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Renormalized Energy and Asymptotic Expansion of Optimal Logarithmic Energy on the Sphere

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Abstract

We study the Hamiltonian of a two-dimensional Coulomb system of n repelling points confined by an external field verifying the weak growth assumption of Hardy and Kuijlaars as in [16, 17]. We prove an asymptotic expansion (as $n \rightarrow +\infty$) for the minimum of this Hamiltonian using the Gamma-Convergence's method of Sandier and Serfaty in [28] and depending on the minimum of a “renormalized energy” W introduced in [27]. We connect our result with the next-order term for optimal logarithmic energy on \mathbb{S}^2 to prove the conjecture of Rakhmanov, Saff and Zhou in [24] about the existence of this term for which we find an upper bound. Finally we prove the equivalence between the conjecture of Brauchart, Hardin and Saff in [7] about the value of this coefficient and the conjecture of the global minimality of the triangular lattice for W among configurations of fixed average density.

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1 Introduction

Let $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ be a configuration subjected to a logarithmic potential and confined by an external field V . We define the Hamiltonian of this system, also called “Coulomb gas”, by

$$w_n(x_1, \dots, x_n) := - \sum_{i \neq j}^n \log \|x_i - x_j\| + n \sum_{i=1}^n V(x_i)$$

where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^2 and V is a potential satisfying some growth assumption and some properties detailed below. The discrete energy w_n is linked to the following continuous problem of logarithmic potential theory : find a probability measure μ_V on \mathbb{R}^2 which minimizes

$$I_V(\mu) := \iint_{\mathbb{R}^2} \left(\frac{V(x)}{2} + \frac{V(y)}{2} - \log \|x - y\| \right) d\mu(x) d\mu(y).$$

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This kind of “equilibrium problem” dates back to Gauss and has been studied among others by Frostman in [15] and by Saff and Totik in [25]. The classical growth assumption for V

$$\lim_{\|x\| \rightarrow +\infty} \{V(x) - 2 \log \|x\|\} = +\infty \quad (1.1)$$

leads to a unique measure μ_V with compact support. Recently, Hardy and Kuijlaars gave in [16, 17] the following weak assumption

$$\liminf_{\|x\| \rightarrow +\infty} \{V(x) - 2 \log \|x\|\} > -\infty \quad (1.2)$$

which will be used throughout this paper, and more precisely

$$\lim_{\|x\| \rightarrow +\infty} \{V(x) - 2 \log \|x\|\} = L \in \mathbb{R}, \quad (1.3)$$

so that the support of μ_V , which exists and is unique, can be unbounded. Moreover Bloom, Levenberg and Wielonsky proved in [3] classical Frostman inequalities for this case.

Sandier and Serfaty introduced in [27] a Coulombian interaction for an infinity of points in the plane with a uniform neutralized background called “renormalized energy” W . This energy can be seen as an energy of planar interactions between vortices in the Ginzburg-Landau theory of superconductors. In [28] a “splitting formula” is used to connect w_n and W in case of compact support for μ_V . Moreover the authors give an asymptotic expansion as $n \rightarrow +\infty$, of $w_n(x_1, \dots, x_n)$, when (x_1, \dots, x_n) is a minimizer, which is equal – up to lower order terms – to $n^2 I_V(\mu_V) - \frac{n}{2} \log n$ plus a term of order n which depends on μ_V and on the minimum of W among the configurations of fixed density one. In this paper we prove the very same formula when V satisfies the weak growth assumption (1.3). We reuse the method from [27, 28] and we combine it with the compactification approach of [16, 17, 3] to connect this asymptotic expansion with the equilibrium problem on the sphere by an inverse stereographic projection.

More precisely, on the unit sphere, Brauchart, Hardin and Saff, in [7], give important conjectures about the asymptotic of the logarithmic energy¹

$$E_{\log}(y_1, \dots, y_n) := - \sum_{i \neq j}^n \log \|y_i - y_j\|$$

for its minimizer on the unit sphere. This problem is linked with various topics studied in the literature as explained in [18], like the existence of large stable molecules of spherical points (for example the C₆₀ buckminsterfullerene), or the computations problems from analysis of satellite data by arithmetic averages at some well-chosen points, also called “spherical design” (see, among others, [14, 31, 1, 11, 13, 12]) or finally the 7th “Smale’s Problem for the Twentieth Century” (see [30]), i.e. to find, for any $n \geq 2$, a universal constant $c \in \mathbb{R}$ and a configuration $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$ such that

$$E_{\log}(y_1, \dots, y_n) - \min_{\{y_i\} \in \mathbb{S}^2} E_{\log}(y_1, \dots, y_n) \leq c \log n.$$

In this paper we focus on the term of order n which contributes to the understanding of the Smale’s problem. Indeed it is known that if (y_1, \dots, y_n) is a minimizer of E_{\log} on \mathbb{S}^2 then there exist some constants c_1 and c_2 such that

$$\frac{1}{2} - \log 2 - \frac{\log n}{2n} + \frac{c_1}{n} \leq \frac{E_{\log}(y_1, \dots, y_n)}{n^2} \leq \frac{1}{2} - \log 2 - \frac{\log n}{2n} + \frac{c_2}{n},$$

¹where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^3 .

where the lower bound has been derived by Wagner in [32] and where the upper bound is due to Kuijlaars and Saff in [19]. Therefore we have the following asymptotic expansion for a minimizer (y_1, \dots, y_n) of E_{\log} :

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2\right) n^2 - \frac{n}{2} \log n + O(n).$$

The two following conjectures, given by Saff *et al.*, are about the next-order term in this asymptotic expansion, i.e. the expansion of the $O(n)$. The first one comes from [24] and concerns the existence of the term of order n , i.e. the existence of the following limit for a minimizer (y_1, \dots, y_n) of E_{\log} :

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[E_{\log}(y_1, \dots, y_n) - \left(\frac{1}{2} - \log 2\right) n^2 + \frac{n}{2} \log n \right].$$

CONJECTURE 1 ([24, 7]): There exists a constant C , independent of n , such that, if $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$ minimizes E_{\log} for any n ,

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2\right) n^2 - \frac{n}{2} \log n + Cn + o(n) \text{ as } n \rightarrow +\infty.$$

As we prove an asymptotic expansion in the whole plane for w_n , we transport this expansion on \mathbb{S}^2 by an inverse stereographic projection to find C , which depends on the minimum of the energy W . The second conjecture for this asymptotic expansion on the sphere is about the exact value of C which comes from an other conjecture (Conjecture 3 of [7]) for the term of order n in the expansion of optimal Riesz energy on the unit sphere by analytic continuation combined with the fact that the derivative of the Riesz potential is $\lim_{s \rightarrow 0} (\|x\|^{-s} - 1) / s = -\log \|x\|$.

CONJECTURE 2 ([7]): $C = C_{BHS} := 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}$.

Because our order n term depends on the global minimizer of W , the value of this minimum is a key point. The following conjecture is linked with the Ginzburg-Landau theory of superconductors (see [27] or the review [29] of Serfaty for the link between W and the vortices in the Ginzburg-Landau theory).

CONJECTURE 3 ([27]): The triangular lattice is a global minimizer of W for a fixed density of points.

With a formula from [27] and the Chowla-Selberg formula, we compute the exact value of the renormalized energy for the triangular lattice and we find the value of Conjecture 2 if Conjecture 3 is true. Thus our results can be summarized by the following theorem:

THEOREM 1.1.

1. Let V be an admissible potential², then we have the asymptotic expansion, as $n \rightarrow +\infty$,

$$\min_{(\mathbb{R}^2)^n} w_n = I_V(\mu_V) n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx \right) n + o(n)$$

where $d\mu_V(x) = m_V(x) dx$ is the equilibrium measure associated to the external field V , W the renormalized energy and \mathcal{A}_1 is the set of configurations of unit average density³.

²See Section 3.3 for assumptions on V .

³See Section 2 for precise definitions of W and \mathcal{A}_1 .

2. There exists $C \neq 0$ independent of n such that, as $n \rightarrow +\infty$,

$$\min_{(\mathbb{S}^2)^n} E_{\log} = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + Cn + o(n)$$

and more precisely $C = \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2$.

3. We have the following upper bound : $C \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} =: C_{BHS}$.

4. $\min_{\mathcal{A}_1} W$ is achieved for the triangular lattice of density one $\iff C = C_{BHS}$.

This last result, that is the equivalence of Conjectures 2 and 3, is somehow surprising as it links two different domains of analysis and provides another good motivation to prove one of these conjectures.

We proceed as follows. In Section 2 we recall the definition of W and important results from [27]. In Section 3 we recall results about existence, uniqueness and variational Forstman inequalities for μ_V . Moreover, we connect between equilibrium problems and Moebius transformations and we give assumptions on admissible potential V . In Section 4 we derive the fundamental splitting formula which connects w_n with the renormalized energy W . Three useful lemmas are given in Section 5 : a mass spreading result, its link with W and an Ergodic Theorem. The main theorem of this paper is stated and proved in Section 6 and gives the asymptotic expansion of w_n . Finally we prove in Section 7 the Conjecture 1 about the existence of C , the upper bound for this coefficient and the equivalence between Conjectures 2 and 3.

2 Renormalized Energy

Here we recall the definition of the renormalized energy W (see [28] for more details).

Definition 2.1. Let m be a nonnegative number. For any continuous function χ and any vector field E in \mathbb{R}^2 such that

$$\operatorname{div} E = 2\pi(\nu - m) \text{ and } \operatorname{curl} E = 0 \quad (2.1)$$

where ν has the form

$$\nu = \sum_{p \in \Lambda} \delta_p, \text{ for some discrete set } \Lambda \subset \mathbb{R}^2, \quad (2.2)$$

we let

$$W(E, \chi) = \lim_{\eta \rightarrow 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p, \eta)} \chi(x) \|E(x)\|^2 dx + \pi \log \eta \sum_{p \in \Lambda} \chi(p) \right). \quad (2.3)$$

Definition 2.2. Let m be a nonnegative number and B_R denotes the closed ball of radius R centred at the origin and we write $|B_R|$ its Lebesgue measure. Let E be a vector field in \mathbb{R}^2 . We say E belongs to the admissible class \mathcal{A}_m if (2.1), (2.2) hold and

$$\frac{\nu(B_R)}{|B_R|} \text{ is bounded by a constant independent of } R > 1.$$

Remark 2.1. The real m is the average density of the points of Λ when $E \in \mathcal{A}_m$.

We use the notation χ_{B_R} for positive cutoff functions satisfying, for some constant C independent of R

$$\|\nabla\chi_{B_R}\| \leq C, \quad \text{Supp}(\chi_{B_R}) \subset B_R, \quad \chi_{B_R}(x) = 1 \text{ if } d(x, B_R^c) \geq 1. \quad (2.4)$$

Definition 2.3. The renormalized energy W is defined, for $E \in \mathcal{A}_m$ and $\{\chi_{B_R}\}_R$ satisfying (2.4), by

$$W(E) = \limsup_{R \rightarrow +\infty} \frac{W(E, \chi_{B_R})}{|B_R|}.$$

Remark 2.2. In [27, Theorem 1], Sandier and Serfaty introduced W as being computed with averages over general shapes and showed that the minimum of W over \mathcal{A}_m does not depend on the shape used. It was shown in the same paper that :

- the value of W does not depend on the choice of the cutoff functions as long as it satisfies (2.4),
- W is bounded below and admits a minimizer over \mathcal{A}_1 ,
- (see [27, Eq. (1.9),(1.12)]) if $E \in \mathcal{A}_m$, $m > 0$, then

$$E' = \frac{1}{\sqrt{m}}E(\cdot/\sqrt{m}) \in \mathcal{A}_1 \quad \text{and} \quad W(E) = m \left(W(E') - \frac{\pi}{2} \log m \right).$$

Consequently, we have

$$\min_{\mathcal{A}_m} W = m \left(\min_{\mathcal{A}_1} W - \frac{\pi}{2} \log m \right) \quad (2.5)$$

therefore E is a minimizer of W over \mathcal{A}_m if and only if E' minimizes W over \mathcal{A}_1 .

In the periodic case, we have the following important minimality result due to Sandier and Serfaty in [27, Theorem 2] :

Theorem 2.3. The unique minimizer of W , up to rotation, over Bravais lattices⁴ of fixed density m is the Abrikosov triangular lattice

$$\Lambda_m = \sqrt{\frac{2}{m\sqrt{3}}} \left(\mathbb{Z}(1, 0) \oplus \mathbb{Z} \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right).$$

Proof. It is proved by Sandier and Serfaty but we purpose to give here an alternative short proof of this minimality. Indeed, it is not difficult to prove, for any Bravais lattice Λ , that $W(\Lambda) = ah(\Lambda) + b$ where $a > 0$, b a real and $h(\Lambda)$ is the height of the flat torus \mathbb{C}/Λ (see [23, 8, 12] for more details). In [23, Section 4, page 205], Osgood, Phillips and Sarnak proved, for $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau = a + ib$, that

$$h(\Lambda) = -\frac{1}{\pi} \log(b|\eta(\tau)|^4) + C, \quad C \in \mathbb{R},$$

where η is the Dedekind eta function⁵. As Sandier and Serfaty proved in [27] that, up to a constant,

$$W(\Lambda) = -\frac{1}{2} \log \left(\sqrt{2\pi b} |\eta(\tau)|^2 \right),$$

the relation is clear. It is proved in [23, Corollary 1(b)] that Abrikosov lattice minimizes h among Bravais lattices with fixed density, hence it minimizes W . \square

This supports our Conjecture 3, i.e. the Abrikosov triangular lattice is a global minimizer of W among all configurations with fixed (average) density.

3 Equilibrium Problem in the Whole Plane

In this Section we recall results on existence, uniqueness and characterization of equilibrium measure μ_V . Furthermore we prove regularity results for its logarithmic potential, we introduce inversion i and we give assumptions on admissible external field V .

⁴A Bravais lattice of \mathbb{R}^2 , also called "simple lattice" is $L = \mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v}$ where (\vec{u}, \vec{v}) is a basis of \mathbb{R}^2 .

⁵See Section 7.2.

3.1 Equilibrium measure, Frostman inequalities and differentiation of U^{μ_V}

Definition 3.1. ([3]) Let $K \subset \mathbb{R}^2$ be a compact set and let $\mathcal{M}_1(K)$ be the set of probability measure on K . Let

$$I_0(\mu) := - \iint_{K \times K} \log \|x - y\| d\mu(x) d\mu(y)$$

be the **logarithmic energy** of μ on K . We say that K is **log-polar** if $I_0(\mu) = +\infty$ for any $\mu \in \mathcal{M}_1(K)$ and we say that a Borel set E is log-polar if every compact subset of E is polar. We write

$$U^\mu(x) := - \int_{\mathbb{R}^2} \log \|x - y\| d\mu(y)$$

for the **logarithmic potential** of μ .

Moreover, we say that an equality holds **quasi-everywhere** (q.e.) on $A \subset \mathbb{R}^2$ if it holds on $A \setminus P$ where P is log-polar.

Remark 3.1. We recall that Lebesgue measure of a log-polar set is zero.

Now we recall results about existence, uniqueness and characterization of equilibrium measure μ_V proved in [15, 25] for classical growth assumption (1.1) and by Hardy and Kuijlaars in [16, 17] (for existence and uniqueness) and Bloom, Levenberg and Wielonsky in [3] (for Frostman variational inequalities) for weak growth assumption (1.2).

Theorem 3.2. ([15, 25, 16, 17, 3]) Let V a lower semicontinuous function on \mathbb{R}^2 with $\{x \in \mathbb{R}^2; V(x) < +\infty\}$ a non log-polar subset of \mathbb{R}^2 and such that

$$\liminf_{\|x\| \rightarrow +\infty} \{V(x) - 2 \log \|x\|\} > -\infty,$$

then we have :

1. $\inf_{\mu \in \mathcal{M}_1(\mathbb{R}^2)} I_V(\mu)$ is finite, where

$$I_V(\mu) := \iint_{\mathbb{R}^2} \left(\frac{V(x)}{2} + \frac{V(y)}{2} - \log \|x - y\| \right) d\mu(x) d\mu(y);$$

2. there exists a unique equilibrium measure $\mu_V \in \mathcal{M}_1(\mathbb{R}^2)$ with $I_V(\mu_V) = \inf_{\mu \in \mathcal{M}_1(\mathbb{R}^2)} I_V(\mu)$ and the logarithmic energy $I_0(\mu_V)$ is finite ;
3. the support Σ_V of μ_V is contained in $\{x \in \mathbb{R}^2; V(x) < +\infty\}$ and Σ_V is not log-polar;
4. let $c_V := I_V(\mu_V) - \int_{\mathbb{R}^2} \frac{V(x)}{2} d\mu_V(x)$ denote the Robin constant. Then we have the following Frostman variational inequalities :

$$U^{\mu_V}(x) + \frac{V(x)}{2} \geq c_V \quad \text{q.e. on } \mathbb{R}^2 \tag{3.1}$$

$$U^{\mu_V}(x) + \frac{V(x)}{2} \leq c_V \quad \text{for all } x \in \Sigma_V. \tag{3.2}$$

Remark 3.3. In particular we have $U^{\mu_V}(x) + \frac{V(x)}{2} = c_V$ q.e. on Σ_V .

Remark 3.4. We can replace \mathbb{R}^2 by a non log-polar closed subset K to restrict the minimization problem to $\inf_{\mu \in \mathcal{M}_1(K)} I_V(\mu)$ and the result is always true, but integrals and support are restricted to closed set K .

3.2 Moebius Transformations and Energy

In this part and throughout this paper, $|z|$ denote the modulus of complex number z . Depending on context we will use $|x|$ or $\|x\|$ if x is considered as a complex number or a point of \mathbb{R}^2 .

Definition 3.2. Let $\varphi : z \mapsto \frac{az+b}{cz+d}$ be such that $ad-bc=1$ defined on $\mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$, i.e. $\varphi \in PSL_2(\mathbb{C})$ is a Moebius transformation, and $\mu \in \mathcal{M}_1(\mathbb{C})$ be a probability measure, then $\varphi\#\mu$ denote its push-forward by φ , i.e. the measure characterized by

$$\int_{\mathbb{C}} f(z) d(\varphi\#\mu)(z) = \int_{\mathbb{C}} f(\varphi(x)) d\mu(x)$$

for every Borel function on \mathbb{C} .

Moreover, for $V : \mathbb{C} \rightarrow \mathbb{R}$ a function and $\varphi \in PSL_2(\mathbb{C})$, we define

$$V_{\varphi}(z) := V(\varphi(z)) + 2 \log |cz + d|.$$

Lemma 3.5. Let $\varphi \in PSL_2(\mathbb{C})$, $V : \mathbb{C} \rightarrow \mathbb{R}$ be a function and $\mu \in \mathcal{M}(\mathbb{C})$, then

$$I_V(\mu) = I_{V_{\varphi}}(\varphi\#\mu).$$

Proof. We have

$$\begin{aligned} I_V(\mu) &= \iint_{\mathbb{C}^2} \left(\frac{V(x)}{2} + \frac{V(y)}{2} - \log |x - y| \right) d\mu(x) d\mu(y) \\ &= \iint_{\mathbb{C}^2} \left(\frac{V(\varphi(x))}{2} + \frac{V(\varphi(y))}{2} - \log |\varphi(x) - \varphi(y)| \right) d(\varphi\#\mu)(x) d(\varphi\#\mu)(y) \\ &= \iint_{\mathbb{C}^2} \left(\frac{V(\varphi(x))}{2} + \frac{V(\varphi(y))}{2} - \log \left| \frac{(ad-bc)(x-y)}{(cx+d)(cy+d)} \right| \right) d(\varphi\#\mu)(x) d(\varphi\#\mu)(y) \\ &= \iint_{\mathbb{C}^2} \left(\frac{V(\varphi(x))}{2} + \frac{V(\varphi(y))}{2} - \log |x - y| + \log |cx + d| + \log |cy + d| \right) d(\varphi\#\mu)(x) d(\varphi\#\mu)(y) \\ &= \iint_{\mathbb{C}^2} \left(\frac{V_{\varphi}(x)}{2} + \frac{V_{\varphi}(y)}{2} - \log |x - y| \right) d(\varphi\#\mu)(x) d(\varphi\#\mu)(y) \\ &= I_{V_{\varphi}}(\varphi\#\mu). \end{aligned}$$

□

Proposition 3.6. Let $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a $C^3(\mathbb{R}^2)$ function. If, for any $\varphi \in PSL_2(\mathbb{C})$,

1. the set $\{x \in \mathbb{R}^2; V_{\varphi}(x) < +\infty\}$ is not log-polar,
2. function V_{φ} is bounded at the neighbourhood of 0,

then for each $\varphi \in PSL_2(\mathbb{C})$ there exists a unique equilibrium measure $\mu_{V_{\varphi}}$ in the sense of Theorem 3.2.

Proof. By 2., if $d = 0$ then $-bc = 1 \neq 0$, and for any $b \neq 0$, $c \neq 0$ and $a \in \mathbb{C}$,

$$\lim_{|x| \rightarrow 0} V_{\varphi}(x) = \lim_{|x| \rightarrow 0} \left\{ V \left(\frac{a}{c} + \frac{b}{cx} \right) + 2 \log |x| \right\} \quad (3.3)$$

is finite. Now, we have

$$\lim_{|x| \rightarrow +\infty} \{V_{\varphi}(x) - 2 \log |x|\} = \lim_{|x| \rightarrow +\infty} \left\{ V \left(\frac{ax+b}{cx+d} \right) + 2 \log \left| c + \frac{d}{x} \right| \right\}$$

and it follows that, if $c \neq 0$, then

$$\lim_{|x| \rightarrow +\infty} \{V_\varphi(x) - 2 \log |x|\} = V\left(\frac{a}{c}\right) + 2 \log |c|$$

which is finite because V is C^3 and $|c| \neq 0$. Moreover, if $c = 0$ then $ad = 1 \neq 0$, and it follows that, for any $a \neq 0$, $d \neq 0$ and $b \in \mathbb{C}$,

$$\lim_{|x| \rightarrow +\infty} \{V_\varphi(x) - 2 \log |x|\} = \lim_{|x| \rightarrow +\infty} \left\{ V\left(\frac{ax+b}{d}\right) - 2 \log |x| \right\} = \lim_{|y| \rightarrow 0} \left\{ V\left(\frac{b}{d} + \frac{a}{dy}\right) + 2 \log |y| \right\}$$

is finite by (3.3). Thus assumptions 1. and 2. imply, by Theorem 3.2, existence and uniqueness of equilibrium measures μ_{V_φ} for any φ . \square

3.3 Admissible potential V

Definition 3.3. We say that $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is **admissible** if, for any $\varphi \in PSL_2(\mathbb{C})$, we have :

- **(H1)** : the set $\{x \in \mathbb{R}^2; V_\varphi(x) < +\infty\}$ is not log-polar;
- **(H2)** : V_φ is bounded at the neighbourhood of 0;
- **(H3)** : equilibrium measure μ_{V_φ} is given by $d\mu_{V_\varphi}(x) = m_{V_\varphi}(x) \mathbf{1}_{\Sigma_{V_\varphi}}(x) dx$ where m_{V_φ} is C^1 ;
- **(H4)** : there exist $m_\varphi, M_\varphi \in \mathbb{R}$ such that, for any $x \in B_1 \cap \Sigma_{V_\varphi}$,

$$0 < m_\varphi \leq m_{V_\varphi}(x) \leq M_\varphi;$$

- **(H5)** : V_φ is C^3 on \mathbb{R}^2 and $\partial \Sigma_V$ is C^1 .

Remark 3.7. Assumptions related to $\varphi(z) \neq az+b$ will use for construction of Gamma-convergence's upper bound.

Remark 3.8. Actually, given $d\mu_V(x) = m_V(x) \mathbf{1}_{\Sigma_V}(x) dx$, we have, by Lemma 3.5, $\mu_{V_\varphi} = \varphi\# \mu_V$ with

$$m_{V_\varphi}(x) = \frac{m_V(\varphi(x))}{|cx+d|^4}$$

because Jacobian's determinant of φ is $|cx+d|^{-4}$, hence assumption **(H4)** implies that, for any $\varphi \in PSL_2(\mathbb{C})$ and any $x \in B_1 \cap \Sigma_{V_\varphi}$,

$$0 < m_\varphi \leq \frac{m_V(\varphi(x))}{|cx+d|^4} \leq M_\varphi. \quad (3.4)$$

In particular, for $(a, b, c, d) = (1, 0, 0, 1)$, i.e. $V_\varphi = V$, we write $m_1 = m_\varphi$ and $M_1 = M_\varphi$, hence for any $x \in B_1 \cap \Sigma_V$,

$$0 < m_1 \leq m_V(x) \leq M_1. \quad (3.5)$$

Taking $x = -y^{-1}$ in (3.4), it follows that, for any φ and any $y \in B_1^c \cap \Sigma_{V_\varphi}$,

$$0 < \frac{m_\varphi}{|y|^4} \leq \frac{m_V(\varphi(-y^{-1}))}{|-c+dy|^4} \leq \frac{M_\varphi}{|y|^4}.$$

As $\varphi(-y^{-1}) = \frac{by-a}{dy-c}$, taking $(a, b, c, d) = (0, 1, -1, 0)$, there exist $m_2, M_2 \in \mathbb{R}$ such that, for any $y \in B_1^c \cap \Sigma_V$,

$$0 < \frac{m_2}{|y|^4} \leq m_V(y) \leq \frac{M_2}{|y|^4} \quad (3.6)$$

which gives the behaviour of μ_V at infinity. Hence by (3.5) and (3.6) there exist $m, M \in \mathbb{R}_+$ such that, for any $x \in \mathbb{R}^2$,

$$0 < \frac{m}{1 + \|x\|^4} \leq m_V(x) \leq \frac{M}{1 + \|x\|^4}. \quad (3.7)$$

Furthermore, the same argument allows to prove existence of $\tilde{m}_\varphi, \tilde{M}_\varphi \in \mathbb{R}_+$ such that, for any $x \in \mathbb{R}^2$

$$0 < \frac{\tilde{m}_\varphi}{1 + \|x\|^4} \leq m_{V_\varphi}(x) \leq \frac{\tilde{M}_\varphi}{1 + \|x\|^4}.$$

4 Splitting Formula

As in [28] we denote the blown-up quantities by primes :

$$x' = \sqrt{n}x, \quad m'_V(x) = m_V\left(\frac{x}{\sqrt{n}}\right), \quad d\mu'_V(x') = m'_V(x')dx'$$

and we define

$$\zeta(x) := U^{\mu_V}(x) + \frac{V(x)}{2} - c_V,$$

then by (3.1) and (3.2), $\zeta(x) = 0$ q.e. in Σ_V and $\zeta(x) > 0$ q.e. in $\mathbb{R}^2 \setminus \Sigma_V$. For $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$, we define $\nu_n = \sum_{i=1}^n \delta_{x_i}$ and

$$H_n := -2\pi\Delta^{-1}(\nu_n - m_V) = - \int_{\mathbb{R}^2} \log \|\cdot - y\| d(\nu_n - n\mu_V)(y) = - \sum_{i=1}^n \log \|\cdot - x_i\| - nU^{\mu_V}$$

where Δ^{-1} is the convolution's operator with $\frac{1}{2\pi} \log \|\cdot\|$, hence such that $\Delta \circ \Delta^{-1} = I_2$ where Δ denote the usual laplacian. Moreover we set

$$H'_n := -2\pi\Delta^{-1}(\nu'_n - \mu'_V)$$

where $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$.

Lemma 4.1. *We have*

$$\lim_{R \rightarrow +\infty} \int_{B_R} H_n(x) d\mu_V(x) = \int_{\mathbb{R}^2} H_n(x) d\mu_V(x) \quad \text{and} \quad \lim_{R \rightarrow +\infty} W(\nabla H_n, \mathbf{1}_{B_R}) = W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2}). \quad (4.1)$$

Proof. By (3.7) we get⁶

$$\int_{\mathbb{R}^2} \log(1 + \|y\|) d\mu_V(y) < +\infty \quad (4.2)$$

therefore by dominated convergence argument from [22, Theorem 9.1, Chapter 5] (used for the continuity of U^{μ_V}), we have

$$H_n(x) = \sum_{i=1}^n \int_{\mathbb{R}^2} \log \left(\frac{\|x - y\|}{\|x - x_i\|} \right) d\mu_V(y) \quad (4.3)$$

⁶Actually, the fact that μ_V is an equilibrium measure is sufficient to obtain (4.2), as explained by Mizuta in [22, Theorem 6.1, Chapter 2] or by Bloom, Levenberg and Wielonsky in [3, Lemma 3.2]

and it follows that $H_n(x) = O(\|x\|^{-1})$ as $\|x\| \rightarrow +\infty$ which implies first equality because $I_0(\mu_V)$ is finite.

The second equality follows from dominated convergence argument of Mizuta in [21, Theorem 1], because we remark that, for $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x_i = (x_{i,1}, x_{i,2})$, for any $1 \leq i \leq n$ and any $j \in \{1, 2\}$, we have

$$\partial_{x_j} \left(\log \left(\frac{\|x - y\|}{\|x - x_i\|} \right) m_V(y) \right) = \left(\frac{x_j - y_j}{\|x - y\|^2} - \frac{x_j - x_{i,j}}{\|x - x_i\|^2} \right) m_V(y).$$

Hence we get $\nabla H_n(x) = O(\|x\|^{-2})$ as $\|x\| \rightarrow +\infty$ and we have integrability of $\|\nabla H_n\|^2$ at infinity. \square

Lemma 4.2. *Let V admissible then, for any $n \geq 2$ and for any configuration $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$, we have*

$$w_n(x_1, \dots, x_n) = n^2 I_V(\mu_V) - \frac{n}{2} \log n + \frac{1}{\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i). \quad (4.4)$$

Proof. Exactly the same proof as in [28, Lemma 3.1], because we have Frostman inequalities (3.1), (3.2) and Lemma 4.1. In particular we have $H_n(x) = O(\|x\|^{-1})$ and $\nabla H_n(x) = O(\|x\|^{-2})$ as $\|x\| \rightarrow +\infty$ which implies, exactly like in the compact support case, that

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R} H_n(x) \nabla H_n(x) \cdot \vec{\nu}(x) dx = 0$$

where $\vec{\nu}(x)$ is the outer unit normal vector at $x \in \partial B_R$. \square

5 Useful Lemmas

Here we give essential results to prove the lower bound of our main theorem. Indeed our goal is to write $W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2})$ like an integral and to use an Ergodic Theorem to bound by below the next-order term in the previous splitting formula, as in [28].

5.1 Mass spreading result and modified density g

By a mass displacement method introduced in [26], we have the following result, essentially from [28, Proposition 3.4] :

Lemma 5.1. *Let V be admissible and assume (ν, E) are such that $\nu = \sum_{p \in \Lambda} \delta_p$ for some finite subset*

$\Lambda \subset \mathbb{R}^2$ and $\operatorname{div} E = 2\pi(\nu - m_V)$, $\operatorname{curl} E = 0$ in \mathbb{R}^2 . Then, given any $\rho > 0$ there exists a signed measure g supported on \mathbb{R}^2 and such that

- there exists a family \mathcal{B}_ρ of disjoint closed balls covering $\operatorname{Supp}(\nu)$, with the sum of the radii of the balls in \mathcal{B}_ρ intersecting with any ball of radius 1 bounded by ρ , and such that

$$g(A) \geq -C(\|a\|_\infty + 1) + \frac{1}{4} \int_A \|E(x)\|^2 \mathbb{1}_{\Omega \setminus \mathcal{B}_\rho}(x) dx, \quad \text{for any } A \subset \mathbb{R}^2,$$

where C depends only on ρ ;

- we have

$$dg(x) = \frac{1}{2} \|E(x)\|^2 dx \quad \text{outside } \bigcup_{p \in \Lambda} B(p, \lambda)$$

where λ depends only on ρ ;
- for any function χ compactly supported in \mathbb{R}^2 we have

$$\left| W(E, \chi) - \int \chi dg \right| \leq CN(\log N + \|m_V\|_\infty) \|\nabla \chi\|_\infty \quad (5.1)$$

where $N = \#\{p \in \Lambda; B(p, \lambda) \cap \text{Supp}(\nabla \chi) \neq \emptyset\}$ for some λ and C depending only on ρ ;
- for any $U \subset \Omega$

$$\#(\Lambda \cap U) \leq C(1 + \|m_V\|_\infty^2 |\hat{U}| + g(\hat{U})) \quad (5.2)$$

where $\hat{U} := \{x \in \mathbb{R}^2; d(x, U) < 1\}$.

Proof. As explained in the proof of Lemma 4.1, as V is admissible, m_V decays sufficiently quickly at infinity such that, for sufficiently large M ,

$$\int_{B_M^c} dg(x) = \frac{1}{2} \int_{B_M^c} \|E(x)\|^2 dx < +\infty.$$

Hence, by the fact that $\#\Lambda$ is finite, we can apply Proposition 3.4 from [28] with $\hat{\Omega} = \Omega = \mathbb{R}^2$ and $a = m_V$. \square

Definition 5.1. Assume $\nu_n = \sum_{i=1}^n \delta_{x_i}$. Letting $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ be the measure in blown-up coordinates and $E_{\nu_n} = \nabla H'_n$, we denote by g_{ν_n} the result of applying the previous proposition to (ν'_n, E_{ν_n}) .

Moreover we need the following result, essentially [28, Lemma 3.7], which connects g and renormalized energy:

Lemma 5.2. ([28]) For any $\nu_n = \sum_{i=1}^n \delta_{x_i}$, we have

$$W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) = \int_{\mathbb{R}^2} dg_{\nu_n}. \quad (5.3)$$

Proof. We apply the inequality (5.1) to χ_{B_R} defined in (2.4) for R sufficiently large in order to have $N = \#\{p \in \Lambda; B(p, \lambda) \cap \text{Supp}(\nabla \chi_{B_R}) \neq \emptyset\} = 0$. Hence we obtain

$$W(\nabla H'_n, \chi_{B_R}) = \int_{\mathbb{R}^2} \chi_{B_R} dg_{\nu_n}$$

and, because, by Lemma 4.1 and definition of g_{ν_n} ,

$$\lim_{R \rightarrow +\infty} W(\nabla H'_n, \chi_{B_R}) = W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) \quad \text{and} \quad \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^2} \chi_{B_R} dg_{\nu_n} = \int_{\mathbb{R}^2} dg_{\nu_n},$$

we get (5.3). \square

5.2 Ergodic Theorem

We recall exactly the notations of Sandier and Serfaty from [28, Section 4.1] for the two-parameter groups acting continuously on a metric Polish space X which is a space of functions :

- we define θ_λ acting on X by $\theta_\lambda u(x) = u(x + \lambda)$ for any $x, \lambda \in \mathbb{R}^2$;
- we also define T_λ^ε and T_λ acting on $\mathbb{R}^2 \times X$ by $T_\lambda^\varepsilon(x, u) := (x + \varepsilon\lambda, \theta_\lambda u)$ and $T_\lambda(x, u) := (x, \theta_\lambda u)$.

For a probability measure P on $\mathbb{R}^2 \times X$ we say that P is $T_{\lambda(x)}$ -invariant if for every function $\lambda(x)$ of class C^1 , it is invariant under the mapping $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$.

Let V be admissible and $\mu_V^R := \frac{\mu_V}{\mu_V(B_R)}$ the normalized measure on B_R of density m_V^R . Let $(f_\varepsilon)_\varepsilon$ and f be positive measurable functions on $\mathbb{R}^2 \times X$.

We assume that for any $\{x_\varepsilon, u_\varepsilon\}_\varepsilon$ such that for any $R > 0$, $\limsup_{\varepsilon \rightarrow 0} \int_{B_R} f_\varepsilon(x_\varepsilon + \varepsilon\lambda, \theta_\lambda u_\varepsilon) d\lambda < +\infty$,

we have the following properties :

- 1) (Coercivity) $\{u_\varepsilon\}_\varepsilon$ has a convergent subsequence,
- 2) (Γ -liminf) if $\{x_\varepsilon, u_\varepsilon\}_\varepsilon$ converge to $\{x, u\}$ then $\liminf_{\varepsilon \rightarrow 0} f_\varepsilon(x_\varepsilon, u_\varepsilon) \geq f(x, u)$.

Remark 5.3. We needn't convergent subsequence of $(x_\varepsilon)_\varepsilon$ in property of coercivity because we use same arguments as in [28, Theorem 6] only for marginals.

Theorem 5.4. Let $V, X, (f_\varepsilon)_\varepsilon$ and f be as above. We define $F_\varepsilon(u) := \int_{\mathbb{R}^2} f_\varepsilon(x, \theta_{\frac{x}{\varepsilon}} u) d\mu_V(x)$ and let $(u_\varepsilon)_\varepsilon \in X$ be a sequence such that $F_\varepsilon(u_\varepsilon) \leq C$ for any $\varepsilon > 0$. Let P_ε be the image of μ_V by $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$, then

1. $(P_\varepsilon)_\varepsilon$ admits a convergent's subsequence to a probability measure P ,
2. the first marginal of P is μ_V ,
3. P is $T_{\lambda(x)}$ -invariant,
4. $(x, u) = \lim_{\varepsilon \rightarrow 0} (x_\varepsilon, \theta_{\frac{x_\varepsilon}{\varepsilon}} u_\varepsilon)$ P -a.e.,
5. $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u)$.
6. Moreover we have

$$\int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u) = \int_{\mathbb{R}^2 \times X} \left(\lim_{R \rightarrow +\infty} \int_{B_R} f(x, \theta_\lambda u) d\lambda \right) dP(x, u). \quad (5.4)$$

where \int_{B_R} denote the integral average over B_R .

Proof. The method is the same as [27, 28] but where the normalized Lebesgue measure on Σ_V – which is compact in these papers – is replaced by μ_V . We rewrite details for a better understanding.

STEP 1 : Convergence of a subsequence of (P_ε) to a probability measure P

For any $R > 0$ and $\varepsilon > 0$, we define $F_\varepsilon^R(u) = \int_{B_R} f_\varepsilon(x, \theta_{\frac{x}{\varepsilon}} u) d\mu_V^R(x)$. Let $(u_\varepsilon)_\varepsilon$ be such that $F_\varepsilon(u_\varepsilon) \leq C$. For R large enough, $F_\varepsilon^R(u_\varepsilon) \leq C$ for any $\varepsilon > 0$.

We define $X_{K,R}^\varepsilon := \left\{ (\lambda, u) \in B_R \times X; \int_{B_R} f_\varepsilon(\lambda + \varepsilon x, \theta_x u) dx > K \right\}$ and for $\varepsilon < 1$, Y_R^ε the image of $B_{R(1-\varepsilon)}$ by $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$. Let P_ε^R be the image of μ_V^R by the same map. We have, by (3.7),

$$\begin{aligned} \int_{Y_R^\varepsilon} \int_{B_R} f_\varepsilon(\lambda + \varepsilon x, \theta_x u) dx dP_\varepsilon^R(\lambda, u) &= \int_{B_{R(1-\varepsilon)}} \int_{B_R} f_\varepsilon(\lambda + \varepsilon x, \theta_{\frac{\lambda + \varepsilon x}{\varepsilon}} u_\varepsilon) dx d\mu_V^R(\lambda) \\ &= \frac{1}{\varepsilon^2 |B_R|} \int_{\mathbb{R}^2} \left[\mathbf{1}_{\varepsilon B_R} * \mathbf{1}_{B_{R(1-\varepsilon)}} m_V^R \right] (y) f_\varepsilon(y, \theta_{\frac{y}{\varepsilon}} u_\varepsilon) dy \\ &\leq \frac{M}{m} C =: C' \end{aligned}$$

Therefore by the Markov inequality $P_\varepsilon^R(X_{K,R}^\varepsilon \cap Y_R^\varepsilon) \leq \frac{C'}{K}$ and we get

$$P_\varepsilon^R(X_{K,R}^\varepsilon) \leq P_\varepsilon^R(X_{K,R}^\varepsilon \cap Y_R^\varepsilon) + P_\varepsilon^R((Y_R^\varepsilon)^c) \leq \frac{C'}{K} + 1 - \mu_V^R(B_{R(1-\varepsilon)}).$$

Let $\delta > 0$ be a real, $(R_n)_n$ (which goes to $+\infty$ at infinity) and $(\varepsilon_n)_n$ (which goes to 0 at infinity) such that for any $n \in \mathbb{N}$, $1 - \mu_V^{R_n}(B_{R_n(1-\varepsilon_n)}) < \delta 2^{-n}$, then

$$P_{\varepsilon_n}^{R_n} \left(\bigcup_{k=1}^n X_{2^k/\delta, R_k}^{\varepsilon_k} \right) \leq \sum_{k=1}^n P_{\varepsilon_n}^{R_n} \left(X_{2^k/\delta, R_k}^{\varepsilon_k} \right) \leq \sum_{k=1}^n \left[\frac{C'\delta}{2^k} + \frac{\delta}{2^k} \right] \leq C''\delta.$$

We define $K_n := \left(\bigcup_{k=1}^n X_{2^k/\delta, R_k}^{\varepsilon_k} \right)^c$, then we have $P_{\varepsilon_n}^{R_n}(K_n) \geq 1 - C''\delta$. If $\{\lambda_n, u_n\} \in K_n$ for every n , then it follows that, for any $R > R_0$ and any $n > R$ such that $R_n > R$,

$$\int_{B_R} f_\varepsilon(\lambda_n + \varepsilon x, \theta_x u_n) dx \leq \frac{2^R}{\delta} < +\infty$$

and by the property 1) of coercivity, $(u_n)_n$ has a convergent's subsequence. Now we use the following simple lemma [27, Lemma 2.1] :

Lemma 5.5. (E. Lesigne, [27]) *Assume $(P_n)_n$ are Borel probability measures on a Polish metric space X and that for any $\delta > 0$ there exists $(K_n)_n$ such that $P_n(K_n) \geq 1 - \delta$ for every n and such that if $(x_n)_n$ satisfies for every n that $x_n \in K_n$, then any subsequence of $(x_n)_n$ admits a convergent subsequence. Then $(P_n)_n$ admits a subsequence which converges weakly to a probability measure P .*

Applying this result to the second marginal of $P_{\varepsilon_n}^{R_n}$ and by the fact that its first marginal is also tight – because it converges to μ_V –, $(P_{\varepsilon_n}^{R_n})_n$ is tight and has a convergent subsequence such that $P_{\varepsilon_n}^{R_n} \rightarrow P$ as $n \rightarrow +\infty$.

STEP 2 : P is $T_{\lambda(x)}$ -invariant. Let λ be a C^1 function on \mathbb{R}^2 , Φ a bounded continuous function on $\mathbb{R}^2 \times X$ and P_λ the image of P by $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$. By the change of variables $y = \varepsilon\lambda(x) + x = (\varepsilon\lambda + I_2)(x)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) &= \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP_\varepsilon(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi(x, \theta_{\lambda(x) + \frac{x}{\varepsilon}} u_\varepsilon) d\mu_V(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi(x, \theta_{\frac{\varepsilon\lambda(x) + x}{\varepsilon}} u_\varepsilon) d\mu_V(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{\Phi \left((\varepsilon\lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon \right) m_V((\varepsilon\lambda + I_2)^{-1}(y)) dy}{|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon\lambda)^{-1}(y)))|} \end{aligned}$$

where $D\lambda$ is the differential of λ . We define

$$f(y, \varepsilon\lambda + I_2) := \frac{\Phi \left((\varepsilon\lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon \right) m_V((\varepsilon\lambda + I_2)^{-1}(y))}{|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon\lambda)^{-1}(y)))|}.$$

For $R > 0$ and ε sufficiently small such that $|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon\lambda)^{-1}(y)))| \geq 1/2$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} f(y, \varepsilon\lambda + I_2) dy = \lim_{\varepsilon \rightarrow 0} \int_{B_R} f(y, \varepsilon\lambda + I_2) dy + \lim_{\varepsilon \rightarrow 0} \int_{B_R^c} f(y, \varepsilon\lambda + I_2) dy$$

$$= \int_{B_R} \lim_{\varepsilon \rightarrow 0} f(y, \varepsilon\lambda + I_2) dy + \lim_{\varepsilon \rightarrow 0} \int_{B_R^c} f(y, \varepsilon\lambda + I_2) dy$$

because Φ and m_V are bounded. We get

$$\lim_{\varepsilon \rightarrow 0} \int_{B_R^c} f(y, \varepsilon\lambda + I_2) dy \leq \lim_{\varepsilon \rightarrow 0} K \int_{B_R^c} m_V((\varepsilon\lambda + I_2)^{-1}(y)) dy \longrightarrow 0 \text{ as } R \rightarrow +\infty.$$

and it follows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} f(y, \varepsilon\lambda + I_2) dy = \int_{B_R} \lim_{\varepsilon \rightarrow 0} f(y, \varepsilon\lambda + I_2) dy + \lim_{\varepsilon \rightarrow 0} o_R(1).$$

Thus we can exchange the limit and the integral and we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) &= \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi\left((\varepsilon\lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon\right) d\mu_V(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi\left((\varepsilon\lambda + I_2)^{-1}(y), u\right) dP_\varepsilon(y, u). \end{aligned}$$

The first step gives that $(P_\varepsilon)_\varepsilon$ is tight hence for any $\delta > 0$ there exists a compact set $K_\delta \subset \mathbb{R}^2 \times X$ such that $P_\varepsilon(K_\delta^c) < \delta$. Φ is continuous and bounded on K_δ therefore Φ is uniformly continuous on this compact set and $\Phi\left((\varepsilon\lambda + I_2)^{-1}(y), u\right)$ converges uniformly to $\Phi(y, u)$ on K_δ . Now, as $\delta \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi\left((\varepsilon\lambda + I_2)^{-1}(y), u\right) dP_\varepsilon(y, u) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi(y, u) dP_\varepsilon(y, u) = \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP(x, u).$$

We proved that $\int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) = \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP(x, u)$, i.e. P is $T_{\lambda(x)}$ -invariant.

STEP 3 : $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u)$. It is clear with the Γ -liminf property and the following result (see [27, Lemma 2.2] for the proof) :

Lemma 5.6. ([27]) *Assume that X is a Polish metric space, that $\{P_\varepsilon\}_{\varepsilon > 0}$, P are Borel probability measures on X such that $P_\varepsilon \rightarrow P$ as $\varepsilon \rightarrow 0$, and that $\{f_\varepsilon\}_{\varepsilon > 0}$ and f are positive measurable functions on X such that $\liminf_{\varepsilon \rightarrow 0} f_\varepsilon(x_\varepsilon) \geq f(x)$ whenever $x_\varepsilon \rightarrow x$. Then $\liminf_{\varepsilon \rightarrow 0} \int f_\varepsilon dP_\varepsilon \geq \int f dP$.*

Results 2 and 4 are obvious by construction of P and the point *vi*) follows from the Wiener's Ergodic Theorem proved by Becker in [2, Theorem 3] as in [27, Part 1, Section 2] and [28, Theorem 6] and because P is $T_{\lambda(x)}$ -invariant. \square

6 Asymptotic Expansion of the Hamiltonian

We define $\alpha := \frac{1}{\pi} \int_{\mathbb{R}^2} \min_{\mathcal{A}_{m_V(x)}} W dx = \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx$ and

$$F_n(\nu) = \begin{cases} \frac{1}{n} \left(\frac{1}{\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) + 2n \int \zeta d\nu \right) & \text{if } \nu \text{ is of the form } \sum_{i=1}^n \delta_{x_i} \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear that α is finite because, as recall in Section 2, $\min_{\mathcal{A}_1} W$ is achieved and, by (3.7) and (4.2), we have

$$\int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx \leq \int_{\mathbb{R}^2} m_V(x) \log(M(1 + \|x\|)) dx < +\infty.$$

For $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ let E_{ν_n} be a solution of $\operatorname{div} E_{\nu_n} = 2\pi(\nu'_n - m'_V)$, $\operatorname{curl} E_{\nu_n} = 0$ and we set

$$P_{\nu_n} := \int_{\mathbb{R}^2} \delta_{(x, E_{\nu_n}(x\sqrt{n}+))} d\mu_V(x).$$

With the following result we generalize [28, Theorem 2] for a broader class of equilibrium measures (for example, with unbounded support). We use a Gamma-Convergence method (see [4] or [20] for details) as in [28] hence we show a lower bound, by Ergodic Theorem, and an upper bound by the compact case seen in [28].

These two bounds give the convergence of $\frac{1}{n} \left[w_n(x_1, \dots, x_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right]$ to α for a minimizer (x_1, \dots, x_n) of w_n .

6.1 Main result

Theorem 6.1. *Let $1 < p < 2$ and $X = \mathbb{R}^2 \times L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. Let V be an admissible function.*

A. Lower bound : *Let $(\nu_n)_n$ such that $\frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) \leq C$, then*

1. P_{ν_n} is a probability measure on X and admits a subsequence which converges to a probability measure P on X ,
2. the first marginal of P is μ_V ,
3. P is $T_{\lambda(x)}$ -invariant,
4. $E \in \mathcal{A}_{m_V(x)}$ P -a.e.,
5. we have the lower bound

$$\liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E) \geq \alpha. \quad (6.1)$$

B. Upper bound. *Conversely, assume P is a $T_{\lambda(x)}$ -invariant probability measure on X whose first marginal is μ_V and such that for P -almost every (x, E) we have $E \in \mathcal{A}_{m_V(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of measures on \mathbb{R}^2 and a sequence $\{E_n\}_n$ in $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{div} E_n = 2\pi(\nu'_n - m'_V)$ and such that defining $P_n = \int_{\mathbb{R}^2} \delta_{(x, E_n(x\sqrt{n}+))} d\mu_V(x)$, we have $P_n \rightarrow P$ as $n \rightarrow +\infty$ and*

$$\limsup_{n \rightarrow +\infty} F_n(\nu_n) \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E). \quad (6.2)$$

C. Consequences for minimizers. *If (x_1, \dots, x_n) minimizes w_n for every n and $\nu_n = \sum_{i=1}^n \delta_{x_i}$, then :*

1. for P -almost every (x, E) , E minimizes W over $\mathcal{A}_{m_V(x)}$,

2. we have

$$\lim_{n \rightarrow +\infty} F_n(\nu_n) = \lim_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E) = \alpha \quad (6.3)$$

hence we obtain the following asymptotic expansion, as $n \rightarrow +\infty$:

$$\min_{(\mathbb{R}^2)^n} w_n = I_V(\mu_V)n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx \right) n + o(n). \quad (6.4)$$

Remarks 6.2. - The lower bound (6.1) is valid for any configuration.

- The following equality from (6.3)

$$\lim_{n \rightarrow +\infty} \frac{1}{n} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) = \int_{\mathbb{R}^2} \min_{\mathcal{A}_{m_V(x)}} W dx$$

is explained by Serfaty in [29, Theorem 5] as follows : “minimizers of w_n provide configurations of points in the plane whose associated vector fields E minimize, after blow-up and taking the limit $n \rightarrow +\infty$, the renormalized energy (heuristically) for almost every blow-up center”.

6.2 Proof of the lower bound with Ergodic Theorem

We follow the same lines as in [28, Section 4.2]. Let χ be a C^∞ cutoff function with support the unit ball B_1 and integral equal to 1. We define

$$\mathbf{f}_n(x, \nu, E, g) := \begin{cases} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} dg(y) & \text{if } (\nu, E, g) = \theta_{\sqrt{n}x}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n) \\ +\infty & \text{otherwise} \end{cases}$$

where $\bar{\nu}_n$ is a subsequence such that $F_n(\bar{\nu}_n) \leq C$, which we assume to exist. Hence $\bar{\nu}'_n, \bar{E}_n, \bar{g}_n$ are respectively its expression in blow-up coordinates, its associated vector field and the corresponding signed Radon measure by Definition 5.1. We define and bound by above, as in [28, Section 4.2, Step 1], the following expression

$$\begin{aligned} \mathbf{F}_n(\nu, E, g) &:= \int_{\mathbb{R}^2} \mathbf{f}_n(x, \theta_{x\sqrt{n}}(\nu, E, g)) d\mu_V(x) \\ &= \int_{\mathbb{R}^2} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} d(\theta_{x\sqrt{n}} \# g) d\mu_V(x) \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi(y - x\sqrt{n}) dx d\bar{g}_n(y) \\ &\leq \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) + \frac{\bar{g}_n^-(U^c)}{n\pi} \end{aligned}$$

by (5.3), where $U = \{x' : d(x', \mathbb{R}^2 \setminus \Sigma) \geq 1\}$. As in [28], we have $\bar{g}_n^-(U^c) = o(n)$. Hence, if $(\nu, E, g) = (\bar{\nu}'_n, \bar{E}_n, \bar{g}_n)$, as $n \rightarrow +\infty$:

$$\mathbf{F}_n(\nu, E, g) \leq \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) + o(1)$$

and $\mathbf{F}_n(\nu, E, g) = +\infty$ otherwise.

Now, as in [28], we want to use Ergodic Theorem 5.4 with $\varepsilon = \frac{1}{\sqrt{n}}$ and $X = \mathcal{M}_+ \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2) \times \mathcal{M}$ where $p \in]1, 2[$, \mathcal{M}_+ is the set of nonnegative Radon measures on \mathbb{R}^2 and \mathcal{M} the set of Radon measures bounded below by $-C_V := -C(\|m_V\|_\infty^2 + 1)$. Let Q_n be the image of μ_V by $x \mapsto (x, \theta_{x\sqrt{n}}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n))$. We have :

1) \mathbf{f}_n is coercive by [28, Lemma 6.3]. Indeed, if $(x_n, \nu_n, E_n, g_n)_n$ is so that, for any $R > 0$,

$$\limsup_{n \rightarrow +\infty} \int_{B_R} \mathbf{f}_n \left(x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, E_n, g_n) \right) d\lambda < +\infty$$

then the integrand is bounded for a.e. λ . By assumption on \mathbf{f}_n , $\theta_\lambda(\nu_n, E_n, g_n) = \theta_{x_n\sqrt{n}+\lambda}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n)$, hence it follows that

$$(\nu_n, E_n, g_n) = \theta_{x_n\sqrt{n}}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n),$$

and for any $R > 0$ there exists $C_R > 0$ such that for any $n > 0$

$$\begin{aligned} \int_{B_R} \mathbf{f}_n \left(x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, E_n, g_n) \right) d\lambda &= \int_{B_R} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V \left(x_n + \frac{\lambda}{\sqrt{n}} \right)} d(\theta_{\lambda+x_n\sqrt{n}} \# \bar{g}_n(y)) d\lambda \\ &= \frac{1}{\pi} \int_{B_R} \int_{\mathbb{R}^2} \frac{\chi(y - x_n\sqrt{n} - \lambda)}{m_V \left(x_n + \frac{\lambda}{\sqrt{n}} \right)} d\bar{g}_n(y) d\lambda \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \chi * \left(\mathbf{1}_{B_R(x_n\sqrt{n})} \frac{1}{m_V(\cdot/\sqrt{n})} \right) d\bar{g}_n(y) < C_R. \end{aligned}$$

As, by (3.7), $m_V(x) \leq M$ for any $x \in \mathbb{R}^2$, we get

$$\frac{1}{M\pi} \int_{\mathbb{R}^2} \chi * \mathbf{1}_{B_R(x_n\sqrt{n})} d\bar{g}_n(y) < C_R.$$

This and the fact that \bar{g}_n is bounded below implies that $\bar{g}_n(B_R(x_n\sqrt{n}))$ is bounded independently of n . Hence by the same argument as in [28, Lemma 6.3] we have the convergence of a subsequence of (ν_n, E_n, g_n) . Remark that we don't want the convergence of a subsequence of $(x_n)_n$.

2) We have the Γ -liminf property : if $(x_n, \nu_n, E_n, g_n) \rightarrow (x, \nu, E, g)$ as $n \rightarrow +\infty$ then

$$\liminf_{n \rightarrow +\infty} \mathbf{f}_n(x_n, \nu_n, E_n, g_n) \geq f(x, \nu, E, g) := \frac{1}{\pi} \int \frac{\chi(y)}{m_V(x)} dg(y)$$

obviously if the left-hand side is finite, by Fatou Lemma. Therefore Ergodic Theorem 5.4 implies that:

1. Q_n admits a subsequence which converges to Q which has μ_V for first marginal,
2. $(x, \nu, E, g) = \lim_{n \rightarrow +\infty} (x_n, \theta_{x_n\sqrt{n}}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n))$ Q -a.e.,
3. Q is $T_{\lambda(x)}$ -invariant,
4. $\liminf_{n \rightarrow +\infty} \mathbf{F}_n(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} dg(y) \right) dQ(x, \nu, E, g)$.
5. $\frac{1}{\pi} \int \int \frac{\chi(y)}{m_V(x)} dg(y) dQ(x, \nu, E, g) = \int \left(\lim_{R \rightarrow +\infty} \int_{B_R} \int \frac{\chi(y - \lambda)}{m_V(x)} dg(y) d\lambda \right) dQ(x, \nu, E, g)$.

Now we can follow exactly the lines of [28, Section 4.2, Step 3] to prove point 4), and to obtain, after noticing that P_n is the marginal of Q_n corresponding to the variables (x, E) which converge to a $T_{\lambda(x)}$ -invariant probability measure,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) &\geq \int \left(\int \chi dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} \\ &= \int \lim_{R \rightarrow +\infty} \left(\frac{1}{\pi R^2} \int \chi * \mathbf{1}_{B_R} dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} \\ &\geq \frac{1}{\pi} \int W(E) \frac{dQ(x, \nu, E, g)}{m_V(x)} = \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP(x, E). \end{aligned}$$

Thus the lower bound (6.1) is proved. The fact that the right-hand side is larger than α is obvious because the first marginal of $\frac{dP}{m_V}$ is the Lebesgue measure.

6.3 Proof of the upper bound by compactification and conclusion

Here we assume $\Sigma_V = \mathbb{R}^2$ because the study on this case is sufficient. Indeed, if $\Sigma_V \neq \mathbb{R}^2$ is unbounded, there exists $x_0 \in \mathbb{R}^2 \setminus \Sigma_V$ such that $B(x_0, b) \not\subset \Sigma_V$ and we take $\varphi(z) = \frac{b}{z - x_0}$. Moreover, as V is admissible, $\partial\Sigma_V$ is C^1 and $\varphi(\partial\Sigma_V) = \partial\Sigma_{V_\varphi}$ is C^1 . Furthermore equilibrium measure μ_{V_φ} exists and we can apply Sandier-Serfaty's construction from [28, Corollary 4.6] (see Step 1 below) in Σ_{V_φ} in order to obtain wanted sequences, in Σ_V and $L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$, as described below.

Let

$$\varphi(z) = -\frac{1}{z}$$

be the transformation of $PSL_2(\mathbb{C})$ which corresponds to $(a, b, c, d) = (0, 1, -1, 0)$ and we have

$$V_\varphi(x) = V(\varphi(x)) + 2 \log \|x\|.$$

Without loss of generality, we can assume that $\mu_V(B_1) = \mu_V(B_1^c) = 1/2$. Otherwise, there exists R such that $\mu_V(B_R) = \mu_V(B_R^c) = 1/2$ and we should use inversion $\varphi_R(z) = -Rz^{-1}$, with $(a, b, c, d) = (0, R^{1/2}, -R^{-1/2}, 0)$, which nothing change in our proof because V is admissible.

Our idea is to cut $\Sigma_V = \mathbb{R}^2$ into two parts in order to construct a sequence of $2n$ points associated to a sequence of vector fields. We will study only this case of $2n$ points because the method is exactly the same for any number $n = m + p$ of points but we need to cut \mathbb{R}^2 into two parts with measures proportional to m and p . We will cut $\Sigma_V = \mathbb{R}^2$ as follows :

- B_1 where we will construct sequences of n points and vector fields such that we have upper bound (6.2) for the problem – associated to V – in B_1 ;
- B_1^c that we will transport in B_1 by inversion φ in order to construct sequences of n points and vector fields for this problem – associated to V_φ – in B_1 , which we will deduce points and vector fields in B_1^c by inversion φ .

STEP 1 : Recall of compact case and notations

We need [28, Corollary 4.6] when K is a compact set of \mathbb{R}^2 :

Theorem 6.3. ([28]) *Let P be a $T_{\lambda(x)}$ -invariant probability measure on $X = K \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ with first marginal $dx|_K/|K|$ and such that for P almost every (x, E) we have $E \in \mathcal{A}_{m(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of empirical measures on K and a sequence $\{E_n\}_n$ in $L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{div} E_n = 2\pi(\nu_n' - m')$, $P_n := \int_K \delta_{(x, E_n(\sqrt{n}x+))} dx \rightarrow P$ as $n \rightarrow +\infty$ and*

$$\limsup_{n \rightarrow +\infty} F_n(\nu_n) \leq \frac{|K|}{\pi} \int W(E) dP(x, E).$$

We write $\mu_V = \mu_V^1 + \mu_V^2$ where $\mu_V^1 := \mu_V|_{B_1}$ and $\mu_V^2 := \mu_V|_{B_1^c}$. Let $\tilde{\mu}_V^2 := \varphi\#\mu_V^2$, then, with suitable notations m_V^1 and $m_{V_\varphi}^2$,

$$d\mu_V^1(x) = m_V(x)\mathbb{1}_{B_1}(x)dx = m_V^1(x)dx \quad \text{and} \quad d\tilde{\mu}_V^2(x) = m_{V_\varphi}(x)\mathbb{1}_{B_1}(x)dx = m_{V_\varphi}^2(x)dx.$$

By assumption **(H4)**, we have for any $x \in B_1$,

$$0 < m \leq m_V(x) \leq M, \quad \text{and} \quad 0 < m_\varphi \leq m_{V_\varphi}(x) \leq M_\varphi$$

and ∂B_1 is C^1 .

Let P be a $T_{\lambda(x)}$ -invariant probability measure on X whose first marginal is μ_V and such that for P -almost every (x, E) we have $E \in \mathcal{A}_{m_V(x)}$. We can write

$$P = P^1 + P^2$$

where P^1 is the restriction of P to $B_1 \times L_{loc}^p(B_1, \mathbb{R}^2)$ with first marginal μ_V^1 , and P^2 is the restriction of P to $B_1^c \times L_{loc}^p(B_1^c, \mathbb{R}^2)$ with first marginal μ_V^2 . We set

$$dP^1(x, u) = m_V(x)|B_1|d\tilde{P}^1(x, u)$$

then \tilde{P}^1 is a $T_{\lambda(x)}$ -invariant probability measure on $B_1 \times L_{loc}^p(B_1, \mathbb{R}^2)$ with first marginal $dx|_{B_1}/|B_1|$ and such that, for \tilde{P}^1 a.e. (x, E) , $E \in \mathcal{A}_{m_V^1(x)}$. We denote by $\varphi\#P^2$ the pushforward of P^2 by

$$(x, E) \mapsto (\varphi(x), \tilde{E} = \varphi\#E) \quad \text{where} \quad \tilde{E}(x) = (\varphi\#E)(x) := ((D_x\varphi)^T)^{-1}E(\varphi(x)). \quad (6.5)$$

for any $x \in B_1$ where $D_x\varphi$ is the differential of φ at point x and $(D_x\varphi)^T$ its transpose.

We set

$$d(\varphi\#P^2)(x, u) = m_{V_\varphi}(x)|B_1|d\tilde{P}^2(x, u)$$

and \tilde{P}^2 is a $T_{\lambda(x)}$ -invariant probability measure on $B_1 \times L_{loc}^p(B_1, \mathbb{R}^2)$ with first marginal $dx|_{B_1}/|B_1|$ and such that, for \tilde{P}^2 a.e. (\tilde{x}, \tilde{E}) , $\tilde{E} \in \mathcal{A}_{m_{V_\varphi}^2(\tilde{x})}$.

STEP 2 : Application of Theorem 6.3

Our idea is to apply Theorem 6.3 to \tilde{P}^1 and \tilde{P}^2 in order to construct a set of points and a set of vector fields as we want for upper bound (6.2).

Applying Theorem 6.3 to \tilde{P}^1 we construct a sequence $\left\{ \nu_n^1 := \sum_{i=1}^n \delta_{x_i^1} \right\}$ of empirical measures on B_1 and a sequence $\{E_n^1\}_n$ in $L_{loc}^p(B_1, \mathbb{R}^2)$ such that

$$\operatorname{div} E_n^1 = 2\pi((\nu_n^1)' - (m_V^1)') \quad \text{and} \quad \tilde{P}_n^1 := \int_{B_1} \delta_{(x, E_n^1(\sqrt{n}x+))} dx \rightarrow \tilde{P}^1$$

as $n \rightarrow +\infty$ and

$$\limsup_{n \rightarrow +\infty} F_n^1(\nu_n^1) \leq \frac{|B_1|}{\pi} \int W(E) d\tilde{P}^1(x, E). \quad (6.6)$$

where

$$F_n^1(\nu_n^1) = \frac{1}{n} \left(\frac{1}{\pi} W(E_n^1, \mathbb{1}_{\mathbb{R}^2}) + 2n \int \zeta^1 d\nu_n^1 \right)$$

with $\zeta^1(x) = U^{\mu_V^1}(x) + \frac{V(x)}{2} - c_V^1$ and c_V^1 is the Robin constant for the equilibrium problem on B_1 associated to V .

Applying now the same Theorem to \tilde{P}^2 , we construct a sequence $\left\{ \tilde{\nu}_n^2 := \sum_{i=1}^n \delta_{\tilde{x}_i^2} \right\}$ of empirical measures on B_1 and a sequence $\{\tilde{E}_n^2\}_n$ in $L_{loc}^p(B_1, \mathbb{R}^2)$ such that

$$\operatorname{div} \tilde{E}_n^2 = 2\pi((\tilde{\nu}_n^2)' - (m_{V_\varphi}^2)') \quad \text{and} \quad \tilde{P}_n^2 := \int_{B_1} \delta_{(x, \tilde{E}_n^2(\sqrt{n}x+))} dx \rightarrow \tilde{P}^2$$

as $n \rightarrow +\infty$ and

$$\limsup_{n \rightarrow +\infty} \tilde{F}_n^2(\tilde{\nu}_n^2) \leq \frac{|B_1|}{\pi} \int W(E) d\tilde{P}^2(x, E). \quad (6.7)$$

where

$$\tilde{F}_n^2(\tilde{\nu}_n^2) = \frac{1}{n} \left(\frac{1}{\pi} W(\tilde{E}_n^2, \mathbb{1}_{\mathbb{R}^2}) + 2n \int \tilde{\zeta}^2 d\tilde{\nu}_n^2 \right)$$

with $\tilde{\zeta}^2(x) = U^{\mu_{V_\varphi}^2}(x) + \frac{V_\varphi(x)}{2} - c_{V_\varphi}^2$ and $c_{V_\varphi}^2$ is the Robin constant for the equilibrium problem on B_1 associated to V_φ .

STEP 3: Construction of sequences and conclusion

It is not difficult to see that we can assume $\tilde{x}_j^2 \neq 0$ for any j and any $n \geq 2$ (otherwise we translate a little bit the Sandier-Serfaty construction). Now we set, by (6.5), for any n ,

$$\nu_n^2 := \varphi\#\tilde{\nu}_n^2 = \sum_{j=1}^n \delta_{x_j^2} \quad \text{and} \quad \forall x \in B_1^c, E_n^2(x) := (D_x \varphi)^T \tilde{E}_n^2(\varphi(x)),$$

with $x_j^2 = \varphi(\tilde{x}_j^2)$ for any $1 \leq j \leq n$. Hence, we have a sequence of vector-fields $E_n^2 \in L_{loc}^p(B_1^c, \mathbb{R}^2)$ such that

$$\operatorname{div} E_n^2 = 2\pi((\nu_n^2)' - (m_V^2)')$$

where $m_V^2(x) = m_V(x) \mathbb{1}_{\tilde{B}_1^c}(x)$ is the density of μ_V^2 . Moreover we have, for sufficiently small η ,

$$\begin{aligned} \int_{B_1 \setminus \bigcup_{i=1}^n B(\tilde{x}_i^2, \eta)} \|\tilde{E}_n^2(x)\|^2 dx &= \int_{B_1 \setminus \bigcup_{i=1}^n B(x_i^2, \tilde{\eta})} \|\tilde{E}_n^2(\varphi(x))\|^2 |\det D_x \varphi| dx \\ &= \int_{B_1^c \setminus \bigcup_{i=1}^n B(x_i^2, \tilde{\eta})} \|((D_x \varphi)^T)^{-1} E_n^2(x)\|^2 |\det D_x \varphi| dx \\ &= \int_{B_1^c \setminus \bigcup_{i=1}^n B(x_i^2, \tilde{\eta})} \|E_n^2(x)\|^2 dx \end{aligned}$$

hence we get

$$W(\tilde{E}_n^2, \mathbb{1}_{\mathbb{R}^2}) = W(E_n^2, \mathbb{1}_{\mathbb{R}^2}). \quad (6.8)$$

Now we will prove that, for any $x \in \bar{B}_1^c$,

$$\tilde{\zeta}^2(\varphi(x)) = \zeta^2(x) \quad (6.9)$$

where $\zeta^2(x) = U\mu_V^2(x) + \frac{V(x)}{2} - c_V^2$ and c_V^2 is the Robin constant for the equilibrium problem on \bar{B}_1^c associated to V . Indeed, we have

$$\begin{aligned} c_{V_\varphi}^2 &= I_{V_\varphi}(\tilde{\mu}_V^2) - \int_{B_1} \frac{V_\varphi(x)}{2} d\tilde{\mu}_V^2(x) \\ &= I_V(\mu_V^2) - \int_{B_1} \left(\frac{V(\varphi(x)) + 2 \log \|x\|}{2} \right) d\tilde{\mu}_V^2(x) \\ &= I_V(\mu_V^2) - \int_{\bar{B}_1^c} V(x) d\mu_V^2(x) + \int_{\bar{B}_1^c} \log \|x\| d\mu_V^2(x) \\ &= c_V^2 + \int_{\bar{B}_1^c} \log \|x\| d\mu_V^2(x). \end{aligned}$$

Hence, for any $x \in \bar{B}_1^c$,

$$\begin{aligned} \tilde{\zeta}^2(\varphi(x)) &= U\tilde{\mu}_V^2(\varphi(x)) + \frac{V_\varphi(\varphi(x))}{2} - c_{V_\varphi}^2 \\ &= - \int_{B_1} \log \|\varphi(x) - y\| d\tilde{\mu}_V^2(y) + \frac{V(x)}{2} - \log \|x\| - c_V^2 - \int_{\bar{B}_1^c} \log \|x\| d\mu_V^2(x) \\ &= - \int_{\bar{B}_1^c} \log \|\varphi(x) - \varphi(y)\| d\mu_V^2(y) + \frac{V(x)}{2} - \log \|x\| - c_V^2 - \int_{\bar{B}_1^c} \log \|x\| d\mu_V^2(x) \\ &= U\mu_V^2(x) + \int_{\bar{B}_1^c} (\log \|x\| + \log \|y\|) d\mu_V^2(y) + \frac{V(x)}{2} - \log \|x\| - c_V^2 - \int_{\bar{B}_1^c} \log \|x\| d\mu_V^2(x) \\ &= U\mu_V^2(x) + \frac{V(x)}{2} - c_V^2 \\ &= \zeta^2(x). \end{aligned}$$

By (6.8) and (6.9), we have

$$\tilde{F}_n^2(\tilde{\nu}_n^2) = F_n^2(\nu_n^2)$$

where

$$F_n^2(\nu_n^2) = \frac{1}{n} \left(\frac{1}{\pi} W(E_n^2, \mathbb{1}_{\mathbb{R}^2}) + 2n \int \zeta^2 d\nu_n^2 \right)$$

and by (6.7) we get

$$\limsup_{n \rightarrow +\infty} F_n^2(\nu_n^2) \leq \frac{|B_1|}{\pi} \int W(E) d\tilde{P}^2(x, E). \quad (6.10)$$

Finally me set

$$\nu_{2n} := \nu_n^1 + \nu_n^2 \quad \text{and} \quad E_{2n} := E_n^1 + E_n^2$$

and by (6.6) and (6.10), we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} F_{2n}(\nu_{2n}) &\leq \frac{|B_1|}{\pi} \int W(E) dP^1(x, E) + \frac{|B_1|}{\pi} \int W(E) d\tilde{P}^2(x, E) \\ &= \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP^1(x, E) + \frac{1}{\pi} \int \frac{W(E)}{m_{V_\varphi}(x)} d(\varphi\#P^2)(x, E) \\ &= \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP^1(x, E) + \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP^2(x, E) \end{aligned}$$

$$= \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP(x, E)$$

which prove our upper bound (6.2). Furthermore, by changes of variable,

$$P_n^1 := \int_{B_1} \delta_{(x, E_n^1(x\sqrt{n+}))} d\mu_V(x) \rightarrow P^1 \quad \text{and} \quad P_n^2 := \int_{B_1^c} \delta_{(x, E_n^2(x\sqrt{n+}))} d\mu_V(x) \rightarrow P^2$$

in the weak sense of measure, and it follows that

$$P_n = P_n^1 + P_n^2 \rightarrow P^1 + P^2 = P.$$

Part **C** follows from **A** and **B** : inequalities (6.1) and (6.2) become equalities. By Gamma-Convergence we minimize $\frac{1}{\pi} \int \frac{W}{m_V} dP$ over vector fields of \mathcal{A}_{m_V} and we find α .

7 Consequence : the Logarithmic Energy on the Sphere

As we have asymptotic expansion of the minimum of Hamiltonian w_n where minimizers can be in the whole plane – not only in a compact set as in classical case – we will use inverse stereographic projection from \mathbb{R}^2 to a sphere in order to determine asymptotics of optimal logarithmic energy on sphere.

7.1 Inverse stereographic projection

Here we recall properties of the inverse stereographic projection used by Hardy and Kuijlaars in [16, 17] and by Bloom, Levenberg and Wielonsky in [3] in order to prove Theorem 3.2.

Let \mathcal{S} be the sphere of \mathbb{R}^3 centred in $(0, 0, 1/2)$ of radius $1/2$, Σ a unbounded closed set of \mathbb{R}^2 and $T : \mathbb{R}^2 \rightarrow \mathcal{S}$ the associated inverse stereographic projection defined by

$$T(x_1, x_2) = \left(\frac{x_1}{1 + \|x\|^2}, \frac{x_2}{1 + \|x\|^2}, \frac{\|x\|^2}{1 + \|x\|^2} \right), \text{ for any } x = (x_1, x_2) \in \mathbb{R}^2$$

where $\|\cdot\|$ is the Euclidean norm of \mathbb{R}^3 , with $\mathbb{R}^2 := \{(x_1, x_2, 0); x_1, x_2 \in \mathbb{R}\}$. We know that T is a conformal homeomorphism from \mathbb{C} to $\mathcal{S} \setminus \{N\}$ where $N := (0, 0, 1)$ is the North pole of \mathcal{S} .

We have the following identity :

$$\|T(x) - T(y)\| = \frac{\|x - y\|}{\sqrt{1 + \|x\|^2} \sqrt{1 + \|y\|^2}}, \text{ for any } x, y \in \mathbb{R}^2$$

and if $\|y\| \rightarrow +\infty$ we obtain, for any $x \in \mathbb{R}^2$:

$$\|T(x) - N\| = \frac{1}{\sqrt{1 + \|x\|^2}}. \tag{7.1}$$

We note $\Sigma_{\mathcal{S}} = T(\Sigma) \cup \{N\}$ the closure of $T(\Sigma)$ in \mathcal{S} . Let $\mathcal{M}_1(\Sigma)$ be the set of probability measures on Σ . For $\mu \in \mathcal{M}_1(\Sigma)$, we denote by $T\#\mu$ its push-forward measure by T characterized by

$$\int_{\Sigma_{\mathcal{S}}} f(z) dT\#\mu(z) = \int_{\Sigma} f(T(x)) d\mu(x)$$

for every Borel function $f : \Sigma_{\mathcal{S}} \rightarrow \mathbb{R}$. We have the following important result due to Hardy :

Lemma 7.1. ([16, Lemma 2.1]) *Application $T\#$ is an homeomorphism from the space $\mathcal{M}_1(\Sigma)$ to the set $\{\mu \in \mathcal{M}_1(\Sigma_{\mathcal{S}}); \mu(N) = 0\}$.*

7.2 Asymptotic of the optimal logarithmic energy on the unit sphere

An important case is the equilibrium measure associated to the potential

$$V(x) = \log(1 + \|x\|^2)$$

corresponding to the external field $\mathcal{V} \equiv 0$ on \mathcal{S} and where $T\sharp\mu_V$ is the uniform probability measure on \mathcal{S} . Hence (see [16, Remark 2.2]) we find

$$d\mu_V(x) = \frac{dx}{\pi(1 + \|x\|^2)^2} \quad \text{and} \quad \Sigma_V = \mathbb{R}^2.$$

Moreover, V is clearly admissible because, for any $\varphi \in PSL_2(\mathbb{C})$,

- $V_\varphi(x) = \log(|cx + d|^2 + |ax + b|^2)$ which satisfies **(H1)** and $V_\varphi(0) = \log(|d|^2 + |b|^2)$ is finite because $ad - bc = 1$, then **(H2)** is also satisfied.
- Density $m_{V_\varphi}(x) = \frac{m_V(\varphi(x))}{|cx + d|^4} = \frac{1}{\pi(|ax + b|^2 + |cx + d|^2)^2}$ is C^1 .
- As $x \mapsto |ax + b|^2 + |cx + d|^2 \neq 0$ is continuous on compact B_1 , **(H4)** is satisfied.

We define

$$\bar{w}_n(x_1, \dots, x_n) := - \sum_{i \neq j}^n \log \|x_i - x_j\| + (n-1) \sum_{i=1}^n \log(1 + \|x_i\|^2)$$

and the logarithmic energy of a configuration $(y_1, \dots, y_n) \in \mathcal{S}^n$ is

$$E_{\log}(y_1, \dots, y_n) := - \sum_{i \neq j}^n \log \|y_i - y_j\|.$$

Lemma 7.2. *For any $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$, we have the following equalities*

$$\bar{w}_n(x_1, \dots, x_n) = E_{\log}(T(x_1), \dots, T(x_n)) \quad \text{and} \quad w_n(x_1, \dots, x_n) = E_{\log}(T(x_1), \dots, T(x_n), N)$$

which imply that

$$\begin{aligned} (x_1, \dots, x_n) \text{ minimizes } \bar{w}_n &\iff (T(x_1), \dots, T(x_n)) \text{ minimizes } E_{\log} \\ (x_1, \dots, x_n) \text{ minimizes } w_n &\iff (T(x_1), \dots, T(x_n), N) \text{ minimizes } E_{\log}. \end{aligned}$$

Proof. For any $1 \leq i \leq n$, we set $y_i := T(x_i)$, hence we get, by (7.1),

$$\begin{aligned} E_{\log}(y_1, \dots, y_n) &:= - \sum_{i \neq j}^n \log \|y_i - y_j\| \\ &= - \sum_{i \neq j}^n \log \|T(x_i) - T(x_j)\| \\ &= - \sum_{i \neq j}^n \log \left(\frac{\|x_i - x_j\|}{\sqrt{1 + \|x_i\|^2} \sqrt{1 + \|x_j\|^2}} \right) \\ &= - \sum_{i \neq j}^n \log \|x_i - x_j\| + (n-1) \sum_{i=1}^n \log(1 + \|x_i\|^2) \\ &= \bar{w}_n(x_1, \dots, x_n). \end{aligned}$$

Furthermore, by (7.1), we obtain

$$\begin{aligned} w_n(x_1, \dots, x_n) &= \bar{w}_n(x_1, \dots, x_n) + \sum_{i=1}^n \log(1 + \|x_i\|^2) \\ &= - \sum_{i \neq j} \log \|y_i - y_j\| - 2 \sum_{i=1}^n \log \|y_i - N\| = E_{\log}(y_1, \dots, y_n, N). \end{aligned}$$

□

Lemma 7.3. *If (x_1, \dots, x_n) minimizes w_n or \bar{w}_n , then, for $\nu_n := \sum_{i=1}^n \delta_{x_i}$, we have*

$$\frac{\nu_n}{n} \rightarrow \mu_V \quad \text{as } n \rightarrow +\infty,$$

in the weak sense of measures.

Proof. Let (x_1, \dots, x_n) be a minimizer of \bar{w}_n , then $(T(x_1), \dots, T(x_n))$ is a minimizer of E_{\log} . Brauchart, Dragnev and Saff proved in [6, Proposition 11] that

$$\frac{1}{n} \sum_{i=1}^n \delta_{T(x_i)} \rightarrow T\#\mu_V.$$

As $T\#\mu_V(N) = 0$, by Lemma 7.1 we get the result.

If (x_1, \dots, x_n) is a minimizer of w_n , then $(T(x_1), \dots, T(x_n), N)$ minimizes E_{\log} and we can use our previous argument because

$$\frac{1}{n+1} \left(\sum_{i=1}^n \delta_{T(x_i)} + \delta_N \right) = \frac{1}{n} \sum_{i=1}^n \delta_{T(x_i)} \left(\frac{n}{n+1} \right) + \frac{\delta_N}{n+1} \rightarrow T\#\mu_V$$

in the weak sense of measures and it follows that $\frac{1}{n} \sum_{i=1}^n \delta_{T(x_i)} \rightarrow T\#\mu_V$ in the weak sense of measures. □

Lemma 7.4. *If (x_1, \dots, x_n) is a minimizer of \bar{w}_n or w_n and if $\nu_n := \sum_{i=1}^n \delta_{x_i}$ then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) = \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x)$$

Proof. Let (x_1, \dots, x_n) be a minimizer of \bar{w}_n . We define $y_i := T(x_i)$ for any i and we notice that

$$\frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) = -\frac{2}{n} \int_{\mathbb{R}^2} \log \left(\frac{1}{\sqrt{1 + \|x\|^2}} \right) d\nu_n(x) = -\frac{2}{n} \int_{\mathcal{S}} \log \|y - N\| dT\#\nu_n(y),$$

and by the previous Lemma, (y_1, \dots, y_n) is a minimizer of E_{\log} on \mathcal{S} . Now we use [6, Theorem 15] about the optimal point separation which yields the existence of constants C and n_0 such that for any $n \geq n_0$ and any minimizer $\{y_1, \dots, y_n\} \in \mathcal{S}^n$ of the logarithmic energy on the sphere, we have

$$\min_{i \neq j} \|y_i - y_j\| > \frac{C}{\sqrt{n-1}}.$$

Without loss of generality, we can assume that N is such that, for any $n \geq n_0$ and any $1 \leq i \leq n$

$$\|y_i - N\| > \frac{C}{\sqrt{n-1}}. \quad (7.2)$$

For $n \geq n_0$ and $\delta > 0$ sufficiently small we define, for any $0 < r \leq \delta$,

$$n(r) := \#\{y_i; y_i \in B(N, r) \cap \mathcal{S}\}$$

and $r_i = \|y_i - N\|$ where $y_i \in B(N, \delta) \cap \mathcal{S}$. We notice that there exists a constant C such that $n(r) \leq Cr^2n$ for any r . Hence we have, by integration by parts and the separation (7.2) :

$$\begin{aligned} -\sum_{i=1}^{n_\delta} \log r_i &= -\int_{1/\sqrt{n-1}}^{\delta} \log rn'(r)dr \\ &= -\log \delta n(\delta) + \int_{1/\sqrt{n-1}}^{\delta} \frac{n(r)}{r} dr \\ &\leq -C\delta^2 \log \delta n + Cn \int_{1/\sqrt{n-1}}^{\delta} r dr \\ &\leq C\delta^2 |\log \delta| n. \end{aligned}$$

Thus for $(y_1, \dots, y_{n_\delta}) \in B(N, \delta) \cap \mathcal{S}$, it follows that

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B(N, \delta) \cap \mathcal{S}} \log \|y - N\| dT\#\nu_n(y) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} -\sum_{i=1}^{n_\delta} \frac{1}{n} \log \|y_i - N\| = 0. \quad (7.3)$$

By Lemma 7.3, $\frac{\nu_n}{n}$ goes weakly to the measure μ_V on B_R for any R , hence we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) &= \lim_{n \rightarrow +\infty} \frac{1}{n} \left(\int_{B_R} \log(1 + \|x\|^2) d\nu_n(x) + \int_{B_R^c} \log(1 + \|x\|^2) d\nu_n(x) \right) \\ &= \int_{B_R} \log(1 + \|x\|^2) d\mu_V(x) + \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B_R^c} \log(1 + \|x\|^2) d\nu_n(x). \end{aligned}$$

Therefore it follows from (7.3) that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) &= \lim_{R \rightarrow +\infty} \left(\int_{B_R} \log(1 + \|x\|^2) d\mu_V(x) + \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B_R^c} \log(1 + \|x\|^2) d\nu_n(x) \right) \\ &= \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x). \end{aligned}$$

If (x_1, \dots, x_n) is a minimizer of w_n , by Lemmas 7.2 and 7.3, we can use exactly the same argument of separation, therefore the convergence is proved. \square

The following result proves the existence of the constant C in the Conjecture 1 of Rakhmanov, Saff and Zhou.

Theorem 7.5. *If $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$ is a minimizer of E_{\log} then, as $n \rightarrow +\infty$*

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 \right) n + o(n).$$

Proof. As E_{\log} is invariant by translation of the 2-sphere, we work on the sphere $\tilde{\mathbb{S}}^2$ of radius 1 and centred in $(0, 0, 1/2)$. Let $(y_1, \dots, y_n) \in \tilde{\mathbb{S}}^2$ be a minimizer of E_{\log} . Without loss of generality, for any n we can choose this configuration such that $y_i \neq N$ for any $1 \leq i \leq n$, hence there exists (x_1, \dots, x_n) such that $\frac{y_i}{2} = T(x_i)$ for any i and we get

$$E_{\log}(y_1, \dots, y_n) = -\sum_{i \neq j}^n \log \|y_i - y_j\|$$

$$\begin{aligned}
&= - \sum_{i \neq j}^n \log \|T(x_i) - T(x_j)\| - n(n-1) \log 2 \\
&= \bar{w}_n(x_1, \dots, x_n) - n(n-1) \log 2.
\end{aligned}$$

and by Lemma 7.2, (y_1, \dots, y_n) is a minimizer of E_{\log} if and only if (x_1, \dots, x_n) is a minimizer of \bar{w}_n . By the lower bound (6.1) and the convergence of Lemma 7.4 we have, for a minimizer $(\bar{x}_1, \dots, \bar{x}_n)$ of \bar{w}_n :

$$\begin{aligned}
&\liminf_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
&= \liminf_{n \rightarrow +\infty} \frac{1}{n} \left[w_n(\bar{x}_1, \dots, \bar{x}_n) - \sum_{i=1}^n \log(1 + \|\bar{x}_i\|^2) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
&\geq \alpha - \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x).
\end{aligned}$$

Upper bound (6.2) and Lemma 7.3 yield, for $(\bar{x}_1, \dots, \bar{x}_n)$ a minimizer of \bar{w}_n and (x_1, \dots, x_n) a minimizer of w_n :

$$\begin{aligned}
&\limsup_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
&\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(x_1, \dots, x_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
&= \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[w_n(x_1, \dots, x_n) - \sum_{i=1}^n \log(1 + \|x_i\|^2) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
&= \alpha - \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x).
\end{aligned}$$

Thus we get

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] = \alpha - \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x)$$

and we have the following asymptotic expansion, as $n \rightarrow +\infty$, when $(\bar{x}_1, \dots, \bar{x}_n)$ is a minimizer of \bar{w}_n :

$$\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) = n^2 I_V(\mu_V) - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx - \int_{\mathbb{R}^2} V(x) d\mu_V(x) \right) n + o(n).$$

We know that $I_V(\mu_V) = \frac{1}{2}$ (see [5, Eq. (2.26)]) and

$$\begin{aligned}
\int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x) &= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\log(1 + \|x\|^2)}{(1 + \|x\|^2)^2} \\
&= 2 \int_0^{+\infty} \frac{r \log(1 + r^2)}{(1 + r^2)^2} dr \\
&= - \left[\frac{\log(1 + r^2)}{1 + r^2} \right]_0^{+\infty} + \int_{\mathbb{R}^2} \frac{2r}{(1 + r^2)^2} dr \\
&= - \left[\frac{1}{1 + r^2} \right]_0^{+\infty} \\
&= 1.
\end{aligned}$$

Hence we obtain, as $n \rightarrow +\infty$,

$$\begin{aligned}\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) &= \frac{n^2}{2} - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{1}{2} \int \log(\pi(1 + \|x\|^2)^2) d\mu_V(x) - 1 \right) n + o(n) \\ &= \frac{n^2}{2} - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \int \log(1 + \|x\|^2) d\mu_V(x) - 1 \right) n + o(n) \\ &= \frac{n^2}{2} - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} \right) n + o(n)\end{aligned}$$

and the asymptotic expansion of E_{\log} , for its minimizer (y_1, \dots, y_n) , is, as $n \rightarrow +\infty$:

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 \right) n + o(n).$$

□

Remark 7.6. It follows from lower bound proved by Rakhmanov, Saff and Zhou in [24, Theorem 3.1], that

$$\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 = \lim_{n \rightarrow +\infty} \frac{1}{n} \left[E_{\log}(y_1, \dots, y_n) - \left(\frac{1}{2} - \log 2 \right) n^2 + \frac{n}{2} \log n \right] \geq -\frac{1}{2} \log \left[\frac{\pi}{2} (1 - e^{-a})^b \right]$$

where $a := \frac{2\sqrt{2\pi}}{\sqrt{27}} \left(\sqrt{2\pi + \sqrt{27}} + \sqrt{2\pi} \right)$ and $b := \frac{\sqrt{2\pi + \sqrt{27}} - \sqrt{2\pi}}{\sqrt{2\pi + \sqrt{27}} + \sqrt{2\pi}}$ and we get

$$\min_{\mathcal{A}_1} W \geq -\frac{\pi}{2} \log \left[2\pi^2 (1 - e^{-a})^b \right] \approx -4.6842707.$$

7.3 Computation of renormalized energy for the triangular lattice and upper bound for the term of order n

Sandier and Serfaty proved in [27, Lemma 3.3] that

$$W(\Lambda_{1/2\pi}) = -\frac{1}{2} \log \left(\sqrt{2\pi b} |\eta(\tau)|^2 \right)$$

where $\Lambda_{1/2\pi}$ is the triangular lattice corresponding to the density $m = 1/2\pi$, $\tau = a + ib = 1/2 + i\frac{\sqrt{3}}{2}$ and η is the Dedekind eta function defined, with $q = e^{2i\pi\tau}$, by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

We recall Chowla-Selberg formula (see [9] or [10, Proposition 10.5.11] for details) :

$$\prod_{m=1}^{|D|} \Gamma \left(\frac{m}{|D|} \right)^{\frac{w}{2} \left(\frac{D}{m} \right)} = 4\pi \sqrt{-D} b |\eta(\tau)|^4$$

for τ a root of the integral quadratic equation $\alpha z^2 + \beta z + \gamma = 0$ where $D = b^2 - 4ac < 0$, $\left(\frac{D}{m} \right)$ is the Kronecker symbol, w the number of roots of unity in $\mathbb{Q}(i\sqrt{-D})$ and when the class number of $\mathbb{Q}(i\sqrt{-D})$ is equal to 1. In our case $b = \sqrt{3}/2$, $w = 6$, $\alpha = \beta = \gamma = 1$ because τ is a root of unity, hence $D = -3$, $\left(\frac{-3}{1} \right) = 1$ and $\left(\frac{-3}{2} \right) = -1$ by the Gauss Lemma. Therefore we obtain

$$\Gamma(1/3)^3 \Gamma(2/3)^{-3} = 4\pi \sqrt{3} \frac{\sqrt{3}}{2} |\eta(\tau)|^4$$

and by Euler's reflection formula $\Gamma(1 - 1/3)\Gamma(1/3) = \frac{\pi}{\sin(\pi/3)}$ we get

$$\frac{\Gamma(1/3)^6 3\sqrt{3}}{8\pi^3} = \frac{4\pi\sqrt{3} \times \sqrt{3}|\eta(\tau)|^4}{2}$$

and finally we have

$$|\eta(\tau)|^4 = \frac{\Gamma(1/3)^6 \sqrt{3}}{16\pi^4}.$$

Now it is possible to find the exact value of the renormalized energy of the triangular lattice Λ_1 of density $m = 1$:

$$\begin{aligned} W(\Lambda_1) &= 2\pi W(\Lambda_{1/2\pi}) - \pi \frac{\log(2\pi)}{2} \\ &= -\pi \log\left(\sqrt{2\pi b}|\eta(\tau)|^2\right) - \pi \frac{\log(2\pi)}{2} \\ &= \pi \log \pi - \frac{\pi}{2} \log 3 - 3\pi \log(\Gamma(1/3)) + \frac{3}{2}\pi \log 2 \\ &= \pi \log\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\Gamma(1/3)^3}\right) \\ &\approx -4.1504128. \end{aligned}$$

Thus we get

$$\begin{aligned} &\frac{1}{\pi}W(\Lambda_1) + \frac{\log \pi}{2} + \log 2 \\ &= \frac{1}{\pi}\left(\pi \log \pi - \frac{\pi}{2} \log 3 - 3\pi \log(\Gamma(1/3)) + \frac{3}{2}\pi \log 2\right) + \frac{\log \pi}{2} + \log 2 \\ &= 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} = C_{BHS} \approx -0.0556053 \end{aligned}$$

and we find exactly the value C_{BHS} conjectured by Brauchart, Hardin and Saff in [7, Conjecture 4]. Therefore Conjecture 2 is true if and only if the triangular lattice Λ_1 is a global minimizer of W among vector fields in \mathcal{A}_1 , i.e.

$$\min_{\mathcal{A}_1} W = W(\Lambda_1) = \pi \log\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\Gamma(1/3)^3}\right).$$

Thus we obtain the following result

Theorem 7.7. *We have :*

1. *If (y_1, \dots, y_n) is a minimizer of E_{\log} , then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[E_{\log}(y_1, \dots, y_n) - \left(\frac{1}{2} - \log 2\right) n^2 + \frac{n}{2} \log n \right] \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}$$

2. *Conjectures 2 and 3 are equivalent, i.e. $\min_{\mathcal{A}_1} W = W(\Lambda_1) \iff C = C_{BHS}$.*

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