis implies that all the base points of $\tilde{f}^{-1}$ and $\tilde{g}$ including infinitely near ones are distinct so that a resolution $\tilde{S} \xleftarrow{\sigma} X \xrightarrow{\sigma'} \tilde{S}'$ of the birational map $\tilde{S} \dashrightarrow \tilde{S}'$ induced by $g\circ f$ is obtained from $\tilde{S}''$ by simultaneously resolving the base points of $\tilde{f}^{-1}$ and $\tilde{g}$:



The surface X dominates the minimal resolution X_f of $\tilde{f}$ and X_g of $\tilde{g}$. We denote by E_0, E_0', E_0'' the strict transforms of B_S, B_S' and B_S'' in X (or in X_f, X_g). By construction the chain joining E_0 to E_0' in X is the union of the strict transform of the chain joining E_0 to E_0'' in X_f with the strict transform of the chain joining E_0'' to E_0' in X_g. Since $B_{S''}$ is contracted by f^{-1}, its strict transform E_0'' in X_f has negative self-intersection. Furthermore since $\mathcal{B}(g)$ is a smooth point of S'', the lift of g to X_f has a proper base point on E_0'' and so the strict transform of E_0'' in X has self-intersection ≤ -2. Since E_0'' is the only curve that could have been a (-1)-curve simultaneously exceptional for σ and σ', we conclude that X is a minimal resolution of $g\circ f$.

Now the first part of the assertion follows directly from the construction of the factorization. The second part follows from the fact that since the image $\mathcal{B}(g) \in B_{S''}$ of $B_{S'}$ by g^{-1} is distinct from $\mathcal{B}(f^{-1})$, the image $\mathcal{B}(g\circ f)$ of $B_{S'}$ by $(g\circ f)^{-1}$ coincides with the image $\mathcal{B}(f) \in B_S$ of $B_{S''}$ by f^{-1}. For the same reason, $\mathcal{B}((g\circ f)^{-1}) = \mathcal{B}(g^{-1})$. □

Remark 8. The assumption that $\mathcal{B}(f^{-1})$ or $\mathcal{B}(g)$ is a smooth point of S'' implies in particular that $B_{S''}$ supports at most a singular point of S'' (Proposition 3, assertion (3)). So the only situation in which the conclusion of the Corollary above could fail is when $B_{S''}$ supports exactly two singular points which are

the proper base points of $\mathcal{B}(f^{-1})$ and $\mathcal{B}(g)$ respectively. The following example, which was pointed out to us by the referee, shows that this phenomenon can indeed occur.

Consider $S = \mathbb{P}^2$, with boundary B_S equal to a line. We construct a surface X by blowing-up three points: first a point on S producing an exceptional divisor E; then the intersection point $E \cap E_0$ (where E_0 is the strict transform of B_S) producing the exceptional divisor E_0'; and finally blowing-up $E_0 \cap E_0'$ producing E_0''.

We construct a surface S'' from X by contracting the curves E, E_0 and E_0'; similarly we construct S' by contracting E, E_0 and E_0''. These surfaces are singular, we have $\tilde{S}'' = X$, and $\tilde{S}'$ is the surface obtained from X by contracting E_0 and E_0''. Denote by f, g the birational maps $S \dashrightarrow S''$ and $S'' \dashrightarrow S'$ (see Figure 4). Then the factorization of $g \circ f$ is not the concatenation of the factorizations of f and g. What's going wrong here is that X is not a minimal resolution of $g \circ f$, indeed E_0'' is a (-1)-curve on X which is exceptional for both σ and σ'.

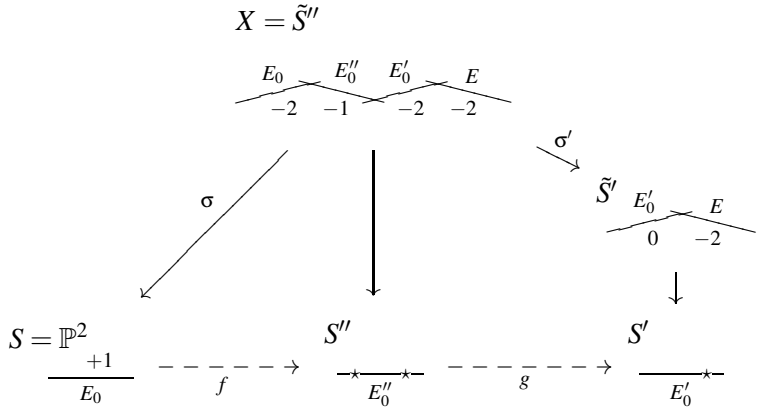


FIGURE 4. The counter-example in Remark 8 ($\star$ denotes a singularity, numbers are self-intersections)

2.3. **Additional properties of the factorization.** Noting that the definition of the maximal multiplicity λ (see §1.1) makes sense even when S is not a Mori fiber space, we observe that our algorithm retains one aspect of the log Sarkisov program of Bruno and Matsuki [6], namely the fact that the first divisorial contraction involved in each link is a maximal extraction:

Proposition 9. *The birational morphism $Z \to S$ with exceptional divisor E_1 constructed in the proof of the theorem is a maximal extraction.*

Proof. A maximal extraction (see [23, prop. 13-1-8] and [6, p. 485] for the logarithmic case) is obtained from a smooth surface which dominates S and S' by a process of the $K+B$-MMP. So we may use the surface X from the proof of the theorem. The precise procedure consists in two steps (we use the notations λ and H that have been defined in §1.1): Running first a $K + B + \frac{1}{\lambda}H$-MMP over S until we reach a log minimal model, then running a $K+B$-MMP over S; the last contraction gives a maximal extraction. The crucial observation is that each extremal divisorial contraction of the log MMP in the first step is also a contraction for the genuine $K+B$-MMP. The fact that we are running a log MMP over

S guarantees that the only curves affected by the procedure are contained in the boundary. By Lemma 4, as long as E_1 admits two neighboring components (E_0 and another one), it cannot correspond to a $K+B$ negative extremal ray. Remark also that if B_S supports a singularity q which is not a proper base point for f, then all exceptional divisors of the resolution of q have multiplicities $\lambda_i = 0$ and thus are contracted in the first step. It follows that the maximal extraction we constructed, which is the last divisorial contraction $Z \to S$, must have E_1 as exceptional divisor. □

Remark 10. In contrast with the log Sarkisov algorithm of Bruno and Matsuki, we did not assume from the beginning that the pairs (S,B_S) and $(S',B_{S'})$ were log-MMP related. In our situation, this property is automatic: this is probably a well-known fact, but we can also obtain it as a by-product of the proof of Theorem 1. Indeed, letting again $E_0,\dots,E_n$ be the subchain of rational curves in the boundary B_X of X defined in the proof, Lemma 4 guarantees that all the irreducible components of B_X except the ones contained in that chain can be successively contracted by a process of the $K+B$ MMP. The surface W obtained by this procedure has boundary $B_W = \sum_{i=0}^{n} E_i$ and dominates both S and S' by a sequence of $K+B$ divisorial contractions.

On the other hand, the elementary example of the identity map of $\mathbb{A}^2$ viewed as a rational map from $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ with a unique proper base point p located at the intersection of the two rules at infinity and for which the blow-up of p is not a $K+B$ extremal contraction shows that arbitrary dlt completions of a given quasi-projective surface need not be log-MMP related in general. So if one wants to extend our factorization result to pairs with reducible boundaries, it becomes necessary to at least require from the very beginning that the pairs under consideration are log-MMP related.

3. Quasi-projective surfaces with smooth completions

In this section we derive from our factorization theorem a general description of the automorphism group of V when V admits a smooth completion (S,B_S) with irreducible boundary $B_S \simeq \mathbb{P}^1$. In what follows, such pairs (S,B_S) are simply referred to as *smooth completions*, the inner smooth quasi-projective surface $V = S \smallsetminus B_S$ being implicit.

Smooth completions (S,B_S) for which $B_S^2 < 0$ can be quite arbitrary since for instance any blow-up $\sigma : S \to S'$ of a point on a smooth projective surface S' with exceptional divisor B_S gives rise to such a pair (S,B_S). In contrast, the possible structures of pairs (S,B_S) with $B_S^2 \geq 0$ are much more constrained, as summarized by the following Proposition:

Proposition 11. *If (S,B_S) is a smooth completion with $B_S^2 \geq 0$, then after the contraction of finitely many (-1)-curves contained in V, we reach a pair of the following type:*

(1) $(\mathbb{P}^2,B)$ where B is either a line or a smooth conic,

(2) $(\mathbb{F},B)$ where $p\colon \mathbb{F} \to D$ is a ruled surface over a smooth projective curve D and where B is either a fiber or a section. Furthermore, if $B^2 \neq 0$ then $\mathbb{F}$ is a Hirzebruch surface $\mathbb{F}_n$, for some $n \geq 0$, and B is a section.

Proof. Up to replacing (S,B_S) by a pair obtained by successively contracting all possible (-1)-curves in $S \smallsetminus B_S$ and having the strict transform of B_S for its boundary, we may assume from the very beginning that $S \smallsetminus B_S$ does not contain a (-1)-curve. Since $(K_S+B_S)\cdot B_S = -2$ by adjunction formula, it follows that K_S+B_S is not nef and so there exists a K_S+B_S-negative extremal rational curve C on S. Since $B_S^2 \geq 0$, the conditions $(K_S+B_S)\cdot C = (K_S+C)\cdot C + B_S\cdot C - C^2 = -2 + B_S\cdot C - C^2 < 0$ and $C^2 < 0$ would imply that C is a (-1)-curve disjoint from B_S, which is impossible. Thus (S,B_S) is a log Mori fiber space $g : S \to D$. If D is a point then S is smooth log del Pezzo surface of rank 1, whence is isomorphic to $\mathbb{P}^2$ and $B_S \simeq \mathbb{P}^1$ is either a line or a smooth conic. Otherwise, since S is smooth $g : S \to D$

is a $\mathbb{P}^1$-bundle and the condition $(K_S+B_S)\cdot F=(K_S+F)\cdot F+B_S\cdot F=-2+B_S\cdot F<0$ for a fiber $F\simeq\mathbb{P}^1$ of g implies that $B_S\cdot F=0$ or 1. Thus B_S is either a fiber if $B_S\cdot F=0$ or a section otherwise. This immediately implies the remaining assertions. □

3.1. **Triangular birational maps between smooth completions.** Let us first observe that if (S,B_S) is a smooth completion with $B_S^2<0$ then every birational map of smooth completions $f:(S,B_S)\dashrightarrow(S',B_{S'})$ is in fact an isomorphism. Indeed, otherwise it would have a proper base point on B_S, and since $B_S^2<0$ it would follow that the total transform of B_S in the minimal resolution of f contains no (-1)-curve except the strict transform of $B_{S'}$, in contradiction with the fact that S' is smooth. It follows in particular that if a smooth quasi-projective surface V admits a smooth completion (S,B_S) with $B_S^2<0$ then the automorphism group of V coincides with the subgroup $\mathrm{Aut}(S,B_S)$ of $\mathrm{Aut}(S)$ consisting of automorphisms preserving the boundary B_S. In contrast, if (S,B_S) and $(S',B_{S'})$ are smooth completions with $B_S^2\geq 0$ or $B_{S'}^2\geq 0$, then strictly birational maps of smooth completions $(S,B_S)\dashrightarrow(S',B_{S'})$ may exist in general.

3.1.1. *Structure of intermediate pairs.* Given such a strictly birational map, we prove in the next lemma that the dlt pairs (S_i,B_{S_i}) which appear in the factorization of f as in Theorem 1 have at most one singularity. So the following definition makes sense: If S_i is singular, then we say that it has *index* k if in the minimal resolution of its singularities the exceptional curve which intersects the strict transform of B_{S_i} has self-intersection $-k$. Otherwise, if S_i is smooth then we say that S_i has index 1. We note $\mathrm{ind}(S_i)$ the index of S_i.

Lemma 12. *Let $f:(S,B_S)\dashrightarrow(S',B_{S'})$ be a strictly birational map of smooth completions and let $S=S_0\dashrightarrow S_1\dashrightarrow\cdots\dashrightarrow S_n=S'$ be its factorization into elementary links given by Theorem 1. Then the following holds :*

1) If $B_S^2=0$ then each S_i is smooth with $B_{S_i}^2=B_S^2=0$,

2) If $B_S^2>0$ then each S_i has at most one singularity. Furthermore:

a) If S_i is smooth then $B_{S_i}^2=B_S^2$ whereas if S_i is singular, the boundary of a minimal resolution of S_i is a chain of B_S^2+1 rational curves with self-intersections $(0,-k_i,-2,\ldots,-2)$ where $k_i=\mathrm{ind}(S_i)$;

b) For all $i=0,\ldots,n-1$ the indexes of S_i and S_{i+1} differ exactly by 1 and if $\mathrm{ind}(S_i)\geq 2$ and $\mathrm{ind}(S_i)=\mathrm{ind}(S_{i-1})-1$ then $\mathrm{ind}(S_{i+1})=\mathrm{ind}(S_i)-1$.

Proof. Let (S_j,B_j) be one of the intermediate dlt completions, and let $f_j:(S_j,B_j)\dashrightarrow(S',B_{S'})$ be the induced birational map. Suppose S_j is smooth, with $B_{S_j}^2=B_S^2=d\geq 0$ and consider as in the proof of Theorem 1 the surface Y containing the strict transforms E_j and E_{j+1} of the boundaries of S_j and S_{j+1}. Since S' is smooth, the strict transform of B_j in the minimal resolution X_j of f_j is a (-1)-curve. It follows that the boundary of Y is equal to a chain of $d+2$ curves with self-intersections $(-1,-1,-2,\ldots,-2)$. If $d=0$ then S_{j+1} is again smooth with $B_{S_{j+1}}^2=0$ and so, 1) follows by induction. Otherwise, if $d>0$ then S_{j+1} has a unique singularity and the boundary of the minimal resolution of S_{j+1} is a chain of $d+1$ curves with self-intersections $(0,-2,\ldots,-2)$. In particular, S_{j+1} has index 2 (see Figure 5, (a), with $k=2$). Now we proceed by induction, assuming that S_i has exactly one singularity, and that the boundary of the minimal resolution $\tilde{S}_i$ of S_i is a chain of $d+1$ rational curves with self-intersections $(0,-(k-1),-2,\ldots,-2)$, where $k-1=\mathrm{ind}(S_i)\geq 2$. We denote by C the second irreducible component of this chain which has thus self-intersection $-(k-1)$. Let $\tilde{S}_i\leftarrow X_i\rightarrow S'$ be the minimal resolution of the induced birational map $\tilde{S}_i\dashrightarrow S'$. Since the strict transform E_i of B_{S_i} is a 0-curve on $\tilde{S}_i$ and a (-1)-curve on X_i as S' is smooth, we see that there is exactly one blow-up on E_i, which by definition produces the divisor E_{i+1}. Then there are two cases :

a) If the proper base point on E_i coincides with the intersection point of E_i and C, then the boundary of Y is a chain of curves with self-intersections $(-1,-1,-k,-2,\ldots,-2)$, where the first three are E_i and E_{i+1} and C. Thus in this case S_{i+1} has again exactly one singularity and has index k (the picture is again Figure 5, (a)).

b) Otherwise, if the proper base point on E_i is any other point, then the boundary of Y is a chain of curves with self-intersections $(-2,\ldots,-2,-(k-1),-1,-1)$, where the last three are C, E_i and E_{i+1}. In this case S_{i+1} has again at most one singularity and has index $k-2$ (see Figure 5, (b)). It is smooth if and only if $k-1=2$ and in this case its boundary $B_{S_{i+1}}$, which is the strict transform of E_{i+1} has again self-intersection $B^2_{S_{i+1}} = d$.

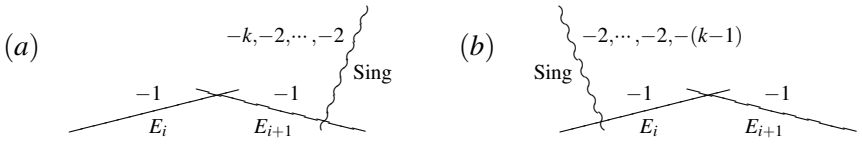


FIGURE 5. Boundary of Y in the proof of Lemma 12.

The last assertion follows from the fact that by construction the center of the blow-up on E_{i+1} producing the next divisor E_{i+2} does not coincide with the intersection point of E_i and E_{i+1}. □

3.1.2. *Triangular birational maps.*

Definition 13. A strictly birational map of smooth completions $\phi : (S,B_S) \dashrightarrow (S',B_{S'})$ is called *triangular* if all the intermediate surfaces S_i that appear in the factorization produced by Theorem 1 are singular.

Given a smooth pair (S,B_S) with $B_S^2 \geq 0$, it follows from Proposition 11 that S dominates birationally a surface $\mathbb{F}$ which is either $\mathbb{P}^2$ or a ruled surface.

First we discuss the case where $B_S^2 = 0$. Then the strict transform of B_S in $\mathbb{F}$ still have self-intersection 0, so $\mathbb{F}$ is a ruled surface $p\colon \mathbb{F} \to D$ and the strict transform of B_S is either a fiber F or a section of p. Note that in the second case $\mathbb{F}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ in such a way that p coincides with the first projection while the strict transform of B_S is a fiber F of the second projection: up to changing the projection we can assume that B_S is a fiber, as in the first case. Then, it follows from Lemma 12 that the notion of a triangular map coincides with that of a link and that every such link consists of the blow-up of a point on F followed by the contraction of its strict transform. Assume now that $f\colon (S,B_S) \dashrightarrow (S',B_{S'})$ is a strictly birational map of smooth completions, where (S,B_S) and $(S',B_{S'})$ dominate some ruled surfaces $p\colon \mathbb{F} \to D$ and $p' : \mathbb{F}' \to D'$ respectively. Then $B^2_{S'} = 0$, we can assume that the strict transforms of B_S and $B_{S'}$ are fibers of p and p' respectively, and the birational transformation $\mathbb{F} \dashrightarrow \mathbb{F}'$ induced by f consists of elementary transformations between ruled surfaces. It follows that f preserves the $\mathbb{P}^1$-fibrations $\rho : S \to D$ and $\rho' : S' \to D'$ induced by these rulings hence induces an isomorphism $f : V = S \smallsetminus B_S \to V' = S' \smallsetminus B_{S'}$ of $\mathbb{P}^1$-fibered quasi-projective surfaces

$$\begin{array}{ccc} V = S \smallsetminus B_S & \xrightarrow[f]{\sim} & V' = S' \smallsetminus B_{S'} \\ \downarrow{\scriptstyle \rho|_V} & & \downarrow{\scriptstyle \rho'|_{V'}} \\ D \smallsetminus \rho(B_S) & \xrightarrow{\sim} & D' \smallsetminus \rho'(B_{S'}). \end{array}$$

Next we consider the case of a triangular map $\phi\colon (S,B_S) \dashrightarrow (S',B_{S'})$ between smooth completions with $B_S^2 = B_{S'}^2 = d > 0$. Note that since B_S and $B_{S'}$ are smooth rational curves, it follows form Noether's Lemma that the surfaces S and S' are rational. We deduce from the description given in the proof of Lemma 12 that the total transform of B_S in the minimal resolution X of ϕ is a tree of rational curves with the dual graph pictured in Figure 6.

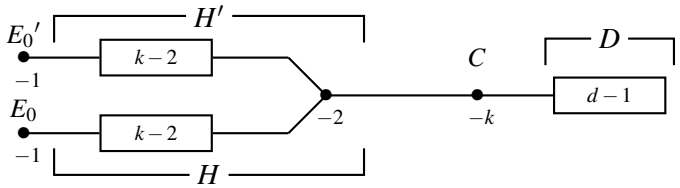


FIGURE 6. Minimal resolution of a triangular map.

Here E_0 and $E_0' = E_n$ denote the strict transforms of B_S and $B_{S'}$ respectively, the two boxes on the left represent chains of $k-2$ rational curves with self-intersection (-2), and the one on the right a chain D of $d-1$ such curves. Note also that the proper base point of ϕ coincides with the proper base point of the first elementary link $S = S_0 \dashrightarrow S_1$ while the one of its inverse coincides with the proper base point of the inverse of the last one $S_{n-1} \dashrightarrow S_n = S'$ (see Corollary 7).

Let $\delta : X \to \hat{S}$ and $\delta' : X \to \hat{S}'$ be the morphisms given by the smooth contractions of the sub-trees $H \cup H' \cup E_0'$ and $H \cup H' \cup E_0$ onto $q = E_0 \cap C$ and $q' = E_0' \cap C$. Since S and S' are rational and $E_0^2 = 0$ and $E_0'^2 = 0$ on $\hat{S}$ and $\hat{S}'$, it follows from Riemann-Roch Theorem that the complete linear systems $|E_0|$ and $|E_0'|$ are base point free and define $\mathbb{P}^1$-fibrations $\hat{\rho}\colon \hat{S} \to \mathbb{P}^1$ and $\hat{\rho}'\colon \hat{S}' \to \mathbb{P}^1$ both having the image of C as a section. Note further that the image of D in $\hat{S}$ and $\hat{S}'$ is a proper subset of a fiber of $\hat{\rho}$ and $\hat{\rho}'$ respectively: indeed, if not empty, the image of D has negative definite self-intersection matrix and hence cannot be equal to a full fiber of a $\mathbb{P}^1$-fibration. By contracting the remaining exceptional divisors, we see that $|E_0|$ (resp. $|E_0'|$) coincides with the strict transform on $\hat{S}$ (resp. $\hat{S}'$) of the rational subpencil $\mathcal{P}_{\mathcal{B}(\phi)} \subset |B_S|$ (resp. $\mathcal{P}'_{\mathcal{B}(\phi^{-1})} \subset |B_{S'}|$) consisting of curves having local intersection number with B_S (resp. $B_{S'}$) at $\mathcal{B}(\phi)$ (resp. $\mathcal{B}(\phi^{-1})$) equal to d. Equivalently, the fibrations $\hat{\rho}$ and $\hat{\rho}'$ coincide respectively with the minimal resolution of the rational maps $\rho\colon S \dashrightarrow \mathbb{P}^1$ and $\rho'\colon S' \dashrightarrow \mathbb{P}^1$ defined by $\mathcal{P}_{\mathcal{B}(\phi)}$ and $\mathcal{P}'_{\mathcal{B}(\phi^{-1})}$. These two maps restrict on $V = S \smallsetminus B_S$ and $V' = S' \smallsetminus B_{S'}$ to quasi-projective $\mathbb{A}^1$-fibrations $\rho\,|_V : V \to \mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \rho(B_S)$ and $\rho'\,|_{V'}\colon V' \to \mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \rho'(B_{S'})$, i.e., surjective morphisms with general fiber isomorphic to the affine line $\mathbb{A}^1$.

The birational map $\phi\colon S \dashrightarrow S'$ lifts to $\hat{\phi}\colon \hat{S} \dashrightarrow \hat{S}'$ mapping $\hat{S} \smallsetminus E_0$ isomorphically onto $\hat{S}' \smallsetminus E_0'$, having q as unique proper base point while its inverse has q' as a unique proper base point. Since the total transforms of E_0 and E_0' in X coincide, the lifted $\mathbb{P}^1$-fibrations $\hat{\rho} \circ \delta$ and $\hat{\rho}' \circ \delta'$ on X coincide. This implies that $\hat{\phi}$ restricts to an isomorphism of $\mathbb{A}^1$-fibered quasi-projective surfaces

$$\begin{array}{ccc} V = S \smallsetminus B_S = \hat{S} \smallsetminus \tau^{-1}(B_S) & \xrightarrow[\hat{\phi}]{\sim} & V' = S' \smallsetminus B_{S'} = \hat{S}' \smallsetminus \tau'^{-1}(B_{S'}) \\ \downarrow{\scriptstyle \hat{\rho}|_V} & & \downarrow{\scriptstyle \hat{\rho}'|_{V'}} \\ \mathbb{A}^1 & \xrightarrow{\quad\sim\quad} & \mathbb{A}^1 \end{array}$$

where $\tau : \hat{S} \to S$ and $\tau' : \hat{S}' \to S'$ denote the contraction of C and the right chain D of $d-1$ curves with self-intersection -2 pictured in Figure 6 above.

A birational map $\hat{\phi}\colon \hat{S} \dashrightarrow \hat{S}'$ restricting to an isomorphism of $\mathbb{A}^1$-fibered surfaces as above is called a *fibered modification* (see also [5, 2.2.1]).

In general, if (S, B_S) is a smooth completion with $B_S^2 = d > 0$ and p is a point of B_S then the base locus of the linear subsystem $\mathcal{P}_p \subset |B_S|$ consisting of curves having a local intersection number with B_S at p equal to d is solved as follows. We perform d successive blow-ups with centers on the successive strict transforms of B_S, until we reach a surface $\hat{S}$ on which the strict transform of $\mathcal{P}_p$ is equal to the complete linear system $|E_0|$ generated by the strict transform E_0 of B_S. Since E_0 is a smooth rational curve with $E_0^2 = 0$, $\mathcal{P}_p$ defines a rational pencil $\rho_p\colon S \dashrightarrow \mathbb{P}^1$ which restricts on $V = S \smallsetminus B_S$ to a quasi-projective $\mathbb{A}^1$-fibration $\rho_p|_V\colon V \to \mathbb{A}^1$. This leads to the following alternative characterization of triangular maps:

Lemma 14. *For a strictly birational map of smooth completions $\phi : (S, B_S) \dashrightarrow (S', B_{S'})$ with $B_S^2 = B_{S'}^2 > 0$, the following are equivalent :*

a) ϕ is a triangular map;
b) There exist points $p \in B_S$ and $p' \in B_{S'}$ such that ϕ maps the pencil $\mathcal{P}_p$ onto the pencil $\mathcal{P}'_{p'}$; If so, the points p and p' are equal to $\mathcal{B}(\phi)$ and $\mathcal{B}(\phi^{-1})$ respectively.
c) ϕ maps the pencil $\mathcal{P}_{\mathcal{B}(\phi)}$ onto the pencil $\mathcal{P}'_{\mathcal{B}(\phi^{-1})}$;
d) ϕ induces an isomorphism of $\mathbb{A}^1$-fibered quasi-projective surfaces $(S \smallsetminus B_S, \rho_{\mathcal{B}(\phi)}) \xrightarrow{\sim} (S' \smallsetminus B_{S'}, \rho'_{\mathcal{B}(\phi^{-1})})$.

Proof. Properties c) and d) are clearly equivalent. If b) holds and the proper base point of ϕ is distinct from p then all infinitely near base points of ϕ are also distinct from p. Since ϕ contracts B_S, the strict transform in S' of a general member of $\mathcal{P}_p$ has self-intersection strictly bigger than $B_S^2 = B_{S'}^2$ hence cannot be a general member of a pencil of the form $\mathcal{P}'_{p'}$. So $\mathcal{B}(\phi) = p$ and for the same reason $\mathcal{B}(\phi^{-1}) = p'$ which proves the equivalence of b) and c). The fact that a triangular map $\phi : (S, B_S) \dashrightarrow (S', B_{S'})$ maps $\mathcal{P}_{\mathcal{B}(\phi)}$ onto $\mathcal{P}'_{\mathcal{B}(\phi^{-1})}$ follows from the above discussion.

It remains to prove that c) implies a). If c) holds then since S and S' are both smooth, the strict transforms of B_S and $B_{S'}$ in the minimal resolution $S \xleftarrow{\sigma} X \xrightarrow{\sigma'} S'$ of ϕ are both (-1)-curves. So ϕ and ϕ^{-1} both have at least $d+1$ base points including infinitely near ones and their first $d+1$ base points are supported on B_S and $B_{S'}$ respectively. This implies in turn that σ and σ' factor respectively through the minimal resolutions $\pi\colon \hat{S} \to S$ and $\pi'\colon \hat{S}' \to S'$ of the base points of $\mathcal{P}_{\mathcal{B}(\phi)}$ and $\mathcal{P}'_{\mathcal{B}(\phi^{-1})}$ and that the induced birational map $\hat{\phi} : \hat{S} \dashrightarrow \hat{S}'$ is a fibered modification. By virtue of [5, 2.2.4], the dual graph of the total transform of B_S in X looks like the one pictured in Figure 6 for which it is straightforward to check that all intermediate surfaces occurring in the decomposition are singular. Thus ϕ is a triangular map. □

The following Corollary, which is an immediate consequence of the previous Lemma, will be frequently used in the sequel:

Corollary 15. *If $\phi\colon (S, B_S) \dashrightarrow (S'', B_{S''})$ and $\phi''\colon (S'', B_{S''}) \dashrightarrow (S', B_{S'})$ are triangular maps of smooth completions with $B_S^2 > 0$ and ϕ, ϕ'' in special position, then the composition $\phi' = \phi'' \circ \phi\colon (S, B_S) \dashrightarrow (S', B_{S'})$ is either an isomorphism of pairs mapping $\mathcal{B}(\phi)$ on $\mathcal{B}(\phi''^{-1})$ or a triangular map with $\mathcal{B}(\phi') = \mathcal{B}(\phi)$ and $\mathcal{B}(\phi'^{-1}) = \mathcal{B}(\phi''^{-1})$.*

3.2. **Automorphisms of quasi-projective surfaces with smooth completions.**

3.2.1. *Decomposition into triangular maps and normal forms.* Given a strictly birational map $f : (S, B_S) \dashrightarrow (S', B_{S'})$ of smooth completions with $B_S^2 = B_{S'}^2 > 0$, Lemma 12 provides a decomposition

of f into a finite sequence

$$f = \phi_n \dots \phi_1 \colon (S, B_S) = (S_0, B_{S_0}) \overset{\phi_1}{\dashrightarrow} (S_1, B_{S_1}) \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} (S_n, B_{S_n}) = (S', B_{S'})$$

of triangular maps between smooth completions. Such a decomposition of f is called *minimal* if there does not exist any other decomposition with strictly less than n terms. The following Proposition provides a characterization of these minimal decompositions.

Proposition 16. *A composition*

$$f = \phi_n \dots \phi_1 \colon (S, B_S) = (S_0, B_{S_0}) \overset{\phi_1}{\dashrightarrow} (S_1, B_{S_1}) \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} (S_n, B_{S_n}) = (S', B_{S'})$$

of triangular maps between smooth completions with $B_S^2 = B_{S'}^2 > 0$ is minimal if and only if for every $i = 1, \dots, n-1$, the maps ϕ_i and ϕ_{i+1} are in general position.

Furthermore, if these conditions are satisfied, then the following holds:

a) The map f is strictly birational with $\mathcal{B}(f) = \mathcal{B}(\phi_1)$ and $\mathcal{B}(f^{-1}) = \mathcal{B}(\phi_n^{-1})$,

b) For every other minimal decomposition

$$f = \phi'_n \dots \phi'_1 \colon (S, B_S) = (S'_0, B_{S'_0}) \overset{\phi'_1}{\dashrightarrow} (S'_1, B_{S'_1}) \overset{\phi'_2}{\dashrightarrow} \cdots \overset{\phi'_n}{\dashrightarrow} (S'_n, B_{S'_n}) = (S', B_{S'})$$

of f there exists isomorphisms of pairs $\alpha_0 = \mathrm{id}_S$, $\alpha_i \colon (S_i, B_{S_i}) \xrightarrow{\sim} (S'_i, B_{S'_i})$, $i = 1, \dots n-1$ and $\alpha_n = \mathrm{id}_{S'}$, such that $\alpha_i \phi_i = \phi'_i \alpha_{i-1}$ for every $i = 1, \dots, n$.

Proof. First note that by virtue of Corollary 15, the composition of two triangular maps in special position is either triangular or an isomorphism of pairs. Therefore a composition $\phi_n \dots \phi_1$ in which for some i the maps ϕ_i and ϕ_{i+1} are in special position cannot be minimal.

Next assume that $f = \phi_n \cdots \phi_1$ is a composition for which any two successive triangular maps are in general position. To prove a), up to changing f with its inverse, it is enough to check that f is strictly birational with $\mathcal{B}(f^{-1}) = \mathcal{B}(\phi_n^{-1})$. We proceed by induction on n, the case $n = 1$ being obvious. If $n > 1$ then by induction hypothesis $f_{n-1} = \phi_{n-1} \dots \phi_1$ is a strictly birational map which contracts the curve B_{S_0} to the proper base point $p \in B_{S_{n-1}}$ of ϕ_{n-1}^{-1}. The curve $B_{S_{n-1}}$ is contracted in turn by ϕ_n onto the proper base point of ϕ_n^{-1}. But since ϕ_{n-1} and ϕ_n are in general position, p is not a base point of ϕ_n and so, $f = \phi_n f_{n-1}$ contracts B_{S_0} onto $\phi_n(B_{S_{n-1}}) = \mathcal{B}(\phi_n^{-1})$. This shows that f is strictly birational and that $\mathcal{B}(f^{-1}) = \mathcal{B}(\phi_n^{-1})$.

Now let $f = \phi'_m \dots \phi'_1 \colon (S, B_S) \dashrightarrow (S', B_{S'})$, $m \leq n$, be a minimal decomposition of f into triangular maps. By Corollary 15 again, any two successive triangular maps must be in general position. If ϕ_1^{-1} and ϕ'_1 were in general position, then by a) $\phi'_m \dots \phi'_1 \phi_1^{-1} \dots \phi_n^{-1}$ would be a strictly birational map restricting to the identity on $S' \setminus B_{S'}$, whence on S', which is absurd. Therefore, ϕ_1^{-1} and ϕ'_1 are in special position and it follows from Corollary 15 that $\alpha_1 = \phi'_1 \phi_1^{-1} \colon (S_1, B_{S_1}) \dashrightarrow (S'_1, B_{S'_1})$ is either a triangular map or an isomorphism of pairs. But if α_1 is triangular, then, again by Corollary 15, we would have $\mathcal{B}(\alpha_1) = \mathcal{B}(\phi_1^{-1})$ and $\mathcal{B}(\alpha_1^{-1}) = \mathcal{B}((\phi'_1)^{-1})$. The pairs of maps α_1, ϕ'_2 and ϕ_2^{-1}, α_1 would then be both in general position and $\phi'_m \dots \phi'_2 \alpha_1 \phi_2^{-1} \dots \phi_n^{-1}$ would again be strictly birational. So α_1 is an isomorphism of pairs and writing $\psi'_2 = \phi'_2 \alpha_1$, which is again a triangular map, we deduce in a similar way that $\psi'_2 \phi_2^{-1} \colon (S_2, B_{S_2}) \dashrightarrow (S'_2, B_{S'_2})$ is an isomorphism, that we denote by α_2. By induction, we define $\psi'_r = \phi'_r \alpha_{r-1}$ and obtain an isomorphism $\alpha_r = \psi'_r \phi_r^{-1} \colon (S_r, B_{S_r}) \dashrightarrow (S'_r, B_{S'_r})$ for $r = 2, \dots, m$. The last relation obtained is $\alpha_m \phi_{m+1}^{-1} \dots \phi_n^{-1} = \mathrm{id}_{S'} : (S', B_{S'}) \dashrightarrow (S', B_{S'})$ from which we deduce that $m = n$, and $\alpha_n = \mathrm{id}_{S'}$. Choosing $\alpha_0 = \mathrm{id}_S$ we find that $\alpha_i \phi_i = \phi'_i \alpha_{i-1}$ for every $i = 1, \dots, n$.

This proves on the one hand that the decomposition $f = \phi_n \cdots \phi_1$ was minimal and that b) holds for this decomposition. □

Definition 17. The number $\ell(f)$ of triangular maps occurring in a minimal decomposition of a birational map $f : (S,B) \dashrightarrow (S',B_{S'})$ of smooth completions is called the *length* of f.

Corollary 18. *Let $f = \phi_n \dots \phi_1 : (S,B_S) \dashrightarrow (S',B_{S'})$ be a strictly birational composition of $n \geq 2$ triangular maps. If $\mathcal{B}(f) \neq \mathcal{B}(\phi_1)$ then there exists an index $i \in \{2,\dots,n-1\}$ such that $\phi_i \dots \phi_1$ is an isomorphism.*

Proof. We proceed by induction on the number of triangular maps in the composition. If $n = 2$ then if ϕ_2 and ϕ_1 are in general position or if ϕ_2 and ϕ_1 are in special position and $\phi_2\phi_1$ is triangular then $\mathcal{B}(\phi_2\phi_1) = \mathcal{B}(\phi_1)$ by Proposition 16 and Corollary 15. So $\mathcal{B}(\phi_2\phi_1) = \mathcal{B}(\phi_1)$ unless $\phi_2\phi_1$ is an isomorphism.

Now suppose that $n > 2$.

If ϕ_2 and ϕ_1 are in special position then either $\phi_2\phi_1$ is an isomorphism and we are done, or $\phi_2' = \phi_2\phi_1$ is a triangular map with proper base point equal to that of ϕ_1. Since $f = \phi_n \dots \phi_2'$ with $\mathcal{B}(f) \neq \mathcal{B}(\phi_2')$ the induction hypothesis implies that there exists $i \in \{3,\dots,n-1\}$ such that $\phi_i \dots \phi_2' = \phi_i \dots \phi_2\phi_1$ is an isomorphism.

If ϕ_2 and ϕ_1 are in general position and either $\phi_n \dots \phi_2$ is an isomorphism or $\mathcal{B}(\phi_n \dots \phi_2) = \mathcal{B}(\phi_2)$. Then in both cases $\mathcal{B}(\phi_n \dots \phi_2\phi_1)$ would be equal to $\mathcal{B}(\phi_1)$. So $\phi_n \dots \phi_2$ is not an isomorphism and its proper base point is different from that of ϕ_2. By induction hypothesis, there exists an index $j \in \{3,\dots,n-1\}$ such that $\alpha = \phi_j \dots \phi_2$ is an isomorphism. Replacing ϕ_{j+1} by the triangular map $\phi'_{j+1} = \phi_{j+1}\alpha$, we have $f = \phi_n \dots \phi'_{j+1}\phi_1$ and we are done by induction. □

One can think of Proposition 16 as a kind of presentation by generators and relations, the second part saying in particular that there is essentially no relation except the obvious ones given by Corollary 15. However, even if $(S,B_S) = (S',B_{S'})$ and f is the birational map induced by an automorphism of $V = S \smallsetminus B_S$, in general the triangular maps ϕ_i are not birational transformations between isomorphic smooth completions of V (see §4.3 and 4.4 for illustrations of such situations). If one insists in having generators that live on a particular model, one possibility is to fix a rule to pass from each possible model to the distinguished one (S,B_S). This is what is done in [14], where the relations are then expressed in terms of (intricate) amalgamated products.

Another consequence of Proposition 16 is that it enables to obtain *normal forms* for automorphisms of quasi-projective surfaces admitting a smooth completion. In the following result, and in the rest of the paper, we shall use the notation f^ψ to denote the conjugate $\psi f \psi^{-1}$.

Corollary 19. *Let $f : (S,B_S) \dashrightarrow (S,B_S)$ be a birational self-map of a smooth completion. Then there exists a birational map of smooth completions $\psi : (S,B_S) \dashrightarrow (S',B_{S'})$ such that the conjugate f^ψ has one of the following properties :*

a) f^ψ is a biregular automorphism of the pair $(S',B_{S'})$,

b) f^ψ is a triangular self-map of $(S',B_{S'})$ with the pair f^ψ, f^ψ in special position,

c) The pair f^ψ, f^ψ is in general position.

Proof. Suppose f is not biregular, and consider a minimal factorization $f = \phi_n \dots \phi_1$ into triangular maps given by Proposition 16. If the pair f, f is in special position and f is not triangular (that is, $n \geq 2$), then we consider the conjugate $f_{n-1} = \phi_n^{-1} f \phi_n : (S_{n-1},B_{S_{n-1}}) \dashrightarrow (S_{n-1},B_{S_{n-1}})$. By hypothesis, ϕ_1 and ϕ_n are in special position and so, by Corollary 15, $\phi_1\phi_n$ is either an isomorphism or a triangular map. Thus $f_{n-1} = \phi_{n-1} \dots \phi_2(\phi_1\phi_n)$ has length at most $n-1$ and we are done by induction. □

The existence of normal forms up to conjugacy for automorphisms of $\mathbb{A}^2$ was first noticed by Friedland and Milnor [10] as a consequence of Jung's Theorem. This was the starting point for an exhaustive

study of the possible dynamical behavior of these automorphisms (see [3] and references therein). In particular, if $g = f^{\Psi}$ satisfies Property c) in the conclusion of Corollary 19 and has length n then its iterates g^k, $k \in \mathbb{Z}$, have length $n|k|$. Such a map is thus similar to a composition of generalized Hénon maps and so one can expect that these maps will always present a chaotic dynamical behavior. On the other hand, any finite automorphism of V, any one-parameter flow f_t of automorphisms of V, or more generally every automorphism contained in an algebraic subgroup of $\mathrm{Aut}(V)$ (see Proposition 21 below) always corresponds to Case a) or b) in Corollary 19.

3.2.2. *Tame automorphisms.* Given any smooth completion (S, B_S) of V, we denote by $\mathrm{Aut}(S, B_S)$ the group of automorphisms of the pair (S, B_S). Note that since B_S is the support of an ample divisor on S, $\mathrm{Aut}(S, B_S)$ is an algebraic group.

For every point $p \in B_S$, Corollary 15 implies that the set of triangular self-maps $\phi\colon (S, B_S) \dashrightarrow (S, B_S)$ with $\mathcal{B}(\phi) = \mathcal{B}(\phi^{-1}) = p$ and automorphisms of the pair (S, B_S) fixing p is a group, which we shall denote by $\mathrm{Tr}(S, B_S, p)$. By Lemma 14, the latter coincides with the subgroup of automorphisms of $V = S \smallsetminus B_S$ preserving the quasi-projective $\mathbb{A}^1$-fibration $\rho_p : V \to \mathbb{A}^1$ induced by the rational pencil $\mathcal{P}_p$. The groups $\mathrm{Tr}(S, B_S, p)$ are not algebraic, but they are countable increasing unions of algebraic subgroups. More precisely, see [5, Lemma 2.2.3], there exists a birational map $\tau : S \dashrightarrow \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x, y])$ restricting to a morphism on V such that $\mathrm{Tr}(S, B_S, p)$ is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ consisting of automorphisms of the form $(x, y) \mapsto (ax + b, cy + P(x))$, where $a, c \in \mathbb{C}^*$, $P(x) \in \mathbb{C}[x]$, preserving the points blown-up by τ, including infinitely near ones. For every $d \geq 0$, the set of all automorphisms $(x, y) \mapsto (ax + b, cy + P(x))$ of $\mathbb{A}^2$ with $P(x)$ of degree $\leq d$ is an algebraic group and those preserving the points blown-up by τ form of closed subgroup of it, whence an algebraic group.

Definition 20. Let V be a quasi-projective surface admitting a smooth completion (S, B_S), let $\mathcal{M} = \{(\psi_\alpha, p_\alpha)\}_{\alpha \in A}$ be a nonempty collection of pairs consisting for each $\alpha \in A$ of a birational map of smooth completions $\psi_\alpha\colon (S, B_S) \dashrightarrow (S_\alpha, B_{S_\alpha})$ and a point $p_\alpha \in B_{S_\alpha}$. An automorphism of V considered as a birational self-map f of (S, B_S) is called:

a) *$\mathcal{M}$-affine* (short for *affine relatively to the models in $\mathcal{M}$*) if there exists α such that f^{ψ_α} is an element of $\mathrm{Aut}(S_\alpha, B_{S_\alpha})$;

b) *$\mathcal{M}$-Jonquières* if there exists α such that f^{ψ_α} is an element of $\mathrm{Tr}(S_\alpha, B_{S_\alpha}, p_\alpha)$.

We denote by $\mathcal{M}\mathrm{TA}(V)$ the subgroup of $\mathrm{Aut}(V)$ generated by $\mathcal{M}$-affine and $\mathcal{M}$-Jonquières automorphisms. We call it the group of *$\mathcal{M}$-tame automorphisms* of V.

This notion of tameness depends a priori on the choice of the collection $\mathcal{M}$. However by taking the family $\mathcal{M}_{\mathrm{can}}$ consisting of all pairs (Ψ, p') where $\Psi\colon (S, B_S) \dashrightarrow (S', B_{S'})$ is a birational map of smooth completions and p' is a point of $B_{S'}$, we obtain a canonical intrinsic notion of tameness.

An automorphism of V with associated birational self-map f of (S, B_S) is said to be *generalized affine* (resp. *generalized Jonquières*) if f is $\mathcal{M}_{\mathrm{can}}$-affine (resp. $\mathcal{M}_{\mathrm{can}}$-Jonquières). We denote

$$\mathrm{GTA}(V) = \mathcal{M}_{\mathrm{can}}\mathrm{TA}(V)$$

the subgroup of $\mathrm{Aut}(V)$ generated by generalized Jonquières and generalized affine automorphisms. Its elements will be called *generalized tame* automorphisms of V.

In other words, $\mathrm{GTA}(V)$ is generated by automorphisms of V which either preserve a quasi-projective $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ induced by a pencil of the form $\mathcal{P}_p$ on a suitable smooth completion of V or extend to biregular automorphisms of suitable smooth completions of V. In fact, since for an element $f \in \mathrm{Aut}(S, B_S)$ the induced action of f on $B_S \simeq \mathbb{P}^1$ always has a fixed point p, it follows that every generalized affine automorphism is also generalized Jonquières. So $\mathrm{GTA}(V)$ coincides with the normal subgroup of $\mathrm{Aut}(V)$ generated by automorphisms preserving an $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ as above.

Proposition 21. *Let V be a quasi-projective surface admitting a smooth completion. Then for every algebraic subgroup G of $\mathrm{Aut}(V)$, there exists a smooth completion (S,B_S) of V such that G is a subgroup of $\mathrm{Aut}(S,B_S)$ or of $\mathrm{Tr}(S,B_S,p)$. In particular, every algebraic subgroup of $\mathrm{Aut}(V)$ consists of generalized tame automorphisms of V.*

Proof. Let $(S',B_{S'})$ be a smooth completion of V. By Sumihiro equivariant completion theorem [26], there exists a smooth projective surface Z on which G acts biregularly and a G-equivariant open embedding $V \hookrightarrow Z$. The induced birational map $\tau\colon Z \dashrightarrow S'$ has finitely many base points, including infinitely near ones and similarly for its inverse. It follows that the number of base points of an element g of G considered as a birational self-map $g\colon (S',B_{S'}) \dashrightarrow (S',B_{S'})$ is bounded by the sum of the number of base points of τ and its inverse. This implies in turn that there exists a minimal integer $M(S') \geq 0$ such that the length of any such g is at most $M(S')$ (in the sense of Definition 17). The bound $M(S')$ depends on the particular completion we choose to realize the birational action of G. Now we choose a smooth completion (S,B_S) of V such that $M(S)$ is minimal. If $M(S)=0$ the birational self-map $g\colon (S,B_S) \dashrightarrow (S,B_S)$ associated to every element of G is an automorphism of pairs and hence $G \subset \mathrm{Aut}(S,B_S)$.

Assume now that $M(S) \geq 1$, and let $g\colon (S,B_S) \dashrightarrow (S,B_S)$ be the birational self-map associated to an element of G which realizes the bound $M(S)$, that is $g = \psi_M \circ \cdots \circ \psi_1$ is a composition of $M = M(S)$ triangular maps. Let $h\colon (S,B_S) \dashrightarrow (S,B_S)$ be the birational self-map associated to another element of G.

Suppose first that $\ell(h) \geq 1$ and let $h = \phi_m \circ \cdots \circ \phi_1$, where $m = \ell(h)$, be a minimal decomposition of h into triangular maps. If h^{-1} and g were in general position then, by Proposition 16 $\psi_M \cdots \psi_1 \phi_1^{-1} \cdots \phi_m^{-1}$ would be a minimal decomposition of $g \circ h^{-1}$, and we would have $\ell(g \circ h^{-1}) = \ell(g) + \ell(h) > M(S)$ in contradiction with the definition of $M(S)$. So $\mathcal{B}(\phi_1) = \mathcal{B}(h) = \mathcal{B}(g) = \mathcal{B}(\psi_1)$, and $\psi_1 \circ \phi_1^{-1}$ is either triangular or biregular. If $\psi_1 \circ \phi_1^{-1}$ is triangular, then by Corollary 15 the pairs of maps $(\psi_1 \circ \phi_1^{-1}), \psi_2$ and $\phi_2^{-1}, (\psi_1 \circ \phi_1^{-1})$ are both in general position, and we have $\ell(g \circ h^{-1}) = M + m - 1$, so $m = 1$. This implies that if $\ell(h) \geq 2$ then $\psi_1 \circ \phi_1^{-1}$ is biregular, and applying the same reasoning to h^{-1} instead of h we also get that $\psi_1 \circ \phi_m$ is biregular in this case. Finally observe that in the case $\ell(h) = 1$ we have $\ell(\psi_1 h \psi_1^{-1}) = \ell(\psi_1 h^{-1} \psi_1^{-1}) \leq 1$: indeed either $\psi_1 \circ \phi_1^{-1}$ is biregular and this is clear, or $\psi_1 \circ \phi_1^{-1}$ is triangular with base point equal to $\mathcal{B}(\phi_1^{-1}) = \mathcal{B}(\psi_1)$ and so ψ_1^{-1} and $\psi_1 \circ \phi_1^{-1}$ are in special position.

Consider now the case $\ell(h) = 0$. We claim that h fixes $\mathcal{B}(g) = \mathcal{B}(\psi_1)$: Otherwise g^{-1} and gh would be in general position and we would have $\ell(ghg^{-1}) = 2\ell(g) > M(S)$, a contradiction. It follows that $\psi_1 h \psi_1^{-1}$ is triangular or biregular in this case.

In conclusion, conjugating the group G by the birational map $\psi_1 : (S,B_S) \dashrightarrow (S_1,B_{S_1})$, we obtain that $\ell(\psi_1 h \psi_1^{-1}) \leq 1$ if $\ell(h) \leq 1$ whereas $\ell(\psi_1 h \psi_1^{-1}) = l(h) - 2$ if $\ell(h) \geq 2$. So $M(S) = 1$ for otherwise we would have $M(S_1) < M(S)$, in contradiction with the minimality of $M(S)$. This shows that all elements in G extend to biregular or triangular maps from S to itself. The argument above shows that the point $p = \mathcal{B}(g)$ is fixed by all biregular elements of G and is the proper base point of all triangular elements in G, that is, $G \subset \mathrm{Tr}(S,B_S,p)$. □

3.3. **Automorphisms of affine surfaces with smooth completions.** Here we consider the particular case of affine surfaces V admitting smooth completions (S,B_S). By Proposition 11, every such surface is isomorphic to $\mathbb{P}^2 \smallsetminus C$ where C is a line or a smooth conic or to the complement of an ample section C in a Hirzebruch surface $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$. As we saw before, the integer $d = B_S^2$ is an invariant of V and in fact the only invariant except in the case $d = 4$. Indeed, by the Danilov-Gizatullin Isomorphy Theorem [14], the isomorphy type as an abstract affine surface of the complement of an ample section C in a Hirzebruch surface depends neither on the ambient projective surface nor on the choice of the section,

but only on its self-intersection. The following proposition summarizes some of the properties of the automorphism groups of these surfaces.

Proposition 22. *For an affine surface V_d admitting a smooth completion (S,B_S) with $B_S^2 = d > 0$, the following holds :*

1) If $d \leq 4$ then every automorphism of V_d is generalized tame, and one has $\mathrm{Aut}(V_d) = \mathcal{M}\mathrm{TA}(V_d)$ for a finite family $\mathcal{M}$ of completions. In particular, $\mathrm{Aut}(V_d)$ is generated by countably many algebraic subgroups.

2) If $d \geq 5$ then $\mathrm{GTA}(V_d)$ is a proper normal subgroup of $\mathrm{Aut}(V_d)$ and it cannot be generated by countably many algebraic subgroups.

Proof. The fact that for every $d \leq 4$, $\mathrm{Aut}(V_d) = \mathcal{M}\mathrm{TA}(V_d)$ for a natural choice of finitely many smooth completions $\mathcal{M}$ is checked in the examples in Section 4. The second part of assertion 1) then follows from the fact groups of the form $\mathrm{Aut}(S,B_S)$ and $\mathrm{Tr}(S,B_S,p)$ are generated by countable families of algebraic subgroups (see §3.2.2).

For the second assertion, we first need to prove that if $d \geq 5$ then there exist smooth completions (S,B_S) of V_d with the property that the orbits of the induced action of $\mathrm{Aut}(S,B_S)$ on B_S are finite. Namely, if $d > 7$ is even then V_d admits a smooth completion $(\mathbb{F}_2,C)$ where C is a section of $\pi_2 \colon \mathbb{F}_2 \to \mathbb{P}^1$ with self-intersection d intersecting the exceptional section C_0 of π_2 with self-intersection -2 in $(d-2)/2 \geq 3$ distinct points. Similarly, if $d \geq 7$ is odd, then there exists a smooth completion $(\mathbb{F}_1,C)$ of V where C is a section of $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ with self-intersection d intersecting the exceptional section C_0 of π_1 with self-intersection -1 in $(d-1)/2 \geq 3$ distinct points. Since in each case the set $C \cap C_0$ is necessarily globally preserved by the induced action of the automorphism group of the ambient pair on $C \simeq \mathbb{P}^1$, we conclude that the orbits of this group on C are finite. The remaining two cases $d = 5,6$ are treated explicitly in the examples in Section 4, §4.4.1 1-b) and §4.5 respectively.

From now on, we identify $V = V_d$, $d \geq 5$, to $S \setminus B_S$ for a fixed smooth completion (S,B_S) with the property that the orbits of the induced $\mathrm{Aut}(S,B_S)$-action on B_S are finite.

Let us show first that $\mathrm{GTA}(V)$ cannot be generated by a countable family $(G_i)_{i\in\mathbb{N}}$ of algebraic subgroups. By Proposition 21, to each G_i is associated a smooth completion (S_i,B_{S_i}) of V and a birational map of pairs $\psi_i \colon (S,B_S) \dashrightarrow (S_i,B_{S_i})$, such that $\psi_i G_i \psi_i^{-1}$ is contained either in $\mathrm{Aut}(S_i,B_{S_i})$ or in $\mathrm{Tr}(S_i,B_{S_i},p_i)$ for some point $p_i \in B_{S_i}$. For every $i \in \mathbb{N}$, we fix a minimal decomposition $\psi_i = \phi_{i,n_i} \dots \phi_{i,1} \colon (S,B_S) = (S_{i,0},B_{S_{i,0}}) \dashrightarrow (S_{i,n_i},B_{S_{i,n_i}}) = (S_i,B_{S_i})$ of ψ_i into triangular maps $\phi_{i,k} \colon (S_{i,k-1},B_{S_{i,k-1}}) \dashrightarrow (S_{i,k},B_{S_{i,k}})$. Let $q_{ij} = \mathcal{B}(\phi_{i,j+1})$ and $r_{ij} = \mathcal{B}(\phi_{i,j}^{-1})$: Observe that q_{ij} and r_{ij} are both points in $B_{S_{i,j}}$. Then define $\mathcal{C}$ as the subset of B_S consisting of points of the form $\alpha_{i,j}^{-1}(q_{ij})$, $\beta_{i,j}^{-1}(r_{i,j})$ and $\gamma_i^{-1}(p_i)$ for all possible isomorphisms of pairs $\alpha_{i,j} \colon (S,B_S) \to (S_{i,j},B_{S_{i,j}})$, $\beta_{i,j} \colon (S,B_S) \to (S_{i,j},B_{S_{i,j}})$ and $\gamma_i \colon (S,B_S) \to (S_i,B_{S_i})$ respectively.

It then follows from Corollary 18 that the set of possible proper base points of elements of $\mathrm{Aut}(V)$ considered as birational self-maps of (S,B_S) is contained in $\mathcal{C}$. Note that Proposition 16 b) implies that $\mathcal{C}$ does not depend on the choice of the minimal decompositions of the birational maps ψ_i. Our choice of (S,B_S) implies that the $\mathrm{Aut}(S,B_S)$-orbit of $\mathcal{C}$ is countable (in fact one could show that $\mathcal{C}$ is already $\mathrm{Aut}(S,B_S)$-invariant, but this is not necessary for the argument). So for every point p in its complement, a strictly birational element in $\mathrm{Tr}(S,B_S,p)$ is an element of $\mathrm{GTA}(V)$ which does not belong to the subgroup generated by the G_i.

Next, to derive that $\mathrm{GTA}(V)$ is a proper subgroup of $\mathrm{Aut}(V)$, we exploit a more precise version of the Danilov-Gizatullin Isomorphy Theorem (see [8, §3.1], in particular Lemma 3.2 and the discussion just before) which asserts that if $(S',B_{S'})$ and $(S'',B_{S''})$ are smooth completions of V and $p' \in B_{S'}$, $p'' \in B_{S''}$

are general points, then the $\mathbb{A}^1$-fibered affine surfaces $\rho'|_{V'}\colon V' = S' \smallsetminus B_{S'} \to \mathbb{A}^1$ and $\rho''|_{V''}\colon V'' = S'' \smallsetminus B_{S''} \to \mathbb{A}^1$ are isomorphic. More precisely, this holds whenever ρ' and ρ'' have reduced fibers, a property which is always satisfied for general p' and p''. In view of Lemma 14, this implies in particular that for every pair of general points p and p' in B_S, there exists a triangular map $\theta\colon (S,B_S) \dashrightarrow (S,B_S)$ with $\mathcal{B}(\theta) = p$ and $\mathcal{B}(\theta^{-1}) = p'$. By virtue of Lemma 23 below, such a map θ corresponds to an element of $\mathrm{GTA}(V)$ if and only if $\mathcal{B}(\theta)$ and $\mathcal{B}(\theta^{-1})$ belong to a same orbit of the induced action of $\mathrm{Aut}(S,B_S)$ on B_S. Since $\mathrm{Aut}(S,B_S)$ acts on B_S with finite orbits, it follows that we can find two general points p, p' of B_S in distinct orbits; a corresponding triangular self-map θ then induces an element of $\mathrm{Aut}(V)\setminus\mathrm{GTA}(V)$. □

In the proof of the previous theorem, we used the following characterization of generalized tame automorphisms of length 1:

Lemma 23. *A triangular map $\theta\colon (S,B_S) \dashrightarrow (S,B_S)$ of smooth completions is a generalized tame automorphism of $S\setminus B_S$ if and only if $\mathcal{B}(\theta)$ and $\mathcal{B}(\theta^{-1})$ belong to a same orbit of the action of $\mathrm{Aut}(S,B_S)$ on B_S.*

Proof. Clearly if there exists an automorphism $\alpha \in \mathrm{Aut}(S,B_S)$ such that $p = \mathcal{B}(\theta) = \alpha\mathcal{B}(\theta^{-1})$, then $\alpha\theta \in \mathrm{Tr}(S,B_S,p)$, hence θ is generalized tame.

Conversely, let θ be a generalized tame triangular map. For any automorphism $\alpha \in \mathrm{Aut}(S,B_S)$, $\alpha\theta$ is again generalized tame and triangular hence can be written in the form

$$\alpha\theta = \prod_{i=1}^{n} g_i^{\psi_i} = \psi_n g_n \psi_n^{-1} \dots \psi_1 g_1 \psi_1^{-1}$$

where for every $i = 1,\dots,n$, $\psi_i\colon (S_i,B_{S_i}) \dashrightarrow (S,B_S)$ is a birational map of smooth completions, and $g_i \in \mathrm{Aut}(S_i,B_{S_i}) \cup \bigcup_{p_i\in B_{S_i}} \mathrm{Tr}(S_i,B_{S_i},p_i)$. Among all such factorizations (for all choices of α), we choose one with the property that $L = \sum_{i=1}^{n}(\ell(\psi_i)+\ell(g_i)+\ell(\psi_i^{-1}))$ is minimal, which implies that for all i we have $\ell(g_i^{\psi_i}) = 2\ell(\psi_i)+\ell(g_i)$. Furthermore among all factorizations realizing this minimum L, we pick one with the minimal number n of factors.

Now if a composition of the form $\beta = \prod_{i=i_0}^{j_0} g_i^{\psi_i}$, where $1 \le i_0 \le j_0 \le n$, were an isomorphism, then for all $k > j_0$ we could write

$$g_k^{\psi_k}\beta = \beta\beta^{-1} g_k^{\psi_k}\beta = \beta g_k^{\beta^{-1}\psi_k}$$

So we could shift the β all the way to the left, without increasing L since $\ell(\psi_k) = \ell(\beta^{-1}\psi_k)$, and this would contradict the minimality of n. Thus we can assume that no composition of the form $\prod_{i=i_0}^{j_0} g_i^{\psi_i}$ is an isomorphism, or in other words that $\ell(\prod_{i=i_0}^{j_0} g_i^{\psi_i}) \ge 1$. Taking $i_0 = j_0$ we obtain in particular that no $g_i^{\psi_i}$ is an isomorphism. We deduce from Corollary 7 that $\mathcal{B}(g_i^{\psi_i}) = \mathcal{B}(\psi_i^{-1})$ if ψ_i is not an isomorphism, and $\mathcal{B}(g_i^{\psi_i}) = \psi_i(\mathcal{B}(g_i))$ otherwise. We also observe that in both cases $\mathcal{B}(g_i^{\psi_i}) = \mathcal{B}((g_i^{\psi_i})^{-1})$.

Now we check that we are in position to apply Corollary 18. Since we already know that no composition of the form $\prod_{i=i_0}^{j_0} g_i^{\psi_i}$ is an isomorphism, and since $\ell(g_i^{\psi_i}) = 2\ell(\psi_i)+\ell(g_i)$, it remains to exclude the existence of two indexes $j_0 > i_0$ such that $g_{j_0}^{\psi_{j_0}} = fh$ with f,h in general position and $\beta = h\prod_{i=i_0}^{j_0-1} g_i^{\psi_i}$ an isomorphism. In such a case we would have

$$\prod_{i=i_0}^{j_0} g_i^{\psi_i} = f\beta = h^{-1}\beta\beta^{-1}hf\beta = \left(\prod_{i=i_0}^{j_0-1} g_i^{\psi_i}\right)\beta^{-1}hf\beta.$$

But on one hand $\beta^{-1}hf\beta = (\beta^{-1}h)fh(h^{-1}\beta) = \left(\prod_{i=i_0}^{j_0-1} g_i^{\psi_i}\right)^{-1} g_{j_0}^{\psi_{j_0}} \left(\prod_{i=i_0}^{j_0-1} g_i^{\psi_i}\right)$ is a conjugate of g_{j_0}, and on the other hand h, f are in special position. So $\ell(\beta^{-1}hf\beta) = \ell(hf) < \ell(fh)$, in contradiction with the minimality of L.

Therefore, it follows from Corollary 18 and the observation made on the proper base point of $g_i^{\psi_i}$ that for all $i_0 = 2, \ldots, n$:

$$\mathcal{B}\left(g_{i_0}^{\psi_{i_0}}\right) = \mathcal{B}\left(\prod_{i=i_0}^{n} g_i^{\psi_i}\right) \quad \text{and} \quad \mathcal{B}\left(\left(\prod_{i=1}^{i_0-1} g_i^{\psi_i}\right)^{-1}\right) = \mathcal{B}\left(g_{i_0-1}^{\psi_{i_0-1}}\right).$$

Furthermore if $\prod_{i=1}^{i_0-1} g_i^{\psi_i}$ and $\prod_{i=i_0}^{n} g_i^{\psi_i}$ were in general position for some index i_0, then by Corollary 7 we would have

$$\ell\left(\prod_{i=1}^{n} g_i^{\psi_i}\right) = \ell\left(\prod_{i=i_0+1}^{n} g_i^{\psi_i}\right) + \ell\left(\prod_{i=1}^{i_0} g_i^{\psi_i}\right) \geq 2,$$

in contradiction with the fact that $\alpha\theta$ is triangular. So all such compositions are in special position, and we obtain $\mathcal{B}\left(g_{i_0}^{\psi_{i_0}}\right) = \mathcal{B}\left(g_{i_0-1}^{\psi_{i_0-1}}\right)$. Finally $\mathcal{B}\left(g_n^{\psi_n}\right) = \mathcal{B}\left(g_1^{\psi_1}\right)$, and we have

$$\mathcal{B}(\theta) = \mathcal{B}(g_1^{\psi_1}) = \mathcal{B}(g_n^{\psi_n}) = \mathcal{B}(\theta^{-1}\alpha^{-1}) = \alpha\mathcal{B}(\theta^{-1}). \quad \square$$

4. Examples

Here we illustrate our algorithm by describing the automorphism groups of affine varieties admitting a smooth completion with boundary B_S of self-intersection at most 6. We first check that we recover the well-known structure of the automorphism group of $\mathbb{A}^2$. Then we consider the case of the affine quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus \Delta$ started in Section 1 for which we recover in particular the description of its automorphism group given for instance in [21]. We briefly discuss the case of the complement of a smooth conic in $\mathbb{P}^2$ which is similar. As a next step, we describe the situation for affine surfaces admitting smooth completions with boundaries of self-intersection 3 and 4 for which two new phenomena arise successively: the existence of non isomorphic $\mathbb{A}^1$-fibrations associated to rational pencils $\mathcal{P}_p$ with proper base point on the boundary, and the existence of non isomorphic smooth completions of a given affine surface. We observe that all these examples share the common property that their automorphisms are tame. Finally, we consider the more subtle situations of affine surfaces admitting smooth completions with boundary of self-intersection 5 and 6 for which we establish the existence of non-tame automorphisms.

4.1. Automorphisms of $\mathbb{A}^2$, $\mathbb{P}^2 \smallsetminus \{$smooth conic$\}$ and of the smooth affine quadric surface.

4.1.1. *The affine plane.* Here we derive Jung's Theorem from the description of the triangular maps which appear in the factorization of an automorphism of $\mathbb{A}^2$ (see also [20] and [23], which contain proofs of Jung's Theorem derived from the philosophy of the (log) Sarkisov program). We let

$$\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u,v]) = \mathbb{P}^2 \smallsetminus L_0 = \mathrm{Proj}(\mathbb{C}[x,y,z]) \smallsetminus \{z=0\}$$

with $(u,v) = (x/z, y/z)$, and we define the affine and Jonquières automorphisms with respect to this unique completion, with the choice of $p_\infty = [1:0:0]$ (see Definition 20). The restriction to $\mathbb{A}^2$ of the rational pencil $\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ generated by L_0 and the line $y=0$ coincides with the second projection $\mathrm{pr}_v\colon \mathbb{A}^2 \to \mathbb{A}^1$. Since the pairs $(\mathbb{P}^2, L)$ where L is a line are the only smooth completions of $\mathbb{A}^2$, our algorithm leads to a factorization of an arbitrary polynomial automorphism f of $\mathbb{A}^2$ into a finite sequence

of triangular birational maps $\phi_i\colon (\mathbb{P}^2, L_{i-1}) \dashrightarrow (\mathbb{P}^2, L_i)$, where the L_i are lines. These maps are obtained as sequences of $2k_i-2$, $k_i \geq 2$, elementary links as in Theorem 1 of the form

$$(\mathbb{P}^2, L_{i-1}) \dashrightarrow (\mathbb{P}^2(2), B_1) \dashrightarrow \cdots \dashrightarrow (\mathbb{P}^2(k_i), B_{k_i-1}) \dashrightarrow \cdots \dashrightarrow (\mathbb{P}^2(2), B_{2k_i-3}) \dashrightarrow (\mathbb{P}^2, L_i)$$

where $\mathbb{P}^2(d)$ is the weighted projective plane $\mathbb{P}^2(d,1,1)$, obtained from the Hirzebruch surface $\pi_d\colon \mathbb{F}_d \to \mathbb{P}^1$ by contracting the section with self-intersection $-d$, and each intermediate boundary is the image by the contraction of a fiber of π_d. Since $\mathrm{Aut}(\mathbb{P}^2)$ acts transitively on pairs consisting of a line and a point of it, we may associate to each ϕ_i two isomorphisms of pairs $\alpha_i\colon (\mathbb{P}^2, L_0) \to (\mathbb{P}^2, L_{i-1})$ and $\beta_i\colon (\mathbb{P}^2, L_0) \to (\mathbb{P}^2, L_i)$, which map $p_\infty = [1:0:0]$ respectively onto the proper base points of ϕ_i and ϕ_i^{-1}. Then the induced birational map $\beta_i^{-1}\phi_i\alpha_i$ restricts on $\mathbb{A}^2$ to an automorphism commuting with the second projection. Thus each $\beta_i^{-1}\phi_i\alpha_i$ is a triangular automorphism of $\mathbb{A}^2$ in the usual sense (but in this paper, we prefer to use the terminology "Jonquières", since we reserve "triangular" for a more general notion). Thus every automorphism of $\mathbb{A}^2$ is $\mathcal{M}$-tame with respect to the family $\mathcal{M} = \{(\mathrm{id}_{\mathbb{P}^2}, p_\infty)\}$ and via Proposition 16, we recover the classical description of the automorphism group of $\mathbb{A}^2$ as the free product of its affine and Jonquières subgroups amalgamated along their intersection. We also recover via Proposition 21 another classical fact: any algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is conjugated to a subgroup of affine or Jonquières automorphisms.

4.1.2. *The smooth affine quadric surface.* The structure of the automorphism group of the smooth affine quadric surface $Q = \mathrm{Spec}(\mathbb{C}[x,y,z]/(xz-y(y+1))$ is similar to that of the affine plane. Via the open embedding

$$Q \hookrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 = \mathrm{Proj}(\mathbb{C}[u_0:u_1]) \times \mathrm{Proj}(\mathbb{C}[v_0:v_1]),\ (x,y,z) \mapsto ([x:y],[x:y+1]) = ([y+1:z],[y:z]),$$

we identify Q with the complement of the diagonal $D_0 = \{u_0v_1 - u_1v_0 = 0\}$. The rational pencil on $\mathbb{F}_0$ generated by D_0 and the union of the two rules $\{u_1=0\} \cup \{v_1=0\}$ has a unique proper base point $p = ([1:0],[1:0])$ and the restriction to Q of the corresponding rational map $\rho\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^1$ coincides with the $\mathbb{A}^1$-fibration $\mathrm{pr}_z\colon S \to \mathbb{A}^1$. The minimal resolution $X \to \mathbb{F}_0$ is obtained by blowing-up two times the point p with successive exceptional divisors F and C_0. The surface X then dominates the Hirzebruch surface $\pi_1\colon \mathbb{F}_1 \to \mathbb{P}^1$ via the contraction of the strict transforms of $\{u_1=0\}$ and $\{v_1=0\}$. Since $\mathrm{Pic}(Q) \simeq \mathbb{Z}$, it follows from Proposition 11 that the pairs $(\mathbb{F}_0, D)$ where D is a smooth curve of type $(1,1)$ are the only possible smooth completions of Q. Proposition 16 and the description of the resolution of triangular maps given in §3.1.2 lead to a decomposition of every automorphism of Q into a sequence of triangular maps

$$\phi_i\colon (\mathbb{F}_0, D_{i-1}) \dashrightarrow (\hat{\mathbb{P}}^2(2), B_1) \dashrightarrow \cdots \dashrightarrow (\hat{\mathbb{P}}^2(k_i), B_{k_i-1}) \dashrightarrow \cdots \dashrightarrow (\hat{\mathbb{P}}^2(2), B_{2k_i-3}) \dashrightarrow (\mathbb{F}_0, D_i)$$

where the projective surface $\hat{\mathbb{P}}^2(d)$ is obtained from the Hirzebruch surface $\pi_d\colon \mathbb{F}_d \to \mathbb{P}^1$, with negative section C_0, by first blowing-up two distinct points in a fiber $F \smallsetminus C_0$ of $\mathbb{F}_d \to \mathbb{P}^1$ and then contracting successively the strict transforms of F and C_0. The intermediate boundaries are the images by these contraction of a fiber F' of $\mathbb{F}_d$ distinct from F. Remark that $\hat{\mathbb{P}}^2(d)$ dominates the weighted projective plane $\mathbb{P}^2(d)$ via a single divisorial contraction, hence the notation.

Since $\mathrm{Aut}(\mathbb{F}_0)$ acts transitively on the set of pairs consisting of a smooth curve of type $(1,1)$ and a point on it, we may associate to each ϕ_i two isomorphisms of pairs $\alpha_i\colon (\mathbb{F}_0, D_0) \to (\mathbb{F}_0, D_{i-1})$ and $\beta_i\colon (\mathbb{F}_0, D_0) \to (\mathbb{F}_0, D_i)$, which map $p_\infty = ([1:0],[1:0])$ respectively onto the proper base points of ϕ_i and ϕ_i^{-1}. Thus the induced birational map $\beta_i^{-1}\phi_i\alpha_i$ restricts on Q to an automorphism commuting with the $\mathbb{A}^1$-fibration pr_z. Such automorphisms come as the lifts via the morphism $\mathrm{p}_{z,y}\colon Q \to \mathbb{A}^2$ of Jonquières automorphisms of $\mathbb{A}^2$ of the form $(z,y) \mapsto (az, y+zP(z))$, or the form $(z,y) \mapsto (az, -(y+1)+zP(z))$ where $a \in \mathbb{C}^*$ and $P(z)$ is a polynomial. Every automorphism of the second family is obtained from one

of the first family by composing with the affine involution $(z,y)\mapsto(z,-(y+1))$ of $\mathbb{A}^2$ which lifts to the involution Q induced by the "symmetry" with respect to the diagonal D_0 in $\mathbb{P}^1\times\mathbb{P}^1$. We recover in this way the presentation given in [21] of $\mathrm{Aut}(V)$ as the amalgamated product of $\mathrm{Aff}(Q)=\mathrm{Aut}(\mathbb{F}_0,\Delta)\mid_Q$ and $\mathrm{Aut}(Q,\mathrm{pr}_z)=\mathrm{Tr}(\mathbb{F}_0,\Delta,p_\infty)\mid_Q$ over their intersection. In particular, similarly as in the case of $\mathbb{A}^2$, we have $\mathrm{Aut}(Q)=\mathcal{M}\mathrm{TA}(Q)$, with respect to the family $\mathcal{M}=\{(\mathrm{id}_{\mathbb{F}_0},p_\infty)\}$. Proposition 21 says in turn that every algebraic subgroup of $\mathrm{Aut}(V)$ is conjugated to a subgroup of $\mathrm{Aff}(Q)$ or $\mathrm{Aut}(Q,\mathrm{pr}_z)$. For instance, every algebraic action of the additive group $\mathbb{G}_a$ on Q is conjugated to an action preserving the fibration pr_z.

4.1.3. *The complement of a smooth conic.* Since the automorphism group of $\mathbb{P}^2$ acts transitively on the set of pairs consisting of a smooth conic and a point of it, every smooth pair $(\mathbb{P}^2,C)$ where C is a smooth conic is isomorphic to $(\mathbb{P}^2,C_0)$ where $C_0=\{yz-x^2=0\}$, and every rational pencil $\mathcal{P}_p$ associated to a point on C is conjugated to $\mathcal{P}_{p_0}$ generated by C_0 and $2L_0$ where L_0 denotes the tangent to C_0 at the point $p_0=[0:0:1]$. The corresponding $\mathbb{A}^1$-fibration $q=\rho_{p_0}\mid_V\colon V=\mathbb{P}^2\smallsetminus C_0\to\mathbb{A}^1$ has a unique degenerate fiber $L_0\cap V$, of multiplicity 2. The automorphism group of V is then the amalgamated product of $\mathrm{Aff}(V)=\mathrm{Aut}(\mathbb{P}^2,C_0)\mid_V$ and $\mathrm{Aut}(V,q)=\mathrm{Tr}(\mathbb{P}^2,C_0,p_0)\mid_V$ over their intersection. Again, we have $\mathrm{Aut}(V)=\mathcal{M}\mathrm{TA}(V)$, for $\mathcal{M}$ a collection consisting of a unique model. The interested reader can find more details in [22, §1.1].

4.2. **Complement of a section with self-intersection 3 in $\mathbb{F}_1$.** By Proposition 11, a smooth completion of (S,B_S) of an affine surface V with $B_S^2\neq 1,4$ is of the form $(\mathbb{F}_n,C)$ where C is an ample section. Since B_S^2 is an invariant of V, we see that if $B_S^2=3$ the only smooth completions of V are pairs $(\mathbb{F}_1,C)$ where C is a section of self-intersection 3. If we identify $\pi_1\colon\mathbb{F}_1\to\mathbb{P}^1$ with the blow-up σ of $\mathbb{P}^2=\mathrm{Proj}(\mathbb{C}[x,y,z])$ at the point $q=[0:1:0]$ with exceptional divisor C_0 and denote by F_∞ the strict transform of the line $\{z=0\}\subset\mathbb{P}^2$, then every such section C is the strict transform of a smooth conic in $\mathbb{P}^2$ passing through q. The automorphism group of $\mathbb{F}_1$ acts transitively on such sections and so, every smooth completion of V is isomorphic to $(\mathbb{F}_1,D_0)$ where D_0 denotes the strict transform of the conic $\{yz-x^2=0\}\subset\mathbb{P}^2$ tangent to the line $\{z=0\}$ at q. The automorphism group of the pair $(\mathbb{F}_1,D_0)$ acts on D_0 with two orbits: the point $p_\infty=D_0\cap C_0=D_0\cap F_\infty$ and its complement. This implies in turn that every rational pencil $\mathcal{P}_p$ associated to a point on D_0 is conjugated either to $\mathcal{P}_{p_\infty}$ or to $\mathcal{P}_{p_0}$ where $p_0=\sigma^{-1}([0:0:1])$. Both of these pencils have a unique singular member consisting of the divisor C_0+2F_∞ in the first case and $L+F_0$ where L and F_0 are the respective strict transforms of the lines $\{y=0\}$ and $\{x=0\}$ of $\mathbb{P}^2$ in the second case. The induced $\mathbb{A}^1$-fibrations on $V\simeq\mathbb{F}_1\smallsetminus D_0$ are not isomorphic: the one induced by $\mathcal{P}_{p_\infty}$ has a unique degenerate scheme theoretic fiber which consists of the union of two affine lines $C_0\cap V$ and $F_\infty\cap V$ where $F_\infty\cap V$ occurs with multiplicity 2, while the one induced by $\mathcal{P}_{p_0}$ has a unique degenerate fiber consisting of two reduced affine lines $L\cap V$ and $F_0\cap V$. In particular we see from Lemma 14 that any triangular map $V\to V$ is either in $\mathrm{Tr}(\mathbb{F}_1,D_0,p_\infty)$, or up to left-right composition by automorphisms of $(\mathbb{F}_1,D_0)$, in $\mathrm{Tr}(\mathbb{F}_1,D_0,p_0)$.

Now given an automorphism of V considered as a birational self-map f of $(\mathbb{F}_1,D_0)$ with decomposition

$$f=\phi_n\circ\cdots\circ\phi_1\colon(\mathbb{F}_1,D_0)=(S_0,B_{S_0})\dashrightarrow\cdots\dashrightarrow(S_i,B_{S_i})\dashrightarrow\cdots\dashrightarrow(S_n,B_{S_n})=(\mathbb{F}_1,D_0)$$

into triangular maps, we can find isomorphisms $\alpha_i\colon(\mathbb{F}_1,D_0)\to(S_{i-1},B_{S_{i-1}})$, $\beta_i\colon(\mathbb{F}_1,D_0)\to(S_i,B_{S_i})$, $i=1,\dots,n$ such that for every $i=1,\dots,n$, $\psi_i=\beta_i^{-1}\phi_i\alpha_i$ is an element of either $\mathrm{Tr}(\mathbb{F}_1,D_0,p_0)$ or $\mathrm{Tr}(\mathbb{F}_1,D_0,p_\infty)$. Writing f as

$$f=\beta_n\psi_n(\alpha_n^{-1}\beta_{n-1})\dots\beta_2\psi_2(\alpha_2^{-1}\beta_1)\psi_1\alpha_1^{-1}$$

where α_1^{-1}, β_n and $\alpha_i^{-1}\beta_{i-1}$, $i = 2, \ldots, n$ are elements of $\mathrm{Aut}(\mathbb{F}_1, D_0)$, we conclude that $\mathrm{Aut}(V)$ is generated by $\mathrm{Aut}(\mathbb{F}_1, D_0)$, $\mathrm{Tr}(\mathbb{F}_1, D_0, p_0)$ and $\mathrm{Tr}(\mathbb{F}_1, D_0, p_\infty)$. In other words $\mathrm{Aut}(V) = \mathcal{M}\mathrm{TA}(V)$ for the family $\mathcal{M} = \{(\mathrm{id}_{\mathbb{F}_1}, p_0), (\mathrm{id}_{\mathbb{F}_1}, p_\infty)\}$. Remark that in this case we cannot use a single model anymore.

4.3. **Complement of a section with self-intersection 4 in $\mathbb{F}_0$.** Here we consider the case of an affine surface V admitting a smooth completion by a smooth rational curve with self-intersection 4, and which is not isomorphic to the complement of a conic in $\mathbb{P}^2$. According to Proposition 11, the corresponding pairs (S, B_S) are either $(\mathbb{F}_0, B)$ where B is an arbitrary smooth rational curve with self-intersection 4 (which is automatically a section with respect to one of the two rulings) or $(\mathbb{F}_2, B)$ where B is a section of the $\mathbb{P}^1$-bundle structure $\pi_2 \colon \mathbb{F}_2 \to \mathbb{P}^1$. First we review these smooth completions (S, B_S) with a particular emphasis on the rational pencils $\mathcal{P}_p$ related with all possible triangular elementary maps that can occur in the factorization given by Proposition 16. Given such a pencil $\mathcal{P}_p$, we let $\sigma : \hat{S} \to S$ be the minimal resolution of the corresponding rational map $\rho_p \colon S \dashrightarrow \mathbb{P}^1$. The last exceptional divisor extracted by σ is a section C of the induced $\mathbb{P}^1$-fibration $\rho_p\sigma : \hat{S} \to \mathbb{P}^1$ and one can prove (see [5], Section 2) that $\hat{S}$ dominates $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ through a uniquely determined sequence of contractions $\tau : \hat{S} \to \mathbb{F}_1$ in such a way that the general fibers of $\rho_p\sigma$ coincide with that of $\pi_1\tau$ and that the strict transform $\tau_*(C)$ of C coincides with the exceptional section of $\pi_1 \colon \mathbb{F}_1 \to \mathbb{P}^1$ with self-intersection -1. We will use this point of view to give a uniform description of the different pencils involved: see Figure 7.

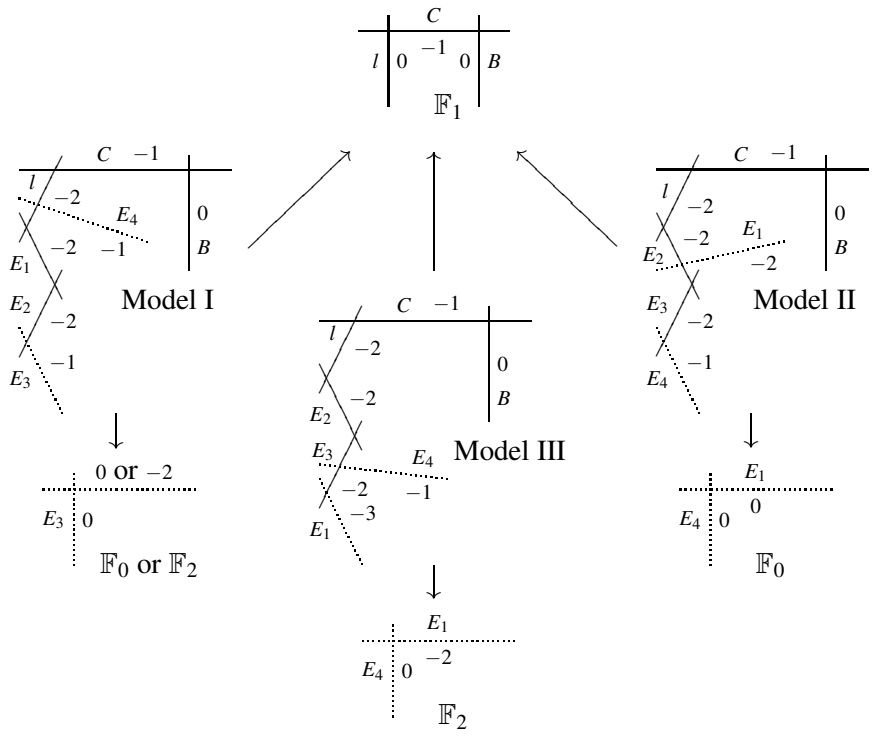


FIGURE 7. The three models of pencils with their resolution (the index of the exceptional divisors E_i corresponds to their order of construction coming from $\mathbb{F}_1$).

4.3.1. *The case* $(\mathbb{F}_0, B)$. With the bi-homogeneous coordinates introduced in §4.1.2, every pair $(\mathbb{F}_0, B)$ where B is an irreducible curve with $B^2 = 4$ is isomorphic to $(\mathbb{F}_0, D_0)$ where $D_0 = \{u_1^2 v_0 - u_0^2 v_1 = 0\}$. Letting $C_0 = \{v_0 = 0\}$ and $F_0 = \{u_0 = 0\}$ we have $D_0 \sim C_0 + 2F_0$. The automorphism group of the pair $(\mathbb{F}_0, D_0)$ acts on D_0 with two orbits : the pair of points $\{p_0', p_\infty'\} = \{([0:1],[0:1]),([1:0],[1:0])\}$ and their complement. This implies in turn that there exist only two models of rational pencils $\mathcal{P}_{p'}$ up to conjugacy by automorphisms of $(\mathbb{F}_0, D_0)$, say $\mathcal{P}_{p_0'}$ and $\mathcal{P}_{p_1'}$ where $p_1' = ([1:1],[1:1])$. They can be described as follows:

a) The pencil $\mathcal{P}_{p_1'}$ is generated by D_0 and $H + F_1$ where $H = \{(u_0 - 3u_1)(v_0 + 3v_1) + 8u_1 v_1 = 0\}$ is the unique irreducible curve of type $(1,1)$ intersecting D_0 only in p_1', with multiplicity 3, and F_1 is the fiber of the first ruling over the point $[1:1]$. The restriction of this pencil to $V_0 = \mathbb{F}_0 \smallsetminus D_0$ is an $\mathbb{A}^1$-fibration $V_0 \to \mathbb{A}^1$ with a unique degenerate fiber consisting of the disjoint union of two reduced affine lines $H \cap V_0$ and $F_1 \cap V_0$. See Figure 7 (model I).

b) The pencil $\mathcal{P}_{p_0'}$ is generated by D_0 and $C_0 + 2F_0$ (note that C_0 is the tangent to D_0 at the point p_0'). Its restriction to V_0 is an $\mathbb{A}^1$-fibration $V_0 \to \mathbb{A}^1$ with a unique degenerate fiber consisting of the disjoint union of a reduced affine line $C_0 \cap V_0$ and a non reduced one $F_0 \cap V_0$, occurring with multiplicity 2. See Figure 7 (model II).

The fact that the $\mathbb{A}^1$-fibrations associated to the pencils $\mathcal{P}_{p_0'}$ and $\mathcal{P}_{p_1'}$ are not isomorphic implies further that every triangular self-map $\phi : (\mathbb{F}_0, D_0) \dashrightarrow (\mathbb{F}_0, D_0)$ is the product of an element of $\mathrm{Tr}(\mathbb{F}_0, D_0, p_0')$ or $\mathrm{Tr}(\mathbb{F}_0, D_0, p_1')$ and an element of $\mathrm{Aut}(\mathbb{F}_0, D_0)$.

4.3.2. *The case* $(\mathbb{F}_2, B)$. Letting C_0 be the exceptional section of $\pi_2 : \mathbb{F}_2 \to \mathbb{P}^1$ with self-intersection -2, a section B of π_2 with self-intersection 4 is linearly equivalent to $C_0 + 3F_\infty$ where F_∞ is a fiber of π_2. In particular B intersects C_0 transversely in a single point, which we can assume to be $C_0 \cap F_\infty$. We identify $\mathbb{F}_2 \smallsetminus (C_0 \cup F_\infty)$ to $\mathbb{A}^2$ with coordinates x and y in such way that the induced ruling on $\mathbb{A}^2$ is given by the first projection and that the closures in $\mathbb{F}_2$ of the level sets of y are sections of π_2 linearly equivalent to $C_0 + 2F_\infty$ (equivalently, the closure of the curve $\{y = 0\}$ in $\mathbb{F}_2$ does not intersect C_0). With this choice, B coincides with the closure of an affine cubic defined by an equation of the form $y = ax^3 + bx^2 + cx + d$. Since any automorphism of $\mathbb{A}^2$ of the form $(x,y) \mapsto (\lambda x + \nu, \mu y + P(x))$ where P is a polynomial of degree at most 2, extends to a biregular automorphism of $\mathbb{F}_2$, it follows that every pair $(\mathbb{F}_2, B)$ where B is a section with self-intersection 4 is isomorphic to $(\mathbb{F}_2, D_2)$ where D_2 is the closure in $\mathbb{F}_2$ of the affine cubic in $\mathbb{A}^2$ with equation $y = x^3$. Furthermore, the automorphism group of the pair $(\mathbb{F}_2, D_2)$ acts on D_2 with two orbits: the fixed point $p_\infty = D_2 \cap C_0 = D_2 \cap F_\infty$ and its complement. Again, we have two possible models of rational pencils $\mathcal{P}_p$ up to conjugacy by automorphisms of $(\mathbb{F}_2, D_2)$, say $\mathcal{P}_{p_0}$ where $p_0 = (0,0) \subset \mathbb{A}^2 \subset \mathbb{F}_2$ and $\mathcal{P}_{p_\infty}$:

a) The pencil $\mathcal{P}_{p_0}$ is generated by D_2 and $H + F_0$ where $H \sim C_0 + 2F_\infty$ is the closure in $\mathbb{F}_2$ of the affine line $\{y = 0\} \subset \mathbb{A}^2$ which intersects D_2 only in p_0, with multiplicity 3, and where $F_0 = \pi_2^{-1}([0:1]) \subset \mathbb{F}_2$. Its restriction to $W_0 = \mathbb{F}_2 \smallsetminus D_2$ is an $\mathbb{A}^1$-fibration $W_0 \to \mathbb{A}^1$ with a unique degenerate fiber consisting of the disjoint union of two reduced affine lines $H \cap W_0$ and $F_0 \cap W_0$. A minimal resolution of this pencil is given in Figure 7 (model I).

b) The pencil $\mathcal{P}_{p_\infty}$ is generated by D_2 and $C_0 + 3F_\infty$ (remember that D_2 intersects C_0 transversely in p_∞). Its restriction to W_0 is an $\mathbb{A}^1$-fibration $W_0 \to \mathbb{A}^1$ with a unique degenerate fiber consisting of the disjoint union of a reduced affine line $C_0 \cap W_0$ and a non reduced one $F_\infty \cap W_0$, occurring with multiplicity 3. See Figure 7 (model III).

4.3.3. *Connecting triangular maps.* By the Danilov-Gizatullin Theorem, $(\mathbb{F}_0, D_0)$ and $(\mathbb{F}_2, D_2)$ can arise as smooth completions of a same affine surface. However, let us briefly explain how to derive this

fact directly by constructing appropriate triangular maps $\phi\colon (\mathbb{F}_2, D_2) \dashrightarrow (\mathbb{F}_0, D_0)$. In view of Lemma 14 and of the description of the rational pencils given above, the only possibility is that the proper base points of such a map ϕ and its inverse belong respectively to the open orbits of the actions of $\mathrm{Aut}(\mathbb{F}_2, D_2)$ on D_2 and of $\mathrm{Aut}(\mathbb{F}_0, D_0)$ on D_0. Let us construct a particular quadratic triangular map Φ_0 with $\mathcal{B}(\phi) = p_0$ and $\mathcal{B}(\phi^{-1}) = p_1'$ (see Figure 8 for the notations).

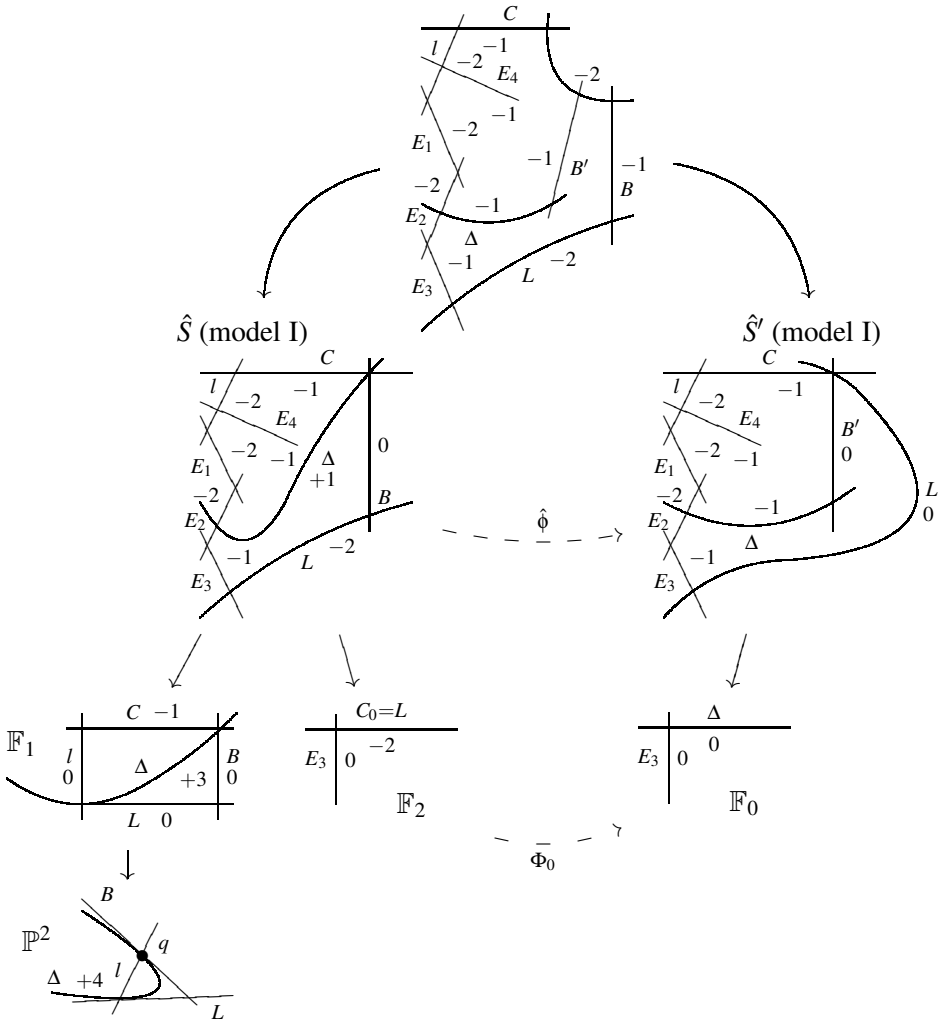


FIGURE 8. Quadratic triangular map $\Phi_0\colon (\mathbb{F}_2, D_2) \dashrightarrow (\mathbb{F}_0, D_0)$.

Let $\hat{S} \to \mathbb{F}_2$ be the minimal resolution of the base points of the rational pencil $\mathcal{P}_{p_0}$ as in §4.3.2.a) above. The surface $\hat{S}$ can also be obtained from $\mathbb{P}^2$ by a sequence of blow-ups with successive exceptional divisors C, E_1, E_2, E_3 and E_4 in such a way that curves l and $B = D_2$ correspond to the strict transforms of a pair of lines in $\mathbb{P}^2$ intersecting at the center q of the first blow-up. In this setting, the strict transform of $C_0 \subset \mathbb{F}_2$ in $\hat{S}$ coincides with the strict transform of a certain line L in $\mathbb{P}^2$ intersecting B in a point distinct from q. Let $\hat{\psi} : \hat{S} \dashrightarrow \hat{S}'$ be any fibered modification of degree 2 and let B' be the

strict transform in $\hat{S}'$ of the second exceptional divisor produced. Then one checks that there exists a unique smooth conic Δ in $\mathbb{P}^2$ tangent to B in q and to L at the point $L\cap l$ such that its strict transform in $\hat{S}'$ is a (-1)-curve which intersects transversely the strict transforms of E_2 and B' in general points. By successively contracting $E_4,\dots,E_1$, we arrive at a new projective surface S' in which the strict transform of B' is a smooth rational curve with self-intersection 4 and such that the strict transforms of Δ and E_3 are smooth rational curves with self-intersection 0, intersecting transversely in a single point. Thus $S'\simeq\mathbb{F}_0$ and $\hat{\phi}:\hat{S}\dashrightarrow\hat{S}'$ descends to a triangular map $\psi:(\mathbb{F}_2,D_2)\dashrightarrow(\mathbb{F}_0,B')$. Moreover, the proper base point of ψ^{-1} is located at a point where B' intersects the two rulings transversely. So there exists an isomorphism of pairs $\beta\colon(\mathbb{F}_0,B')\to(\mathbb{F}_0,D_0)$ such that $\Phi_0=\beta\psi\colon(\mathbb{F}_2,D_2)\dashrightarrow(\mathbb{F}_0,D_0)$ is triangular and maps $\mathcal{P}_{p_0}$ onto $\mathcal{P}_{p'_1}$.

4.3.4. *The automorphism group.* To determine the automorphism group of an affine surface V admitting a smooth completion (S,B_S) with $B_S^2=4$ we can proceed as follows. First we may assume up to isomorphism that $V=\mathbb{F}_2\smallsetminus D_2$. Then given an automorphism ξ of V we consider a minimal factorization of the associated birational self-map f of $(\mathbb{F}_2,D_2)$ into triangular maps

$$f=\phi_n\circ\cdots\circ\phi_1\colon(\mathbb{F}_2,D_2)=(S_0,B_{S_0})\dashrightarrow\cdots\dashrightarrow(S_i,B_{S_i})\dashrightarrow\cdots\dashrightarrow(S_n,B_{S_n})=(\mathbb{F}_2,D_2)$$

where each S_i is isomorphic either to $\mathbb{F}_0$ or $\mathbb{F}_2$.

If the intermediate surfaces S_j are not all isomorphic to $\mathbb{F}_2$, then we let $j\in\{1,\dots,n-1\}$ and $k\in\{j+1,\dots,n\}$ be minimal with the property that $S_j\simeq S_{k-1}\simeq\mathbb{F}_0$ and $S_k\simeq\mathbb{F}_2$. Replacing if necessary ϕ_{j-1}, ϕ_j and ϕ_{j+1} by $\alpha\phi_{j-1}$, $\beta\phi_j\alpha^{-1}$ and $\phi_{j+1}\beta^{-1}$ for isomorphisms $\alpha\colon(S_{j-1},B_{S_{j-1}})\to(\mathbb{F}_2,D_2)$ and $\beta\colon(S_j,B_{S_j})\to(\mathbb{F}_0,D_0)$, we may assume from the beginning that $(S_{j,-1},B_{j-1})=(\mathbb{F}_2,D_2)$ and $(S_j,B_j)=(\mathbb{F}_0,D_0)$. We may assume similarly that $(S_{k-1},B_{S_{k-1}})=(\mathbb{F}_0,D_0)$ and $(S_k,B_{S_k})=(\mathbb{F}_2,D_2)$. Now consider the triangular maps $\phi_j:(\mathbb{F}_2,D_2)\dashrightarrow(\mathbb{F}_0,D_0)$ and $\phi_k:(\mathbb{F}_0,D_0)\dashrightarrow(\mathbb{F}_2,D_2)$. Since the $\mathbb{A}^1$-fibrations induced by the pencils $\mathcal{P}_{p_\infty}$ on $\mathbb{F}_2$ and $\mathcal{P}_{p'_0}$ on $\mathbb{F}_0$ are not isomorphic and not isomorphic to those associated to points in $D_2\setminus\{p_\infty\}$ and $D_0\setminus\{p'_0\}$ it must be that $\mathcal{B}(\phi_j)\in D_2\setminus\{p_\infty\}$ and $\mathcal{B}(\phi_j^{-1})\in D_0\setminus\{p'_0\}$ (see §4.3.1 and 4.3.2). It follows that there exist automorphisms $\alpha_j\in\mathrm{Aut}(\mathbb{F}_2,D_2)$ and $\beta_j\in\mathrm{Aut}(\mathbb{F}_0,D_0)$ mapping $\mathcal{B}(\phi_j)$ onto p_0 and $\mathcal{B}(\phi_j^{-1})$ onto p'_1 respectively. So replacing ϕ_{j-1}, ϕ_j and ϕ_{j+1} by $\alpha_j\phi_{j-1}$, $\beta_j\phi_j\alpha_j^{-1}$ and $\phi_{j+1}\beta_j$ respectively, we may assume from the beginning that $\mathcal{B}(\phi_j)=p_0$ and $\mathcal{B}(\phi_j^{-1})=p'_1$. For the same reason, we may assume that $\mathcal{B}(\phi_k)=p'_1$ and $\mathcal{B}(\phi_k^{-1})=p_0$. Strictly speaking, in the case $k=j+1$, we have to insert an automorphism $\alpha\in\mathrm{Aut}(\mathbb{F}_0,D_0)$ between ϕ_j and ϕ_{j+1}, which will play the same role as $\phi_{k-1}\dots\phi_{j+1}$ in the sequel. Recall that by construction, the particular triangular map $\Phi_0:(\mathbb{F}_2,D_2)\dashrightarrow(\mathbb{F}_0,D_0)$ constructed in §4.3.3 has $\mathcal{B}(\Phi_0)=p_0$ and $\mathcal{B}(\Phi_0^{-1})=p'_1$. It then follows from Corollary 15 that $\Phi_0^{-1}\phi_j:(\mathbb{F}_2,D_2)\dashrightarrow(\mathbb{F}_2,D_2)$ is an element of $\mathrm{Tr}(\mathbb{F}_2,D_2,p_0)$ while $\Phi_0\phi_k:(\mathbb{F}_0,D_0)\dashrightarrow(\mathbb{F}_0,D_0)$ belongs to $\mathrm{Tr}(F_0,D_0,p'_1)$. Summing up, we can rewrite f in the form

$$\begin{aligned}f=\phi_n\cdots\phi_{k+1}\Phi_0^{-1}\left[(\Phi_0\phi_k)\phi_{k-1}\cdots\phi_{j+1}\right]\Phi_0\left[(\Phi_0^{-1}\phi_j)\cdots\phi_1\right]&\\ =f'\Phi_0^{-1}\left[(\Phi_0\phi_k)\phi_{k-1}\cdots\phi_{j+1}\right]\Phi_0\left[(\Phi_0^{-1}\phi_j)\cdots\phi_1\right]&\end{aligned}$$

where $f':(\mathbb{F}_2,D_2)\dashrightarrow(\mathbb{F}_2,D_2)$ has length $\ell(f')<\ell(f)$ and where the sequences $(\Phi_0\phi_k)\cdots\phi_{j+1}$ and $(\Phi_0^{-1}\phi_j)\cdots\phi_1$ only involve intermediate surfaces isomorphic to $\mathbb{F}_0$ and $\mathbb{F}_2$ respectively.

Now we deduce as in §4.2 that $(\Phi_0^{-1}\phi_j)\cdots\phi_1$ can be written as a sequence of elements in $\mathrm{Aut}(\mathbb{F}_2,D_2)$, $\mathrm{Tr}(\mathbb{F}_2,D_2,p_0)$ and $\mathrm{Tr}(\mathbb{F}_2,D_2,p_\infty)$. Similarly, $(\Phi_0\phi_k)\cdots\phi_{j+1}$ can be decomposed into a sequence of elements in $\mathrm{Aut}(\mathbb{F}_0,D_0)$, $\mathrm{Tr}(\mathbb{F}_0,D_0,p'_0)$ and $\mathrm{Tr}(\mathbb{F}_0,D_0,p'_1)$ and so $\Phi_0^{-1}\left[(\Phi_0\phi_k)\cdots\phi_{j+1}\right]\Phi_0$ can be written

as a composition of elements of the conjugates of these groups by $\Phi_0 : (\mathbb{F}_2, D_2) \dashrightarrow (\mathbb{F}_0, D_0)$. We conclude by induction on the length that $\mathrm{Aut}(V) = \mathcal{M}\mathrm{TA}(V)$ with

$$\mathcal{M} = \left\{(\mathrm{id}_{\mathbb{F}_2}, p_0), (\mathrm{id}_{\mathbb{F}_2}, p_\infty), (\Phi_0, p_0'), (\Phi_0, p_1')\right\}.$$

4.4. **Automorphisms of the complement of a section with self-intersection 5 in $\mathbb{F}_1$.** While it could seem at first glance similar to the previous ones, this case exhibits a new behavior which is more representative of the general situation: the existence of non-tame automorphisms.

4.4.1. *Possible models and associated rational pencils.* In view of Proposition 11 there exists only two possible types of smooth completions (S, B_S) with $B_S^2 = 5$ and $S \smallsetminus B_S$ affine: the complements of sections with self-intersection 5 in either $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ or $\pi_3 : \mathbb{F}_3 \to \mathbb{P}^1$.

1) In the first case, every such section B is linearly equivalent to $C_0 + 3F$ where C_0 is the exceptional section of π_1 and F a fiber. In particular, $B \cdot C_0 = 2$ and with the notations of §4.2, we have two possible pairs up to isomorphisms: first $(\mathbb{F}_1, D_1)$ where D_1 is the strict transform of the nodal cubic $C_1 = \{x^3 - z^3 = xyz\} \subset \mathbb{P}^2$ with tangents $\{z = 0\}$ and $\{x = 0\}$ at $q_0 = [0 : 1 : 0]$; and second $(\mathbb{F}_1, D_2)$ where D_2 is the strict transform of the cuspidal cubic $C_2 = \{x^3 = z^2 y\} \subset \mathbb{P}^2$ tangent to $\{z = 0\}$ at q_0.

a) The automorphism group of $(\mathbb{F}_1, D_2)$ acts on D_2 with three orbits : $p_{\infty,2} = C_0 \cap F_\infty$, $p_{0,2} = \sigma^{-1}([0 : 0 : 1])$ and their complement. The pencil $\mathcal{P}_{p_{\infty,2}}$ is generated by D_2 and $C_0 + 3F_\infty$ and it restricts on $V_{0,2} = \mathbb{F}_1 \smallsetminus D_2$ to an $\mathbb{A}^1$-fibration with a unique degenerate fiber consisting of two affine lines $C_0 \cap V_{0,2}$ and $F_\infty \cap V_{0,2}$, the second one occurring with multiplicity 3. The pencil $\mathcal{P}_{p_{0,2}}$ is generated by D_2 and $L + 2F_0$ where $L \sim C_0 + F_\infty$ is the strict transform of the tangent line to C_2 at $[0 : 0 : 1]$ and $F_0 = \pi_1^{-1}\pi_1(p_{0,2})$. Its restriction to $V_{0,2}$ is an $\mathbb{A}^1$-fibration with a unique degenerate fiber consisting of two affine lines $L \cap V_{0,2}$ and $F_0 \cap V_{0,2}$, the second one occurring with multiplicity 2. Finally, for every $p \in D_2 \smallsetminus (p_{0,2} \cup p_{\infty,2})$, the pencil $\mathcal{P}_p$ is generated by D_2 and $H + F_p$ where $F_p = \pi_1^{-1}(\pi_1(p))$, and $H \sim C_0 + 2F_\infty$ is the strict transform of the unique smooth conic in $\mathbb{P}^2$ intersecting C_2 with multiplicity 4 at $\sigma(p)$ and 2 at q_0. The induced $\mathbb{A}^1$-fibration on $V_{0,2}$ has a unique degenerate fiber consisting of two reduced affine lines $H \cap V_{0,2}$ and $F_p \cap V_{0,2}$.

b) The automorphism group of $(\mathbb{F}_1, D_1)$ acts on D_1 via the dihedral group of order 6 generated by the symmetry with respect to the point $p_1 = \sigma^{-1}([1 : 0 : 1])$ and the lift of the $\mathbb{Z}_3$-action on C_1 defined by $\varepsilon \cdot [x : y : z] = [\varepsilon x, \varepsilon^{-1} y : z]$. In particular, the induced action has no open orbit.

For the pair $(\mathbb{F}_1, D_1)$, we have two types of pencils : the first family consists of the pencils $\mathcal{P}_{p_{\varepsilon^k}}$ at the points $p_{\varepsilon^k} = \sigma^{-1}([1 : 0 : \varepsilon^k])$, $k = 0, 1, 2$. These are generated respectively by D_1 and $L_{\varepsilon^k} + 2F_{\varepsilon^k}$ where $L_{\varepsilon^k} \sim C_0 + F$ is the strict transform of the tangent line to C_1 at the point $[1 : 0 : \varepsilon^k]$ and $F_{\varepsilon^k} = \pi_1^{-1}(\pi_1(p_{\varepsilon^k}))$. The induced $\mathbb{A}^1$-fibrations on $V_{0,1} = \mathbb{F}_1 \smallsetminus D_1$ have a unique degenerate fiber consisting of the disjoint union of two affine lines $L_{\varepsilon^k} \cap V_{0,1}$ and $F_{\varepsilon^k} \cap V_{0,1}$, the second occurring with multiplicity 2.

On the other hand, for every point $p \in D_1 \smallsetminus \{p_1, p_\varepsilon, p_{\varepsilon^2}\}$, the pencil $\mathcal{P}_p$ is generated by D_1 and $H_p + F_p$ where $F_p = \pi_1^{-1}(\pi_1(p))$, and H_p is the strict transform of the unique smooth conic in $\mathbb{P}^2$ intersecting C_1 with multiplicity 4 at $\sigma(p)$ and 2 at q_0 if $p \in D_1 \smallsetminus C_0$ or the strict transform of one of the two smooth conics intersecting C_1 with multiplicity 6 at q_0 otherwise. In each case the induced $\mathbb{A}^1$-fibration on $V_{0,1}$ has unique degenerate fiber consisting of the disjoint union of two reduced affine lines $H \cap V_{0,1}$ and $F_p \cap V_{0,1}$.

In contrast with the previous case, the description of the action of $\mathrm{Aut}(\mathbb{F}_1, D_1)$ on D_1 implies that even though the $\mathbb{A}^1$-fibrations on $\mathbb{F}_1 \setminus D_1$ induced by the pencils $\mathcal{P}_p$, $p \in D_1 \smallsetminus \{p_1, p_\varepsilon, p_{\varepsilon^2}\}$ are abstractly

isomorphic, they are no longer pairwise conjugate via elements of $\mathrm{Aut}(\mathbb{F}_1, D_1)$.

2) In the second case $(\mathbb{F}_3, B)$, a section B of π_3 with self-intersection 5 is linearly equivalent to $C_0 + 4F$ where C_0 is the exceptional section of π_3 with self-intersection -3 and F is a fiber of π_3. Since the automorphism group of $\mathbb{F}_3$ acts transitively on such sections, there exists a unique model $(\mathbb{F}_3, D_3)$ up to isomorphism of pairs. Furthermore, the automorphism group of $(\mathbb{F}_3, D_3)$ acts on D_3 with two orbits: the point $p_\infty = D_3 \cap C_0$ and its complement. The pencil $\mathcal{P}_{p_\infty}$ is generated by D_3 and $C_0 + 4F_\infty$ where $F_\infty = \pi_3^{-1}(\pi_3(p_\infty))$ and it restricts on $W_0 = \mathbb{F}_3 \smallsetminus D_3$ to an $\mathbb{A}^1$-fibration over $\mathbb{A}^1$ with a unique degenerate fiber consisting of two affine lines $C_0 \cap W_0$ and $F_\infty \cap W_0$, the second one occurring with multiplicity 4. For every other point $p \in D_3 \smallsetminus p_\infty$, the rational pencil $\mathcal{P}_p$ is generated by D_3 and $H + F_p$ where $F_p = \pi_3^{-1}(\pi_3(p))$, and $H \sim C_0 + 3F$ is the unique section of π_3 intersecting D_3 at p only with multiplicity 4. The induced $\mathbb{A}^1$-fibration on W_0 has a unique degenerate fiber consisting of two reduced affine lines $H \cap W_0$ and $F_p \cap W_0$.

4.5. **Automorphisms of the complement of a section with self-intersection 6 in $\mathbb{F}_0$.** In this case a further new phenomenon occurs: the existence of uncountably many isomorphy types of smooth completions (S, B_S), only finitely many of these having non-trivial automorphism groups. Below we only summarize these possible abstract isomorphy types and observe what is strictly necessary to finish the proof of Proposition 22. The three types of possible models of smooth completions (S, B_S) of an affine surface with $B_S^2 = 6$ are $(\mathbb{F}_0, C)$, $(\mathbb{F}_2, C)$ and $(\mathbb{F}_4, C)$ where C is each time an ample section with self-intersection 6.

1) The case $(\mathbb{F}_4, C)$: a section C with $C^2 = 6$ is linearly equivalent to $C_0 + 5F$ where C_0 is the exceptional section of $\pi_4 \colon \mathbb{F}_4 \to \mathbb{P}^1$ with self-intersection -4 and F is a fiber of π_4. Note that C intersects C_0 transversally in a unique point $p_{\infty,4}$. The automorphism group of $\mathbb{F}_4$ acts transitively on the set of such sections and, identifying $\mathbb{F}_4 \smallsetminus (C_0 \cup F_\infty)$ where $F_\infty = \pi_4^{-1}(\pi_4(p_{\infty,4}))$ with $\mathbb{A}^2$ in a similar way as in §4.3, we may assume that $C = C_4$ is the closure of the affine quintic $\{y = x^5\} \subset \mathbb{A}^2$. The automorphism group of $(\mathbb{F}_4, C_4)$ acts on C_4 with two orbits: the point $p_{\infty,4} = C_4 \cap C_0$ and its complement.

2) The case $(\mathbb{F}_2, C)$: a section C with $C^2 = 6$ is linearly equivalent to $C_0 + 4F$ where C_0 is the exceptional section of $\pi_2 \colon \mathbb{F}_2 \to \mathbb{P}^1$ with self-intersection -2 and F is a fiber of π_2. Such a section intersects C_0 either in a single point with multiplicity two or transversally in two distinct points.

a) In the first case, up to an automorphism of $\mathbb{F}_2$ we may assume that $C = C_{2,1}$ is the closure in $\mathbb{F}_2$ of the intersection of the quartic $\{yz^3 = x^4\} \subset \mathbb{P}^2$ with $\mathbb{A}^2$. The group $\mathrm{Aut}(\mathbb{F}_2, C_{2,1})$ acts on $C_{2,1}$ with three orbits: the point $p_{\infty,2} = C_{2,1} \cap C_0$, the point $p_{0,2} = (0,0) \subset \mathbb{A}^2 \subset \mathbb{F}_2$ and their complement.

b) In the second case, up to an automorphism of $\mathbb{F}_2$ we may assume that $C = C_{2,2}$ is the closure in $\mathbb{F}_2$ of the intersection of the quartic $\{xyz^2 = x^4 - z^4\} \subset \mathbb{P}^2$ with $\mathbb{A}^2$. The group $\mathrm{Aut}(\mathbb{F}_2, C_{2,2})$ acts on $C_{2,2}$ via the dihedral group of order 8 generated by the symmetry with center at the point $p_s = [1:0:1]$ and the lift of the $\mathbb{Z}_4$-action on $\{xyz^2 = x^4 - z^4\}$ defined by $\varepsilon \cdot [x:y:z] = [\varepsilon x : \varepsilon^{-1} y : z]$.

3) The case $(\mathbb{F}_0, C)$: a section C of the first projection $\pi_0 = \mathrm{pr}_1 \colon \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with $C^2 = 6$ is linearly equivalent to $C_0 + 3F$ where C_0 is a fiber of pr_2 and F a fiber of π_0. Such sections can be first roughly divided into three classes according to the number of fibers of the second projection which intersect C with multiplicity 3.

a) If there exist at least two such fibers intersecting C with multiplicity 3 then the pair $(\mathbb{F}_0, C)$ is isomorphic to $(\mathbb{F}_0, C_{0,0})$ where $C_{0,0} = \{u_1^3 v_0 + u_0^3 v_1 = 0\}$. The group $\mathrm{Aut}(\mathbb{F}_0, C_{0,0})$ is then isomorphic

to $\mathbb{C}^* \times \mathbb{Z}_2$ where $\mathbb{C}^*$ acts by $\lambda \cdot ([u_0 : u_1], [v_0 : v_1]) = ([\lambda u_0 : u_1], [\lambda^3 v_0 : v_1])$ and where $\mathbb{Z}_2$ exchanges u_0, v_0 with u_1, v_1.

b) If there exists a unique fiber of pr_2 intersecting C with multiplicity 3, then the pair $(\mathbb{F}_0, C)$ is isomorphic to $(\mathbb{F}_0, C_{0,1})$ where $C_{0,1} = \{u_1^3 v_0 + u_0^2(u_0 + u_1)v_1 = 0\}$. Its automorphism group is isomorphic to $\mathbb{Z}_2$, acting via $([u_0 : u_1], [v_0 : v_1]) \mapsto ([-u_0 - 2u_1/3 : u_1], [-v_0 - 4v_1/27 : v_1])$.

c) Finally, if there is no fiber of pr_2 intersecting C with multiplicity 3 then the pair $(\mathbb{F}_0, C)$ is isomorphic to a pair of the form $(\mathbb{F}_0, C_{1,b})$ where $C_{1,b} = \{u_1^2(u_0 + u_1)v_0 + u_0^2(u_0 + bu_1)v_1 = 0\}$ for some $b \in \mathbb{C} \setminus \{0, 1\}$ such that the polynomial $s(t) = 2t^2 + (b+3)t + 2b$ has simple roots (this last condition guarantees precisely that $C_{1,b}$ cannot intersect a fiber of pr_2 with multiplicity 3). Furthermore, such a curve $C_{1,b}$ has exactly four horizontal tangents at the following points $P_i(b) = (p_i(b), q_i(b))$: $P_1(b) = ([0 : 1], [0 : 1])$, $P_2(b) = ([1 : 0], [1 : 0])$, $P_3(b) = ([r_1 : 1], [r_1^2(r_1 + b)/(r_1 + 1) : 1])$ and $P_4(b) = ([r_2 : 1], [r_2^2(r_2 + b)/(r_2 + 1) : 1])$, where $r_1, r_2 \in \mathbb{C} \setminus \{-1\}$ are the roots of $s(t)$. It follows from this description that two pairs $(\mathbb{F}_0, C_{1,b})$ and $(\mathbb{F}_0, C_{1,b'})$ are isomorphic only if there exists a permutation $\sigma \in \mathfrak{S}_4$ such that the cross-ratios of $(p_1(b), p_2(b), p_3(b), p_4(b))$ (resp. $(q_1(b), q_2(b), q_3(b), q_4(b))$) and $(p_{\sigma(1)}(b'), p_{\sigma(2)}(b'), p_{\sigma(3)}(b'), p_{\sigma(4)}(b'))$ (resp. $(q_{\sigma(1)}(b'), q_{\sigma(2)}(b'), q_{\sigma(3)}(b'), q_{\sigma(4)}(b'))$) are equal. A direct computation implies in turn that there exists uncountably many isomorphy classes of such pairs all having a finite group of automorphism of order at most 24 and that this group is in fact trivial except for finitely many of these.

REFERENCES

1. Maria Alberich-Carramiñana, *Geometry of the plane Cremona maps*, Lecture Notes in Mathematics, vol. 1769, Springer-Verlag, Berlin, 2002.
2. Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, second ed., vol. 4, Springer-Verlag, Berlin, 2004.
3. Eric Bedford and John Smillie, *Polynomial diffeomorphisms of $\mathbb{C}^2$. VIII. Quasi-expansion*, Amer. J. Math. **124** (2002), no. 2, 221–271.
4. Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.
5. Jérémy Blanc and Adrien Dubouloz, *Automorphisms of $\mathbb{A}^1$-fibered surfaces*, Trans. Amer. Math. Soc. (2011), no. 363, 5887–5924.
6. Andrea Bruno and Kenji Matsuki, *Log Sarkisov program*, Internat. J. Math. **8** (1997), no. 4, 451–494.
7. Alessio Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254.
8. Adrien Dubouloz and David Finston, *On exotic affine 3-spheres*, Preprint arXiv:1106.2900, to appear in Journal of Algebraic Geometry (2014).
9. Adrien Dubouloz and Stéphane Lamy, *Variations on the log-Sarkisov program for surfaces*, Preprint arXiv:0802.2441.
10. Shmuel Friedland and John Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems **9** (1989), no. 1, 67–99.
11. Osamu Fujino, *Minimal model theory for log surfaces*, Publ. Res. Inst. Math. Sci. **48** (2012), no. 2, 339–371.
12. M. H. Gizatullin, *Quasihomogeneous affine surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1047–1071.
13. Marat Gizatullin and Vladimir Danilov, *Automorphisms of affine surfaces. I*, Math. USSR Izvestiya **9** (1975), 493–534.
14. ———, *Automorphisms of affine surfaces. II*, Math. USSR Izvestiya **11** (1977), 51–98.
15. Christopher D. Hacon and James McKernan, *The Sarkisov program*, J. Algebraic Geom. **22** (2013), no. 2, 389–405.
16. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
17. János Kollár (ed.), *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).
18. János Kollár and Sándor Kovács, *Singularities of the Minimal Model Program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013.

19. János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
20. Stéphane Lamy, *Une preuve géométrique du théorème de Jung*, Enseign. Math. (2) **48** (2002), no. 3-4, 291–315.
21. ———, *Sur la structure du groupe d'automorphismes de certaines surfaces affines*, Publ. Mat. **49** (2005), no. 1, 3–20.
22. ———, *Groupes de transformations birationnelles de surfaces*, Mémoire d'habilitation (2010).
23. Kenji Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.
24. Yu. M. Polyakova, *A family of categories of log terminal pairs and automorphisms of surfaces*, Izvestiya Mathematics **74** (2010), no. 3, 541–593.
25. Yuri G. Prokhorov, *Lectures on complements on log surfaces*, MSJ Memoirs, vol. 10, Mathematical Society of Japan, Tokyo, 2001.
26. Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UNIVERSITÉ DE BOURGOGNE, 9 AVENUE ALAIN SAVARY - BP 47870, 21078 DIJON CEDEX, FRANCE

E-mail address: adrien.dubouloz@u-bourgogne.fr

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, 31062 TOULOUSE CEDEX 9, FRANCE

E-mail address: slamy@math.univ-toulouse.fr