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Abstract. We study the behavior of the critical price of an American put option near maturity in the Jump diffusion model when the underlying stock pays dividends at a continuous rate and the limit of the critical price is smaller than the stock price. In particular, we prove that, unlike the case where the limit is equal to the strike price, jumps can influence the convergence rate.

Introduction

The behavior of the critical price of the American put near maturity has been deeply investigated. Its limit was characterized in the Black Scholes model (see [5, 13]) by

$$b(T) := \lim_{t\to T} b(t) = \min\left(\frac{r}{\delta}K, K\right),$$

where r and δ denote the interest rate and the dividend rate and $b(t)$ is the critical price at time t.

This result was generalized to more general exponential Lévy models in [7]. In fact, denoting $\bar{d} = r - \delta - \int (e^y - 1)^+ \nu(dy)$ [1], with ν the Lévy measure of the underlying Lévy process, we have

$$b(T) = K \text{ , if } \quad \bar{d} \geq 0,$$

and

$$b(T) = \xi \text{ , if } \quad \bar{d} < 0,$$

where ξ is the unique solution, in $[0, K]$, of

$$rK - \delta x - \int (xe^y - K)^+ \nu(dy) = 0. \tag{1}$$

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[1]The quantity $\bar{d}$ is denoted by d^+ in [7]

In the Black Scholes Model, the quantity $\bar{d}$ reduces to $\bar{d} = r - \delta$ and we distinguish, according as $\bar{d} > 0$, $\bar{d} = 0$ and $\bar{d} < 0$, different behaviors of the critical price near maturity. In fact, Barles et al in [1] (see also D. Lamberton [6]) established, in the case where $\bar{d} > 0$ (which implies $b(T) = K$), that

$$\frac{K - b(t)}{\sigma K} \sim_{t\to T} \sqrt{(T-t)|\ln(T-t)|}, \tag{2}$$

where the expression $f \sim_{t\to a} g$ (or $f \sim_a g$) is equivalent to $\lim_{t\to a} \frac{f(t)}{g(t)} = 1$. The cases $\bar{d} < 0$ and $\bar{d} = 0$ were investigated by D. Lamberton and S. Villeneuve in [14] and they obtained :
If $\bar{d} = 0$ (which also implies $b(T) = K$)

$$\frac{K - b(t)}{\sigma K} \sim_{t\to T} \sqrt{2(T-t)|\ln(T-t)|}.$$

If $\bar{d} < 0$ ($b(T) < K$), there exists $y_0 \in (0, 1)$, which is characterized thanks to an auxiliary optimal stopping problem, such that

$$\frac{b(T) - b(t)}{\sigma b(T)} \sim_{t\to T} y_0\sqrt{(T-t)}.$$

The critical price has also been studied in the Jump diffusion model. In fact, Pham proved in [11] that the result (2), obtained in [1, 6], remains exactly the same in the Jump diffusion model, in the case where $\bar{d} > 0$ and $\delta = 0$. This remains true if $\delta > 0$ (see [10]).

The purpose of this paper is to study the convergence rate of the critical price of the American put, in the Jump diffusion model, with $\bar{d} \leq 0$. Considering the results of Pham in [11], we expect to obtain the same results as the study performed by Lamberton and Villeneuve in the Black-Scholes model when ($\bar{d} = r - \delta \leq 0$), meaning that jumps do not have any influence on the convergence rate. Surprisingly, we obtain the expected result only for the case $\bar{d} = 0$. Indeed, we obtain for $\bar{d} = 0$ (see Theorem 4.1),

$$\frac{K - b(t)}{\sigma K} \sim_{t\to T} \sqrt{2(T-t)|\ln(T-t)|},$$

and for $\bar{d} < 0$ (see Theorem 3.2),

$$\frac{b(T) - b(t)}{\sigma b(T)} \sim_{t\to T} y_{\lambda,\beta}\sqrt{(T-t)},$$

where $y_{\lambda,\beta}$ is a real umber satisfying $y_{\lambda,\beta} \geq y_0$, and depending on $\nu(\{\frac{K}{b(T)}\})$ we can have $y_{\lambda,\beta} > y_0$. This point will be discussed in more details in section 3.3.

This study is composed of four sections. In Section 1, we recall some useful results on the American put which will be used throughout this study. In Section 2, we give some results on the regularity of the American put price and the early exercise premium. In Section 3, we investigate the case where the limit of the critical price is far from the singularity K. Therefore, we have enough regularity to give an expansion of the American put price near maturity from which the critical price behavior will be deduced. The method is similar to the one used in [14] and is based on an expansion of the American put price along

parabolas. However, the possibility that the stock price jumps into a neighborhood of the exercise price produces a contribution of the local time in the expansion. Section 4 is devoted to the study of the case $\bar{d} = 0$. In this case $b(T) = K$, hence we have no longer enough smoothness to obtain an expansion around the limit point $(T, b(T))$. Then we will study the behavior of the European critical price $b_e(t)$ instead of $b(t)$. Thereafter, we prove that $b(t)$ and $b_e(t)$ have the same behavior.

1. Preliminary

In the Jump Diffusion model, under a risk-neutral probability, the risky asset price is modelized by $(S_t)_{t\geq 0}$ given by

$$S_t = S_0 e^{\tilde{X}_t}, \qquad \text{with } \tilde{X}_t = (r-\delta)t + \sigma B_t - \frac{\sigma^2}{2}t + Z_t - t\int (e^y - 1)\nu(dy)$$

where $r > 0$ is the interest rate, $\delta \geq 0$ the dividend rate, $(B_t)_{t\geq 0}$ a Standard Brownian Motion and $(Z_t)_{t\geq 0}$ a Compound Poisson Process and ν its Levy measure. We then have

$$dS_t = S_{t^-}\left(\gamma_0 dt + \sigma dB_t + d\bar{Z}_t\right), \text{ with } \bar{Z}_t = \Sigma_{0<s\leq t}(e^{\Delta Z_s} - 1) \text{ and } \gamma_0 = r - \delta - \int (e^y - 1)\nu(dy).$$

Denote by $\mathbb{F}$ the completed natural filtration of the process $\tilde{X}_t$ and suppose all over this paper that the following assumptions are satisfied

$$\sigma > 0, \qquad \nu(\mathbb{R}) < \infty, \qquad \int e^y \nu(dy) < \infty \qquad \text{and} \qquad \bar{d} = r - \delta - \int_{y>0} (e^y - 1)\nu(dy) \leq 0.$$

The price of an American put with maturity $T > 0$ and strike price $K > 0$ is given, at $t \in [0, T]$, by $P(t, S_t)$ with P defined for all $(t, x) \in [0, T] \times \mathbb{R}^+$ by

$$P(t, x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}(e^{-r\tau}(K - xe^{\tilde{X}_\tau})_+),$$

where $\mathcal{T}_{0,T-t}$ is the set of all $\mathbb{F}$-stopping times taking values in $[0, T-t]$. The value function P can also be characterized (see [7]) as the unique continuous and bounded solution of the following variational inequality

$$\max\{\psi - P; \frac{\partial P}{\partial t} + \mathcal{A}P - rP\} = 0, \quad \text{(in the sense of distributions)},$$

with the terminal condition $P(T, .) = \psi$. Here $\mathcal{A}$ is the infinitesimal generator of the process S. The free boundary of this variational inequality is called the exercise boundary, and at each $t \in [0, T]$, the critical price is given by

$$b(t) = \inf\left\{x > 0 \mid P(t, x) > (K - x)^+\right\}.$$

It was proved in [7] that, if $\bar{d} \leq 0$, then

$$\lim_{t\to T} b(t) = \xi := b(T), \tag{3}$$

where ξ is the unique solution, in $[0, K]$, of $rK = \delta x + \int (xe^y - K)^+ \nu(dy)$. Note that, if $\bar{d} = 0$, then $b(T) = \xi = K$.

Finally, recall that the price of a European put with maturity T and strike price K is given, at time t, by

$$P_e(t,x) = \mathbb{E}\left(e^{-r(T-t)}(K - S_{T-t})_+ \mid S_0 = x\right).$$

The quantity $(P - P_e)$ is called the early exercise premium, we then have $P(t,x) = P_e(t,x) + e(T-t,x)$. Setting $\theta = T - t$, then the early exercise prime, $e(\theta, x)$, is characterized for the American put in the exponential Levy model as follows (see[10])

$$e(\theta,x) = \mathbb{E}\left\{\int_0^\theta e^{-rs}\left(rK - \delta S_s^x - \int_{y>0}\left[P(t+s, S_s^x e^y) - (K - S_s^x e^y)\right]\nu(dy)\right) 1_{\{S_s^x < b(t+s)\}} ds\right\}.$$

We also define, for all $t \in (0,T)$, the European critical price, $b_e(t)$, as the unique solution of

$$F(t,x) = P_e(t,x) - (K - x) = 0.$$

It easy to check that, for all $t \in (0,T)$, $b_e(t)$ is well defined, $b_e(t) \in (0,K)$. It is also straightforward that $P_e \leq P$, therefore $b(t) \leq b_e(t) \leq K$.

2. Regularity estimate for the value function in the jump diffusion model

In this section, we study the spatial derivatives behavior of P, P_e and $e(\theta,x)$ near $(T, b(T))$. We also give a lower bound for the second spatial derivative near $(T, b(T))$. These results will be proved in Appendix 1.

Lemma 2.1. *Under the model assumption, we have*

(1) *For all $x \in (0, b_e(t) \wedge b(T)]$, we have, as $\theta (= T - t)$ goes to 0,*

$$\left|\frac{\partial e}{\partial x}(\theta, x)\right| = \frac{1}{x} o(\sqrt{\theta}),$$

with $o(\sqrt{\theta})$ uniform with respect to x.

(2) *For all $x \in (0, b(T) \wedge b_e(t)]$, we have*

$$\frac{\partial P}{\partial x}(t,x) + 1 = (1 + \frac{1}{x}) o(\sqrt{\theta}),$$

with $o(\sqrt{\theta})$ uniform with respect to x.

Lemma 2.2. *According to the hypothesis of the model, we have, for all $b(t) \leq x < b(T) \wedge b_e(t)$ and for all $\theta = T - t$ small enough, the following inequality*

$$\inf_{b(t)<u<x} \frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) \geq \left(\bar{\delta} - \epsilon(\theta)\right)(b(T) - x) - \lambda\beta\mathbb{E}\left(\sigma B_\theta - \ln(\frac{b(T)}{x})\right)^+ + o(\sqrt{\theta}),$$

with $\lim_{\theta \downarrow 0} \epsilon(\theta) = 0$, $\bar{\delta} = \delta + \int_{\{y > \ln(\frac{K}{b(T)})\}} e^y \nu(dy)$, $\lambda = \nu\left\{\ln\left(\frac{K}{b(T)}\right)\right\}$ *and* $\beta = \frac{K}{\delta b(T)}$.

3. Regular case

We begin this section with introducing an auxiliary optimal stopping problem which will be needful for deriving the expansion of the American put price near maturity along a parabolic branch. Once we have this expansion we will be able to derive the convergence rate of the critical price.

3.1. **An auxiliary optimal stopping problem.** Let β be a non-negative number, $(B_s)_{s\geq 0}$ be a standard Brownian motion with local time at x denoted by $\tilde{L}^x$. We denote by $\mathcal{T}_{0,1}$ the set of all $\sigma\left(B_t\ ;\ t\geq 0\right)$-stopping times with values in $[0,1]$. Consider also a Poisson process $(N_s)_{s\geq 0}$, independent of B, with intensity λ, we denote by $\hat{T}_1$ its first jump time and by $\hat{\mathcal{T}}_{0,1}$ the set of all $\sigma\left((N_t,B_t)\ ;\ t\geq 0\right)$-stopping times with values in $[0,1]$. We define the functions $v_{\lambda,\beta}$ as follows

$$v_{\lambda,\beta}(y) \quad = \quad \sup_{\tau\in\hat{\mathcal{T}}_{0,1}} \mathbb{E}\left[e^{\lambda\tau}1_{\{\hat{N}_\tau=0\}}\int_0^\tau f_{\lambda\beta}(y+B_s)ds+\frac{\beta}{2}e^{\lambda\tau}1_{\{\hat{N}_\tau=1\}}\left(L_\tau^{-y}(B)-L_{\hat{T}_1}^{-y}(B)\right)\right],$$

where $f_a(x)=x+ax^+$. Notice that $v_{\lambda,\beta}$ is a non negative function. Moreover, we have

Lemma 3.1. *Define*

$$y_{\lambda,\beta}=-\inf\{x\in\mathbb{R}\mid v_{\lambda,\beta}(x)>0\}.$$

We have $0<y_{\lambda,\beta}<1+\lambda\beta(2+e^\lambda)$ *and*

$$\forall y<-y_{\lambda,\beta},\quad v_{\lambda,\beta}(y)=0.$$

We finish this paragraph with an inequality, which will be used to derive a lower bound for the second derivative of P (see the proof of the upper bound in Theorem 3.2).
We define the function C on $\mathbb{R}$ by $C(x)=x-\lambda\beta\ \mathbb{E}(B_1-x)^+$ and we have the following lemma,

Lemma 3.2. *For all* $x>y_{\lambda,\beta}$*, we have*

$$C(x)>0.$$

These results will be proved in *Appendix 2: A study of* $v_{\lambda,\beta}$.

3.2. **American put price expansion.** Throughout this section, we assume $\bar{d}<0$, so that $b(T)<K$. We then have enough regularity of the American put price to derive an expansion of P around $b(T)$ along a certain parabolic branch.

Theorem 3.1. *Let* a *be a negative number* ($a<0$) *and* $b(T)$ *denote the limit of* $b(t)$ *when* t *goes to* T, $b(T)=\lim_{t\to T}b(t)$. *If* $\bar{d}<0$, *we have*

$$P(T-\theta,b(T)e^{a\sqrt{\theta}})=(K-b(T)e^{a\sqrt{\theta}})^+ + C\theta^{\frac{3}{2}}v_{\lambda,\beta}(\frac{a}{\sigma})+o(\theta^{\frac{3}{2}}),$$

where $C=\sigma b(T)\bar{\delta}e^\lambda$, *with* $\lambda=\nu\{\ln\frac{K}{b(T)}\}$, $\bar{\delta}=\delta+\int_{y>\ln(K/b(T))}e^y\nu(dy)$ *and* $v_{\lambda,\beta}(y)$ *as defined in the previous section with* $\beta=\frac{K}{b(T)\bar{\delta}}$.

Remark 1. *Notice that if ν does not charge $\left\{\ln(\frac{K}{b(T)})\right\}$, meaning that $\lambda = 0$ and $\hat{T}_1 = \infty$ a.s, then,*

$$\upsilon_{\lambda,\beta}(a) = \upsilon_0(a) = \sup_{\tau \in \mathcal{T}_{0,1}} \mathbb{E}\left(\int_0^\tau (a + B_s)\, ds\right).$$

In this case, the American put price will have the same expansion as in the Black&Scholes model, (see [9]).

Before proving Theorem 3.1, we state an elementary estimate for the expectation of the local time of Brownian motion.

Lemma 3.3. *For all real number a and for all $t > 0$, we have*

$$0 \leq \mathbb{E}(a - B_t)_+ - a_+ \leq \sqrt{t}\frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi}}.$$

Proof of lemma 3.3:
The first inequality follows from Jensen's inequality. For the other inequality, we have

$$\begin{aligned}\mathbb{E}(a - B_t)_+ &= \int_{-\infty}^{a/\sqrt{t}} (a - \sqrt{t}y)e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\ &= a\int_{-\infty}^{a/\sqrt{t}} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} + \sqrt{t}\frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi}}\end{aligned}$$

Then, if $a \leq 0$,

$$\mathbb{E}(a - B_t)_+ \leq \sqrt{t}\frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi}}.$$

If $a \geq 0$, we can write

$$\mathbb{E}(a - B_t)_+ - a = -\int_{a/\sqrt{t}}^{+\infty} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} + \sqrt{t}\frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi}} \leq \sqrt{t}\frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi}}.$$

□

In order to derive the expansion of the American put price, we start from the Meyer-Ito formula (see [12]):
(4)

$$(K-S_t)_+ = (K-S_0)_+ + \int_0^t (-1_{\{S_s \leq K\}})S_s(\gamma_0 ds + \sigma dB_s) + \sum_{0<s\leq t} (K-S_s)_+ - (K-S_{s^-})_+ + \frac{1}{2}L_t^K,$$

where L_t^K is the local time of the process at K until the date t. We give, in the following lemma, an estimation of $\mathbb{E}L_t^K$, for small times t, which will allow us to neglect a part of the contribution of the local time in the expansion of $P(t, x)$, near maturity.

Lemma 3.4. *Let a be a negative number, $a < 0$ and $S_0 = b(T)e^{a\sqrt{\theta}}$. If $b(T) < K$, then we have, for all $\mathbb{F}$-stopping time τ with values in $[0, \theta]$,*

$$\mathbb{E}\left(L_\tau^K\right) = 2K\mathbb{E}\left[\left((-a\sqrt{\theta} - \sigma B_\tau)^+ - (-a\sqrt{\theta} - \sigma B_{\hat{T}_1})^+\right) 1_{\{\hat{T}_1 < \tau\}}\right] + o(\theta^{\frac{3}{2}}) \leq w_0\theta^{3/2},$$

where $\hat{T}_1 = \inf\{s \geq 0\ ;\ \Delta X_s = \ln(\frac{K}{b(T)})\}$ and w_0 a non-negative constant independent of a.

Proof of Lemma 3.4:
Let T_1 be the first jump time of the process Z and τ an $\mathbb{F}$-stopping time with values in $[0, \theta]$. We have, the local time being a nondecreasing process,

$$\begin{aligned} L_\tau^K = L_{\tau\wedge T_1}^K + L_\tau^K - L_{\tau\wedge T_1}^K &= L_{\tau\wedge T_1}^K + 1_{\{T_1<\tau\}}\left(L_\tau^K - L_{T_1}^K\right) \\ &\leq L_{\theta\wedge T_1}^K + 1_{\{T_1<\theta\}}\left(L_{T_1+\theta}^K - L_{T_1}^K\right). \end{aligned}$$

Estimating $\mathbb{E}L_{\theta\wedge T_1}^K$

In the stochastic interval $[0, T_1[$, the process (S_t) matches with the process $(\check{S}_t)$ defined by

$$\check{S}_t = S_0 e^{(\gamma_0 - \frac{\sigma^2}{2})t + \sigma B_t}.$$

We deduce (when observing that the process L^K is continuous) that

$$L_{\theta\wedge T_1}^K = \check{L}_{\theta\wedge T_1}^K \leq \check{L}_\theta^K,$$

where $\check{L}^K$ is the local time at K of the process $\check{S}$. Note that

$$\frac{1}{2}\check{L}_\theta^K = (K - \check{S}_\theta)_+ - (K - S_0)_+ - \int_0^\theta (-1_{\{\check{S}_s \leq K\}})\check{S}_s(\gamma_0 ds + \sigma dB_s).$$

As the process $(\check{L}_\theta^K)$ increases only on $\{\check{S}_t = K\}$, we have

$$\check{L}_\theta^K = \check{L}_\theta^K 1_{\{\tau_K < \theta\}},$$

where $\check{\tau}_K = \inf\{t \geq 0; \check{S}_t > K\}$. By Hölder,

$$\mathbb{E}\check{L}_\theta^K \leq \left(\mathbb{P}(\check{\tau}_K < \theta)\right)^{1-\frac{1}{p}} ||\check{L}_\theta^K||_p, \quad p > 1.$$

We easily deduce that $\mathbb{E}\check{L}_\theta^K = o(\theta^n)$, for all $n > 0$.

Estimating $\mathbb{E}\left[1_{\{T_1<\tau\}}\left(L_\tau^K - L_{T_1}^K\right)\right]$
Notice that we have

$$\mathbb{E}\left[1_{\{T_1<\tau\}}\left(L_\tau^K - L_{T_1}^K\right)\right] \leq \mathbb{E}\left[1_{\{T_1<\theta\}}\left(L_{T_1+\theta}^K - L_{T_1}^K\right)\right],$$

and by the strong Markov property, we obtain

$$\mathbb{E}\left[1_{\{T_1<\theta\}}\left(L_{T_1+\theta}^K - L_{T_1}^K\right)\right] = \mathbb{E}\left(1_{\{T_1<\theta\}}\mathbb{E}_{S_{T_1}}(L_\theta^K)\right), \tag{5}$$

where $\mathbb{E}_x$ is the expectation associated to $\mathbb{P}_x$ and $\mathbb{P}_x$ defines the law of S_t when $S_0 = x$.

Estimating $\mathbb{E}_x\left(L_\theta^K\right)$

Let T_1 be the first jump time of the process Z. We then have

$$L_\theta^K = L_{\theta\wedge T_1}^K + L_\theta^K - L_{\theta\wedge T_1}^K$$

According to equality (4) we deduce, using the compensation formula (see [3])

$$\begin{aligned}\frac{1}{2}L_\theta^K &= (K-S_\theta)_+ - (K-S_0)_+ + \int_0^\theta 1_{\{S_{s^-}\le K\}} S_{s^-}(\gamma_0 ds + \sigma dB_s) \\ &\quad - \int_0^\theta ds \int \Phi(S_{s^-}, y)\nu(dy) + M_\theta,\end{aligned}$$

where $\Phi(x,y) = (K - xe^y)_+ - (K-x)_+$ and (M_t) is a martingale which vanishes at 0. Taking expectations, we have

$$\frac{1}{2}\mathbb{E}\left(L_\theta^K\right) = \mathbb{E}(K-S_\theta)_+ - (K-S_0)_+ + \mathbb{E}\int_0^\theta \left(\gamma_0 S_s 1_{\{S_s\le K\}} - \int \Phi(S_s,y)\nu(dy)\right) ds.$$

We deduce easily from this equality that

$$\frac{1}{2}\mathbb{E}_x\left(L_\theta^K\right) = \mathbb{E}_x(K-S_\theta)_+ - (K-x)_+ + xO(\theta)$$

with $O(\theta)$ independent of x. We have

$$\mathbb{E}_x(K-S_\theta)_+ - (K-x)_+ = \mathbb{E}_x(K - xe^{(r-\delta-\frac{\sigma^2}{2})\theta+\sigma B_\theta+\tilde{Z}_\theta})_+ - (K-x)_+$$

We also have

$$\begin{aligned}\mathbb{E}\left|e^{(r-\delta-\frac{\sigma^2}{2})\theta+\sigma B_\theta+\tilde{Z}_\theta} - e^{\sigma B_\theta}\right| &= e^{\sigma^2\theta/2}\mathbb{E}\left|e^{(r-\delta-\frac{\sigma^2}{2})\theta+\tilde{Z}_\theta} - 1\right| \\ &= O(\theta)\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}_x(K-S_\theta)_+ - (K-x)_+ &= \mathbb{E}(K - xe^{\sigma B_\theta})_+ - (K-x)_+ + xO(\theta) \\ &= \mathbb{E}(K - x(1+\sigma B_\theta))_+ - (K-x)_+ + xO(\theta) \\ &= x\sigma\left(\mathbb{E}\left(\frac{K-x}{x\sigma} - B_\theta\right)_+ - \left(\frac{K-x}{x\sigma}\right)_+\right) + xO(\theta).\end{aligned}$$

Hence, using lemma 3.3 above,

$$\mathbb{E}_x(K-S_\theta)_+ - (K-x)_+ \le x\sigma\sqrt{\theta/(2\pi)}\exp\left(-\frac{(K-x)^2}{2x^2\sigma^2\theta}\right) + xO(\theta).$$

Going back to (5), we obtain

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[1_{\{T_1<\theta\}}\left(L^K_{T_1+\theta}-L^K_{T_1}\right)\right]\\
&\leq \sigma\sqrt{\frac{\theta}{2\pi}}\mathbb{E}\left(1_{\{T_1<\theta\}}S_{T_1}\exp\left(-\frac{(K-S_{T_1})^2}{2S^2_{T_1}\sigma^2\theta}\right)\right)+\mathbb{E}\left(1_{\{T_1<\theta\}}S_{T_1}\right)O(\theta)\\
&= \sigma\sqrt{\frac{\theta}{2\pi}}S_0\mathbb{E}\left(1_{\{T_1<\theta\}}e^{(\gamma_0-\frac{\sigma^2}{2})T_1+\sigma B_{T_1}+Z_{T_1}}\exp\left(-\frac{(K-S_{T_1})^2}{2S^2_{T_1}\sigma^2\theta}\right)\right)+O(\theta^2).
\end{aligned}$$

At this stage, we notice that $\mathbb{P}(T_1\leq\theta)=1-e^{-\lambda\theta}=O(\theta)$ and that, conditionally on $\{T_1\leq\theta\}$, T_1 is uniformly distributed on $[0,\theta]$.

As Z_{T_1} is independent of both T_1 and B, we see that, conditionally to $\{T_1<\theta\}$, S_{T_1} has the same law as

$$K\exp\left\{\left(V-\ln(\frac{K}{b(T)})\right)+\sqrt{\theta}\left(a+(\gamma_0-\frac{\sigma^2}{2})\sqrt{\theta}U+\sigma g\sqrt{U}\right)\right\},$$

where U, g and V are two independent random variables, U is uniform on $[0,1]$, g standard Gaussian and V has the same law as Z_{T_1}. Therefore, we can state that there exists a non negative constant independent of a such that

$$\mathbb{E}(L^K_\theta \mid S_0=b(T)e^{a\sqrt{\theta}})\leq w_0\theta^{3/2}=O(\theta^{3/2}).$$

Estimating $\mathbb{E}\left[L^K_\tau-L^K_{\tau\wedge T_1}\right]$, in the case where $\nu\{\ln(\frac{K}{b(T)})\}=0$:

If we assume $\nu\{\ln(\frac{K}{b(T)})\}=0$, which means that $V-\ln(\frac{K}{b(T)})\neq 0$ a.s , we obtain, by dominated convergence that

$$\lim_{t\downarrow 0}\mathbb{E}\left(S_{T_1}\exp\left(-\frac{K^2(1-\frac{S_{T_1}}{K})^2}{2S^2_{T_1}\sigma^2\theta}\right)\mid T_1<\theta\right)=0.$$

Therefore $\mathbb{E}\left[1_{\{T_1<\theta\}}\left(L^K_{T_1+\theta}-L^K_{T_1}\right)\right]=o(\theta^{3/2})$, hence

$$\mathbb{E}\left(L^K_\theta \mid S_0=b(T)e^{a\sqrt{\theta}}\right)=o(\theta^{3/2})$$

Estimating $\mathbb{E}\left[L^K_\tau-L^K_{\tau\wedge T_1}\right]$, in the case where $\nu\{\ln(\frac{K}{b(T)})\}>0$:

Let us introduce the processes $\hat{X}$ and $\hat{Z}$ such that

$$\hat{Z}_t=\sum_{s<t}\Delta\tilde{X}_s 1_{\{\Delta\tilde{X}_s=\ln\frac{K}{b(T)}\}}\quad\text{and}\quad\hat{X}=\tilde{X}-\hat{Z},$$

and $\hat{T}_1=\inf\{s\geq 0,\quad \hat{Z}_t\neq 0\}$. Then, since $\tau\leq\theta$, we have

$$\mathbb{E}\left[L^K_{\tau\wedge\hat{T}_1}-L^K_{\tau\wedge T_1}\right]=\mathbb{E}\left[(L^K_{\tau\wedge\hat{T}_1}-L^K_{\tau\wedge T_1})1_{\{T_1<\tau\wedge\hat{T}_1\}}\right]=o(\theta^{\frac{3}{2}}).$$

Indeed, on $\{\tau<\hat{T}_1\}$, the process $\tilde{X}$ matches with the process $\hat{X}$ whose Lévy measure does not charge the point $\{\ln(\frac{K}{b(T)})\}$, (we are in the same case as $\nu\{\ln(\frac{K}{b(T)})\}=0$). And on

$\{T_1 < \hat{T}_1 \leq \tau\} \subset \{T_1 < \hat{T}_1 \leq \theta\}$, the process Z has jumped two times before θ, however, $\mathbb{P}\left(\sum_{s\leq\theta} 1_{\{\Delta Z_s \neq 0\}} \geq 2\right) = O(\theta^2)$. Thus,

$$\mathbb{E}\left[L^K_\tau - L^K_{\tau\wedge T_1}\right] = \mathbb{E}\left[L^K_\tau - L^K_{\tau\wedge\hat{T}_1}\right] + o(\theta^{3/2}).$$

Besides,

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left[L^K_\tau - L^K_{\tau\wedge\hat{T}_1}\right] &= \frac{1}{2}\mathbb{E}\left[(L^K_\tau - L^K_{\tau\wedge\hat{T}_1})1_{\hat{T}_1<\tau}\right] \\
&= \mathbb{E}\left[\left((K-S_\tau)^+ - (K-S_{\hat{T}_1})^+\right)1_{\hat{T}_1<\tau}\right] + o(\theta^{3/2}) \\
&= \mathbb{E}\left[\left((K-\frac{S_0K}{b(T)}e^{\hat{X}_{\hat{T}_1}-\tilde{X}_{\hat{T}_1}+\tilde{X}_\tau})^+ - (K-\frac{S_0K}{b(T)}e^{\hat{X}_{\hat{T}_1}})^+\right)1_{\hat{T}_1<\tau}\right] + o(\theta^{\frac{3}{2}}).
\end{aligned}$$

Since $\mathbb{P}\left(\sum_{s\leq\theta} 1_{\{\Delta\tilde{X}_s\neq 0\}} \geq 2\right) = O(\theta^2)$, conditionally on $\{\hat{T}_1 < \tau\}$, we can assume that $N_\theta = 1$, where N_θ denotes the number of jumps of $\tilde{X}$ up to θ, $N_\theta = \sum_{s\leq\theta} 1_{\{\Delta\tilde{X}_s\neq 0\}} = \sum_{s\leq\theta} 1_{\{\Delta Z_s\neq 0\}}$. Noticing that $\frac{S_0K}{b(T)} = Ke^{a\sqrt{\theta}}$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[L^K_\tau - L^K_{\tau\wedge\hat{T}_1}\right] \\
&= 2\mathbb{E}\left[\left((K-\frac{S_0K}{b(T)}e^{\hat{X}_{\hat{T}_1}-\tilde{X}_{\hat{T}_1}+\tilde{X}_\tau})^+ - (K-\frac{S_0K}{b(T)}e^{\hat{X}_{\hat{T}_1}})^+\right)1_{\{N_\theta=1\}}1_{\{\hat{T}_1<\tau\}}\right] + o(\theta^{\frac{3}{2}}) \\
&= 2K\mathbb{E}\left[\left((-a\sqrt{\theta}-\mu\tau-\sigma B_\tau)^+ - (-a\sqrt{\theta}-\mu\hat{T}_1-\sigma B_{\hat{T}_1})^+\right)1_{\{\hat{T}_1<\tau\}}\right] + o(\theta^{\frac{3}{2}}) \\
&= 2K\mathbb{E}\left[\left((-a\sqrt{\theta}-\sigma B_\tau)^+ - (-a\sqrt{\theta}-\sigma B_{\hat{T}_1})^+\right)1_{\{\hat{T}_1<\tau\}}\right] + o(\theta^{\frac{3}{2}})
\end{aligned}$$

The last two equalities follow from $\mathbb{P}(\hat{T}_1 < \tau) = O(\theta)$, $\left|(1-e^x+x)1_{\{x\leq 0\}}\right| \leq \frac{x^2}{2}$ and the fact that, for all stropping time ϱ with values in $[0,\theta]$, we have

$$\theta\mathbb{E}\left(a + \mu\frac{\varrho}{\sqrt{\theta}} + \frac{\sigma}{\sqrt{\theta}}B_\varrho\right)^2 \leq C\theta.$$

□

Proof of Theorem 3.1: First of all, we recall our notation $\check{X}_t = \tilde{X}_t - Z_t$, $\check{S}_t = \tilde{S}_t/e^{Z_t}$ (i.e the continuous part of the processes) and T_1 the first jump time $T_1 = \inf\{t > 0 | Z_t \neq 0\}$ and from now on, we consider S_0 as a function of θ. More precisely, we denote by $S_0^\theta = b(T)e^{a\sqrt{\theta}} = e^{x_0+a\sqrt{\theta}}$, with $a < 0$ and $x_0 = \ln(b(T))$.

According to equation (4), we have for all stopping times $\tau \in \mathcal{T}_{0,\theta}$,

$$\begin{aligned}
&\mathbb{E}\left[e^{-r\tau}(K-S_\tau)_+\right]-(K-S_0)^+ \\
&= \mathbb{E}\left[\int_0^\tau \left(e^{-rs}1_{\{S_s\le K\}}\left(-rK+\delta S_s+S_s\int(e^y-1)\nu(du)\right)\right.\right. \\
&\quad \left.\left.+e^{-rs}\int\left[(K-S_se^y)^+-(K-S_s)^+\right])\nu(dy)\right)ds\right]+\frac{1}{2}\mathbb{E}\left(\int_0^\tau e^{-rs}dL_s^K\right) \\
&= \mathcal{I}^a(\tau)+\mathcal{J}^a(\tau), \qquad (6)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}^a(\tau) &= \mathbb{E}\left[\int_0^\tau \left(e^{-rs}1_{\{S_s\le K\}}\left(-rK+\delta S_s+S_s\int(e^y-1)\nu(du)\right)\right.\right. \\
&\quad \left.\left.+e^{-rs}\int\left[(K-S_se^y)^+-(K-S_s)^+\right])\nu(dy)\right)ds\right]
\end{aligned}$$

and

$$\mathcal{J}^a(\tau)=\frac{1}{2}\mathbb{E}\left(\int_0^\tau e^{-rs}dL_s^K\right).$$

At this stage, since $\mathcal{J}^a \ge 0$, we can state that , given $S_0 = b(T)e^{\lambda\sqrt{\theta}}$, we have

$$\mathcal{I}^a(\tau) \le \mathbb{E}\left[e^{-r\tau}(K-S_\tau)_+\right]-(K-S_0)^+ = \mathcal{I}^a(\tau)+\mathcal{J}^a(\tau) \le \mathcal{I}^a(\tau)+w_0\theta^{\frac{3}{2}}, \qquad (7)$$

the last inequality follows from Lemma 3.4. In what follows, we will express $\mathcal{I}^a$ and $\mathcal{J}^a$ in more appropriate forms. Let us start with $\mathcal{J}^a$.

Estimating $\mathcal{J}^a$:
Recall that $\hat{T}_1 = \inf\{t \ge 0 \ ; \ \Delta\tilde{X}_t = \ln\frac{K}{b(T)}\}$ and set$S_0 = b(T)e^{\lambda\sqrt{\theta}}$ with $\lambda < 0$, then according to Lemma 3.4, we have $\mathbb{E}\left(L_\theta^K\right) = O(\theta^{3/2})$, therefore

$$\begin{aligned}
\mathcal{J}^a(\tau) &= \frac{1}{2}\mathbb{E}\left(L_\tau^K\right)+o(\theta^{\frac{3}{2}}) \\
&= K\mathbb{E}\left[\left((-a\sqrt{\theta}-\sigma B_\tau)^+-(-a\sqrt{\theta}-\sigma B_{\hat{T}_1})^+\right)1_{\{\hat{T}_1<\tau\}}\right]+o(\theta^{\frac{3}{2}}). \qquad (8)
\end{aligned}$$

Estimating of $\mathcal{I}^a$:
First of all, remark that we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\tau \left(e^{-rs}1_{\{S_s>K\}}\int \left[(K-S_se^y)^+ - (K-S_s)^+\right]\nu(dy)\right)ds\right]\\
&\leq\ K\nu(\mathbb{R})\int_0^\theta \mathbb{P}\{S_s>K\}ds\\
&\leq\ K\nu(\mathbb{R})\int_0^\theta \mathbb{P}\{S_s>K,T_1>\theta\}+\mathbb{P}\{S_s>K,T_1\leq\theta\}ds \qquad (9)\\
&\leq\ K\nu(\mathbb{R})\left(\int_0^\theta \mathbb{P}\{\check{S}_s>K\}ds+\theta\mathbb{P}\{T_1\leq\theta\}\right)=O(\theta^2).
\end{aligned}
$$

And noticing that

$$
1_{\{x\leq K\}}\left(x(e^y-1)+\left[(K-xe^y)^+-(K-x)^+\right]\right)=(xe^y-K)^+1_{\{x\leq K\}},
$$

we thus obtain

$$
\mathcal{I}^a(\tau)=\mathbb{E}\left(\int_0^\tau e^{-rs}1_{\{S_s\leq K\}}\left(-rK+\delta S_s+\int(S_se^y-K)^+\nu(dy)\right)ds\right)+o(\theta^{\frac{3}{2}}).
$$

We can also omit e^{-rs} in the expression as an error of the order of $O(\theta^2)$. Then we obtain, for all stopping times τ with values in $[0,\theta]$

$$
\mathcal{I}^a(\tau)=\mathbb{E}\left(\int_0^\tau 1_{\{S_s\leq K\}}\left(-rK+\delta S_s+\int(S_se^y-K)^+\nu(dy)\right)ds\right)+o(\theta^{\frac{3}{2}}).
$$

We denote

$$
h(x)=-rK+\delta e^x+\int(e^xe^y-K)^+\nu(dy),
$$

and recall that $S_t=S_0^\theta e^{\tilde{X}_t}=b(T)e^{a\sqrt{\theta}+\tilde{X}_t}=b(T)e^{\tilde{X}_t^{a\sqrt{\theta}}}=e^{x_0+\tilde{X}_t^{a\sqrt{\theta}}}$, where $\tilde{X}_t^y=y+\tilde{X}_t$. We thus have

$$
\mathcal{I}^a(\tau)\ =\ \underbrace{\mathbb{E}\left(\int_0^\tau 1_{\{a\sqrt{\theta}+\tilde{X}_s\leq\ln\frac{K}{b(T)}\}}h(x_0+a\sqrt{\theta}+\tilde{X}_s)ds\right)}_{(I)}+o(\theta^{\frac{3}{2}}). \qquad (10)
$$

Now, we will try to express the quantity (I) under a more appropriate form. The first step is to neglect the contribution of the finite variation part of the process $\tilde{X}$. Notice that

$$
\left|1_{\{x\leq\ln(K)\}}h(x)\right|\leq K(r\vee|\bar{d}|)\quad\text{and}\quad|h(x)-h(y)|\leq|e^x-e^y|\left(\delta+\int_{y>0}e^y\nu(du)\right).
$$

Moreover, for all $(x,y)\in\mathbb{R}^2$, we have

$$
\begin{aligned}
&\left|1_{\{x\leq\ln(K)\}}h(x)-1_{\{y\leq\ln(K)\}}h(y)\right|\\
&=\ \left|(h(x)-h(y))\,1_{\{x\vee y\leq\ln(K)\}}+h(x)1_{\{x\leq\ln(K)<y\}}-h(y)1_{\{y\leq\ln(K)<x\}}\right|\\
&\leq\ A_0\left|e^x-e^y\right|1_{\{x\vee y\leq\ln(K)\}}A_1\left(1_{\{\ln(K)<y\}}+1_{\{\ln(K)<x\}}\right),
\end{aligned}
$$

where $A_1 = K(r \vee |\bar{d}|) > 0$ and $A_0 = \delta + \int_{y>0} e^y \nu(du)$. Let $k_b = \ln\left(\frac{K}{b(T)}\right) > 0$ and recall that $\tilde{X}_t - \sigma B_t = (\gamma_0 - \frac{\sigma^2}{2})t + Z_t$, then

$$\begin{aligned}
&\left|1_{\{x_0+a\sqrt{\theta}+\tilde{X}_s \le \ln K\}} h(x_0 + a\sqrt{\theta} + \tilde{X}_s) - 1_{\{x_0+a\sqrt{\theta}+\sigma B_s \le \ln K\}} h(x_0 + a\sqrt{\theta} + \sigma B_s)\right| \\
&\quad \le \quad A_0 \left|e^{x_0+a\sqrt{\theta}+\tilde{X}_s} - e^{x_0+a\sqrt{\theta}+\sigma B_s}\right| 1_{\{\tilde{X}_s \vee \sigma B_s \le k_b - a\sqrt{\theta}\}} + C(1_{\{k_b - a\sqrt{\theta} < \sigma B_s\}} + 1_{\{k_b - a\sqrt{\theta} < \tilde{X}_s\}}) \\
&\quad \le \quad A_0 b(T) e^{\sigma B_s} \left|e^{(\gamma_0 - \frac{\sigma^2}{2})s + Z_s} - 1\right| + C(1_{\{k_b < \sigma B_s\}} + 1_{\{k_b < \tilde{X}_s\}}),
\end{aligned}$$

the last inequality is due to $a < 0$ and $e_0^x = B(T)$.
Taking the expectation, we obtain, for all $s \in [0, \theta]$

$$\mathbb{P}(k_b < \sigma B_s) \le \mathbb{P}(\frac{k_b}{\sigma\sqrt{\theta}} < \sigma B_1) \le C\sqrt{\theta} e^{-\frac{k_b^2}{2\sigma^2\theta}},$$

for θ small enough, we have $\frac{k_b}{2} < k_b - (\gamma_0 - \frac{\sigma^2}{2})s$, then

$$\mathbb{P}(k_b < \tilde{X}_s) \le \mathbb{P}(\frac{k_b - (\gamma_0 - \frac{\sigma^2}{2})s}{\sigma\sqrt{\theta}} < B_1) + \mathbb{P}(T_1 \le \theta) \le C\sqrt{\theta} e^{-\frac{k_b^2}{8\sigma^2\theta}} + A\theta$$

and

$$\mathbb{E}\left(e^{\sigma B_s}\left|e^{(\gamma_0 - \frac{\sigma^2}{2})s + Z_s} - 1\right|\right) \le e^{\frac{\sigma^2}{2}s}\left|e^{(\gamma_0 - \frac{\sigma^2}{2})s} - 1\right| + e^{\gamma_0 s}\mathbb{E}\left|e^{Z_s} - 1\right| \le D\theta.$$

Hence,

$$\begin{aligned}
\int_0^\theta \mathbb{E}\Big(&\Big|1_{\{x_0+a\sqrt{\theta}+\tilde{X}_s \le \ln K\}} h(x_0 + a\sqrt{\theta} + \tilde{X}_s) \\
&- 1_{\{x_0+a\sqrt{\theta}+\sigma B_s \le \ln K\}} h(x_0 + a\sqrt{\theta} + \sigma B_s)\Big|\Big) ds = O(\theta^2).
\end{aligned}$$

Thanks to this estimation, equation (10) becomes

$$\mathcal{I}^a(\tau) = \mathbb{E}\left(\int_0^\tau 1_{\{a\sqrt{\theta}+\sigma B_s \le \ln \frac{K}{b(T)}\}} h(x_0 + a\sqrt{\theta} + \sigma B_s) ds\right) + o(\theta^{\frac{3}{2}}). \tag{11}$$

The function h is convex, therefore it is right and left differentiable. Particularly, we have all $x < \ln(K)$,

$$h'_g(x) = e^x\left(\delta + \int e^y 1_{\{y > \ln(K) - x\}} \nu(dy)\right)$$

and

$$h'_d(x) = e^x\left(\delta + \int e^y 1_{\{y \ge \ln(K) - x\}} \nu(dy)\right).$$

Hence, we can write

$$h'_d(x_0)(x - x_0)^+ - h'_g(x_0)(x - x_0)^- \le h(x) - h(x_0) \le h'_g(x)(x - x_0)^+ - h'_d(x)(x - x_0)^-,$$

hence

$$\begin{aligned} 0 &\leq h(x) - \left(h(x_0) + h'_d(x_0)(x-x_0)^+ - h'_g(x_0)(x-x_0)^-\right) \\ &\leq \left(h'_g(x) - h'_d(x_0)\right)(x-x_0)^+ + \left(h'_g(x_0) - h'_d(x)\right)(x-x_0)^- \\ &= \left(h'_g(x \vee x_0) - h'_d(x \wedge x_0)\right)|x - x_0|. \end{aligned}$$

Thanks to the equation characterizing $b(T)$ when $\bar{d} < 0$, we have $h(x_0) = h(\ln(b(T)) = 0$. We thus obtain, by setting $\Delta h'(x_0) = h'_d(x_0) - h'_g(x_0)$,

$$h(x_0 + x) = \Delta h'(x_0)x^+ + h'_g(x_0)x + |x|\,\tilde{R}(x),$$

where $\tilde{R}(x) \longrightarrow_{x\to 0} 0$, and

$$\begin{aligned} 0 \leq \tilde{R}(x) &\leq \left(h'_g(x_0 + x^+) - h'_d(x_0 - x^-)\right) \\ &\leq L\left(1 + e^x\right), \end{aligned}$$

with L a positive constant. We can then write

$$\begin{aligned} & 1_{\{a\sqrt{\theta}+\sigma B_s \leq \ln \frac{K}{b(T)}\}} h(x_0 + a\sqrt{\theta} + \sigma B_s) \\ &= \left(\Delta h'(x_0)(a\sqrt{\theta} + \sigma B_s)^+ + h'_g(x_0)a\sqrt{\theta} + \sigma B_s\right)\left(1 - 1_{\{a\sqrt{\theta}+\sigma B_s > \ln \frac{K}{b(T)}\}}\right) \\ &+ \left|a\sqrt{\theta} + \sigma B_s\right| \tilde{R}(a\sqrt{\theta} + \sigma B_s) 1_{\{\tilde{X}_s^{a\sqrt{\theta}} \leq \ln \frac{K}{b(T)}\}}. \end{aligned} \tag{12}$$

We state that

$$\left|\mathbb{E}\int_0^\tau \left|a\sqrt{\theta} + \sigma B_s\right| \tilde{R}(a\sqrt{\theta} + \sigma B_s) 1_{\{\tilde{X}_s^{a\sqrt{\theta}} \leq \ln \frac{K}{b(T)}\}} ds\right| = o(\theta^{\frac{3}{2}}) \tag{13}$$

$$\left|\mathbb{E}\int_0^\tau \left(\Delta h'(x_0)(a\sqrt{\theta} + \sigma B_s)^+ + h'_g(x_0)a\sqrt{\theta} + \sigma B_s\right) 1_{\{a\sqrt{\theta}+\sigma B_s > \ln \frac{K}{b(T)}\}} ds\right| = o(\theta^{\frac{3}{2}}), \tag{14}$$

Indeed, we have for (13), by setting $s = u\theta$,

$$\begin{aligned} & \left|\mathbb{E}\left(\int_0^\tau \left|a\sqrt{\theta} + \sigma B_s\right| \tilde{R}(a\sqrt{\theta} + \sigma B_s) 1_{\{a\sqrt{\theta}+\sigma B_s \leq \ln \frac{K}{b(T)}\}} ds\right)\right| \\ &= \theta^{\frac{3}{2}} \int_0^1 \mathbb{E}\left[|a + \sigma B_s|\,\tilde{R}(\sqrt{\theta}(a + \sigma B_s)) 1_{\{a\sqrt{\theta}+\sigma\sqrt{\theta}B_s \leq \ln \frac{K}{b(T)}\}}\right] ds. \end{aligned}$$

As $|\tilde{R}(x)| \leq L(e^x + 1)$ and $|\tilde{R}(x)| \longrightarrow_{x\to 0} 0$, we have by bounded convergence

$$\int_0^1 \mathbb{E}\left[|a + \sigma B_s|\,\tilde{R}(\tilde{X}_{\theta s}^{a\sqrt{\theta}}) 1_{\{a\sqrt{\theta}+\sigma B_s \leq \ln \frac{K}{b(T)}\}}\right] ds \quad \longrightarrow_{\theta\to 0} \quad 0.$$

And for the estimate in (14), we have

$$\begin{aligned}
&\left|\mathbb{E}\int_0^\tau \left(\Delta h'(x_0)(a\sqrt{\theta}+\sigma B_s)^+ + h'_g(x_0)a\sqrt{\theta}+\sigma B_s\right) 1_{\{a\sqrt{\theta}+\sigma B_s>\ln\frac{K}{b(T)}\}} ds\right| \\
&\leq\ C\sqrt{\theta}\int_0^\theta \mathbb{E}\left[\left(|a|+\sigma\sqrt{\frac{s}{\theta}}\,|B_1|\right) 1_{\{a+\sigma B_1>\frac{1}{\sqrt{\theta}}\ln\frac{K}{b(T)}\}}\right] ds \\
&\leq\ C\theta^{\frac{3}{2}}\sqrt{\mathbb{E}\left(|a|+|B_1|\right)^2}\sqrt{\mathbb{P}\{a+\sigma B_1>\frac{1}{\sqrt{\theta}}\ln\frac{K}{b(T)}\}} \\
&=\ O(\theta^n).
\end{aligned}$$

Therefore, taking the expectation of the integral of (12) between 0 and all stopping time $\tau\in\mathcal{T}_{0,\theta}$ gives

$$\begin{aligned}
\mathcal{I}^a(\tau) &= \mathbb{E}\int_0^\tau 1_{\{a\sqrt{\theta}+\sigma B_s\leq\ln\frac{K}{b(T)}\}} h(x_0+a\sqrt{\theta}+\sigma B_s)ds \\
&= h'_g(x_0)\mathbb{E}\int_0^\tau\Big(a\sqrt{\theta}+\sigma B_s\Big)\,ds + \Delta h'(x_0)\mathbb{E}\int_0^\tau\Big(a\sqrt{\theta}+\sigma B_s\Big)^+ ds + o(\theta^{\frac{3}{2}}), \\
&= b(T)\bar{\delta}\mathbb{E}\int_0^\tau\Big(a\sqrt{\theta}+\sigma B_s\Big) + \lambda\beta\left(a\sqrt{\theta}+\sigma B_s\right)^+ ds + o(\theta^{\frac{3}{2}}), \qquad (15)
\end{aligned}$$

with $\bar{\delta}=\delta+\int_{y>\ln\frac{K}{b(T)}} e^y\nu(dy)$, $\beta=\frac{K}{b(T)\delta}$, $\lambda=\nu\{\ln\frac{K}{b(T)}\}$ and we recall that $h'_g(x_0)=b(T)\bar{\delta}$ and $\Delta h'(x_0)=K\nu\{\ln\frac{K}{b(T)}\}$ then $(\lambda\beta=\frac{\Delta h'(x_0)}{h'_g(x_0)})$.
Comming back to (6) and using (8) and (15), we obtain

$$\begin{aligned}
\mathbb{E}\left(e^{-r\tau}(K-S_\tau)^+\right) &= (K-S_0)^+ + \mathbb{E}\left(b(T)\bar{\delta}\int_0^\tau\left(a\sqrt{\theta}+\sigma B_s+\lambda\beta(a\sqrt{\theta}+\sigma B_s)^+\right)ds\right) \\
&\quad + K1_{\{\hat{T}_1<\tau\}}\left((a\sqrt{\theta}+\sigma B_\tau)^+ - (a\sqrt{\theta}+\sigma B_{\hat{T}_1})^+\right)\Big) + o(\theta^{3/2}),
\end{aligned}$$

with $o(\theta^{3/2})$ independent of τ. Hence

$$P(T-\theta, b(T)e^{a\sqrt{\theta}}) = (K-b(T)e^{a\sqrt{\theta}})^+ + \sigma b(T)\bar{\delta}\bar{v}_{\lambda,\beta,\theta}(a/\sigma) + o(\theta^{3/2}),$$

where $\bar{v}_{\lambda,\beta,\theta}$ defined by

$$\bar{v}_{\lambda,\beta,\theta}(y) = \sup_{\tau\in\mathcal{T}_{0,\theta}}\mathbb{E}\left(\int_0^\tau f_{\lambda\beta}(y\sqrt{\theta}+B_s)ds + \beta 1_{\{\hat{T}_1<\tau\}}\left((y\sqrt{\theta}+B_\tau)^+ - (y\sqrt{\theta}+B_{\hat{T}_1})^+\right)\right),$$

with $f_a(x)=x+ax^+$. To simplify the expression of $\bar{v}_{\lambda,\beta,\theta}$, we notice first that, if we set $B^\theta_t=B_{\theta t}/\sqrt{\theta}$, we can write

$$\begin{aligned}
\bar{v}_{\lambda,\beta,\theta} &= \sqrt{\theta}\sup_{\tau\in\mathcal{T}_{0,\theta}}\mathbb{E}\left(\int_0^\tau f_{\lambda\beta}(y+B^\theta_{s/\theta})ds + \beta 1_{\{\hat{T}_1<\tau\}}\left((y+B^\theta_{\tau/\theta})^+ - (y+B^\theta_{\hat{T}_1/\theta})^+\right)\right) \\
&= \sqrt{\theta}\sup_{\tau\in\mathcal{T}_{0,\theta}}\mathbb{E}\left(\theta\int_0^{\tau/\theta} f_{\lambda\beta}(y+B^\theta_s)ds + \beta 1_{\{\hat{T}_1<\tau\}}\left((y+B^\theta_{\tau/\theta})^+ - (y+B^\theta_{\hat{T}_1/\theta})^+\right)\right)
\end{aligned}$$

We also notice that $\tau \in \mathcal{T}_{0,\theta}$ if and only if $\tau/\theta \in \mathcal{T}^{\theta}_{0,1}$, where $\mathcal{T}^{\theta}_{0,1}$ is the set of the stopping times of the filtration $(\mathcal{F}_{\theta t})_{t\geq 0}$, with values in $[0,1]$, then

$$\bar{v}_{\lambda,\beta,\theta} = \sqrt{\theta} \sup_{\tau\in\mathcal{T}^{\theta}_{0,1}} \mathbb{E}\left(\theta\int_0^{\tau} f_{\lambda\beta}(y+B^{\theta}_s)ds + \beta 1_{\{\hat{T}_1<\theta\tau\}}\left((y+B^{\theta}_{\tau})^+ - (y+B^{\theta}_{\hat{T}_1/\theta})^+\right)\right)$$

Note that $\bar{v}_{\lambda,\beta,\theta}(y)$ does not change if we replace $\mathcal{T}^{\theta}_{0,1}$ by $\hat{\mathcal{T}}_{0,1}$ the set of the stopping times of the natural filtration of the couple $(B^{\theta}_t, \hat{N}_{\theta t})$, where $\hat{N}$ is defined by

$$\hat{N}_t = \sum_{0<s\leq t} 1_{\{\Delta Z_s = \ln(K/b(T))\}}.$$

The processes $(\hat{N}_{\theta t})_{t\geq 0}$ is a Poisson process with intensity $\theta\lambda$, where $\lambda = \nu\{\ln(K/b(T))\}$. Under the probability $\hat{\mathbb{P}}$, defined by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \theta^{\hat{N}_1} e^{-\lambda(\theta-1)},$$

the process $(B_t, \hat{N}_t)_{0\leq t\leq 1}$ has the same law as $(B^{\theta}_t, \hat{N}_{\theta t})_{0\leq t\leq 1}$. Hence,

$$\begin{aligned}
\bar{v}_{\lambda,\beta,\theta}(y) &= \sqrt{\theta}\sup_{\tau\in\mathcal{T}_{0,1}} \mathbb{E}\left[\theta^{\hat{N}_1} e^{-\lambda(\theta-1)}\left(\theta\int_0^{\tau} f_{\lambda\beta}(y+B_s)ds + \beta 1_{\{\hat{T}_1<\tau\}}\left((y+B_{\tau})^+ - (y+B_{\hat{T}_1})^+\right)\right)\right] \\
&= \sqrt{\theta}\sup_{\tau\in\mathcal{T}_{0,1}} \mathbb{E}\left[\theta^{\hat{N}_{\tau}} e^{-\lambda\tau(\theta-1)}\left(\theta\int_0^{\tau} f_{\lambda\beta}(y+B_s)ds + \frac{\beta}{2} 1_{\{\hat{T}_1<\tau\}}\left(L^{-y}_{\tau}(B) - L^{-y}_{\hat{T}_1}(B)\right)\right)\right],
\end{aligned}$$

where $L^{-y}(B)$ denotes the local time of B at $-y$. We have for $\tau\in\mathcal{T}_{0,1}$,

$$\mathbb{E}\left[\theta^{\hat{N}_{\tau}} e^{\lambda\tau(\theta-1)}\left(\theta\int_0^{\tau} f_{\lambda\beta}(y+B_s)ds\right)\right] = \theta\mathbb{E}\left[1_{\{\hat{N}_{\tau}=0\}} e^{-\lambda\tau(\theta-1)}\left(\int_0^{\tau} f_{\lambda\beta}(y+B_s)ds\right)\right] + \theta R_{\tau},$$

and if $\theta\leq 1$

$$|R_{\tau}| \leq \theta\mathbb{E}\left[1_{\{\hat{N}_{\tau}\geq 1\}} e^{-\lambda\tau(\theta-1)}\left(\int_0^1 |f_{\lambda\beta}(y+B_s)|ds\right)\right] = O(\theta).$$

Hence,

$$\mathbb{E}\left[\theta^{\hat{N}_{\tau}} e^{-\lambda\tau(\theta-1)}\left(\theta\int_0^{\tau} f_{\lambda\beta}(y+B_s)ds\right)\right] = \theta\mathbb{E}\left[1_{\{\hat{N}_{\tau}=0\}} e^{\lambda\tau}\left(\int_0^{\tau} f_{\lambda\beta}(y+B_s)ds\right)\right] + O(\theta^2),$$

Besides,

$$\begin{aligned}
\mathbb{E}\left[\theta^{\hat{N}_1} e^{-\lambda(\theta-1)} 1_{\{\hat{T}_1<\tau\}}\left(L^{-y}_{\tau}(B) - L^{-y}_{\hat{T}_1}(B)\right)\right] &= \mathbb{E}\left[\theta^{\hat{N}_{\tau}} e^{-\lambda\tau(\theta-1)} 1_{\{\hat{T}_1<\tau\}}\left(L^{-y}_{\tau}(B) - L^{-y}_{\hat{T}_1}(B)\right)\right] \\
&= \theta\mathbb{E}\left[e^{\lambda\tau} 1_{\{\hat{N}_{\tau}=1\}}\left(L^{-y}_{\tau}(B) - L^{-y}_{\hat{T}_1}(B)\right)\right] + O(\theta^2).
\end{aligned}$$

We then have

$$\bar{v}_{\lambda,\beta,\theta}(y) = \theta^{3/2} v_{\lambda,\beta}(y) + o(\theta^{3/2}),$$

with

$$v_{\lambda,\beta}(y) \quad = \quad \sup_{\tau\in\mathcal{T}_{0,1}} \mathbb{E}\left[e^{\lambda\tau}1_{\{\hat{N}_\tau=0\}}\int_0^\tau f_{\lambda\beta}(y+B_s)ds + \frac{\beta}{2}e^{\lambda\tau}1_{\{\hat{N}_\tau=1\}}\left(L_\tau^{-y}(B) - L_{\hat{T}_1}^{-y}(B)\right)\right].$$

Finally, we obtain

$$P(T-\theta, b(T)e^{a\sqrt{\theta}}) - (K - b(T)e^{a\sqrt{\theta}}) \quad = \quad \theta^{\frac{3}{2}}(\sigma b(T)\bar{\delta}e^{\lambda})v_{\lambda,\beta}\left(\frac{a}{\sigma}\right) + o(\theta^{\frac{3}{2}}),$$

□

3.3. **Convergence rate of the critical price.** Thanks to the expansion given in Theorem 3.1, we are now able to state the first main result of this paper.

Theorem 3.2. *Under the hypothesis of the model and $\bar{d} < 0$, we have :*
If $\nu\{\ln\frac{K}{b(T)}\} = 0$, then we have

$$\lim_{t\to T}\frac{b(T)-b(t)}{\sigma b(T)\sqrt{(T-t)}} = y_0,$$

with $y_0 = -\sup\{x\in\mathbb{R}\ ;\ v_0(x) = \sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}(\int_0^\tau(x+B_s)ds) = 0\}$.
If $\nu\{\ln\frac{K}{b(T)}\} > 0$, we then have

$$\lim_{t\to T}\frac{b(T)-b(t)}{\sigma b(T)\sqrt{(T-t)}} = y_{\lambda,\beta},$$

with $y_{\lambda,\beta}$ as defined in Lemma 3.1, with

$$\lambda = \nu\{\ln\frac{K}{b(T)}\}\ ,\ \beta = \frac{K}{b(T)\bar{\delta}} \text{ and } \quad \bar{\delta} = \delta + \int_{y>\ln(K/b(T))} e^y\nu(dy).$$

Proofof Theorem 3.2:
According to Theorem 3.1, we have for all $a < 0$,

$$P(T-\theta, b(T)e^{a\sqrt{\theta}}) = (K - b(T)e^{a\sqrt{\theta}})^+ + C\theta^{\frac{3}{2}}v_{\lambda,\beta}(\frac{a}{\sigma}) + o(\theta^{\frac{3}{2}}).$$

Lower bound for $b(T) - b(t)$
Specifically, we have for all $a > -\sigma y_{\lambda,\beta}$, where $y_{\lambda,\beta}$ is defined by Lemma 3.1,

$$v_{\lambda,\beta}(\frac{a}{\sigma}) > 0,$$

we thus obtain for θ close to 0,

$$P(t, b(T)e^{a\sqrt{\theta}}) > (K - b(T)e^{a\sqrt{\theta}}),$$

and then

$$\ln(b(T)) + a\sqrt{\theta} > \ln(b(t)),$$

hence

$$\frac{b(T)-b(t)}{b(t)\sqrt{\theta}} > -a.$$

Noting that since $r > 0$ we have $b(T) > 0$, and by making t tend to T then a to $-\sigma y_{\lambda,\beta}$, we obtain

$$\liminf_{t\to T} \frac{b(T)-b(t)}{b(T)\sqrt{T-t}} \geq \sigma y_{\lambda,\beta}.$$

Upper bound for $b(T)-b(t)$

Let's consider $a \leq -\sigma y_{\lambda,\beta}$, we have thus $\upsilon_{\lambda,\beta}(\frac{a}{\sigma}) = 0$ and consequently,

$$P(t, b(T)e^{a\sqrt{\theta}}) - (K - b(T)e^{a\sqrt{\theta}}) = g(\theta),$$

with $g(\theta) = o(\theta^{\frac{3}{2}})$.

In addition, we have for all $b(t) < x < K$,

$$P(t,x) - P(t,b(t)) - (x-b(t))\frac{\partial P}{\partial x}(t,b(t)) = \int_{b(t)}^{x} (u-b(t))\frac{\partial^2 P}{\partial x^2}(t,du),$$

since $\frac{\partial^2 P}{\partial x^2}(t,du)$ is a positive measure on $]0,+\infty[$. As the smooth-fit is satisfied, $\frac{\partial P}{\partial x}(t,b(t)) = -1$ (see [8]), we have for all $b(t) < x < K$,

$$P(t,x) - (K-x) \quad = \quad \int_{b(t)}^{x} (u-b(t))\frac{\partial^2 P}{\partial x^2}(t,du).$$

Then, for $b(t) < x = b(T)e^{a\sqrt{\theta}}$, we have according to Lemma 2.2,

$$\begin{aligned}
\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) &\geq b(T)\bar{\delta}\left((1-e^{a\sqrt{\theta}}) - \lambda\beta\sqrt{\theta}\sigma\mathbb{E}\left(B_1 + \frac{a}{\sigma}\right)^+\right) + o(\sqrt{\theta}) \\
&\geq b(T)\bar{\delta}\sqrt{\theta}\sigma\left(-\frac{a}{\sigma} - \lambda\beta\mathbb{E}\left(B_1 + \frac{a}{\sigma}\right)^+\right) + o(\sqrt{\theta})
\end{aligned}$$

Hence

$$P(t,x) - (K-x) \geq [(x-b(t))^+]^2\left(\frac{C(-\frac{a}{\sigma})}{b(0)^2\sigma^2}\sqrt{\theta} + o(\sqrt{\theta})\right),$$

where $C(x) = x - \lambda\beta\mathbb{E}(B_1 - x)^+$. Due to lemma 3.2 and to the continuity of $C(x)$, we have, for $\frac{-a}{\sigma}$ close enough to $y_{\lambda,\beta}$, $C(-\frac{a}{\sigma}) > 0$. Moreover,

$$P(t, b(T)e^{a\sqrt{\theta}}) - (K - b(T)e^{a\sqrt{\theta}}) = g(\theta) = o(\theta^{\frac{3}{2}}).$$

Therefore, for θ small enough, there exists a positive constant A such that

$$[(b(T)e^{a\sqrt{\theta}} - b(t))^+]^2 \leq Ab(0)^2\sigma^2\frac{g(\theta)}{C(-\frac{a}{\sigma})\sqrt{\theta}} = o(\theta) \tag{16}$$

$$(b(T)e^{a\sqrt{\theta}} - b(t))^+ = o(\sqrt{\theta}),$$

and then, for θ small enough,

$$\frac{b(T)-b(t)}{b(T)\sqrt{\theta}} \leq -a + o(1).$$

Finally, by making a tend to $-\sigma y_{\lambda,\beta}$, we obtain

$$\limsup_{t\to T} \frac{b(T)-b(t)}{b(T)\sqrt{T-t}} \leq \sigma y_{\lambda,\beta}.$$

□

4. Limit case

In this part, we consider the limit case where $\bar{d} = r - \delta - \int_{y>0}(e^y - 1)\nu(dy) = 0$, we then have

Theorem 4.1. *According to the model hypothesis, if $\bar{d} = 0$, then, we have*

$$\lim_{t\to T} \frac{K - b(t)}{\sigma K\sqrt{(T-t)|\ln(T-t)|}} = \sqrt{2}.$$

The method for proving Theorem 4.1 consists of analysing the behavior of the European critical price $b_e(t)$ introduced in section 1, afterwards we prove that the behavior of the critical price $b(t)$ is similar by controling the difference $b(t) - b_e(t)$.

Let us denote by

$$\alpha(\theta) = \frac{\ln(\frac{K}{b_e(t)}) - \mu\theta}{\sigma\sqrt{\theta}},$$

where $\mu = \gamma_0 - \frac{\sigma^2}{2} = r - \delta - \int(e^y - 1)\nu(dy) - \frac{\sigma^2}{2}$.

Proposition 4.1. *Under the model hypothesis, if $\bar{d} = 0$, then we have*

i) $\alpha(\theta) \sim \sqrt{2\ln(\frac{1}{\theta})}$

ii) $\lim_{\theta\to 0} \frac{K - b_e(t)}{\sigma K\sqrt{|\theta\ln(\theta)|}} = \sqrt{2}$

Proof of Proposition 4.1:
Since $b(t) \leq b_e(t) \leq K$ and $b(t) \to K$, we clearly have

$$\sqrt{\theta}\alpha(\theta) \longrightarrow_{\theta\to 0} 0.$$

We will first prove that $\alpha(\theta) \longrightarrow_{\theta\to 0} +\infty$, or equivalently.

$$\lim_{\theta\to 0} \frac{K - b_e(t)}{\sigma\sqrt{\theta}} = +\infty. \tag{17}$$

We have

$$K - b_e(t) = e^{-r\theta}\mathbb{E}\left[\left(K - b_e(t)e^{\tilde{X}_\theta}\right)^+\right] \tag{18}$$

Therefore

$$\begin{aligned}\frac{K-b_e(t)}{\sqrt{\theta}} &= e^{-r\theta}\mathbb{E}\left[\left(\frac{K-b_e(t)}{\sqrt{\theta}}+b_e(t)\frac{1-e^{\tilde{X}_\theta}}{\sqrt{\theta}}\right)^+\right]\\ &= e^{-r\theta}\mathbb{E}\left[\left(\frac{K-b_e(t)}{\sqrt{\theta}}+b_e(t)\frac{1-e^{\sigma\sqrt{\theta}B_1+\mu\theta+Z_\theta}}{\sqrt{\theta}}\right)^+\right]\end{aligned}$$

Now, if we notice that $\frac{1-e^{\sigma\sqrt{\theta}B_1+\mu\theta+Z_\theta}}{\sqrt{\theta}} \longrightarrow^{p.s}_{\theta\to 0} -\sigma B_1$, we have by Fatou lemma

$$\begin{aligned}\liminf_{\theta\to 0}\frac{K-b_e(t)}{\sqrt{\theta}} &\geq \mathbb{E}\left[\left(\liminf_{\theta\to 0}\frac{K-b_e(t)}{\sqrt{\theta}}-\sigma K B_1\right)^+\right]\\ &= \liminf_{\theta\to 0}\frac{K-b_e(t)}{\sqrt{\theta}}+\mathbb{E}\left[\left(\sigma K B_1-\liminf_{\theta\to 0}\frac{K-b_e(t)}{\sqrt{\theta}}\right)^+\right]\end{aligned}$$

which is equivalent to

$$\mathbb{E}\left[\left(\sigma K B_1-\liminf_{\theta\to 0}\frac{K-b_e(t)}{\sqrt{\theta}}\right)^+\right]\leq 0.$$

This gives (17) which yields the wanted result.

i) We now rewrite equation (18) to obtain

$$K-b_e(t)=e^{-r\theta}K-b_e(t)e^{-\delta\theta}+e^{-r\theta}\mathbb{E}\left[\left(b_e(t)e^{\tilde{X}_\theta}-K\right)^+\right],$$

therefore

$$e^{-r\theta}\mathbb{E}\left[\left(e^{\tilde{X}_\theta}-e^{\ln(\frac{K}{b_e(t)})}\right)^+\right]=\frac{K}{b_e(t)}(1-e^{-r\theta})-(1-e^{-\delta\theta}). \tag{19}$$

We will give an expansion for each side of the equation. For the left hand side of the equation, we have

$$\begin{aligned}&e^{-r\theta}\mathbb{E}\left[\left(e^{\tilde{X}_\theta}-e^{\ln(\frac{K}{b_e(t)})}\right)^+\right]\\ &= e^{-r\theta+\theta\mu+\sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left[\left(e^{\sigma\sqrt{\theta}B_1+Z_\theta-\sigma\alpha(\theta)\sqrt{\theta}}-1\right)^+\right]\\ &= e^{-r\theta+\theta\mu+\sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left[\left(U_\theta e^{Z_\theta}-1\right)^+\right],\end{aligned}$$

where $U_\theta = e^{\sigma\sqrt{\theta}B_1 - \sigma\alpha(\theta)\sqrt{\theta}}$. Since the process Z_t is independent of U_θ, we can write

$$\begin{aligned}
&\mathbb{E}\left[\left(U_\theta e^{Z_\theta} - 1\right)^+ |U_\theta\right] \\
&= (U_\theta - 1)^+ + \mathbb{E}\left[\int_0^\theta ds \int \left(\left(U_\theta e^{Z_s + y} - 1\right)^+ - \left(U_\theta e^{Z_s} - 1\right)^+\right)\nu(dy)|U_\theta\right] \\
&= (U_\theta - 1)^+ + \int_0^\theta ds \int \left((U_\theta e^y - 1)^+ - (U_\theta - 1)^+\right)\nu(dy) + U_\theta O(\theta^2),
\end{aligned}$$

where $O(\theta^2)$ is detrministic. Indeed,

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_0^\theta ds \int \left(\left(U_\theta e^y e^{Z_s} - 1\right)^+ - \left(U_\theta e^y - 1\right)^+\right)\nu(dy)|U_\theta\right]\right| \\
&\leq U_\theta \int e^y \nu(du) \int_0^\theta \mathbb{E}\left|e^{Z_s} - 1\right| ds = U_\theta O(\theta^2).
\end{aligned}$$

Taking the expectation, we thus obtain,

$$\begin{aligned}
&\mathbb{E}\left[\left(U_\theta e^{Z_\theta} - 1\right)^+\right] \\
&= \mathbb{E}\left[(U_\theta - 1)^+\right] + \theta \int \mathbb{E}\left[(U_\theta e^y - 1)^+\right]\nu(dy) - \nu(\mathbb{R})\theta\mathbb{E}\left[(U_\theta - 1)^+\right] + O(\theta^2).
\end{aligned}$$

Since $\alpha(\theta) \to \infty$, we have like in [6]

$$\mathbb{E}\left[(U_\theta - 1)^+\right] \sim \sigma\sqrt{\theta}\mathbb{E}\left(B_1 - \alpha(\theta)\right)^+ = o(\sqrt{\theta}),$$

then

$$\mathbb{E}\left[\left(U_\theta e^{Z_\theta} - 1\right)^+\right] = \mathbb{E}\left[(U_\theta - 1)^+\right] + \theta \int \mathbb{E}\left[(U_\theta e^y - 1)^+\right]\nu(dy) + o(\theta^{\frac{3}{2}}).$$

We recall that $U_\theta = e^{\sigma\sqrt{\theta}B_1 - \sigma\alpha(\theta)\sqrt{\theta}}$, then

$$\begin{aligned}
&\mathbb{E}\left[(U_\theta e^y - 1)^+\right] - \left(e^{y - \sigma\alpha(\theta)\sqrt{\theta}} - 1\right)^+ \\
&\leq e^{y - \sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left|e^{\sigma\sqrt{\theta}B_1} - 1\right| = e^y O(\sqrt{\theta}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left(U_\theta e^{Z_\theta} - 1\right)^+\right] \\
&= \mathbb{E}\left[(U_\theta - 1)^+\right] + \theta \int \left(e^{y - \sigma\alpha(\theta)\sqrt{\theta}} - 1\right)^+ \nu(dy) + O(\theta^{\frac{3}{2}}) \\
&= \mathbb{E}\left[(U_\theta - 1)^+\right] + \theta \int_{y>0} \left(e^{y - \sigma\alpha(\theta)\sqrt{\theta}} - 1\right)\nu(dy) \\
&\quad -\theta \int_{0<y<\sigma\alpha(\theta)\sqrt{\theta}} \left(e^{y - \sigma\alpha(\theta)\sqrt{\theta}} - 1\right)\nu(dy) + O(\theta^{\frac{3}{2}}).
\end{aligned}$$

Since $(1-e^{-x}) \leq x$, we then have

$$\left| \int_{(0,\sigma\alpha(\theta)\sqrt{\theta})} (e^{y-\sigma\alpha(\theta)\sqrt{\theta}} - 1)\nu(dy) \right| \leq \underbrace{\overbrace{\nu\{0<y<\sigma\alpha(\theta)\sqrt{\theta}\}}^{\longrightarrow_{\theta\to 0}0}\sigma\alpha(\theta)\sqrt{\theta}}_{=o(\alpha(\theta)\theta^{\frac{1}{2}})},$$

and noticing that $\theta^{\frac{3}{2}} = o(\alpha(\theta)\theta^{\frac{3}{2}})$, we obtain

$$\mathbb{E}\left(U_\theta e^{Z_\theta} - 1\right)^+ = \mathbb{E}\left(U_\theta - 1\right)^+ + \theta\int (e^y-1)^+\nu(dy) - \alpha(\theta)\theta^{\frac{3}{2}}\sigma\int_{y>0} e^y\nu(dy) + o(\alpha(\theta)\theta^{\frac{3}{2}}).$$

The left hand side of equation (19) becomes

$$\begin{aligned} e^{-r\theta}\mathbb{E}\left[\left(e^{\tilde{X}_\theta} - e^{\ln(\frac{K}{b_e(t)})}\right)^+\right] &= e^{-r\theta+\theta\mu+\sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left[\left(U_\theta e^{Z_\theta}-1\right)^+\right] \\ &= e^{-r\theta+\theta\mu+\sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left[(U_\theta-1)^+\right] \\ &+ \left(1+\sigma\alpha(\theta)\sqrt{\theta}+o(\alpha(\theta)\sqrt{\theta})\right)\left(\theta\int (e^y-1)^+\nu(dy) - \alpha(\theta)\theta^{\frac{3}{2}}\sigma\int_{y>0} e^y\nu(dy) + o(\alpha(\theta)\theta^{\frac{3}{2}})\right) \\ &= e^{-r\theta+\theta\mu+\sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left[(U_\theta-1)^+\right] + \theta\int (e^y-1)^+\nu(dy) - \nu(\mathbb{R}^+)\sigma\alpha(\theta)\theta^{\frac{3}{2}} + o(\alpha(\theta)\theta^{\frac{3}{2}}). \end{aligned} \tag{20}$$

Besides, the right hand side of (19)

$$\begin{aligned} \frac{K}{b_e(t)}(1-e^{-r\theta}) - (1-e^{-\delta\theta}) &= e^{\sigma\sqrt{\theta}\alpha(\theta)+\mu\theta}r\theta - \delta\theta + O(\theta^2) \\ &= (r-\delta)\theta + r\sigma\theta^{\frac{3}{2}}\alpha(\theta) + o(\theta^{\frac{3}{2}}\alpha(\theta)) \\ &= \left(\int (e^y-1)^+\nu(dy)\right)\theta + r\sigma\alpha(\theta)\theta^{\frac{3}{2}} + o(\theta^{\frac{3}{2}}\alpha(\theta)). \end{aligned} \tag{21}$$

Thanks to (20) and (21), equation (19) becomes,

$$e^{-r\theta+\theta\mu+\sigma\alpha(\theta)\sqrt{\theta}}\mathbb{E}\left[(U_\theta-1)^+\right] = \sigma\left(r+\nu(\mathbb{R}^+)\right)\alpha(\theta)\theta^{\frac{3}{2}} + o(\theta^{\frac{3}{2}}\alpha(\theta)).$$

Hence,

$$\mathbb{E}\left[(U_\theta-1)^+\right] \sim \sigma\left(r+\nu(\mathbb{R}^+)\right)\alpha(\theta)\theta^{\frac{3}{2}}.$$

As explained above, thanks to proposition 2.1 in [14], we have

$$\mathbb{E}\left[(U_\theta-1)^+\right] \sim \sigma\sqrt{\theta}\mathbb{E}(B_1-\alpha(\theta))^+ \sim \frac{\sigma\sqrt{\theta}}{\sqrt{2\pi}\alpha^2(\theta)e^{\frac{\alpha^2(\theta)}{2}}}.$$

Thus, we have

$$\frac{1}{\sqrt{2\pi}\alpha^2(\theta)e^{\frac{\alpha^2(\theta)}{2}}} \sim \left(r+\nu(\mathbb{R}^+)\right)\theta\alpha(\theta), \tag{22}$$

hence

$$\alpha(\theta) \sim \sqrt{2\ln(\frac{1}{\theta})}. \tag{23}$$

ii) Since $\frac{K-b_e(t)}{K\sigma\sqrt{\theta}} \sim \alpha(\theta)$, we obtain

$$\frac{K-b_e(t)}{\sigma K} \sim \sqrt{2\theta\ln(\frac{1}{\theta})}.$$

□

To compare the behaviors of $b(t)$ and $b_e(t)$, we have to control the difference between them.

Proposition 4.2. *According to the model hypothesis, if $\bar{d} = 0$, then there exists $C > 0$ such that*

$$0 \leq \frac{b_e(t) - b(t)}{\sqrt{T-t}} \leq C.$$

Before proving Proposition 4.2, we need to prove the non decreasing of $b_e(t)$ near maturity which is the purpose of this following lemma

Lemma 4.1. *The critical European put price, $b_e(t)$, is differentiable on $(0,T)$ and for t close to T, we have*

$$b_e'(t) \geq 0.$$

Proof of Lemma 4.1:
We recall that F is the function defined by $F(t,x) = P_e(t,x) - (K-x)$, F is $\mathcal{C}^1$ on $(0,T)\times(0,K)$ and satisfies $\frac{\partial F}{\partial x}(t,x) = \frac{\partial P_e}{\partial x}(t,x) + 1 > 0$. Due to its definition, $b_e(t)$ satisfies the following equation, $P_e(t,b_e(t)) - (K - b_e(t)) = 0$. Then, thanks to the implicit function theorem, $b_e(t)$ is differentiable on $(0,T)$ and

$$b_e'(t) = -\frac{\frac{\partial F}{\partial t}(t,b_e(t))}{\frac{\partial F}{\partial x}(t,b_e(t))} = -\frac{\frac{\partial P_e}{\partial t}(t,b_e(t))}{\frac{\partial P_e}{\partial x}(t,b_e(t)) + 1},$$

which means that

$$-b_e'(t)\frac{\partial P_e}{\partial t}(t,b_e(t)) \leq 0.$$

We will study the sign of $\frac{\partial P_e}{\partial t}(t,b_e(t))$ instead of that of $b_e'(t)$.
The European put price satisfies the following equation

$$\begin{aligned}
&\frac{\partial P_e}{\partial t}(t,b_e(t)) \\
&= rP_e(t,b_e(t)) - \frac{\sigma^2 b_e(t)^2}{2}\frac{\partial^2 P_e}{\partial x^2}(t,b_e(t)) - (r-\delta)b_e(t)\frac{\partial P_e}{\partial x}(t,b_e(t)) \\
&\quad - \int \left[P_e(t,b_e(t)e^y) - P_e(t,b_e(t)) - b_e(t)(e^y-1)\frac{\partial P_e}{\partial x}(t,b_e(t))\right]\nu(dy) \\
&= r(K-b_e(t)) - \frac{\sigma^2 b_e(t)^2}{2}\frac{\partial^2 P_e}{\partial x^2}(t,b_e(t)) - \underbrace{\left(r-\delta-\int (e^y-1)^+\nu(dy)\right)}_{\bar{d}=0} b_e(t)\frac{\partial P_e}{\partial x}(t,b_e(t)) \\
&\quad - \int_{y>0} P_e(t,b_e(t)e^y)\nu(dy) + \nu(\mathbb{R}_+)P_e(t,b_e(t)) \\
&\quad - \int_{y<0}\left[P_e(t,b_e(t)e^y) - P_e(t,b_e(t)) - b_e(t)(e^y-1)\frac{\partial P_e}{\partial x}(t,b_e(t))\right]\nu(dy)
\end{aligned}$$

Since $P_e(t,.)$ is a non negative convex function, we have $\int_{y>0}\left[P_e(t,b_e(t)e^y)\right]\nu(dy) \geq 0$ and

$$\int_{y<0}\left[P_e(t,b_e(t)e^y) - P_e(t,b_e(t)) - b_e(t)(e^y-1)\frac{\partial P_e}{\partial x}(t,b_e(t))\right]\nu(dy) \geq 0,$$

so that

$$\frac{\partial P_e}{\partial t}(t,b_e(t)) \leq \left(r+\nu(\mathbb{R}^+)\right)(K-b_e(t)) - \frac{\sigma^2 b_e(t)^2}{2}\frac{\partial^2 P_e}{\partial x^2}(t,b_e(t)).$$

Thanks to lemma 4.1, we have an equivalent for $(K-b_e(t))$. Now, let's have a look at the estimate of $\frac{\partial^2 P_e}{\partial x^2}(t,b_e(t))$ near T. We have

$$\begin{aligned}
\frac{\partial P_e}{\partial x}(t,x) &= -e^{-r(T-t)}\mathbb{E}\left[e^{\tilde{X}_{T-t}}1_{\{K-xe^{\tilde{X}_{T-t}}>0\}}\right] \\
&= -e^{-r(T-t)}\int_{-\infty}^{\ln(\frac{K}{x})} e^u p_{\tilde{X}_{T-t}}(u)du,
\end{aligned}$$

where p_X denotes the density of X and $\tilde{X}_t = \mu(t) + \sigma B_t + Z_t$. Then, we have

$$\begin{aligned}
\frac{\partial^2 P_e}{\partial x^2}(t,x) &= e^{-r(T-t)}\frac{K}{x^2}\, p_{\tilde{X}_{T-t}}\left(\ln(\frac{K}{x})\right) \\
&\geq e^{-r(T-t)}\frac{K}{x^2}p_{\mu(T-t)+\sigma B_{T-t}}\left(\ln(\frac{K}{x})\right)\mathbb{P}(T_1>\theta) \\
&= e^{-r\theta}\frac{K}{x^2\sigma\sqrt{2\pi\theta}}e^{\frac{-1}{2}\left(\frac{\ln(K/x)-\mu\theta}{\sigma\sqrt{\theta}}\right)^2}\mathbb{P}(T_1>\theta).
\end{aligned}$$

Then,

$$\frac{\partial P_e}{\partial t}(t,b_e(t)) \quad \leq \quad \left(r+\nu(\mathbb{R}^+)\right)(K-b_e(t)) - e^{-r\theta}\frac{\sigma K}{2\sqrt{2\pi\theta}}e^{-\frac{\alpha(\theta)^2}{2}}\mathbb{P}(T_1>\theta).$$

We can easily check that $K-b_e(t) \sim \sigma K\sqrt{\theta}\alpha(\theta) = o(\alpha^3(\theta)\sqrt{\theta})$, and we recall the equivalency (22)

$$\frac{1}{\sqrt{2\pi}\alpha^2(\theta)e^{\frac{\alpha^2(\theta)}{2}}} \sim \left(r+\nu(\mathbb{R}^+)\right)\theta\alpha(\theta),$$

which yields

$$e^{-r\theta}\frac{\sigma K}{2\sqrt{2\pi\theta}}e^{-\frac{\alpha(\theta)^2}{2}}\mathbb{P}(T_1>\theta) \sim \sigma K\frac{e^{-\frac{\alpha^2(\theta)}{2}}}{2\sqrt{2\pi\theta}} \sim \frac{\sigma K}{2}\left(r+\nu(R^+)\right)\alpha^3(\theta)\sqrt{\theta}.$$

Then, we have, for θ small enough

$$\frac{\partial P_e}{\partial t}(t,b_e(t)) \quad \leq \quad -\frac{\sigma K}{2}\left(r+\nu(R^+)\right)\alpha^3(\theta)\sqrt{\theta} + o(\alpha^3(\theta)\sqrt{\theta}) < 0,$$

which proves that $b_e'(t)$ is a non decreasing function for t close to T.

□

We are now in a position to prove Proposition 4.2.

Proof of Proposition 4.2:

An expansion of $P(t,x)$ around $(t,b(t))$ gives

$$P(t,x) - P(t,b(t)) - (x-b(t))\frac{\partial P}{\partial x}(t,b(t)) = \int_{b(t)}^{x}(u-b(t))\frac{\partial^2 P}{\partial x^2}(t,du),$$

and thanks to the smooth-fit which is satisfied at $b(t)$, we obtain

$$P(t,x) - (K-x) \quad \geq \quad \frac{(x-b(t))^2}{2}\inf_{b(t)\leq u\leq x}\frac{\partial^2 P}{\partial x^2}(t,u).$$

First, we are going to give, as in Lemma 2.2, a lower bound for $\inf_{b(t)\leq u\leq b_e(t)}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u)$. The variational inequality gives, for $u\in(b(t),K)$,

$$\begin{aligned}
&\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u)\\
&\geq\ rP(t,u) - (r-\delta)u\frac{\partial P}{\partial x}(t,u) - \int\left(P(t,ue^y) - P(t,u) - u(e^y-1)\frac{\partial P}{\partial x}(t,u)\right)\nu(dy)\\
&\geq\ r(K-u) - \left(r-\delta-\int_{y>0}(e^y-1)\nu(dy)\right)u\frac{\partial P}{\partial x}(t,u) - \int_{y>0}P(t,ue^y) - P(t,u)\nu(dy)\\
&\quad - \int_{y<0}\left(P(t,ue^y) - (K-u) - u(e^y-1)\frac{\partial P}{\partial x}(t,u)\right)\nu(dy).
\end{aligned}$$

Since $P(t,.)$ is non increasing and $\bar{d}=0$, we obtain

$$\begin{aligned}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) &\geq r(K-u)-\int_{y<0}\Big(P(t,ue^y)-(K-u)-u(e^y-1)\frac{\partial P}{\partial x}(t,u)\Big)\nu(dy)\\ &= r(K-u)-\int_{y<0}P(t,ue^y)-(K-ue^y)\nu(dy)-u\left(\frac{\partial P}{\partial x}(t,u)+1\right)\left(\int_{y<0}(1-e^y)\nu(dy)\right).\end{aligned}$$

Thanks to the convexity of P, $\frac{\partial P}{\partial x}(t,u)$ is non decreasing and $\frac{\partial P}{\partial x}(t,u)\geq -1$. We then have, for all $t<T$,

$$\begin{aligned}\inf_{b(t)\leq u\leq b_e(t)}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) &\geq r(K-b_e(t))-\int_{y<0}P(t,b_e(t)e^y)-(K-b_e(t)e^y)\nu(dy)\\ &\quad -b_e(t)\left(\frac{\partial P}{\partial x}(t,b_e(t))+1\right)\left(\int_{y<0}(1-e^y)\nu(dy)\right)\\ &\geq r(K-b_e(t))-\int_{y<0}P_e(t,b_e(t)e^y)-(K-b_e(t)e^y)\nu(dy)+o(\sqrt{\theta}).\end{aligned}$$

We obtained the last inquality, using the estimate of $e(\theta,x)=O(\theta)$ and $\frac{\partial P}{\partial x}(t,x)+1=o(\sqrt{\theta})$ (see Lemma 2.1). Since $y<0$, we also have $P_e(t,b_e(t)e^y)-(K-b_e(t)e^y)\leq 0$, thus

$$\inf_{b(t)\leq u\leq b_e(t)}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) \geq r(K-b_e(t))+o(\sqrt{\theta}).$$

Besides, for θ small enough, we have $\sqrt{\theta}\leq K-b_e(t)$, then we obtain

$$P(t,b_e(t))-(K-b_e(t)) \geq \frac{[(b_e(t)-b(t))^+]^2}{b_e^2(t)\sigma^2}r(K-b_e(t))(1+o(1)).$$

Furthermore,

$$\begin{aligned}&P(t,b_e(t))-(K-b_e(t))=e(\theta,b_e(t))\\ &= \mathbb{E}\left\{\int_0^\theta e^{-rs}\Big(rK-\delta S_s^{b_e(t)}-\int_{y>0}P(t+s,S_s^{b_e(t)}e^y)-\Big(K-S_s^{b_e(t)}e^y\Big)\nu(dy)\Big)1_{\{S_s^{b_e(t)}<b(t+s)\}}ds\right\}\\ &\leq \mathbb{E}\left\{\int_0^\theta e^{-rs}\left(rK-\delta S_s^{b_e(t)}-\int_{y>0}\left(S_s^{b_e(t)}e^y-K\right)^+\nu(dy)\right)1_{\{S_s^{b_e(t)}<b(t+s)\}}ds\right\}\end{aligned}$$

Since $\delta=r-\int_{y>0}(e^y-1)\nu(dy)$, we have

$$\begin{aligned}0 &\leq \left(rK-\delta x-\int_{y>0}(xe^y-K)^+\nu(dy)\right)1_{\{x<b(t+s)\}}\\ &\leq \left(r(K-x)-\int_{y>0}(xe^y-K)^+-(xe^y-x)^+\nu(dy)\right)1_{\{x<K\}}\\ &\leq \left(r+\nu(\mathbb{R}^+)\right)(K-x)^+,\end{aligned}$$

thus,

$$\begin{aligned} e(\theta, b_e(t)) &\leq \left(r+\nu(\mathbb{R}^+)\right)\mathbb{E}\left\{\int_0^\theta e^{-rs}\left(K-S_s^{b_e(t)}\right)^+ ds\right\} \\ &= \left(r+\nu(\mathbb{R}^+)\right)\int_0^\theta P_e(T-s, b_e(t))ds \\ &= \left(r+\nu(\mathbb{R}^+)\right)\int_0^\theta P_e(t+u, b_e(t))du. \end{aligned}$$

And as we saw in lemma 4.1, near T, $b_e(t)$ is non-decreasing, then $b_e(t) \leq b_e(t+u)$. Due to the non-decreasing of $P_e(t,x)-(K-x)$ on x, we thus have

$$P_e(t+u, b_e(t)) \leq K - b_e(t),$$

In conclusion, we have

$$e(\theta, b_e(t)) \leq \left(r+\nu(\mathbb{R}^+)\right)\theta(K-b_e(t))$$

and

$$e(\theta, b_e(t)) \geq \frac{[b_e(t)-b(t)]^2}{b^2(t)\sigma^2}\bar{\delta}(K-b_e(t)).$$

Which gives the wanted result: There exists a constant C such that

$$\frac{b_e(t)-b(t)}{\sqrt{\theta}} \leq C.$$

□

5. Appendix 1: Proofs of lemmas

Proof of Lemma 2.1:
According to the early exercise premium formula, we have,

$$P(t,x) = P_e(t,x) + e(T-t,x)$$

and

$$e(\theta, x) = \mathbb{E}\left\{\int_0^\theta e^{-rs}\Phi(t+s, xS_s^1)1_{\{xS_s^1 < b(t+s)\}}ds\right\},$$

with

$$\Phi(t,x) = rK - \delta x - \int_{y>0} P(t, xe^y) - (K-xe^y)\,\nu(dy).$$

Notice that Φ is a continuous function and $\|\Phi'_x\|_\infty \leq \delta + \int_{y>0} e^y \nu(dy)$.
1) It is obvious that $0 \leq e(\theta,x) \leq \theta rK = O(\theta)$, since $0 \leq \Phi(t,x)1_{\{x<b(t+s)\}} \leq rK$.

2) For all random variable X, we denote by p_X its density, we thus have for all fixed $s \in [0,\theta]$,

$$\begin{aligned} p_{-\tilde{X}_s}(x) &= p_{-\mu s-\sigma B_s} * p_{-Z_s}(x) = \frac{1}{\sqrt{s}}\frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{(-x+\mu s-u)^2}{2\sigma^2 s}} p_{-Z_s}(u)du \\ &\leq C^{te}\frac{1}{\sqrt{s}}. \end{aligned} \tag{24}$$

We can state

$$\begin{aligned} \frac{\partial e}{\partial x}(\theta,x) &= \mathbb{E}\left\{\int_0^\theta e^{-rs} S_s^1 \Phi'_x(t+s, xS_s^1) 1_{\{xS_s^1 < b(t+s)\}} ds\right\} \\ &\quad - \int_0^\theta \frac{\Phi(t+s,b(t+s))}{x} p_{-\tilde{X}_s}\left(\ln(\frac{x}{b(t+s)})\right) ds. \end{aligned} \tag{25}$$

Then, we have

$$\begin{aligned} \left|\frac{\partial e}{\partial x}(\theta,x)\right| &\leq \left|\mathbb{E}\left\{\int_0^\theta e^{-rs} S_s^1 \Phi'_x(t+s, xS_s^1) 1_{\{xS_s^1 < b(t+s)\}} ds\right\}\right| \\ &\quad + \left|\int_0^\theta \frac{\Phi(t+s,b(t+s))}{x} p_{-\tilde{X}_s}\left(\ln(\frac{x}{b(t+s)})\right) ds\right| \\ &\leq \|\Phi'_x\|_\infty \frac{b(T)}{x}\theta + \left|\int_0^\theta \frac{\Phi(t+s,b(t+s))}{x} p_{-\tilde{X}_s}\left(\ln(\frac{x}{b(t+s)})\right) ds\right|. \end{aligned}$$

According to inequality (24) , we also have

$$\begin{aligned} &\left|\int_0^\theta \frac{\Phi(t+s,b(t+s))}{x} p_{-\tilde{X}_s}\left(\ln(\frac{x}{b(t+s)})\right) ds\right| \\ &\quad \leq C^{te}\left|\int_0^\theta \frac{\Phi(t+s,b(t+s))}{x\sqrt{s}} ds\right| \\ &\quad = \frac{C^{te}}{x}\theta\left|\int_0^1 \frac{\Phi(t+\theta u, b(t+\theta u))}{\sqrt{\theta u}} du\right| \\ &\quad \leq \frac{C^{te}}{x}\sqrt{\theta}\sup_{t\leq u\leq t+\theta}|\Phi(u,b(u))|\int_0^1 \frac{1}{\sqrt{u}} du \\ &\quad \leq \frac{C^{te}}{x}\sqrt{\theta}\sup_{T-\theta\leq u\leq T}|\Phi(u,b(u))|\int_0^1 \frac{1}{\sqrt{u}} du. \end{aligned}$$

However, thanks to the continuity of $b(u)$ and of $\Phi(t,x)$, we have $\lim_{\theta\to 0}\sup_{T-\theta\leq u\leq T}|\Phi(u,b(u))| = |\Phi(T,b(T))| = 0$. Therefore, we conclude that $\left|\frac{\partial e}{\partial x}(\theta,x)\right| = \frac{1}{x}o(\sqrt{\theta})$.

3) Using the previous point, we have $\left|\frac{\partial e}{\partial x}(\theta,x)\right| = o(\sqrt{\theta})$, then for all $x \leq b_e(t)\wedge b(T)$ and

θ small enough , we have

$$\begin{aligned} 0 \leq 1 + \frac{\partial P}{\partial x}(t,x) &\leq \left(1 + \frac{\partial P_e}{\partial x}(t,x)\right) + o(\sqrt{\theta}) \\ &= 1 - \mathbb{E}\left[(e^{X_\theta})1_{\{X_\theta < \ln \frac{K}{x}\}}\right] + o(\sqrt{\theta}) \\ &\leq 1 - \mathbb{E}\left[(e^{X_\theta})1_{\{X_\theta < \ln \frac{K}{b_e(t)\wedge b(T)}\}}\right] + o(\sqrt{\theta}). \end{aligned}$$

If $b(T) = K$, then for $x \leq b_e(t)$,

$$1 - \mathbb{E}\left[(e^{X_\theta})1_{\{X_\theta < \ln \frac{K}{b_e(t)\wedge b(T)}\}}\right] = \mathbb{P}(B_\theta \geq \sqrt{\theta}\alpha(\theta)) - \mathbb{E}\left(\sigma B_\theta 1_{\{B_\theta < \sqrt{\theta}\alpha(\theta)\}}\right) + o(\sqrt{\theta}).$$

Since $\alpha(\theta) \underset{\theta\to 0}{\longrightarrow} \infty$, we have

$$\left|\mathbb{E}\left(\sigma B_\theta 1_{\{B_\theta < \sqrt{\theta}\alpha(\theta)\}}\right)\right| = \sigma\sqrt{\theta}\mathbb{E}\left(B_1 1_{\{B_1 < \alpha(\theta)\}}\right) = o(\sqrt{\theta}),$$

and using equivalencies (22) and (23), we also have

$$\mathbb{P}\left(B_1 \geq \alpha(\theta)\right) \leq \frac{e^{-\frac{\alpha^2(\theta)}{2}}}{\alpha(\theta)} \leq C\theta\alpha^2(\theta) = O(\theta|\ln\theta|) = o(\sqrt{\theta}).$$

If $b(T) < K$, then for θ small enough $b(T) < b_e(t)$ and

$$\begin{aligned} & 1 - \mathbb{E}\left[(e^{X_\theta})1_{\{X_\theta < \ln\left(\frac{K}{b(T)}\right)\}}\right] \\ &= \mathbb{P}\left(B_1 \geq \frac{1}{\sqrt{\theta}}\ln(\frac{K}{b(T)})\right) + \sigma\sqrt{\theta}\mathbb{E}\left(B_1 1_{\left\{B_1 \geq \frac{\ln \frac{K}{b(T)}}{\sqrt{\theta}}\right\}}\right) + o(\sqrt{\theta}) \\ &\leq \sqrt{\theta}\left(\frac{1}{\ln(\frac{K}{b(T)})} + \sigma\right)\frac{e^{-\frac{1}{2\theta}\ln^2(\frac{K}{b(T)})}}{\sqrt{2\pi}} + o(\sqrt{\theta}) = o(\sqrt{\theta}). \end{aligned}$$

□

Proof of Lemma 2.2:
Let be $x \in (b(t), b(T))$, then the variational inequality gives, for almost $u \in (b(t), x)$,

$$\begin{aligned} \frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) &\geq rP(t,u) - (r-\delta)u\frac{\partial P}{\partial x}(t,u) \\ &\quad - \int\left(P(t,ue^y) - P(t,u) - u(e^y - 1)\frac{\partial P}{\partial x}(t,u)\right)\nu(dy). \end{aligned}$$

Notice that $P(t,u) \geq K-u$, thus

$$\begin{aligned}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) &\geq r(K-u)+(r-\delta)\,u \\ &\quad -\int \left(P(t,ue^y)-(K-u)+u(e^y-1)\right)\nu(dy) \\ &\quad -u\left(\frac{\partial P}{\partial x}(t,u)+1\right)\left((r-\delta)-\int(e^y-1)\nu(dy)\right) \qquad (26)\end{aligned}$$

(27)

And thanks to Lemma 2.1, we also have, for all $b(0) \leq u \leq x \leq b_e(t) \wedge b(T)$,

$$\frac{\partial P}{\partial x}(t,u)+1 = o(\sqrt{\theta}),$$

independently of u, therefore,

$$\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) \geq rK-\delta u-\int \left(P(t,ue^y)-(K-ue^y)\right)\nu(dy)+o(\sqrt{\theta}). \qquad (28)$$

As the right hand side of equality (28) is non increasing in u, we obtain

$$\inf_{b(t)\leq u\leq x}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u) \geq rK-\delta x-\int \left(P(t,xe^y)-(K-xe^y)\right)\nu(dy)+o(\sqrt{\theta}). \qquad (29)$$

Notice that

$$\begin{aligned}\int P(t,xe^y)\nu(dy) &= P_e(t,xe^y)+e(\theta,xe^y) \\ &= \int \mathbb{E}(K-xe^y e^{X_\theta})^+\nu(dy)+o(\sqrt{\theta}) \\ &= \int \mathbb{E}\left(K-xe^y(1+\sigma B_\theta)\right)^+\nu(dy)+o(\sqrt{\theta}) \\ &= \int \mathbb{E}\left((K-xe^y)-xe^y\sigma B_\theta\right)^+\nu(dy)+o(\sqrt{\theta}).\end{aligned}$$

We now consider the integral $\int P(t,xe^y)\nu(dy)$ over the sets $\{y<\ln(\frac{K}{b(T)})\}$, $\{\ln(\frac{K}{b(T)})<y\}$ and $\{y=\ln(\frac{K}{b(T)})\}$. Then, on the set $\{y<\ln(\frac{K}{b(T)})\}$, we have

$$\begin{aligned}
\int_{\{y<\ln(\frac{K}{b(T)}\}} P(t,xe^y)\nu(dy) &= \int_{\{y<\ln(\frac{K}{b(T)}\}} \mathbb{E}\left(K-xe^y)-xe^y\sigma B_\theta\right)^+\nu(dy)+o(\sqrt{\theta}) \\
&= \int_{\{y<\ln(\frac{K}{b(T)}\}} (K-xe^y)\mathbb{P}(xe^y\sigma B_\theta<(K-xe^y))\nu(dy) \\
&\quad -\int_{\{y<\ln(\frac{K}{b(T)}\}} xe^y\sigma\mathbb{E}\left(B_\theta 1_{\{xe^y\sigma B_\theta<(K-xe^y)\}}\right)\nu(dy)+o(\sqrt{\theta}) \\
&\leq \int_{\{y<\ln(\frac{K}{b(T)}\}} (K-xe^y)\nu(dy) - x\sigma\sqrt{\theta}\int_{\{y<\ln(\frac{K}{b(T)}\}} e^y\mathbb{E}\left(B_1 1_{\{B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{x}e^{-y}-1)\}}\right)\nu(dy)+o(\sqrt{\theta}).
\end{aligned}$$

For all $y<\ln(\frac{K}{b(T)})$, we have $\frac{K}{x}e^{-y}-1>\frac{K}{b(T)}e^{-y}-1>0$, therefore

$$\begin{aligned}
0 &\leq -\mathbb{E}\left(B_1 1_{\{B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{x}e^{-y}-1)\}}\right)=\mathbb{E}\left(B_1 1_{\{B_1\geq\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{x}e^{-y}-1)\}}\right) \\
&\leq \mathbb{E}\left(B_1 1_{\{B_1\geq\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{b(T)}e^{-y}-1)\}}\right)\longrightarrow_{\theta\to 0} 0.
\end{aligned}$$

By the dominated convergence we obtain,

$$\int_{\{y<\ln(\frac{K}{b(T)}\}} P(t,xe^y)\nu(dy) \leq \int_{\{y<\ln(\frac{K}{b(T)}\}} (K-xe^y)\nu(dy)+o(\sqrt{\theta}). \tag{30}$$

On the set $\{y>\ln(\frac{K}{b(T)})\}$, we have $K<b(T)e^y$, therefore

$$\begin{aligned}
\int_{\{y>\ln(\frac{K}{b(T)}\}} P(t,xe^y)\nu(dy) &= \int_{\{y>\ln(\frac{K}{b(T)}\}} \mathbb{E}\left((K-xe^y)-xe^y\sigma B_\theta\right)^+\nu(dy)+o(\sqrt{\theta}) \\
&\leq \int_{\{y>\ln(\frac{K}{b(T)}\}} \mathbb{E}\left[\left(b(T)e^y-xe^y-xe^y\sigma\sqrt{\theta}B_1\right)1_{\{xe^y\sigma\sqrt{\theta}B_1<(K-xe^y)\}}\right]\nu(dy)+o(\sqrt{\theta}) \\
&= (b(T)-x)\int_{\{y>\ln(\frac{K}{b(T)}\}} e^y\mathbb{P}\left(B_1<\frac{1}{\sigma\sqrt{\theta}}\left(\frac{K}{x}e^{-y}-1\right)\right)\nu(dy) \\
&\quad -\sqrt{\theta}x\int_{\{y>\ln(\frac{K}{b(T)}\}} e^y\sigma\mathbb{E}\left(B_1 1_{\{B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{x}e^{-y}-1)\}}\right)\nu(dy)+o(\sqrt{\theta})
\end{aligned}$$

Notice that for all $y>\ln(\frac{K}{b(T)})$, we have $\frac{1}{\sigma\sqrt{\theta}}\left(\frac{K}{x}e^{-y}-1\right)\leq\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{b(t)}e^{-y}-1)\to-\infty$, thus

$$\begin{aligned}
\mathbb{P}(B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{x}e^{-y}-1)) &\leq \mathbb{P}(B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{b(t)}e^{-y}-1)) \\
&\underset{\theta\to 0}{\longrightarrow} 0,
\end{aligned}$$

and

$$\mathbb{E}\left(|B_1|1_{\{B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{x}e^{-y}-1)\}}\right) \leq \mathbb{E}\left(|B_1|1_{\{B_1<\frac{1}{\sigma\sqrt{\theta}}(\frac{K}{b(t)}e^{-y}-1)\}}\right) \underset{\theta\to 0}{\longrightarrow} 0.$$

Therefore, by dominated convergence, we obtain

$$\int_{\{y>\ln(\frac{K}{b(T)}\}} e^y \mathbb{P}(B_1 < \frac{1}{\sigma\sqrt{\theta}}(\frac{K}{b(t)}e^{-y}-1))\nu(dy) \underset{\theta\to 0}{\longrightarrow} 0$$

and

$$-\sqrt{\theta}x\int_{\{y>\ln(\frac{K}{b(T)}\}} e^y \sigma\mathbb{E}\left(B_1 1_{\{xe^y\sigma\sqrt{\theta}B_1<(K-xe^y)\}}\right)\nu(dy) = o(\sqrt{\theta})$$

Consequently, if we denote by $\epsilon(\theta) = \int_{\{y>\ln(\frac{K}{b(T)}\}} e^y \mathbb{P}(B_1 < \frac{1}{\sigma\sqrt{\theta}}(\frac{K}{b(t)}e^{-y}-1))\nu(dy)$, we obtain

$$\int_{\{y>\ln(\frac{K}{b(T)}\}} P(t,xe^y)\nu(dy) \leq (b(T)-x)\epsilon(\theta) + o(\sqrt{\theta}), \tag{31}$$

with $\epsilon(\theta) \underset{\theta\to 0}{\longrightarrow} 0$.

Finally, on the set $\{y = \ln(\frac{K}{b(T)})\}$, we have

$$\begin{aligned}
\int_{\{\ln(\frac{K}{b(T)})\}} P(t,xe^y)\nu(dy) &= \int_{\{\ln(\frac{K}{b(T)})\}} \mathbb{E}\left((K-xe^y) - xe^y\sigma B_\theta\right)^+ \nu(dy) + o(\sqrt{\theta}) \\
&= \int_{\{\ln(\frac{K}{b(T)})\}} (K-xe^y)\nu(dy) + \int_{\{\ln(\frac{K}{b(T)})\}} \mathbb{E}\left(xe^y\sigma B_\theta - (K-xe^y)\right)^+ \nu(dy) + o(\sqrt{\theta}) \\
&= \int_{\{\ln(\frac{K}{b(T)})\}} (K-xe^y)\nu(dy) + \int_{\{\ln(\frac{K}{b(T)})\}} xe^y \mathbb{E}\left(\sigma B_\theta - (\frac{K}{x}e^{-y}-1)\right)^+ \nu(dy) \\
&= \int_{\{\ln(\frac{K}{b(T)})\}} (K-xe^y)\nu(dy) + \frac{xK}{b(T)}\nu\left\{\ln\left(\frac{K}{b(T)}\right)\right\}\mathbb{E}\left(\sigma B_\theta - \left(\frac{b(T)}{x}-1\right)\right)^+ \\
&\leq \int_{\{\ln(\frac{K}{b(T)})\}} (K-xe^y)\nu(dy) + K\nu\left\{\ln\left(\frac{K}{b(T)}\right)\right\}\mathbb{E}\left(\sigma B_\theta - \ln\left(\frac{b(T)}{x}\right)\right)^+
\end{aligned}$$

We have thus proved that

$$\begin{aligned}
\int P(t,xe^y)\nu(dy) \leq & \int_{\{y\leq\ln(\frac{K}{b(T)}\}} (K-xe^y)\nu(dy) + K\nu\left\{\ln\left(\frac{K}{b(T)}\right)\right\}\mathbb{E}\left(\sigma B_\theta - \ln\left(\frac{b(T)}{x}\right)\right)^+ \\
& + (b(T)-x)\,\epsilon(\theta) + o(\sqrt{\theta}),
\end{aligned}$$

Coming back to inequality (29), we obtain

$$\begin{aligned}
&\int P(t,xe^y)-(K-xe^y)\nu(dy)\\
&\leq \quad -\int_{\{y>\ln(\frac{K}{b(T)}\}}(K-xe^y)\nu(dy)+K\nu\left\{\ln\left(\frac{K}{b(T)}\right)\right\}\mathbb{E}\left(\sigma B_\theta-\ln\left(\frac{b(T)}{x}\right)\right)^+\\
&\qquad +(b(T)-x)\,\epsilon(\theta)+o(\sqrt{\theta}).
\end{aligned}$$

Finally, since $rK=\delta b(T)+\int(b(t)e^y-K)^+\nu(y)$, we have

$$\begin{aligned}
&\inf_{b(t)\leq u\leq x}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u)\\
&\geq \quad rK-\delta x-\int\left(P(t,xe^y)-(K-xe^y)\right)\nu(dy)+o(\sqrt{\theta})\\
&\geq (b(T)-x)\left(\delta+\int_{\{y>\ln\frac{K}{b(T)}\}}e^y\nu(dy)+\epsilon(\theta)\right)-K\nu\left\{\ln\frac{K}{b(T)}\right\}\mathbb{E}\left(\sigma B_\theta-\ln(\frac{b(T)}{x})\right)^+\!+o(\sqrt{\theta}).
\end{aligned}$$

We note $\alpha=\frac{\nu\{\ln\left(\frac{K}{b(T)}\right)\}}{\delta}\frac{K}{b(T)}$ and $\bar{\delta}=\delta+\int_{\{y>\ln(\frac{K}{b(T)}\}}e^y\nu(dy)$, we then have for all u and all x such that $b(t)\leq u\leq x<b(T)$

$$\begin{aligned}
&\inf_{b(t)\leq u\leq x}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u)\\
&\geq \quad b(T)\bar{\delta}\left(\frac{(b(T)-x)}{b(T)}-\alpha\mathbb{E}\left(\sigma B_\theta-\ln(\frac{b(T)}{x})\right)^+\right)-(b(T)-x)\,\epsilon(\theta)+o(\sqrt{\theta}).
\end{aligned}$$

□

Remark 2. *The expression* $\inf_{b(t)<u<x}\frac{u^2\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,u)$ *is justified thanks to the smoothness of* P *in the continuation region which can be proved thanks to PDE arguments (see for instance* [2]*). Nevertheless, we will only need this lower bound of the second derivative in the distribution sense (*$\frac{\partial^2 P}{\partial x^2}(t,du)$*).*

Appendix 2: A study of $v_{\lambda,\beta}$

***Lemma** 3.1 : There exists* $y_{\lambda,\beta}\in\left(0,(1+\lambda\beta(2+e^\lambda)\right)$ *such that such that*

$$\forall y<-y_{\lambda,\beta},\quad v_{\lambda,\beta}(y)=0.$$
$$y_{\lambda,\beta}=-\inf\{x\in\mathbb{R}\mid v_{\lambda,\beta}(x)>0\}.$$

Proof of Lemma 3.1:
We have

$$v_{\lambda,\beta}(y)=\sup_{\tau\in\mathcal{T}_{0,1}}\left(I_0(\tau)+I_1(\tau)\right),$$

with

$$I_0(\tau) = \mathbb{E}\left(e^{\lambda\tau} 1_{\{\hat{N}_\tau=0\}} \int_0^\tau f_{\lambda\beta}(y+B_s)ds\right),$$

and

$$I_1(\tau) = \beta\mathbb{E}\left(e^{\lambda\tau} 1_{\{\hat{N}_\tau=1\}} \left((y+B_\tau)^+ - (y+B_{\hat{T}_1})^+\right)\right).$$

we will study $I_0(\tau)$ and $I_1(\tau)$. First of all, we note that the process $(M_t^0)_{t\geq 0}$ defined by $M_t^0 = e^{\lambda t} 1_{\{\hat{N}_t=0\}}$ is a non negative martingale with $M_0^0 = 1$. Under the probability $\mathbb{P}^0$ with density M_t^0 on $\mathcal{F}_t$, it is straightforward to check that $(B)_{t\geq 0}$ remains a $\mathbb{F}$-Brownian motion. We have if $y \leq 0$,

$$\begin{aligned}
I_0(\tau) &= \mathbb{E}^0\left(\int_0^\tau f_{\lambda\beta}(y+B_s)ds\right) \\
&= \mathbb{E}^0\left(y\tau + \int_0^\tau B_s ds + \lambda\beta\int_0^\tau (y+B_s)^+ ds\right) \\
&\leq y\mathbb{E}^0(\tau) + (1+\lambda\beta)\mathbb{E}^0\left(\int_0^\tau B_s^+ ds\right) \\
&\leq y\mathbb{E}^0(\tau) + (\lambda\beta+1)\mathbb{E}^0\left(\int_0^\tau \mathbb{E}^0\left(B_\tau^+ \mid \mathcal{F}_s\right) ds\right).
\end{aligned}$$

Notice that, for $\tau \in \mathcal{T}_{0,1}$,

$$\begin{aligned}
\mathbb{E}^0\left(\int_0^\tau \mathbb{E}^0\left(B_\tau^+ \mid \mathcal{F}_s\right) ds\right) &= \mathbb{E}^0\left(\int_0^1 1_{\{\tau>s\}}\mathbb{E}^0\left(B_\tau^+ \mid \mathcal{F}_s\right) ds\right) \\
&= \int_0^1 \mathbb{E}^0\left(1_{\{\tau>s\}}\mathbb{E}^0\left(B_\tau^+ \mid \mathcal{F}_s\right)\right) ds \\
&= \mathbb{E}^0\left(\tau B_\tau^+\right) \\
&\leq \mathbb{E}^0\left(\frac{\tau^2 + B_\tau^2}{2}\right) \leq \mathbb{E}^0(\tau),
\end{aligned}$$

where, we used $0 \leq \tau \leq 1$, for the last inequality. We then have

$$I_0(\tau) \leq (y+\lambda\beta+1)\mathbb{E}^0(\tau).$$

For the study of $I_1(\tau)$, let us introduce the martingale $(M_t^1)_{0\leq t\leq 1}$ defined by

$$\begin{aligned}
M_t^1 &= \mathbb{E}\left(e^\lambda 1_{\{\hat{N}_1=1\}} \mid \mathcal{F}_t\right) \\
&= \mathbb{E}\left(e^\lambda 1_{\{\hat{N}_1=1,\hat{N}_t=0\}} \mid \mathcal{F}_t\right) + \mathbb{E}\left(e^\lambda 1_{\{\hat{N}_1=1,\hat{N}_t=1\}} \mid \mathcal{F}_t\right) \\
&= 1_{\{\hat{N}_t=0\}} e^\lambda \mathbb{P}(\hat{N}_1 - \hat{N}_t = 1) + 1_{\{\hat{N}_t=1\}} e^\lambda \mathbb{P}(\hat{N}_1 - \hat{N}_t = 0) \\
&= 1_{\{\hat{N}_t=0\}} \lambda(1-t)e^{\lambda t} + 1_{\{\hat{N}_t=1\}} e^{\lambda t}.
\end{aligned}$$

Under the probability $\mathbb{P}^1$ with density M_t^1/λ on $\mathcal{F}_t$, it is straightforward to check that $(B_t)_{0\le t\le 1}$ remains a $\mathbb{F}$-Brownian motion. We have for $y<0$,

$$\begin{aligned} I_1(\tau) &= \lambda\beta\mathbb{E}^1\left((y+B_\tau)^+-(y+B_{\hat{T}_1\wedge\tau})^+\right)\\ &\le \lambda\beta\mathbb{E}^1\left((y+B_\tau)^+\right)\\ &\le \lambda\beta\mathbb{E}^1\left(B_\tau 1_{\{B_\tau>-y\}}\right)\\ &\le \lambda\beta\mathbb{E}^1\left(B_\tau^2/|y|\right)=\lambda\beta\mathbb{E}^1\left(\tau\right)/|y|. \end{aligned}$$

Using the two upper bound of $I_0(\tau)$ and $I_1(\tau)$, we obtain

$$\begin{aligned} v_{\lambda,\beta}(y) &\le \sup_{\tau\in\mathcal{T}_{0,1}}\left((y+\lambda\beta+1)\mathbb{E}^0(\tau)+\frac{\lambda\beta}{|y|}\mathbb{E}^1(\tau)\right)\\ &= \sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}\left((y+\lambda\beta+1)\tau M^0(\tau)+\frac{\beta}{|y|}\tau M^1(\tau)\right)\\ &= \sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}\left((y+\lambda\beta+1)\tau e^{\lambda\tau}1_{\{\hat{N}_\tau=0\}}+\frac{\beta}{|y|}\tau\left(1_{\{\hat{N}_\tau=0\}}\lambda(1-\tau)e^{\lambda\tau}+1_{\{\hat{N}_\tau=1\}}e^{\lambda\tau}\right)\right)\\ &\le \sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}\left(f(\tau,\hat{N}_\tau)\right), \end{aligned}$$

with

$$f(t,x)=1_{\{x=0\}}te^{\lambda t}\left(y+1+\lambda\beta(1+\frac{1}{|y|})\right)+1_{\{x=1\}}\beta te^{\lambda t}/|y|.$$

Notice that

$$\sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}\left(f(\tau,\hat{N}_\tau)\right)=\sup_{\tau\in\mathcal{T}_{0,1}(\hat{N})}\mathbb{E}\left(f(\tau,\hat{N}_\tau)\right),$$

where $\mathcal{T}_{0,1}(\hat{N})$ denotes the set of the stopping times of the natural completed filtration of the process $(\hat{N}_t)_{t\ge0}$, with values in $[0,1]$.

Then, if $\tau\in\mathcal{T}_{0,1}(\hat{N})$, there exists , thanks to Lemma 5.1, $t_0\in[0,1]$, such that

$$\tau\wedge\hat{T}_1=t_0\wedge\hat{T}_1.$$

we then have

$$\begin{aligned} \mathbb{E}\left(\tau e^{\lambda\tau}1_{\{\hat{N}_\tau=0\}}\right) &= \mathbb{E}\left(\tau e^{\lambda\tau}1_{\{\hat{T}_1>\tau\}}\right)\\ &= t_0e^{\lambda t_0}\mathbb{P}(\hat{T}_1>\tau)\\ &= t_0, \end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left(\tau e^{\lambda\tau}1_{\{\hat{N}_\tau=1\}}\right) &\leq \mathbb{E}\left(\tau e^{\lambda\tau}1_{\{\hat{T}_1\leq\tau\}}\right)\\ &= \mathbb{E}\left(\tau e^{\lambda\tau}1_{\{\hat{T}_1\leq t_0\}}\right)\\ &\leq e^{\lambda}\mathbb{P}(\hat{T}_1\leq t_0)\\ &= e^{\lambda}(1-e^{-\lambda t_0})\leq \lambda e^{\lambda}t_0.\end{aligned}$$

we deduce that

$$\sup_{\tau\in\mathcal{T}_{0,1}(\hat{N})}\mathbb{E}\left(f(\tau,\hat{N}_\tau)\right) \leq \sup_{0\leq t_0\leq 1}\left(t_0\left(y+1+\lambda\frac{\beta(2+e^{\lambda})}{|y|}\right)\right).$$

The right hand side of this equation will be equal to 0 if

$$y+1+\lambda\frac{\beta(2+e^{\lambda})}{|y|}\leq 0,$$

and particularly, if $y\leq -\left(1+\lambda\beta(2+e^{\lambda})\right)$, then

$$-y_{\lambda,\beta}\geq -\left(1+\lambda\beta(2+e^{\lambda})\right).$$

To prove $-y_{\lambda,\beta}<0$, we consider $y=0$. Since for all stopping time τ,

$$\mathbb{E}\left(e^{\lambda\tau}1_{\{\hat{N}_\tau=0\}}\int_0^\tau\lambda\beta(y+B_s)^+ds+\beta e^{\lambda\tau}1_{\{\hat{N}_\tau=1\}}\left((y+B_\tau)^+-(y+B_{\hat{T}_1})^+\right)\right)\geq 0.$$

we have

$$\upsilon_{\lambda,\beta}(0)\geq\sup_{\tau\in\mathcal{T}_{0,1}}\mathbb{E}\int_0^\tau B_s ds=\upsilon_0(0),$$

and it is proved in [14] or [4], Proposition 2.2.4. $\upsilon_0(0)>0$. □

Lemma 5.1. *Let $N=(N_t)_{t\geq 0}$ a homogenous Poisson process with intensity λ, and T_1 its first jump time. if τ is a stopping time of the natural completed filtration of N such that $\tau\leq T_1$ a.s., then, $\tau=T_1$ a.s., or there exists $t_0\geq 0$, such that $\tau=t_0\wedge T_1$ a.s.*

Proof We denote by $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ the natural completed filtration of N. First of all, notice that for all $t\geq 0$ and $A\in\mathcal{F}_t$,

$$\mathbb{P}(A\mid N_t=0)\in\{0,1\}.$$

Indeed, the A having this property form a sub σ-algebra of $\mathcal{F}_t$ which contains the events of the form $\{N_s=n\}$, with $0\leq s\leq t$ and $n\in\mathbb{N}$.
Now, let τ be a $\mathbb{F}$-stopping time. We have for all $t\geq 0$, $\mathbb{P}(\tau>t\mid N_t=0)\in\{0,1\}$. We set

$$I=\{t\in[0,+\infty[\mid\mathbb{P}(\tau>t\mid N_t=0)=0\}.$$

Notice that $t\in I$ if and only if $\mathbb{P}(\tau>t,T_1>t)=0$, or

$$t\in I\Leftrightarrow\mathbb{P}\left(\tau\wedge T_1\leq t\right)=1.$$

If $\tau \leq T_1$ a.s. and if $\mathbb{P}(\tau < T_1) > 0$, there exists $s > 0$ (rational number) such that $\mathbb{P}(\tau \leq s, s < T_1) > 0$, hence $\mathbb{P}(\tau \leq s \mid N_s = 0) > 0$, and $\mathbb{P}(\tau > s \mid N_s = 0) = 0$. We deduce that I is non empty and we can write

$$I = [t_0, +\infty[, \text{ avec } t_0 = \inf\{t \geq 0 \mid \mathbb{P}(\tau > t \mid N_t = 0) = 0\}.$$

We then have $\tau \wedge T_1 \leq t_0$ a.s., hence $\tau \leq t_0 \wedge T_1$. Moreover, for $s < t_0$, we have $\mathbb{P}(\tau > s \mid N_s = 0) = 1$ and $\mathbb{P}(\tau \leq s \mid N_s = 0) = 0$, hence $\mathbb{P}(\tau \leq s, s < T_1) = 0$.Therefore, $\mathbb{P}(\tau < t_0 \wedge T_1) = 0$ and consequently $\tau = t_0 \wedge T_1$ a.p. □

***Lemma** 3.2 : For all $x > y_{\lambda,\beta}$, we have*

$$C(x) > 0.$$

.

Proof We have $v_{\lambda,\beta}(-y_{\lambda,\beta}) = 0$, considering the stopping time $\tau = 1$, we obtain

$$\mathbb{E}\left[e^{\lambda}1_{\{\hat{N}_1=0\}}\int_0^1 f_{\lambda\beta}(B_s - y_{\lambda,\beta})ds + \beta e^{\lambda}1_{\{\hat{N}_1=1\}}\left((B_1 - y_{\lambda,\beta})^+) - (B_{\hat{T}_1} - y_{\lambda,\beta})^+\right)\right] \leq 0.$$

However, we have, using the independence between $\hat{N}$ and B,

$$\begin{aligned}\mathbb{E}\left[e^{\lambda}1_{\{\hat{N}_1=0\}}\int_0^1 f_{\lambda\beta}(B_s - y_{\lambda,\beta})ds\right] &= e^{\lambda}\mathbb{P}(\hat{N}_1 = 0)\left(-y_{\lambda,\beta} + \lambda\beta\mathbb{E}\int_0^1 (B_s - y_{\lambda,\beta})^+ds\right)\\ &= -y_{\lambda,\beta} + \lambda\beta\mathbb{E}\int_0^1 (B_s - y_{\lambda,\beta})^+ds. \quad (32)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathbb{E}\left[\beta e^{\lambda}1_{\{\hat{N}_1=1\}}\left((B_1 - y_{\lambda,\beta})^+) - (B_{\hat{T}_1} - y_{\lambda,\beta})^+\right)\right]\\ &= \beta e^{\lambda}\mathbb{P}(\hat{N}_1 = 1)\left[\mathbb{E}(B_1 - y_{\lambda,\beta})^+ - \mathbb{E}\left((B_{\hat{T}_1} - y_{\lambda,\beta})^+|\hat{N}_1 = 1\right)\right]\\ &= \beta\lambda\left[\mathbb{E}(B_1 - y_{\lambda,\beta})^+ - \mathbb{E}\left((B_{\hat{T}_1} - y_{\lambda,\beta})^+|\hat{T}_1 \leq 1\right)\right].\end{aligned}$$

Noticing that $\lambda\beta = \lambda\beta$ and that conditionally to $\{\hat{T}_1 \leq 1\}$, $\hat{T}_1$ is uniformly distributed on $[0, 1]$, we obtain

$$\begin{aligned}&\mathbb{E}\left[\beta e^{\lambda}1_{\{\hat{N}_1=1\}}\left((B_1 - y_{\lambda,\beta})^+) - (B_{\hat{T}_1} - y_{\lambda,\beta})^+\right)\right]\\ &= \lambda\beta\left[\mathbb{E}(B_1 - y_{\lambda,\beta})^+ - \mathbb{E}\left(\int_0^1 (B_s - y_{\lambda,\beta})^+ds\right)\right]. \quad (33)\end{aligned}$$

Combining (32) and (33), we have

$$-y_{\lambda,\beta} + \lambda\beta\mathbb{E}(B_1 - y_{\lambda,\beta})^+ = -C(y_{\lambda,\beta}) \leq 0$$

To conclude the proof, we use the strict increasing of C, hence for all $x > y_{\lambda,\beta}$, we have

$$C(x) > C(y_{\lambda,\beta}) \geq 0.$$

□

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