Tunneling between corners for Robin Laplacians

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TUNNELING BETWEEN CORNERS FOR ROBIN LAPLACIANS

BERNARD HELFFER AND KONSTANTIN PANKRASHKIN

Abstract. We study the Robin Laplacian in a domain with two corners of the same opening, and we calculate the asymptotics of the two lowest eigenvalues as the distance between the corners increases to infinity.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open set with a sufficiently regular boundary (e.g. compact Lipschitz or non-compact with a suitable behavior at infinity) and $\beta \in \mathbb{R}$. By the associated Robin Laplacian $H(\Omega, \beta)$ we mean the operator acting in a weak sense as

$$H(\Omega,\beta)f := -\Delta f, \quad \frac{\partial f}{\partial n} = \beta f \text{ at } \partial\Omega\,,$$

where n is the unit outward normal at the boundary; a rigorous definition is given below (Subsection 2.3). In various applications, such as the study of the critical temperature in the enhanced surface superconductivity (and in this context the Robin condition is also called the De Gennes condition, see [15] and references therein) or the analysis of certain reaction-diffusion processes, one is interested in the spectral properties of $H(\Omega, \beta)$, the behavior of the spectrum as $\beta \to +\infty$ being of a particular importance [10,16]. For sufficiently regular Ω, it was shown in [17] that the bottom of the spectrum $E(\beta)$ behaves as

$$E(\beta) = -C_\Omega \beta^2 + o(\beta^2) \text{ as } \beta \to +\infty\,,$$

where $C_\Omega > 0$ is a constant depending on the geometry of the boundary. In particular, $C_\Omega = 1$ for smooth domains, and some information on the subsequent terms of the asymptotics was obtained e.g. in [7,9,19,20]. In the non-smooth case one can have $C_\Omega > 1$, and the constant is understood better in the $2D$ case. If ω denotes the smallest corner at the boundary, then

$$C_\Omega = \frac{2}{1-\cos\omega} \text{ if } \omega < \pi\,, \text{ and } C_\Omega = 1 \text{ otherwise.}$$

In other words, intuitively, each corner at the boundary can be viewed as a geometric well, and it is the deepest well which determines the principal term of the spectral asymptotics, and one may expect that the respective vertices serve as the asymptotic support of the respective eigenfunction. One meets the natural question of what happens if one has several wells of the same depth, i.e. several corners with the same opening. Similar questions appear in various settings: semiclassical limit for multiple wells [1,5,12–14], distant potential perturbations [6], domains coupled by a thin tube [4] or waveguides with distant boundary perturbations [3], in which the interaction between wells gives rise to an exponentially small difference between the lowest eigenvalues. The aim of the present paper is to obtain a result in the same spirit for Robin Laplacians in a class of corner domains. We note that the eigenvalues $E(\Omega, \beta)$ of $H(\Omega, \beta)$ satisfy the obvious scaling relation, $E(\Omega, \ell\beta) = \ell^2 E(\ell\Omega, \beta)$, $\ell > 0$, and the regime $\beta \to +\infty$ with a fixed Ω is essentially equivalent to the study of $E(\ell\Omega, \beta)$ as $\ell \to +\infty$ with a fixed β. We prefer to deal with scaled domains in order to have finite limits.

Let us describe our result. Let $\omega \in (0, \pi)$ and $L > 0$. Denote by $\Omega_L = \Omega_L(\omega)$ the intersection of the two infinite sectors Σ_1 and Σ_2,

$$\Sigma_1 := \Big\{(x_1,x_2) : \arg\big((x_1+L)+ix_2\big) \in (0,\omega)\Big\}, \quad \Sigma_2 := \Big\{(x_1,x_2) : (-x_1,x_2) \in \Sigma_1\Big\},$$

see Fig. 1. Clearly, for $\omega \geq \pi/2$ the set Ω_L is an infinite biangle whose vertices are the points $A_1 = (-L, 0)$ and $A_2 = (L, 0)$, while for $\omega < \pi/2$ we obtain the interior of the triangle whose vertices are the above points A_1 and A_2 and the point $A_3 = (0, L\tan\omega)$, see Figure 2. In what follows we fix a constant β,

$$\beta > 0.$$

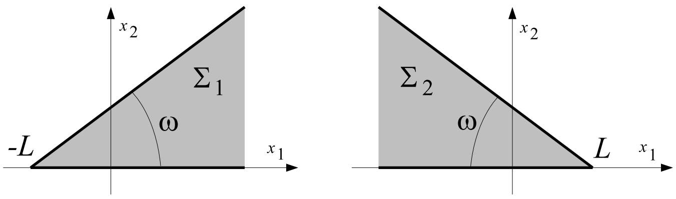

FIGURE 1. The infinite sectors Σ_1 and Σ_2.

The associated Robin Laplacian

$$H_L := H(\Omega_L, \beta)$$

is a self-adjoint operator in $L^2(\Omega_L;\mathbb{R})$, see Subsection 2.3 for the rigorous definition. Indeed, this operator depend on ω through Ω_L but it is not reflected in the notation as ω will be fixed. Elementary considerations show that if $\omega < \pi/2$, then H_L has a compact resolvent, and the spectrum consists of eigenvalues $E_1(L) < E_2(L) \le \dots$. As usually, each eigenvalue may appear several times according to its multiplicity. For $\omega \ge \pi/2$ one has $\operatorname{spec}_{\mathrm{ess}} H_L = [-\beta^2, +\infty)$, so the discrete spectrum consists of eigenvalues $E_1(L) < E_2(L) \le \dots < -\beta^2$.

Our main result is as follows:

Theorem 1.1. *Assume that either $\omega \in \big(0, \frac{\pi}{3}\big)$ or $\omega \in \big[\frac{\pi}{2}, \pi\big)$. As L tends to $+\infty$, the two lowest eigenvalues have the asymptotics*

$$E_1(L) = -\frac{2\beta^2}{1-\cos\omega} - 4\beta^2 \frac{1+\cos\omega}{(1-\cos\omega)^2} \exp\Big(-2\beta\frac{1+\cos\omega}{\sin\omega}L\Big) + O\Big(L^2 \exp\Big(-(2+\delta)\beta\frac{1+\cos\omega}{\sin\omega}L\Big)\Big),$$

$$E_2(L) = -\frac{2\beta^2}{1-\cos\omega} + 4\beta^2 \frac{1+\cos\omega}{(1-\cos\omega)^2} \exp\Big(-2\beta\frac{1+\cos\omega}{\sin\omega}L\Big) + O\Big(L^2 \exp\Big(-(2+\delta)\beta\frac{1+\cos\omega}{\sin\omega}L\Big)\Big),$$

where $\delta = 2\big((\cos\omega)^{-1} - 1\big)$ for $\omega < \pi/3$ and $\delta = 2$ for $\omega \ge \pi/2$. In particular,

$$E_2(L) - E_1(L) = 8\beta^2 \frac{1+\cos\omega}{(1-\cos\omega)^2} \exp\Big(-2\beta\frac{1+\cos\omega}{\sin\omega}L\Big) + O\Big(L^2 \exp\Big(-(2+\delta)\beta\frac{1+\cos\omega}{\sin\omega}L\Big)\Big).$$

Our proof is in the spirit of the scheme developed by Helffer and Sjöstrand for the semiclassical analysis of the multiple well problem [12,13]. In Section 2 we recall the necessary tools and establish some basic properties of the Robin Laplacians in polygons. Section 3 is devoted to the proof of Theorem 1.1. In Appendix A we study the one-dimensional Robin problem which is used to obtain a more precise result for the case $\omega = \frac{\pi}{2}$, which shows that the remainder estimate in Theorem 1.1 is almost optimal. The main difficulties of implementing the Helffer-Sjöstrand approach consist in the geometric and non-additive nature of the interaction and in the presence of the non-trivial boundary condition. Therefore, one needs to find a suitable weak form of some constructions when estimating the eigenfunctions, and the presence of the corners implies the use of rather involved cut-off functions. We tried to make the presentation as linear and self-contained as possible.

Let us add few remarks concerning possible generalizations and improvements.

Remark 1.2. The case $\omega \in \big(\frac{\pi}{3}, \frac{\pi}{2}\big)$ can be considered using the same scheme, one shows that for $L \to +\infty$ one has

$$E_1(L) = -\frac{2\beta^2}{1-\sin\omega} + O\Big(\sqrt{L}\exp\Big(-\beta\frac{\sin\omega}{\cos^2\omega}L\Big)\Big)$$

with the first eigenfunction concentrated near the top corner A_3, and

$$\liminf_{L\to+\infty} E_2(L) > -\frac{2\beta^2}{1-\sin\omega}.$$

Remark 1.3. We remark that the equilateral triangle, i.e. the case $\omega = \frac{\pi}{3}$, was partially studied in [18, Section 7]. It was shown that each eigenvalue $E_j(L)$ can be computed as the solution of some explicit

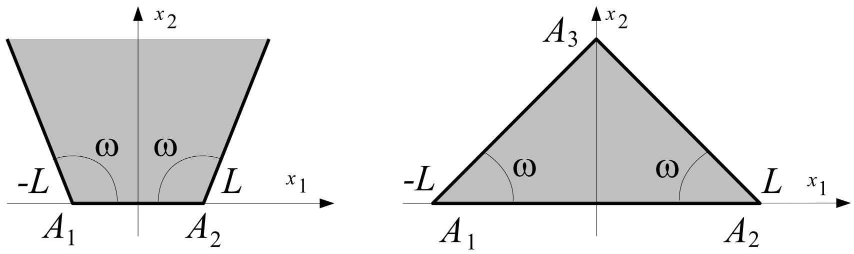

FIGURE 2. The domain Ω_L for $\omega \geq \dfrac{\pi}{2}$ (left) and $\omega < \dfrac{\pi}{2}$ (right).

non-linear system, while the number of equations in the system grows rapidly with j. It was proved that

$$\lim_{L\to+\infty} E_j(L) = -4\beta^2, \quad j = 1,2,3, \quad \liminf_{L\to+\infty} E_4(L) = -\beta^2,$$

$$E_2(L) = E_3(L) \text{ for large } \beta. \tag{1}$$

To obtain more precise estimates using the method of [18] one needs to analyze in detail a non-linear system of nine equations and, to our knowledge, this was not done so far.

We note that our approach is still applicable with suitable modifications, i.e. we need to take into account the interactions between the three corners. One can prove that, for sufficiently large L, there exists a bijection σ between the three lowest eigenvalues of H_L and the three eigenvalues of the matrix

$$N_0 = \begin{pmatrix} E & w & w \\ w & E & w \\ w & w & E \end{pmatrix}, \quad E = -\frac{2\beta^2}{1-\cos\omega} \equiv -4\,\beta^2, \quad w = -24\,\beta^2 e^{-2\sqrt{3}L}\,,$$

such that $\sigma(E) = E + O(L^2 e^{-4\sqrt{3}\beta L})$. Note that the eigenvalues of N_0 are $E_\alpha - w$ (double) and $E_\alpha + 2w$ (simple). Taking into account the inequality $w < 0$ we conclude that the three lowest eigenvalues of H_L behave as

$$E_1(L) = -4\,\beta^2 - 48\,\beta^2 e^{-2\sqrt{3}L} + O(L^2 e^{-4\sqrt{3}\beta L})\,,$$
$$E_j(L) = -4\,\beta^2 + 24\,\beta^2 e^{-2\sqrt{3}L} + O(L^2 e^{-4\sqrt{3}\beta L}), \quad j = 2,3, \quad L \to +\infty,$$

which is in agreement with the degeneracy (1).

Another possible approach is the use of the symmetries in the spirit of [14] and [8, Section 16.2], then one may show directly that the three lowest eigenvalues of H_L are the eigenvalues of the matrix

$$\widetilde{N}_0 := \begin{pmatrix} \widetilde{E} & \widetilde{w} & \widetilde{w} \\ \widetilde{w} & \widetilde{E} & \widetilde{w} \\ \widetilde{w} & \widetilde{w} & \widetilde{E} \end{pmatrix} \text{ with } \widetilde{E} = E + O(L^2 e^{-4\sqrt{3}\beta L}) \text{ and } \widetilde{w} = w + O(L^2 e^{-4\sqrt{3}\beta L}),$$

which gives the degeneracy without using the results of [18].

Remark 1.4. One may see from the proof that the result admits direct extensions to a little bit more general domains. For example, one can take a compact piecewise-smooth Lipschitz domain Ω satisfying the following two conditions:

- for some $\tau \in (1, \tau^*)$ there holds $L\Omega \cap B_L(\tau) = \Omega_L(\omega) \cap B_L(\tau)$, where
$$B_L(\tau) := \big\{(x_1, x_2) : x_2 < \tau L \sin\omega\big\}, \quad \omega \in (0,\pi), \quad \tau^* := \begin{cases} \frac{1}{\cos\omega}\,, & \omega < \frac{\pi}{3}, \\ 2, & \omega \geq \frac{\pi}{3}, \end{cases}$$
i.e. $L\Omega$ coincides with $\Omega_L(\omega)$ in a suitable neighborhood of the corners A_1 and A_2,
- the domain Ω does not contain any further corner whose opening is smaller or equal to ω,

then Theorem 1.1 holds for the eigenvalues $E_j(L)$ of the associated Robin Laplacian $H(L\Omega, \beta)$ with $\delta = 2(\tau - 1)$. In particular, one obtains an admissible domain if one smoothens in a suitable way the top corner of the triangle $\Omega_L(\omega)$ with $\omega \in (\frac{\pi}{3}, \frac{\pi}{2})$.

It would be interesting to know if any result of this kind can be obtained for more general domains and more general relative positions of the corners. For the smooth domains, one may expect that the role of the corners is played by the points of the boundary at which the curvature is maximal [7, 19], which gives rise to similar questions. This is actually the case for various eigenvalue problems arising in the surface superconductivity, see [8] and references therein.

Remark 1.5. For $\omega = \frac{\pi}{2}$, the estimate of Theorem 1.1 takes the form

$$E_1(L) = -2\beta^2 - 4\beta^2 e^{-2\beta L} + O(L^2 e^{-4\beta L}), \quad E_2(L) = -2\beta^2 + 4\beta^2 e^{-2\beta L} + O(L^2 e^{-4\beta L}). \tag{2}$$

On the other hand, one may separate the variables by representing $H_L = A \otimes 1 + 1 \otimes B_L$, where A and B_L are the following operators acting in $L^2(0,\infty)$ and $L^2(-L,L)$ respectively:

$$Au = -u'', \quad D(A) = \big\{u \in H^2(0,\infty) : u'(0) + \beta u(0) = 0\big\},$$
$$B_L v = -v'', \quad D(B_L) = \big\{v \in H^2(-L,L) : v'(-L) + \beta v(-L) = v'(L) - \beta v(L) = 0\big\}.$$

One easily computes $\operatorname{spec} A = \{-\beta^2\} \cup [0,+\infty)$. On the other hand, B_L has a compact resolvent and, if one denotes its eigenvalues by $\varepsilon_j(L)$, then $E_j(L) = -\beta^2 + \varepsilon_j(L)$. The behavior of $\varepsilon_j(L)$, $j = 1,2$, can be studied in a rather explicit way by using the 1D nature of the problem, see Proposition A.3 in the appendix, and one gets

$$E_1(L) = -2\beta^2 - 4\beta^2 e^{-2\beta L} + 8\beta^2(2\beta L - 1)e^{-4\beta L} + O(L^2 e^{-6\beta L}),$$
$$E_2(L) = -2\beta^2 + 4\beta^2 e^{-2\beta L} + 8\beta^2(2\beta L - 1)e^{-4\beta L} + O(L^2 e^{-6\beta L}).$$

One observes that the remainder estimate in our asymptotics (2) only differs by the factor L from the exact one.

Remark 1.6. Our considerations were in part stimulated by the paper [2] which studies the asymptotic behavior of the eigenvalues of the magnetic Neumann Laplacians in curvilinear polygons, but in our case we were able to obtain a more precise result due to the fact that we know the exact eigenfunction of an infinite sector. One may wonder if our machinery can help to progress in the problem of [2]. We note that both the magnetic Neumann Laplacian and the Robin Laplacian appear as approximate models in the theory of surface superconductivity and are closely related to the computation of the critical temperature [10, 13].

2. Preliminaries

2.1. **Basic tools in functional analysis.** Recall first the max-min principle for the eigenvalues of self-adjoint operators.

Proposition 2.1. *Let A be a lower semibounded self-adjoint operator in a Hilbert space $\mathcal{H}$, and let $E := \inf \operatorname{spec}_{\mathrm{ess}} A$ (we use the convention $\inf \emptyset = +\infty$). For $n \in \mathbb{N}$ consider the quantities*

$$E_n := \sup_{\psi_1,\dots,\psi_{n-1} \in \mathcal{H}} \inf_{\substack{u \in D(A),\, u \neq 0 \\ u \perp \psi_1,\dots,\psi_{n-1}}} \frac{\langle u, Au\rangle}{\langle u, u\rangle}.$$

If $E_n < E$, then E_n is the nth eigenvalue of A (if numbered in the non-decreasing order and counted with multiplicities). Furthermore, one obtains an equivalent definition of E_n by setting

$$E_n := \sup_{\psi_1,\dots,\psi_{n-1} \in \mathcal{H}} \inf_{\substack{u \in Q(A),\, u \neq 0 \\ u \perp \psi_1,\dots,\psi_{n-1}}} \frac{a(u,u)}{\langle u, u\rangle},$$

where $Q(A)$ is the form domain of A and a is the associated bilinear form.

Let $\mathcal{H}$ be a Hilbert space. For a closed subspace L of $\mathcal{H}$, we denote by P_L the orthogonal projector on L in $\mathcal{H}$. For an ordered pair (E,F) of closed subspaces E and F of $\mathcal{H}$ we define

$$d(E,F) = \|P_E - P_F P_E\| \equiv \|P_E - P_E P_F\|.$$

The following proposition summarizes some essential properties, cf. [13, Lemma 1.3 and Proposition 1.4]:

Proposition 2.2. *The distance between subspaces has the following properties:*

1. *$d(E,F) = 0$ if and only if $E \subset F$,*
2. *$d(E,G) \le d(E,F) + d(E,G)$ for any closed subspace G of $\mathcal{H}$,*
3. *if $d(E,F) < 1$, then then the map $E \ni f \mapsto P_F f \in F$ is injective, and the map $F \ni f \mapsto P_E f \in E$ has a continuous right inverse,*

(4) *If $d(E,F)<1$ and $d(F,E)<1$, then $d(E,F)=d(F,E)$, the map $F\ni f\mapsto P_E f\in E$ is bijective, and its inverse is continuous.*

The following proposition can be used to estimate $d(E,F)$, see e.g. [13, Proposition 3.5].

Proposition 2.3. *Let A be a self-adjoint operator in $\mathcal{H}$, $I\subset\mathbb{R}$ be a compact interval, $\psi_1,\dots,\psi_n\in D(A)$ be linearly independent, and $\mu_1,\dots,\mu_n\in I$. Denote:*

$$\varepsilon := \max_{j\in\{1,\dots,n\}} \big\|(A-\mu_j)\psi_j\big\|\,,\quad a:=\frac{1}{2}\operatorname{dist}\big(I,(\operatorname{spec}A)\setminus I\big)\,,$$

$$\Lambda := \text{ the smallest eigenvalue of the Gramian matrix } \big(\langle\psi_j,\psi_k\rangle\big)\,.$$

Let E be the subspace spanned by $\psi_1,\dots,\psi_n$ and F be the spectral subspace associated with A and I. If $a>0$, then

$$d(E,F)\le \frac{1}{a}\sqrt{\frac{n}{\Lambda}}\,\varepsilon\,.$$

2.2. **Robin Laplacians in infinite sectors.** For $\alpha\in(0,\pi)$, we define

$$S_\alpha := \big\{(x_1,x_2)\in\mathbb{R}^2:\quad \big|\arg(x_1+ix_2)\big|<\alpha\big\}$$

and consider the associated Robin Laplacian and the bottom of its spectrum:

$$Q_\alpha = H(S_\alpha,\beta)\,,\quad E_\alpha := \inf\operatorname{spec} Q_\alpha\,.$$

The following result is essentially contained in [17]:

Proposition 2.4. *The operator Q_α has the following properties:*

- *If $\alpha<\frac{\pi}{2}$, then*

$$E_\alpha = -\frac{\beta^2}{\sin^2\alpha},$$

and this point is a simple isolated eigenvalue of $\operatorname{spec}Q_\alpha$ *with the associated normalized eigenfunction*

$$U_\alpha(x_1,x_2)=\beta\sqrt{\frac{2\cos\alpha}{\sin^3\alpha}}\exp\Big(-\frac{\beta}{\sin\alpha}x_1\Big)\,. \tag{3}$$

- *If $\alpha\ge\frac{\pi}{2}$, then $E_\alpha=-\beta^2$ and* $\operatorname{spec}Q_\alpha=[E_\alpha,+\infty)$.

In what follows we will use another associated quantity,

$$\Lambda_\alpha := \inf(\operatorname{spec}Q_\alpha)\setminus\{E_\alpha\}.$$

In view of Proposition 2.4 we have:

- if $\alpha<\frac{\pi}{2}$, then $\Lambda_\alpha>E_\alpha$. In this case, if one denotes by P_α the orthogonal projection in $L^2(S_\alpha)$ onto the subspace spanned by U_α, then the spectral theorem implies

$$\langle u,Q_\alpha u\rangle \ge \Lambda_\alpha\|u\|^2+(E_\alpha-\Lambda_\alpha)\big\langle u,P_\alpha u\big\rangle \text{ for all } u\in D(Q_\alpha)\,, \tag{4}$$

- if $\alpha\ge\frac{\pi}{2}$, then $\Lambda_\alpha=E_\alpha$.

2.3. **Robin Laplacians in convex polygons.** In this subsection, let $\Omega_1\subset\mathbb{R}^2$ be a convex polygonal domain, i.e. is the intersection of finitely many half-planes. Assume that Ω_1 has N vertices $B_1,\dots,B_N$, and the corner opening at B_j will be denoted by $2\alpha_j$. We assume that all vertices are non-trivial, which means, due to the convexity, that $\alpha_j\in(0,\frac{\pi}{2})$ for all j. Define

$$\alpha := \min_j \alpha_j\,.$$

Furthermore, we set $\Omega_L:=L\Omega_1$ for some $L>0$ and denote by $A_j:=LB_j$ the vertices of Ω_L. We omit sometimes the reference to L and write more simply Ω. Finally, let us pick some $\beta>0$ and consider the associated Robin Laplacian $H:=H(\Omega,\beta)$. Strictly speaking, H is the operator associated with the bilinear form

$$h_{\Omega,\beta}(u,u)=\iint_\Omega |\nabla u|^2\,dx-\beta\int_{\partial\Omega}|u|^2\,ds\,,\quad u\in H^1(\Omega)\,,$$

where ds means the integration with respect to the length parameter. Using the standard methods we have

$$\operatorname{spec}_{\mathrm{ess}} H = \emptyset \text{ for } \omega < \frac{\pi}{2}, \quad \operatorname{spec}_{\mathrm{ess}} H = [-\beta^2, +\infty) \text{ for } \omega \geq \frac{\pi}{2}.$$

The following proposition is a particular case of a more general result proved in [17]:

Proposition 2.5. $\lim_{L\to+\infty} \inf \operatorname{spec} H = -\dfrac{\beta^2}{\sin^2\alpha} \equiv E_\alpha$.

To describe the domain of H, let us recall first the Green-Riemann formula, which states that, for $f \in H^1(\Omega)$ and $g \in H^2(\Omega)$,

$$\int_{\partial\Omega} f\,\frac{\partial g}{\partial n}\,ds = \iint_\Omega \Big(f\Delta g + \nabla f\cdot\nabla g\Big)\,dx\,, \tag{5}$$

where n is the outward unit normal.

Proposition 2.6. *There holds*

$$D(H) = \Big\{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = \beta u \text{ at } \partial\Omega\Big\} \tag{6}$$

and $Hu = -\Delta u$ for all $u \in D(H)$.

Proof. The claim follows from the general scheme developed for boundary value problems in non-smooth domains [11]. We just explain briefly how this scheme appplies to the Robin boundary condition. We note first that the associated form $h_{\Omega,\beta}$ is semibounded from below and closed due to the standard Sobolev embedding theorems. We note then that for any $u \in D(H)$ one has $Hu = -\Delta u$ in $\mathcal{D}'(\Omega)$. Furthermore, if $\widetilde{D}$ is the set on the right-hand side of (6), then it easily follows from (5) that $\widetilde{D} \subset D(H)$. It follows also that for $f \in H^2(\Omega)$ the inclusion $f \in D(H)$ is equivalent to the equality $\partial f/\partial n = \beta f$ on $\partial\Omega$. In view of these observations, it is sufficient to show that $D(H) \subset H^2(\Omega)$.

Take any $f \in D(H) \subset H^1(\Omega)$ and let $g := Hf \in L^2(\Omega)$. All corners at the boundary of Ω are smaller than π, and the trace of f on $\partial\Omega$ is in $H^{\frac12}(\partial\Omega)$, which means that there exists a solution $u \in H^2(\Omega)$ for the boundary value problem:

$$-\Delta u = g \text{ in } \Omega,\ \frac{\partial u}{\partial n} = \beta f \text{ on } \partial\Omega\,,$$

see [11, Section 2.4] (we are in the case where no singular solutions are present). On the other hand, f is a variational solution of the preceding problem. This means that the function $v := f - u \in H^1(\Omega)$ becomes a variational solution to

$$-\Delta v = 0 \text{ in } \mathcal{D}'(\Omega),\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\,.$$

Again according to [11, Section 2.4] we conclude that the only possible solution is constant, which means that $f = u + v \in H^2(\Omega)$. ☐

Now let us obtain a decay estimate of the eigenfunctions of H corresponding to the lowest eigenvalues as $L \to +\infty$. Let us start with a technical identity.

Lemma 2.7. *Let $u \in H^2(\Omega)$ be real-valued and satisfy the Robin boundary condition $\partial u/\partial n = \beta u$ at $\partial\Omega$. Furthermore, let $\Phi : \Omega \to \mathbb{R}$ be such that $\Phi, \nabla\Phi \in L^\infty(\Omega)$, then*

$$\iint_\Omega \big|\nabla(e^\Phi u)\big|^2 dx - \beta\int_{\partial\Omega} e^{2\Phi}u^2 ds = \iint_\Omega e^{2\Phi}u(-\Delta u)dx + \iint_\Omega |\nabla\Phi|^2 e^{2\Phi}u^2 dx.$$

Proof. We just consider the case $\Phi \in C^2(\bar\Omega)$, then one can pass to the general case using the standard regularization procedure. We have

$$\begin{aligned}\big|\nabla(e^\Phi u)\big|^2 &= \Big(\frac{\partial}{\partial x_1}(e^\Phi u)\Big)^2 + \Big(\frac{\partial}{\partial x_2}(e^\Phi u)\Big)^2 = \Big(\frac{\partial\Phi}{\partial x_1}e^\Phi u + e^\Phi\frac{\partial u}{\partial x_1}\Big)^2 + \Big(\frac{\partial\Phi}{\partial x_2}e^\Phi u + e^\Phi\frac{\partial u}{\partial x_2}\Big)^2\\ &= |\nabla\Phi|^2 e^{2\Phi}u^2 + 2e^\Phi u\nabla\Phi\cdot\nabla u + e^{2\Phi}|\nabla u|^2 = |\nabla\Phi|^2e^{2\Phi}u^2 + \nabla(e^{2\Phi}u)\cdot\nabla u\,.\end{aligned}$$

Integrating this equality in Ω, we arrive at

$$\begin{aligned}\iint_\Omega \left|\nabla(e^\Phi u)\right|^2 dx &= \iint_\Omega |\nabla\Phi|^2 e^{2\Phi} u^2 dx + \iint_\Omega \nabla(e^{2\Phi}u)\cdot\nabla u dx\\ &= \iint_\Omega |\nabla\Phi|^2 e^{2\Phi}u^2 dx + \int_{\partial\Omega} e^{2\Phi} u \frac{\partial u}{\partial n} ds + \iint_\Omega e^{2\Phi} u(-\Delta u) dx\\ &= \iint_\Omega |\nabla\Phi|^2 e^{2\Phi}u^2 dx + \beta\int_{\partial\Omega} e^{2\Phi}u^2 ds + \iint_\Omega e^{2\Phi}u(-\Delta u)\, dx\,.\end{aligned}$$

□

Now, let us choose a constant $b > 0$ such that all corners of Ω are contained in the ball of radius bL centered at the origin, and consider the function $\Phi : \Omega \to \mathbb{R}$ defined by

$$\Phi(x) := \beta \min\Big\{ \min_{j\in\{1,\ldots,N\}} \cot\alpha_j \cdot |x - A_j|, bL\Big\}.$$

For a compact Ω we choose the constant b sufficiently large, so that the exterior minimum can be dropped. The following lemma shows that the eigenfunctions of H corresponding to the lowest eigenvalues are concentrated near the corners with the smallest opening.

Proposition 2.8. *Let $\lambda = \lambda(L) > 0$ be such that $\lim_{L\to+\infty}\lambda(L) = 0$. Then, for any $\varepsilon \in (0, 1)$ there exists $C_\varepsilon > 0$ and $L_\varepsilon > 0$ such that, if $E = E(L)$ is an eigenvalue of H satisfying*

$$E \le -\frac{\beta^2}{\sin^2\alpha} + \lambda\,, \tag{7}$$

and u is an associated normalized eigenfunction, then

$$\left\|e^{(1-\varepsilon)\Phi}u\right\|_{H^1(\Omega)} \le C_\varepsilon e^{\varepsilon L} \quad \textit{for} \quad L \ge L_\varepsilon\,.$$

Proof. Let $r > 0$. Let us pick a C^∞ function $\chi : [0, +\infty) \to [0, 1]$ such that $\chi(t) = 1$ for $t \le r$ and $\chi(t) = 0$ for $t > 2r$, and introduce

$$\widetilde\chi_j(x) = \chi\Big(\frac{|x - A_j|}{L}\Big), \quad j = 1, \ldots, N\,.$$

We assume that r is sufficiently small, which ensures that the supports of $\widetilde\chi_j$ are disjoint and that $\Phi(x) = \beta\cot\alpha_j\,|x - A_j|$ for $x \in \operatorname{supp}\widetilde\chi_j$. An exact value of r will be chosen later. We denote

$$\widetilde\chi_0 := 1 - \sum_{j=1}^N \widetilde\chi_j\,, \quad \chi_j := \widetilde\chi_j \Big/ \sqrt{\sum_{k=0}^N \widetilde\chi_k^2}, \quad j = 0, \ldots, N.$$

We observe that we have the equalities $\operatorname{supp}\chi_j = \operatorname{supp}\widetilde\chi_j$, that each χ_j is C^∞, and that $\sum_{j=0}^N \chi_j^2 = 1$.

For any $v \in H^1(\Omega)$ we also have $\chi_j v \in H^1(\Omega)$, and by a direct computation one obtains

$$h_{\Omega,\beta}(v, v) = \sum_{j=0}^N h_{\Omega,\beta}(\chi_j v, \chi_j v) - \sum_{j=0}^N \left\|v\nabla\chi_j\right\|^2.$$

By construction of χ_j, we one can find a constant $C > 0$ independent of v and L with

$$h_{\Omega,\beta}(v, v) \ge \sum_{j=0}^N h_{\Omega,\beta}(\chi_j v, \chi_j v) - \frac{C}{L^2}\|v\|^2 \text{ for large } L\,.$$

Now let us denote $\Psi := (1 - \varepsilon)\Phi$. By applying the preceding inequality we obtain

$$\begin{aligned}I := \iint_\Omega \left|\nabla(e^\Psi u)\right|^2 dx - \beta\int_{\partial\Omega} |e^\Psi u|^2 ds \ge \delta \iint_\Omega \left|\nabla(e^\Psi u)\right|^2 dx\\ + (1-\delta)\Big[\sum_{j=0}^N \Big(\iint_\Omega \left|\nabla(\chi_j e^\Psi u)\right|^2 dx - \frac{\beta}{1-\delta}\int_{\partial\Omega} \left|\chi_j e^\Psi u\right|^2 ds\Big) - \frac{C}{L^2}\iint_\Omega |e^\Psi u|^2 dx\Big],\end{aligned}$$

where $\delta \in (0, 1)$ is a constant which will be chosen later.

Furthermore, considering $\chi_j e^{\Psi} u$ as a function from $H^1(S_j)$, where S_j is a suitably rotated copy of the sector S_{α_j} (see Subsection 2.2) which coincides with Ω near A_j, we have, for $j = 1, \dots, N$,

$$\iint_\Omega \big|\nabla(\chi_j e^{\Psi} u)\big|^2 dx - \frac{\beta}{1-\delta}\int_{\partial\Omega} \big|\chi_j e^{\Psi} u\big|^2 ds \geq -\frac{\beta^2}{(1-\delta)^2 \sin^2\alpha_j}\iint_\Omega |\chi_j e^{\Psi} u|^2 dx.$$

By the preceding constructions, the support of χ_0 is of the form $\operatorname{supp}\chi_0 = L\Omega'$ with some L-independent Ω'. Furthermore, one can construct a smooth domain D with $L\Omega' \subset LD \subset \Omega$ and such that $\partial(L\Omega')\cap\partial\Omega = \partial(LD)\cap\partial\Omega$. As mentioned in the introduction, the lowest eigenvalue of $H\big(LD, \beta/(1-\delta)\big)$ for large L converges to $-\beta^2/(1-\delta)^2$, i.e. for any $v \in H^1(LD)$ we have

$$\iint_{LD} \big|\nabla v\big|^2 dx - \frac{\beta}{1-\delta}\int_{\partial(LD)} |v|^2 ds \geq -\Big(\frac{\beta^2}{(1-\delta)^2} + \varepsilon_0\Big)\iint_{LD} |v|^2 dx,$$

where $\varepsilon_0 := \varepsilon_0(L,\delta) > 0$ is such that $\lim_{L\to+\infty}\varepsilon_0 = 0$ for any fixed $\delta \in (0,1)$. By taking $v = \chi_0 e^{\Psi} u$ we obtain

$$\iint_\Omega \big|\nabla(\chi_0 e^{\Psi} u)\big|^2 dx - \frac{\beta}{1-\delta}\int_{\partial\Omega} \big|\chi_0 e^{\Psi} u\big|^2 ds \geq -\Big(\frac{\beta^2}{(1-\delta)^2} + \varepsilon_0\Big)\iint_\Omega |\chi_0 e^{\Psi} u|^2 dx.$$

Putting the preceding estimates together we arrive at

$$\begin{aligned}(8)\quad I \geq \delta\iint_\Omega \big|\nabla(e^{\Psi} u)\big|^2 dx &- \Big(\frac{\beta^2}{1-\delta} + \frac{(1-\delta)C}{L^2} + \varepsilon_1\Big)\iint_\Omega |\chi_0 e^{\Psi} u|^2 dx \\ &- \sum_{j=1}^N \Big(\frac{\beta^2}{(1-\delta)\sin^2\alpha_j} + \frac{(1-\delta)C}{L^2}\Big)\iint_\Omega |\chi_j e^{\Psi} u|^2 dx\end{aligned}$$

with $\varepsilon_1 := (1-\delta)\varepsilon_0$. On the other hand, due to Lemma 2.7 we have

$$\begin{aligned}(9)\quad I &= \iint_\Omega e^{2\Psi} u(-\Delta u)dx + \iint_\Omega |\nabla\Psi|^2 e^{2\Psi} u^2\, dx \\ &= E\iint_\Omega e^{2\Psi} u^2 dx + \iint_\Omega |\nabla\Psi|^2 e^{2\Psi} u^2\, dx = \sum_{j=0}^N \iint_\Omega \big(E + |\nabla\Psi|^2\big)|\chi_j e^{\Psi} u|^2\, dx\,.\end{aligned}$$

We estimate as follows:

$$\big|\nabla\Psi(x)\big| \leq (1-\varepsilon)^2\beta^2\cot\alpha \equiv (1-\varepsilon)^2\beta^2\Big(\frac{1}{\sin^2\alpha} - 1\Big), \quad x \in \operatorname{supp}\chi_0\,,$$

$$\big|\nabla\Psi(x)\big| \leq (1-\varepsilon)^2\beta^2\cot\alpha_j \equiv (1-\varepsilon)^2\beta^2\Big(\frac{1}{\sin^2\alpha_j} - 1\Big), \quad x \in \operatorname{supp}\chi_j\,, \quad j = 1, \dots, N\,.$$

Substituting these two inequalities into (9) and using (7) we arrive at

$$\begin{aligned}I \leq &\Big(-\frac{\beta^2}{\sin^2\alpha} + \lambda + (1-\varepsilon)^2\beta^2\Big(\frac{1}{\sin^2\alpha} - 1\Big)\Big)\iint_\Omega |\chi_0 e^{\Psi} u|^2 dx \\ &+ \sum_{j=1}^N \Big(-\frac{\beta^2}{\sin^2\alpha} + \lambda + (1-\varepsilon)^2\beta^2\Big(\frac{1}{\sin^2\alpha_j} - 1\Big)\Big)\iint_\Omega |\chi_j e^{\Psi} u|^2\, dx\,.\end{aligned}$$

Combining with (8) we have:

$$\delta\iint_\Omega \big|\nabla(e^{\Psi} u)\big|^2\, dx + C_0\iint_\Omega |\chi_0 e^{\Psi} u|^2\, dx \leq \sum_{j=1}^N C_j \iint_\Omega |\chi_j e^{\Psi} u|^2\, dx\,,$$

where

$$\begin{aligned}C_0 &:= (2\varepsilon - \varepsilon^2)\Big(\frac{1}{\sin^2\alpha} - 1\Big)\beta^2 - \frac{\delta}{1-\delta}\beta^2 - \frac{(1-\delta)C}{L^2} - \varepsilon_1 - \lambda\,, \\ C_j &:= -\frac{\beta^2}{\sin^2\alpha} + (1-\varepsilon)^2\Big(\frac{1}{\sin^2\alpha_j} - 1\Big)\beta^2 + \frac{\beta^2}{(1-\delta)\sin^2\alpha_j} + \frac{(1-\delta)C}{L^2} + \lambda\,,\ j = 1, \dots, N\,.\end{aligned}$$

As $\varepsilon > 0$ is a fixed positive number and both ε_1 and λ tend to 0 as $L \to +\infty$, we can find $m_\varepsilon > 0$, $\delta > 0$ and $L_0 > 0$ such that $C_0 \geq m_\varepsilon$ for all $L > L_0$. At the same time, for the same δ and L we may estimate $C_j \leq M_\varepsilon$, $j = 1, \dots, N$, which gives

$$\iint_\Omega \left|\nabla(e^\Psi u)\right|^2 dx + \iint_\Omega |\chi_0 e^\Psi u|^2\, dx \leq C_\varepsilon \sum_{j=1}^N |\chi_j e^\Psi u|^2\, dx\,, \quad C_\varepsilon := \frac{M_\varepsilon}{\delta} + \frac{M_\varepsilon}{m_\varepsilon}\,.$$

Now we get the estimate

$$\begin{aligned}\|e^{(1-\varepsilon)\Phi}u\|^2_{H^1(\Omega)} &= \|e^\Psi u\|^2_{H^1(\Omega)} = \iint_\Omega \left|\nabla(e^\Psi u)\right|^2 dx + \iint_\Omega |e^\Psi u|^2\, dx \\ &= \iint_\Omega \left|\nabla(e^\Psi u)\right|^2 dx + \iint_\Omega |\chi_0 e^\Psi u|^2\, dx + \sum_{j=1}^N |\chi_j e^\Psi u|^2\, dx \leq (1 + C_\varepsilon) \sum_{j=1}^N |\chi_j e^\Psi u|^2\, dx \\ &\leq (1 + C_\varepsilon) \exp\Big[(1-\varepsilon) \max_{j\in\{1,\dots,N\}} \sup_{x\in \operatorname{supp}\chi_j} \Phi(x)\Big] \sum_{j=1}^N \iint_\Omega |\chi_j u|^2\, dx\,.\end{aligned}$$

We have

$$\sum_{j=1}^N \iint_\Omega |\chi_j u|^2\, dx \leq \sum_{j=0}^N \iint_\Omega |\chi_j u|^2\, dx = \iint_\Omega |u|^2\, dx = 1\,,$$

and $\max_{j\in\{1,\dots,N\}} \sup_{x\in\operatorname{supp}\chi_j} \Phi(x) \leq 2r\beta(\cot\alpha)L$. Therefore, by taking $r < \varepsilon/(2t\beta\cot\alpha)$, we get the conclusion. □

3. The lowest eigenvalues of H_L

3.1. **Notation.** In this section we study in greater detail the lowest eigenvalues of the operator H_L. We collect first some notation and conventions used below. Note that all the assertions of Section 2 are applicable to H_L as well. Throughout the section we will write

$$\alpha := \frac{\omega}{2} \quad \text{and} \quad \Omega := \Omega_L\,.$$

Furthermore, we introduce the following transformations of $\mathbb{R}^2$:

$$R_1(x_1, x_2) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 + L \\ x_2 \end{pmatrix}, \quad R_2(x_1, x_2) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \begin{pmatrix} L - x_1 \\ x_2 \end{pmatrix}.$$

The geometric meaning of R_j is clear from the equalities $R_j(\Sigma_j) = S_\alpha$, $j = 1, 2$, and we consider the associated rotated eigenfunctions

$$U_j(x) := U_\alpha(R_j x)\,,\ j = 1, 2\,.$$

Recall that S_α and U_α are defined in Subsection 2.2, so we have

$$U_1(x_1, x_2) = \beta\sqrt{\frac{2\cos\alpha}{\sin^3\alpha}}\, e^{-\beta(x_1+L)\cot\alpha - \beta x_2}, \quad U_2(x_1, x_2) = \beta\sqrt{\frac{2\cos\alpha}{\sin^3\alpha}}\, e^{-\beta(L-x_1)\cot\alpha - \beta x_2}. \tag{10}$$

We also recall the notation

$$E_\alpha := -\beta^2/\sin^2\alpha\,.$$

Furthermore, for $j = 1, 2$ we denote by M_j the Robin Laplacian in Σ_j,

$$M_j := H(\Sigma_j, \beta)\,.$$

3.2. **A rough eigenvalue estimate.** Let us obtain some rough information on the behavior of the eigenvalues of H_L as L tends to $+\infty$. Assuming that H_L has at least $n-1$ eigenvalues below the essential spectrum, we denote

$$\widetilde{E}_n(L) := \inf(\operatorname{spec} H_L) \setminus \big\{E_1(L), \dots, E_{n-1}(L)\big\}\,,$$

Lemma 3.1. *Let $\omega \in \big(0, \frac{\pi}{3}\big) \cup \big[\frac{\pi}{2}, \pi\big)$, then for sufficiently large L the operator H_L has at least two eigenvalues below the essential spectrum, and one has*

$$\lim_{L\to+\infty} E_j(L) = E_\alpha, \quad j = 1, 2\,, \tag{11}$$

$$\liminf_{L\to+\infty} \widetilde{E}_3(L) > E_\alpha\,. \tag{12}$$

Proof. For $\delta > 0$, let us pick a C^∞ function $\chi : \mathbb{R}_+ \to [0,1]$ such that $\chi(t) = 1$ for $t \le \delta$ and $\chi(t) = 0$ for $t > 2\delta$. Introduce the functions

$$\widetilde{\chi}_j(x) = \chi\Big(\frac{|x - A_j|}{L}\Big), \quad j = 1, 2.$$

We assume that δ is sufficiently small, which ensures that the supports of $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$ do not intersect, and consider the functions

$$v_j := \widetilde{\chi}_j U_j, \quad j = 1, 2.$$

By a simple computation, as $L \to +\infty$ we have

$$\iint_\Omega v_j v_k \, dx = \delta_{jk} + o(1), \quad \iint_\Omega \nabla v_j \cdot \nabla v_k \, dx - \beta \int_{\partial\Omega} v_j v_k \, ds = E_\alpha \delta_{jk} + o(1), \quad j, k = 1, 2.$$

It follows that, for L large enough,

$$\sup_{0 \not\equiv v \in \mathrm{Span}(v_1, v_2)} \frac{h_{\Omega,\beta}(v,v)}{\langle v, v\rangle} \le E_\alpha + o(1) < -\beta^2 \le \inf \operatorname{spec}_{\mathrm{ess}} H_L = \begin{cases} -\beta^2, & \omega \ge \frac{\pi}{2}, \\ +\infty, & \omega < \frac{\pi}{2}. \end{cases}$$

On the other hand, the functions v_1 and v_2 are linearly independent, and for any $\psi \in L^2(\Omega)$ one can find a non-trivial linear combination $v \in \mathrm{Span}(v_1, v_2)$ which is orthogonal to ψ. Due to the preceding estimate and Proposition 2.1 we obtain then $E_2(L) \le E_\alpha + o(1)$. Combining with $E_2(L) \ge E_1(L)$, and with the result of Proposition 2.5, this gives (11).

Let us now prove (12). Let us introduce $\widetilde{\chi}_0 := 1 - \widetilde{\chi}_1 - \widetilde{\chi}_2$ and set

$$\chi_j := \widetilde{\chi}_j \Big/ \sqrt{\sum_{k=0}^{2} \widetilde{\chi}_k^2}, \quad j = 0, 1, 2.$$

By a direct computation, for any $u \in H^1(\Omega)$ we have

$$h_{\Omega,\beta}(u,u) = \sum_{j=0}^{2} h_{\Omega,\beta}(\chi_j u, \chi_j u) - \sum_{j=0}^{2} \big\| u \nabla \chi_j \big\|^2,$$

and by the construction of χ_j, we can find $L_0 > 0$ and $C > 0$ such that for all u and $L \ge L_0$

$$h_{\Omega,\beta}(u,u) \ge \sum_{j=0}^{2} h_{\Omega,\beta}(\chi_j u) - \frac{C}{L^2} \|u\|^2.$$

Furthermore, we have $\chi_j u \in H^1(\Sigma_j)$, $j = 1, 2$. Consider the orthogonal projections $\Pi_j := \langle U_j, \cdot\rangle U_j$ in $L^2(\Sigma_j)$. By applying the inequality (4) we obtain

$$h_{\Omega,\beta}(\chi_j u, \chi_j u) \ge (E_\alpha - \Lambda_\alpha) \|\Pi_j \chi_j u\|^2_{L^2(\Sigma_j)} + \Lambda_\alpha \|\chi_j u\|^2_{L^2(\Sigma_j)}, \quad j = 1, 2.$$

The norms in $L^2(\Sigma_j)$ can be replaced back by the norms in $L^2(\Omega)$, and we infer

$$h_{\Omega,\beta}(u,u) \ge \langle u, \Pi u\rangle + \Lambda_\alpha \big(\|\chi_1 u\|^2 + \|\chi_2 u\|^2\big) + h_{\Omega,\beta}(\chi_0 u, \chi_0 u) - \frac{C}{L^2} \|u\|^2,$$

where $\Pi := (E_\alpha - \Lambda_\alpha)\big(\chi_1 \Pi_1 \chi_1 + \chi_2 \Pi_2 \chi_2\big)$ is an operator whose range is at most two-dimensional.

To estimate the term with χ_0, we proceed as in the proof of Proposition 2.8. By the preceding constructions, the support of χ_0 has the form $\operatorname{supp} \chi_0 = L\Omega'$ with some L-independent Ω'. Furthermore, one can construct a convex polygonal domain D with $L\Omega' \subset LD \subset \Omega$ such that $\partial(L\Omega') \cap \partial\Omega = \partial(LD) \cap \partial\Omega$ and that the minimal corner θ at the boundary of D is strictly larger than ω. By Proposition 2.5 for any $A < E_{\theta/2}$ and any $v \in H^1(LD)$ we have, as L is sufficiently large, $h_{LD,\beta}(v,v) \ge A\|v\|^2_{L^2(LD)}$. As $E_{\theta/2} > E_{\omega/2} \equiv E_\alpha$, we may assume that $A > E_\alpha$. Using the last equality with $v = \chi_0 u$ we obtain, for large L, $h_{\Omega,\beta}(\chi_0 u, \chi_0 u) \ge A\|\chi_0 u\|^2$.

Putting all together and noting that $\|\chi_0 u\|^2 + \|\chi_1 u\|^2 + \|\chi_2 u\|^2 = \|u\|^2$ we obtain, as L is large,

$$h_{\Omega,\beta}(u,u) \ge \langle u, \Pi u\rangle + \Big(E - \frac{C}{L^2}\Big) \|u\|^2, \quad E = \min(A, \Lambda_\alpha) > E_\alpha.$$

Now take two vectors ψ_1 and ψ_2 spanning the range of Π. For any non-zero $u \in H^1(\Omega)$ which is orthogonal to ψ_1 and ψ_2 we have

$$\frac{h_{\Omega,\beta}(u,u)}{\langle u, u\rangle} \ge E - \frac{C}{L^2},$$

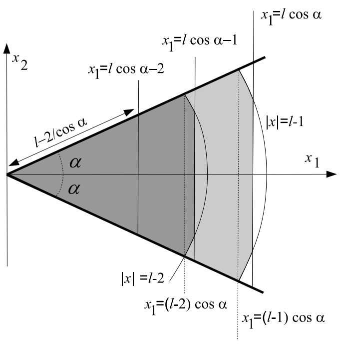

FIGURE 3. The function $\varphi_{\alpha,\ell}$ vanishes outside the shaded domains, and equals 1 in the dark shaded domain.

which gives the announced inequality (12) by the max-min principle. ☐

The following assertion summarizes the preceding considerations:

Proposition 3.2. *Let $\omega \in \big(0, \frac{\pi}{3}\big) \cup \big[\frac{\pi}{2}, \pi\big)$, then there exists $\delta > 0$ and L_0 such that for $L \geq L_0$ the spectrum of H_L in $(E_\alpha - \delta, E_\alpha + \delta)$ consists of exactly two eigenvalues $E_1(L)$ and $E_2(L)$, both converging to E_α as $L \to +\infty$.*

Remark 3.3. Indeed, one can prove an analog of Lemma 3.1 for the remaining ranges of ω in a similar way, and one has:

$$\begin{aligned} &\lim_{L\to+\infty} E_1(L) = E_{\frac{\pi-2\omega}{2}} \quad \text{and} \quad \liminf_{L\to+\infty} \widetilde{E}_2(L) > E_{\frac{\pi-2\omega}{2}} \quad \text{for } \omega \in \Big(\frac{\pi}{3}, \frac{\pi}{2}\Big), \\ &\lim_{L\to+\infty} E_j(L) = E_\alpha, \quad j = 1,2,3, \quad \text{and} \quad \liminf_{L\to+\infty} \widetilde{E}_4(L) > E_\alpha \text{ for } \omega = \frac{\pi}{3}, \end{aligned} \tag{13}$$

and Proposition 3.2 should be suitably reformulated. Concerning the case $\omega = \pi/3$, see also Remark 1.3 above.

For the rest of the section, we assume that

$$\omega \in \Big(0, \frac{\pi}{3}\Big) \cup \Big[\frac{\pi}{2}, \pi\Big).$$

3.3. **Cut-off functions.** We are going to introduce a family of cut-off functions adapted to the geometry of the sector S_α (see Subsection 2.2). Note that our assumptions imply $\alpha < \frac{\pi}{2}$. Pick a function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\chi \in C^\infty(\mathbb{R}), \quad \chi(t) = 1 \text{ for } t \leq -1, \quad \chi(t) = 0 \text{ for } t \geq 0\,,$$

and for $\ell > 0$ we set

$$\varphi_{\alpha,\ell}(x_1, x_2) = \chi(x_1 - \ell\cos\alpha)\chi\big(|x| - (\ell - 1)\big)\,.$$

This function has the following properties for large ℓ, see Figure 3:

$$\begin{aligned} &\varphi_{\alpha,\ell} \in C^\infty(\overline{S}_\alpha)\,, \quad \varphi_{\alpha,\ell}(x) \in [0,1] \text{ for all } x \in S_\alpha\,, \quad \frac{\partial\varphi_{\alpha,\ell}}{\partial n} = 0 \text{ at } \partial S_\alpha\,, \\ &\varphi_{\alpha,\ell}(x) = 1 \text{ for } x = (x_1,x_2) \in \{x_1 \leq \ell\cos\alpha - 2\} \cap S_\alpha\,, \\ &\varphi_{\alpha,\ell}(x) = 0 \text{ for } x = (x_1,x_2) \notin \{x_1 \leq \ell\} \cap S_\alpha\,, \\ &\sum_{|\nu|\leq 2} \|D^\nu \varphi_{\alpha,\ell}\|_\infty \leq c \text{ for some } c > 0 \text{ independent of } \ell\,. \end{aligned} \tag{14}$$

The slightly involved construction of $\varphi_{\alpha,\ell}$ guarantees that for any function $f \in H^2(S_\alpha)$ with $\partial f/\partial n = \beta f$ at the boundary, the product $\varphi_{\alpha,\ell} f$ still satisfies the same boundary condition.

Finally, we set

$$\psi_{\alpha,\ell}(x) := \varphi_{\alpha,\ell}(x) U_\alpha(x)\,,$$

where U_α is defined in (3). Using the properties (14) and a simple direct computation one obtains:

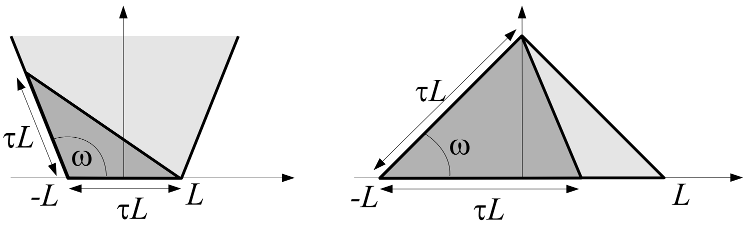

FIGURE 4. The choice of the constant τ.

Lemma 3.4. *The function $\psi_{\alpha,\ell}$ belongs to the domain of the operator Q_α (see subsection 2.2), and the following estimates are valid as $\ell \to +\infty$:*

$$\|\psi_{\alpha,\ell}\|^2_{L^2(S_\alpha)} = 1 + O(\ell e^{-2\beta\ell\cot\alpha}) \quad \textit{and} \quad \big\|(-\Delta - E_\alpha)\psi_{\alpha,\ell}\big\|^2_{L^2(S_\alpha)} = O(\ell e^{-2\beta\ell\cot\alpha}).$$

Now let us choose the maximal constant $\tau > 1$ such that the two isosceles triangles $\Theta_1(\tau L)$ and $\Theta_2(\tau L)$ with the side length τL and the vertex angle ω spanned at the boundary of Ω near respectively A_1 and A_2 are included in Ω. More precisely,

$$\tau := \begin{cases} \dfrac{1}{\cos\omega}, & \omega \in \Big(0, \dfrac{\pi}{3}\Big), \\ 2, & \omega \in \Big[\dfrac{\pi}{2}, \pi\Big) \end{cases} \tag{15}$$

see Figure 4 (cf. the discussion in Remark 1.4). Consider the functions

$$\psi_j(x) = v_j(x) U_j(x) \quad \text{with} \quad v_j(x) := \varphi_{\alpha,\tau L}(R_j x), \quad j = 1, 2.$$

By Proposition 3.2 we can find $\delta > 0$ such that the interval

$$I := (E_\alpha - \delta, E_\alpha + \delta) \tag{16}$$

contains exactly two eigenvalues of H_L and the larger interval $(E_\alpha - 2\delta, E_\alpha + 2\delta)$ does not contain any further spectrum for large L.

Let E denote the subspace spanned by ψ_j, $j = 1, 2$, and F denote the spectral subspace of H_L corresponding to I. We are going to estimate the distances $d(E, F)$ and $d(F, E)$ between these two subspaces, see Subsection 2.1.

Lemma 3.5. *For the Gramian matrix $G := (g_{jk}) = \big(\langle\psi_j, \psi_k\rangle\big)$ we have $g_{jk} = \delta_{jk} + O(Le^{-2\beta L\cot\alpha})$, $j, k = 1, 2$. Furthermore, $g_{11} = g_{22}$ and $g_{12} = g_{21}$.*

Proof. The identities for the coefficients follow from the considerations of symmetry. It follows from Lemma 3.4 that $\|\psi_j\|^2 = 1 + O(Le^{-2\tau\beta L\cot\alpha})$ for $j = 1, 2$. On the other hand, using the explicit expressions (10) for U_j , we obtain

$$\psi_1(x_1, x_2)\psi_2(x_1, x_2) = 2\beta^2 \frac{\cos\alpha}{\sin^3\alpha} \varphi_{\alpha,\tau L}(R_1 x)\varphi_{\alpha,\tau L}(R_2 x) \exp\big(-2\beta L\cot\alpha\big)\exp(-2\beta x_2\big).$$

Using the properties (14) we have $\langle\psi_1, \psi_2\rangle = O(Le^{-2\beta L\cot\alpha})$. As $\tau > 1$ by (15), this gives the result. $\square$

Lemma 3.6. *For large L there holds $d(E, F) = d(F, E) = O(\sqrt{L}e^{-\beta\tau L\cot\alpha})$.*

Proof. Let us show first the desired estimate for $d(E, F)$. By Lemma 3.4, we have

$$\big\|(H_L - E_\alpha)\psi_j\big\| = O(\sqrt{L}e^{-\beta\tau L\cot\alpha}).$$

Using Proposition 2.3 for the previously chosen interval I and applying Lemma 3.5 gives the result.

We will now show that $d(F, E) < 1$ for large L, then by Proposition 2.2 it will follow that $d(F, E) = d(E, F)$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\varphi(t) = 1$ for t near 0 and $\varphi(t) = 0$ for $t > \frac{1}{2}$ and introduce

$$\chi_j(x) := \varphi\Big(\frac{|x - A_j|}{L}\Big), \quad j = 1, 2, \quad \chi_0 := 1 - \chi_1 - \chi_2.$$

Let u_k be a normalized eigenfunction of H_L associated with $E_k(L)$, $k = 1, 2$. We know (Proposition 3.2) that $E_k(L)$ tends to E_α as $L \to +\infty$, so Proposition 2.8 is applicable to u_k. In particular, for some $\sigma > 0$

we have $\|\chi_0 u_k\|_{L^2(\Omega)} = O(e^{-\sigma L})$. Furthermore, using Proposition 2.6 we check that $\chi_j u_k \in D(H_L)$ and that

$$\big\|(H_L - E_\alpha)(\chi_j u_k)\big\|_{L^2(\Omega)} = \big\|(-\Delta - E_\alpha)(\chi_j u_k)\big\|_{L^2(\Omega)} = \big\| - (\Delta\chi_j)u_k - 2\nabla\chi_j \nabla u_k\big\|_{L^2(\Omega)} = O(e^{-\sigma' L}),$$

for some $\sigma' > 0$, and by taking the minimum we may assume that $\sigma = \sigma'$. The last estimate can be also rewritten as an estimate in $L^2(\Sigma_j)$, and we conclude that there exists $L_* > 0$ and $C > 0$ such that

$$\big\|(-\Delta - E_\alpha)(\chi_j u_k)\big\|_{L^2(\Sigma_j)} \le C\, e^{-\sigma L} \quad \text{for } L > L_* .$$

Now let us pick any $\sigma_0 \in (0, \sigma)$ and split the set $\{L : L > L_*\}$ into two disjoint parts I_1 and I_2 as follows. We say that $L \in I_1$ if $\|\chi_j u_k\|_{L^2(\Omega)} \equiv \|\chi_j u_k\|_{L^2(\Sigma_j)} \le e^{-\sigma_0 L}$. Therefore, for $L \in I_2$ we have $\|\chi_j u_k\|_{L^2(\Sigma_j)} \ge e^{-\sigma_0 L}$. We check again that $\chi_j u_k \in D(M_j)$, so by applying Proposition 2.2 to the operator M_j we conclude that $d\big(\operatorname{Span}(\chi_j u_k), \ker(M_j - E_\alpha)\big) \le C_0\, e^{-(\sigma-\sigma_0)L}$, $C_0 > 0$, which means that one can find $a_{jk} \in \mathbb{R}$ such that $\|\chi_j u_k - a_{jk} U_j\|_{L^2(\Sigma_j)} \le C_0\, e^{-(\sigma-\sigma_0)L}$, and $|a_{jk}| \le 1 + C_0\, e^{-(\sigma-\sigma_0)L}$. On the other hand, one can find $\sigma_1 > 0$ such that

$$\|U_j - \psi_j\|_{L^2(\Omega)} \equiv \|U_j - \psi_j\|_{L^2(\Sigma_j)} = \|(1 - v_j)U_j\|_{L^2(\Sigma_j)} \le C_1\, e^{-\sigma_1 L} .$$

Therefore, writing $\sigma_2 := \min(\sigma_1, \sigma - \sigma_0)$, we have

$$\|\chi_j u_k - a_j \psi_j\|_{L^2(\Omega)} = \|\chi_j u_k - a_{jk}\psi_j\|_{L^2(\Sigma_j)} \le C_2\, e^{-\sigma_2 L} \text{ for all } L \in I_2 .$$

By choosing $\sigma_* := \min(\sigma_0, \sigma_2)$, we conclude that, for any sufficiently large L, we can find $a_j \in \mathbb{R}$ with $|a_j| \le 1 + O(e^{-\sigma_* L})$, such that $\|\chi_j u_k - a_{jk}\psi_j\|_{L^2(\Omega)} = O(e^{-\sigma_* L})$. For $L \in I_1$ we can simply take $a_{jk} = 0$. We have then

$$u_k = \sum_{j=0}^{2} \chi_j u_k = \sum_{j=1}^{2} a_{jk}\psi_j + O(e^{-\sigma_* L}) \text{ in } L^2(\Omega) .$$

As the functions u_k, $k = 1, 2$, form an orthonormal basis in F, we have $d(F, E) = O(e^{-\sigma_* L}) < 1$ for large L. ☐

3.4. **Coupling between corners.** Recall that P_E denotes the orthogonal projection on E in $L^2(\Omega)$. In addition, we denote by Π_E the projection on E in $L^2(\Omega)$ along $F^\perp$. The following lemma essentially reproduces Lemma 2.8 in [13]. We give the proof for the sake of completeness.

Lemma 3.7. *For sufficiently large L we have $\|\Pi_E - P_E\| = O(\sqrt{L}e^{-\beta\tau L \cot\alpha})$. Furthermore, we have the following identities:*

(a) $\Pi_E = \Pi_E P_F$,
(b) *the inverse of* $K := (\Pi_E : F \to E)$ *is* $K^{-1} := (P_F : E \to F)$,
(c) $(H_L : F \to F) = K^{-1}(\Pi_E H_L : E \to E)K$.

Proof. By Lemma 3.6 we can write $F = \{x + Ax : x \in E\}$, where A is a bounded linear operator acting from E to $E^\perp$ with $\|A\| = O(\sqrt{L}e^{-\beta c L \cot\alpha})$. Then $F^\perp = \{y - A^* y : y \in E^\perp\}$. Furthermore, if $z = x + y$ with $x \in E$ and $y \in E^\perp$, then $P_E z = x$ and $\Pi_E z = \widetilde{x}$, where $\widetilde{x}$ is the vector from E satisfying $\widetilde{x} - (x + y) \in F^\perp$, which can be rewritten as $\widetilde{x} - (x + y) = A^*\widetilde{y} - \widetilde{y}$ for some $\widetilde{y} \in E^\perp$. Considering separately the terms in E and $E^\perp$ we arrive at the system $\widetilde{x} - x = A^*\widetilde{y}$, $y = \widetilde{y}$, which implies

$$\big\|(P_E - \Pi_E)z\big\| = \|x - \widetilde{x}\| \le \|A\| \cdot \|y\| \le \|A\| \cdot \|z\|$$

and proves the norm estimate.

Let us check the identities. To prove (a) we write $\Pi_E = \Pi_E(P_F + P_{F^\perp})$ and note that $\Pi_E P_{F^\perp} = 0$. To prove (b), we observe first that the existence of the inverses follows from Proposition 2.2. Now let us take any $z \in F$. It is uniquely represented as $z = x + y$ with $x \in E$ and $y \in F^\perp$, and $P_E z = x$. On the other hand, one has $\Pi_F x = z$, which proves the identity (b).

Furthermore, $\Pi_E H_L = \Pi_E H_L(P_F + P_{F^\perp}) = \Pi_E H_L P_F + \Pi_E P_{F^\perp} H_L$. Using again $\Pi_E P_{F^\perp} = 0$, we conclude that $\Pi_E H_L u = \Pi_E H_L P_F u$ for any $u \in E$. Finally, as $H_L P_F u \in F$ for any $u \in E$, we have

$$(\Pi_E H_L : E \to E) = (\Pi_E : F \to E)(H_L : F \to F)(P_F : E \to F) .$$

Combining with (b) leads to (c). ☐

Lemma 3.8. *The matrix M of $\Pi_E H_L : E \to E$ in the basis (ψ_1, ψ_2) is*

$$M = \begin{pmatrix} E_\alpha & w_{12} \\ w_{21} & E_\alpha \end{pmatrix} + O(L^{3/2} e^{-2\beta\tau L \cot\alpha}), \quad L \to +\infty, \quad w_{jk} := \iint_\Omega v_k (U_j \nabla U_k - U_k \nabla U_j) \nabla v_j \, dx \, .$$

Proof. The proof follows the scheme of Theorem 3.9 in [13]. We have $P_E u = \sum_{j,k=1}^2 c_{jk} \langle \psi_k, u\rangle \psi_j$, where c_{jk} are the coefficients satisfying

$$\sum_{j,k=1}^2 c_{jk} \langle \psi_k, \psi_\ell \rangle \psi_j = \psi_\ell \, , \quad j,\ell = 1,2 \, , \text{ i.e. } \sum_{k=1}^2 c_{jk} \langle \psi_k, \psi_\ell \rangle = \delta_{jl} \, , \quad \ell = 1,2 \, .$$

In other words, $(c_{jk}) = G^{-1}$, where G is the Gramian matrix of (ψ_j), and in virtue of Lemma 3.5 we have $c_{jk} = \delta_{jk} + O(Le^{-2\beta L \cot\alpha})$. Therefore, if we introduce another operator $\widehat{\Pi}$ by $\widehat{\Pi} u = \sum_{j=1}^2 \langle \psi_j, u\rangle \psi_j$, we have $\|P_E - \widehat{\Pi}\| = O(Le^{-2\beta L \cot\alpha})$. Combining with Lemma 3.7 we obtain $\|\Pi_E - \widehat{\Pi}\| = O(Le^{-\beta\tau L \cot\alpha})$. Here we used the inequality $\tau \le 2$, see (15).

Now, using the structure of $\psi_j = v_j U_j$ we have $H_L \psi_j = E_\alpha \psi_j - 2\nabla v_j \nabla U_j - (\Delta v_j) U_j$. The $L^2(\Omega)$-norms of two last terms on the right hand side are $O(\sqrt{L} e^{-\beta\tau L \cot\alpha})$, which gives

$$\begin{aligned} \Pi_E H_L \psi_j &= \Pi_E\big(E_\alpha \psi_j\big) + \widehat{\Pi}\big(-2\nabla v_j \nabla U_j - (\Delta v_j)U_j\big) + (\Pi_E - \widehat{\Pi})\big(-2\nabla v_j \nabla U_j - (\Delta v_j) U_j\big) \\ &= E_\alpha \psi_j + \widehat{\Pi}\big(-2\nabla v_j \nabla U_j - (\Delta v_j)U_j\big) + O(L^{3/2} e^{-2\beta\tau L\cot\alpha}) \\ &= E_\alpha \psi_j + \sum_{k=1}^2 b_{jk}\psi_k + O(L^{3/2} e^{-2\beta\tau L \cot\alpha}). \end{aligned} \tag{17}$$

with

$$b_{jk} := -\iint_\Omega \big(2\nabla v_j \nabla U_j + (\Delta v_j) U_j\big)\psi_k \, dx = -\iint_\Omega \big(2\nabla v_j \nabla U_j + (\Delta v_j)U_j\big) v_k U_k \, dx.$$

Using the Green-Riemann formula (5) we have

$$\begin{aligned} \iint_\Omega (-\Delta v_j) U_j v_k U_k \, dx &= \iint_\Omega \nabla v_j \nabla (U_j U_k v_k) \, dx - \int_{\partial\Omega} \frac{\partial v_j}{\partial n} U_j U_k v_k \, ds \\ &= \iint_\Omega U_j U_k \nabla v_j \nabla v_k \, dx + \iint_\Omega U_j v_k \nabla v_j \nabla U_k \, dx + \iint_\Omega v_k U_k \nabla v_j \nabla U_j \, dx \, , \end{aligned}$$

which gives

$$b_{jk} = \delta_{jk} w_{jk} + \varepsilon_{jk}, \quad \varepsilon_{jk} := \iint_\Omega U_j U_k \nabla v_j \nabla v_k \, dx \, . \tag{18}$$

Note that

$$U_1(x_1,x_2) U_2(x_1,x_2) = \frac{2\beta^2 \cos\alpha}{\sin^3\alpha} \exp\big(-2\beta L \cot\alpha\big) \exp(-2\beta x_2\big) \tag{19}$$

and that $\nabla v_1 \nabla v_2$ is supported in a parallelogram of size $O(1)$ in which the value of x_2 is at least

$$S := (\tau - 1) L \cot\alpha - 2/\sin\alpha \, ,$$

see Figure 5. Therefore, $\varepsilon_{12} = \varepsilon_{21} = O(e^{-2\tau\beta L \cot\alpha})$. On the other hand, by Lemma 3.4 we have $\varepsilon_{11} = \varepsilon_{22} = O(Le^{-2\beta\tau L \cot\alpha})$. Substituting these estimates into (18) and then into (17) leads to the conclusion. □

Lemma 3.9. *There holds* $w := w_{12} = w_{21} = -\dfrac{2\beta^2 \cos^2\alpha}{\sin^4\alpha} e^{-2\beta L \cot\alpha} + O\big(Le^{-2\beta\tau L \cot\alpha}\big)$.

Proof. The equality $w_{12} = w_{21}$ follows from the symmetry considerations. Furthermore,

$$U_1 \nabla U_2 - U_2 \nabla U_1 = 2\beta \cot\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} U_1 U_2 \, .$$

Substituting the expression for $U_1 U_2$ from (19) we obtain

$$w_{12} = \frac{4\beta^3 \cos^2\alpha}{\sin^4\alpha} e^{-2\beta L \cot\alpha} A, \quad A := \iint_\Omega e^{-2\beta x_2} v_2 \frac{\partial v_1}{\partial x_1} \, dx \, . \tag{20}$$

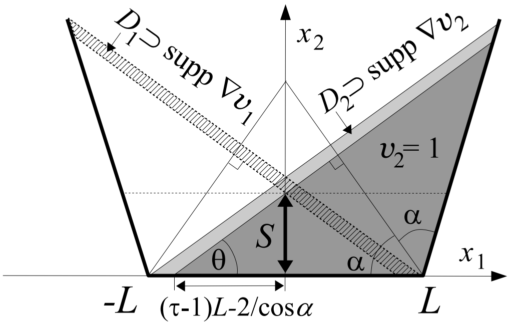

FIGURE 5. Computation of S. In the dark shaded domain there holds $v_2 = 1$, cf. Figure 3. We have $\theta = \frac{\pi}{2} - \alpha$ and, hence, $S = \big((\tau-1)L - 2/\cos\alpha\big)\tan\theta \equiv (\tau-1)L\cot\alpha - 2/\sin\alpha$.

Denote $S := (\tau-1)L\cot\alpha - 2/\sin\alpha$. Using the explicit construction of v_1 and v_2 we can see that for $x_2 < S$ we have the following property (see Figure 5): if $(x_1,x_2) \in \operatorname{supp}\nabla v_1$, then $v_2(x_1,x_2) = 1$. This allows one to estimate A by

$$A = \iint_{\Omega\cap\{x_2\le S\}} e^{-2\beta x_2}\frac{\partial v_1(x_1,x_2)}{\partial x_1}\,dx + O\big(Le^{-2\beta(\tau-1)L\cot\alpha}\big).$$

On the other hand, by Fubini

$$\iint_{\Omega\cap\{x_2\le S\}} e^{-2\beta x_2}\frac{\partial v_1(x_1,x_2)}{\partial x_1}dx = \int_0^S e^{-2\beta x_2}\Big(\int \frac{\partial v_1(x_1,x_2)}{\partial x_1}dx_1\Big)dx_2.$$

The interior integral is equal to (-1) for any x_2, which finally gives

$$A = -\int_0^S e^{-2\beta x_2}dx_2 + O\big(Le^{-2\beta(\tau-1)L\cot\alpha}\big) = -\frac{1}{2\beta} + O\big(Le^{-2\beta(\tau-1)L\cot\alpha}\big).$$

The substitution into (20) gives the result. □

Lemma 3.10. *The matrix N of $\Pi_E H_L : E \to E$ in the orthonormal basis*

$$\phi_k = \sum_{j=1}^{2}\psi_j\sigma_{jk}, \quad k = 1,2, \quad \sigma := (\sigma_{jk}) := \sqrt{G^{-1}}\,,$$

has the form

$$N = N_0 + O(L^2e^{-2\beta\tau L\cot\alpha}) \quad \textit{with} \quad N_0 = \begin{pmatrix} E_\alpha & w \\ w & E_\alpha \end{pmatrix}.$$

Here G is the Gramian matrix from Lemma 3.5.

Proof. Due to Lemma 3.5 we have $G = I + T$ with $T = O(Le^{-2\beta L\cot\alpha})$, which shows that

$$\sigma = I - \frac{1}{2}T + O(L^2e^{-4\beta L\cot\alpha}), \quad \sigma^{-1} = I + \frac{1}{2}T + O(L^2e^{-4\beta L\cot\alpha}).$$

On the other hand, using the matrix M from Lemma 3.8, we have $N = \sigma^{-1}M\sigma$. So we get

$$\begin{aligned} N &= \Big(I + \frac{1}{2}T + O(L^2e^{-4\beta L\cot\alpha}\Big)\Big(E_\alpha + \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} + O(L^{3/2}e^{-2\beta tL\cot\alpha})\Big)\Big(I - \frac{1}{2}T + O(L^2e^{-4\beta L\cot\alpha}\Big) \\ &= \begin{pmatrix} E_\alpha & w \\ w & E_\alpha \end{pmatrix} + \frac{1}{2}\Big[T\begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} - \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}T\Big] + O(L^2e^{-2\beta\tau L\cot\alpha}). \end{aligned}$$

The term in the square brackets equals zero due to Lemma 3.5, and this achieves the proof. □

Proof of Theorem 1.1. Now we are able to finish the proof of the main theorem. The eigenvalues of the matrix N_0 from Lemma 3.10 are $E_\pm := E_\alpha \pm |w|$, and in view of Lemma 3.9 we have

$$E_\pm = -\frac{\beta^2}{\sin^2\alpha} \pm \frac{2\beta^2\cos^2\alpha}{\sin^4\alpha}\,e^{-2\beta L\cot\alpha} + O\big(Le^{-2\beta\tau L\cot\alpha}\big).$$

By Lemma 3.9, the numbers $E_\pm$ coincide up to $O(L^2 e^{-2\beta\tau L\cot\alpha})$ with the eigenvalues of H_L in the interval I from (16), which are exactly $E_1(L)$ and $E_2(L)$. It remains to apply elementary trigonometric identities to pass from $\alpha=\omega/2$ to ω. ☐

Appendix A. 1D Robin problem

In this section, we study the one-dimensional Robin problem. The expressions obtained have their own interest, but some estimates can be used to obtain a better estimate for the analysis of the two-dimensional situation, as explained in Remark 1.5.

Lemma A.1. *For $\beta>0$ and $\ell>0$, denote by $N_{\beta,\ell}$ the operator acting in $L^2(0,\ell)$ as $f\mapsto -f''$ on the functions $f\in H^2(0,\ell)$ satisfying the boundary conditions $f'(0)=0$ and $f'(\ell)=\beta f(\ell)$. Then the lowest eigenvalue $E_N(\beta,\ell)$ is the unique strictly negative eigenvalue, and*

$$E_N(\beta,\ell) = -\beta^2 - 4\beta^2 e^{-2\beta\ell} + 8\beta^2(2\beta\ell-1)e^{-4\beta\ell} + O(\ell^2 e^{-6\beta\ell}) \text{ as } \ell \text{ tends to } +\infty\,,$$

and the associated eigenfunction is $x\mapsto\cosh(\sqrt{-E_N(\beta,\ell)}x)$.

Proof. Let us write $E_N(\beta,\ell)=-k^2$ with $k>0$. The associated eigenfunction f must be of the form $f(x)=Ae^{kx}+Be^{-kx}$ with some $(A,B)\in\mathbb{R}^2\setminus\{(0,0)\}$. Taking into the account the boundary conditions we get the linear system

$$A-B=0\,,\ (k-\beta)e^{k\ell}A-(k+\beta)e^{-k\ell}B=0\,.$$

It follows that $f(x)=2B\cosh(kx)$. The system has non-trivial solutions iff

$$(k-\beta)e^{k\ell}=(k+\beta)e^{-k\ell}\,. \tag{21}$$

This can be rewritten as $k\ell\tanh(k\ell)=\beta\ell$. One easily checks that the function $(0,+\infty)\ni t\mapsto t\tanh t\in(0,+\infty)$ is a bijection, which means that the solution k to (21) is defined uniquely, which shows that we have exactly one negative eigenvalue.

To calculate its asymptotics, we first take into account the signs of all terms in (21), which gives $k>\beta$.

Rewriting (21) in the form

$$(k-\beta)=2\beta e^{-2k\ell}/(1-e^{-2k\ell})=2\beta e^{-2\beta\ell}e^{-2(k-\beta)\ell}/(1-e^{-2(k-\beta)\ell}e^{-2\beta\ell})\,,$$

we get that

$$k-\beta=O(e^{-2\beta\ell})\,. \tag{22}$$

It follows also from (21) that

$$k=\frac{1+e^{-2k\ell}}{1-e^{-2k\ell}}\beta=\big(1+2e^{-2k\ell}+O(e^{-4k\ell})\big)\beta\,,\quad \ell\to+\infty\,. \tag{23}$$

Implementing (22), we infer that

$$k=\big(1+2e^{-2\beta\ell}+O(\ell e^{-4\beta\ell})\big)\beta=\beta+2\beta e^{-2\beta\ell}+O(\ell e^{-4\beta\ell})\,. \tag{24}$$

By taking an additional term in (23),

$$k=\frac{1+e^{-2k\ell}}{1-e^{-2k\ell}}\beta=\big(1+2e^{-2k\ell}+2e^{-4k\ell}+O(e^{-6k\ell})\big)\beta\,,\quad \ell\to+\infty\,,$$

and by using (24) one gets $k=\beta+2\beta e^{-2\beta\ell}+2\beta(1-4\beta\ell)e^{-4\beta\ell}+O(\ell^2e^{-6\beta\ell})$. Computing $E=-k^2$ gives the result. ☐

Lemma A.2. *For $\beta>0$ and $\ell>0$, denote by $D_{\beta,\ell}$ the operator acting in $L^2(0,\ell)$ as $f\mapsto -f''$ on the functions $f\in H^2(0,\ell)$ satisfying the boundary conditions $f(0)=0$ and $f'(\ell)=\beta f(\ell)$, and let $E_D(\beta,\ell)$ denote its lowest eigenvalue. Then $E_D(\beta,\ell)<0$ iff $\beta\ell>1$, and in that case it is the only negative eigenvalue. Furthermore,*

$$E_D(\beta,\ell) = -\beta^2 + 4\beta^2 e^{-2\beta\ell} + 8\beta^2(2\beta\ell-1)e^{-4\beta\ell} + O(\ell^2 e^{-6\beta\ell}) \text{ as } \ell \text{ tends to } +\infty\,,$$

and the associated eigenfunction is $x\mapsto\sinh(\sqrt{-E_D(\beta,\ell)}x)$.

Proof. Let us write $E_D(\beta,\ell) = -k^2$ with $k > 0$. The associated eigenfunction f is of the form $f = Ae^{kx} + Be^{-kx}$ with some $(A,B) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Taking into the account the boundary conditions we get the linear system

$$A + B = 0\,,\ (k-\beta)e^{k\ell}A - (k+\beta)e^{-k\ell}B = 0\,,$$

which gives the representation $f(x) = 2A\sinh(kx)$. Non-trivial solutions exist iff

$$(\beta + k)e^{-k\ell} = (\beta - k)e^{k\ell}\,. \tag{25}$$

The preceding equation can be rewritten as

$$k\ell\coth(k\ell) = \beta\ell\,.$$

One easily checks that the function $(0,+\infty) \ni t \mapsto t\coth t \in (1,+\infty)$ is a bijection, which shows that (25) has a solution iff $\beta\ell > 1$, and if it is the case, the solution is unique, which gives in turn the unicity of the negative eigenvalue. For the rest of the proof we assume that $\beta\ell > 1$. By considering the signs of both sides in (25) we conclude that $k < \beta$. Furthermore, we may rewrite (25) as $g(k) = 0$ with $g(k) = \log(\beta + k) - \log(\beta - k) - 2k\ell$. We have $g(0+) = 0$ and $g(\beta-) = +\infty$. The equation $g'(k) = 0$ takes the form $\beta^2 - k^2 = \beta/\ell$, and its unique solution is $k^* = \beta\sqrt{1 - (\beta\ell)^{-1}}$. It follows that the equation $g(k) = 0$ has a unique solution k in $(0,\beta)$ and that $k \in (k^*,\beta)$. On the other hand, we obtain the estimate $k^* > \beta\Big(1 - (\beta\ell^{-1}\Big) = \beta - 1/\ell$. Hence, the solution of $g(k) = 0$ satisfies

$$\beta - \frac{1}{\ell} < k < \beta\,. \tag{26}$$

We rewrite (25) in the form $\beta - k = \dfrac{2k}{e^{2k\ell} - 1}$, and we deduce with the help of (26) that $\beta - k = O(e^{-2\beta\ell})$ as $\ell \to +\infty$. By going through the same steps as in the proof of Lemma A.1, one gets the result. ☐

Proposition A.3. *For $\beta > 0$ and $\ell > 0$, let B_ℓ denote the operator $f \mapsto -f''$ acting in $L^2(-\ell,\ell)$ on the functions $f \in H^2(-\ell,\ell)$ satisfying the boundary conditions*

$$f'(-\ell) + \beta f(-\ell) = f'(\ell) - \beta f(\ell) = 0\,,$$

and let $E_1(\ell)$ and $E_2(\ell)$ be the two lowest eigenvalues, $E_1(\ell) < E_2(\ell)$. Then:

- *$E_1(\ell) < 0$,*
- *$E_2(\ell) < 0$ iff $\beta\ell > 1$,*
- *all other eigenvalues are non-negative.*

Furthermore,

$$E_1(\ell) = -\beta^2 - 4\beta^2 e^{-2\beta\ell} + 8\beta^2(2\beta\ell - 1)e^{-4\beta\ell} + O(\ell^2 e^{-6\beta\ell})\,,$$
$$E_2(\ell) = -\beta^2 + 4\beta^2 e^{-2\beta\ell} + 8\beta^2(2\beta\ell - 1)e^{-4\beta\ell} + O(\ell^2 e^{-6\beta\ell})\,,$$

as ℓ tends to $+\infty$. The respective eigenfunctions f_1 and f_2 are

$$f_1(x) = \cosh\left(\sqrt{-E_1(\ell)}x\right)\,, \quad f_2(x) = \sinh\left(\sqrt{-E_2(\ell)}x\right)\,.$$

Proof. Let us use the notation of Lemmas A.1 and A.2. Note that:

- B_ℓ commutes with the reflections with respect to the origin,
- its first eigenfunction f_1 is non-vanishing and even, hence, $f_1'(0) = 0$,
- its second eigenfunction f_2 has one zero in $(-\ell,\ell)$ and is odd, hence $f_2(0) = 0$.

Therefore, $E_1(\ell) = E_N(\beta,\ell)$ and $E_2(\ell) = E_D(\beta,\ell)$, and the result follows from Lemmas A.1 and A.2. ☐

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