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Renormalized Energy and Asymptotic Expansion of Optimal Logarithmic Energy on the Sphere

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Abstract

We study the Hamiltonian of a two-dimensional Coulomb system of n repelling points confined by an external field verifying the weak growth assumption of Hardy and Kuijlaars as in [15, 16]. We prove an asymptotic expansion (as $n \rightarrow +\infty$) for the minimum of this Hamiltonian using the Gamma-Convergence's method of Sandier and Serfaty in [24] and depending on the minimum of a “renormalized energy” W introduced in [23]. We connect our result with the next-order term for optimal logarithmic energy on \mathbb{S}^2 to prove the conjecture of Rakhmanov, Saff and Zhou in [20] about the existence of this term for which we find an upper bound. Finally we prove the equivalence between the conjecture of Brauchart, Hardin and Saff in [5] about the value of this coefficient and the conjecture of the global minimality of the triangular lattice for W among configurations of fixed average density.

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1 Introduction

Let $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ be a configuration subjected to a logarithmic potential and confined by an external field V . We define the Hamiltonian of this system, also called “Coulomb gas”, by

$$w_n(x_1, \dots, x_n) := - \sum_{i \neq j}^n \log \|x_i - x_j\| + n \sum_{i=1}^n V(x_i)$$

where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^2 and V is a potential satisfying some growth assumption and some properties detailed below. The discrete energy w_n is linked to the following continuous problem of logarithmic potential theory : find a positive probability measure μ_V on \mathbb{R}^2 which minimizes

$$I_V(\mu) := - \iint_{\mathbb{R}^2} \log \|x - y\| d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} V(x) d\mu(x).$$

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This kind of ‘equilibrium problem’ dates back to Gauss and has been studied among others by Frostman in [13] and by Saff and Totik in [21]. The classical growth assumption for V

$$\lim_{\|x\| \rightarrow +\infty} \frac{V(x)}{2} - \log \|x\| = +\infty$$

leads to a unique measure μ_V with compact support. Recently, Hardy and Kuijlaars gave in [15, 16] the following weak assumption

$$\liminf_{\|x\| \rightarrow +\infty} \{V(x) - \beta \log \|x\|\} > -\infty$$

where $\beta \geq 2$, which will be used throughout this paper, so that the support of μ_V can be unbounded.

Sandier and Serfaty introduced in [23] a Coulombian interaction for an infinity of points in the plane with a uniform neutralized background called ‘renormalized energy’ W . This energy can be seen as an energy of planar interactions between vortices in the Ginzburg-Landau theory of superconductors. In [24] a ‘splitting formula’ is used to connect w_n and W in case of compact support for μ_V . Moreover the authors give an asymptotic expansion as $n \rightarrow +\infty$, of $w_n(x_1, \dots, x_n)$, when (x_1, \dots, x_n) is a minimizer, which is equal to $n^2 I_V(\mu_V) - \frac{n}{2} \log n$ plus a term of order n which depends on μ_V and on the minimum of W among the configurations of fixed density one. In this paper we prove the very same formula when V satisfies the growth assumption on V of Hardy and Kuijlaars in [15, 16] following the same lines as in [24]. As the support of μ_V can be the whole plane, our idea is to connect this asymptotic expansion with the equilibrium problem on the sphere by an inverse stereographic projection.

More precisely, on the unit sphere, Brauchart, Hardin and Saff, in [5], give important conjectures about the asymptotic of the logarithmic energy

$$E_{\log}(y_1, \dots, y_n) := - \sum_{i \neq j}^n \log \|y_i - y_j\|$$

for its minimizer on the unit sphere. This problem is linked with various topics studied in the literature as explained in [17], like the existence of large stable molecules of spherical points (for example the C_{60} buckminsterfullerene), or the computations problems from analysis of satellite data by arithmetic averages at some well-chosen points, also called ‘spherical design’ (see, among others, [11, 27, 1, 8, 10, 9]) or finally the 7th ‘Smale’s Problem for the Twentieth Century’ (see [26]), i.e. to find, for any $n \geq 2$, a universal constant $c \in \mathbb{R}$ and a configuration $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$ such that

$$E_{\log}(y_1, \dots, y_n) - \min_{\{y_i\} \in \mathbb{S}^2} E_{\log}(y_1, \dots, y_n) \leq c \log n.$$

In this paper we focus on the term of order n which contribute to the understanding of the Smale’s problem. Indeed it is known that if (y_1, \dots, y_n) is a minimizer of E_{\log} on \mathbb{S}^2 then there exist some constants c_1 and c_2 such that

$$\frac{1}{2} - \log 2 - \frac{\log n}{2n} + \frac{c_1}{n} \leq \frac{E_{\log}(y_1, \dots, y_n)}{n^2} \leq \frac{1}{2} - \log 2 - \frac{\log n}{2n} + \frac{c_2}{n},$$

where the lower bound has been derived by Wagner in [28] and where the upper bound is due to Kuijlaars and Saff in [18]. Therefore we have the following asymptotic expansion for a minimizer (y_1, \dots, y_n) of E_{\log} :

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2\right) n^2 - \frac{n}{2} \log n + O(n).$$

The two following conjectures, given by Saff *et al.*, are about the next-order term in this asymptotic expansion, i.e. the expansion of the $O(n)$. The first one comes from [20] and concerns the existence of the term of

order n , i.e. the existence of the following limit for a minimizer (y_1, \dots, y_n) of E_{\log} :

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[E_{\log}(y_1, \dots, y_n) - \left(\frac{1}{2} - \log 2 \right) n^2 + \frac{n}{2} \log n \right].$$

CONJECTURE 1 ([20, 5]): There exists a constant C , independent of n , such that, if $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$ minimizes E_{\log} for any n ,

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + Cn + o(n) \text{ as } n \rightarrow +\infty.$$

As we prove an asymptotic expansion in the whole plane for w_n , we transport this expansion on \mathbb{S}^2 by an inverse stereographic projection to find C , which depends on the minimum of the energy W . The second conjecture for this asymptotic expansion on the sphere is about the exact value of C which comes from an other conjecture (Conjecture 3 of [5]) for the term of order n in the expansion of optimal Riesz energy on the unit sphere by analytic continuation combined with the fact that the derivative of the Riesz potential is $\lim_{s \rightarrow 0} (\|x\|^{-s} - 1) / s = -\log \|x\|$.

CONJECTURE 2 ([5]) : $C = 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}$.

Because our order n term depends on the global minimizer of W , the value of this minimum is a key point. The following conjecture is linked with the Ginzburg-Landau theory of superconductors (see [23] or the review [25] of Serfaty for the link between W and the vortices in the Ginzburg-Landau theory).

CONJECTURE 3 ([23]) : The triangular lattice is a global minimizer of W for a fixed density of points.

With a formula from [23] and the Chowla-Selberg formula, we compute the exact value of the renormalized energy for the triangular lattice and we find the value of Conjecture 2 if Conjecture 3 is true. Thus our results can be summarized by the following theorem :

THEOREM 1.1. (1) For (x_1, \dots, x_n) a minimizer of w_n for any n , we have the asymptotic expansion

$$w_n(x_1, \dots, x_n) = I_V(\mu_V)n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx \right) n + o(n) \text{ as } n \rightarrow +\infty$$

where $\mu_V(dx) = m_V(x)dx$ is the equilibrium measure associated to the external field V and \mathcal{A}_1 is the set of configurations of average density 1.

(2) There exists C independent of n such that, for (y_1, \dots, y_n) a minimizer of E_{\log} for any n , as $n \rightarrow +\infty$,

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + Cn + o(n)$$

and more precisely $C = \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2$.

(3) We have the following upper bound : $C \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}$.

(4) $\min_{\mathcal{A}_1} W$ is achieved for the triangular lattice of density one $\iff C = 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}$.

This last result, that is the equivalence of Conjectures 2 and 3, is somehow surprising as it links two different domains of analysis and provides another good motivation to prove one of these conjectures.

We proceed as follows. In Section 2 we recall the definition of W and important results from [23]. We prove classical results of Logarithmic Potential Theory in Section 3 with the weak growth assumption on the external field and we give sufficient assumptions on μ_V to prove our asymptotic formula. In Section 4 we derive the fundamental splitting formula which links w_n with the renormalized energy W . Three useful lemmas are given in Section 5 : a mass spreading result, its link with W and an Ergodic Theorem. The main theorem of this paper is stated and proved in Section 6 and gives the asymptotic expansion of w_n . Finally we prove in Section 7 the Conjecture 1 about the existence of C , the upper bound for this coefficient and the equivalence between Conjectures 2 and 3.

2 Renormalized Energy

Here we recall the definition of the renormalized energy W (see [24] for more details).

Definition 2.1. *Let m be a nonnegative number. For any continuous function χ and any vector field E in \mathbb{R}^2 such that*

$$\operatorname{div} E = 2\pi(\nu - m) \text{ and } \operatorname{curl} E = 0 \quad (2.1)$$

where ν has the form

$$\nu = \sum_{p \in \Lambda} \delta_p, \text{ for some discrete set } \Lambda \subset \mathbb{R}^2, \quad (2.2)$$

we let

$$W(E, \chi) = \lim_{\eta \rightarrow 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p, \eta)} \chi |E|^2 + \pi \log \eta \sum_{p \in \Lambda} \chi(p) \right). \quad (2.3)$$

Definition 2.2. *Let m be a nonnegative number and B_R denotes the closed ball of radius R centred at the origin. Let E be a vector field in \mathbb{R}^2 . We say E belongs to the admissible class \mathcal{A}_m if (2.1), (2.2) hold and*

$$\frac{\nu(B_R)}{|B_R|} \text{ is bounded by a constant independent of } R > 1.$$

Remark 2.1. The real m is the average density of the points of Λ when $E \in \mathcal{A}_m$.

We use the notation χ_{B_R} for positive cutoff functions satisfying, for some constant C independent of R

$$|\nabla \chi_{B_R}| \leq C, \operatorname{Supp}(\chi_{B_R}) \subset B_R, \chi_{B_R}(x) = 1 \text{ if } d(x, B_R^c) \geq 1. \quad (2.4)$$

Definition 2.3. *The renormalized energy W is defined, for $E \in \mathcal{A}_m$ and $\{\chi_{B_R}\}_R$ satisfying (2.4), by*

$$W(E) = \limsup_{R \rightarrow +\infty} \frac{W(E, \chi_{B_R})}{|B_R|}.$$

Remark 2.2. In [23], Sandier and Serfaty introduced W as being computed with averages over general shapes and showed that the minimum of W over \mathcal{A}_m does not depend on the shape used. It was shown in the same paper that :

- the value of W does not depend on the choice of the cutoff functions as long as it satisfies (2.4),
- W is bounded below and admits a minimizer over \mathcal{A}_1 ,
- if $E \in \mathcal{A}_m$, $m > 0$, then $E' = \frac{1}{\sqrt{m}} E(\cdot/\sqrt{m}) \in \mathcal{A}_1$, $W(E) = m \left(W(E') - \frac{\pi}{2} \log m \right)$ and consequently

$$\min_{\mathcal{A}_m} W = m \left(\min_{\mathcal{A}_1} W - \frac{\pi}{2} \log m \right) \quad (2.5)$$

therefore E is a minimizer of W over \mathcal{A}_m if and only if E' minimizes W over \mathcal{A}_1 .

- When Λ is a Bravais lattice $\mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v}$ then W can be expressed explicitly through the Epstein zeta function of this lattice. Moreover, using results from Number Theory, it is proved in Theorem 2 of [23] that the unique minimizer of W over periodic lattice configurations of fixed density m is the Abrikosov triangular lattice

$$\Lambda_m = \sqrt{\frac{2}{m\sqrt{3}}} \left(\mathbb{Z}(1, 0) \oplus \mathbb{Z} \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right).$$

This supports the conjecture that the Abrikosov triangular lattice is a global minimizer of W among all configurations with fixed density.

3 Equilibrium Problem in the Whole Plane

This part follows the work of Hardy and Kuijlaars in [15, 16]. Indeed they prove in their papers the existence and the uniqueness of the equilibrium measure μ_V if the external field V satisfies a weak growth assumption. We prove the classical variational inequalities linking the logarithmic potential U^{μ_V} of μ_V and V , and finally we show the convergence of the empirical measure for minimizers to μ_V . As Brauchart, Dragnev and Saff proved these results, in [12, 4], on the sphere, it is natural to use an inverse stereographic projection to transport our equilibrium problem from the whole plane to the sphere.

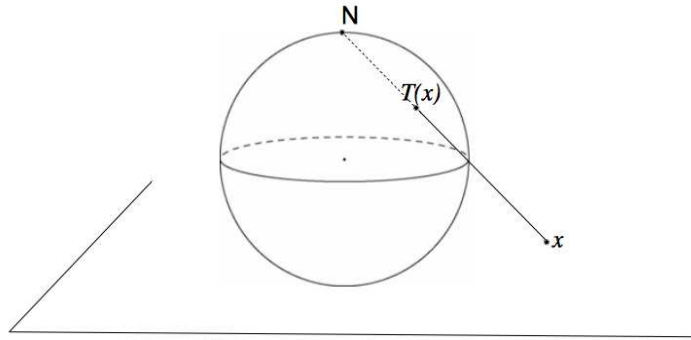
3.1 Inverse stereographic projection and equilibrium measure

Here we recall properties of the inverse stereographic projection used by Hardy and Kuijlaars in [15, 16] in order to generalize the equilibrium results of Saff and Totik from [21] with weak assumptions on the external potential V .

Let \mathcal{S} be the sphere of \mathbb{R}^3 centred in $(0, 0, 1/2)$ of radius $1/2$, Σ a unbounded closed set of \mathbb{R}^2 and $T : \mathbb{R}^2 \rightarrow \mathcal{S}$ the associated inverse stereographic projection defined by

$$T(x_1, x_2) = \left(\frac{x_1}{1 + \|x\|^2}, \frac{x_2}{1 + \|x\|^2}, \frac{\|x\|^2}{1 + \|x\|^2} \right), \text{ for any } x = (x_1, x_2) \in \mathbb{R}^2$$

where $\|\cdot\|$ is the Euclidean norm of \mathbb{R}^3 , with $\mathbb{R}^2 := \{(x_1, x_2, 0); x_1, x_2 \in \mathbb{R}\}$. We know that T is a conformal homeomorphism from \mathbb{C} to $\mathcal{S} \setminus \{N\}$ where $N := (0, 0, 1)$ is the North pole of \mathcal{S} .



Inverse stereographic projection T from \mathbb{R}^2 to \mathcal{S} with point projection N

We have the following identity :

$$\|T(x) - T(y)\| = \frac{\|x - y\|}{\sqrt{1 + \|x\|^2} \sqrt{1 + \|y\|^2}}, \text{ for any } x, y \in \mathbb{R}^2$$

and if $\|y\| \rightarrow +\infty$ we obtain, for any $x \in \mathbb{R}^2$:

$$\|T(x) - N\| = \frac{1}{\sqrt{1 + \|x\|^2}}. \quad (3.1)$$

We note $\Sigma_{\mathcal{S}} = T(\Sigma) \cup \{N\}$ the closure of $T(\Sigma)$ in \mathcal{S} . Let $\mathcal{M}_1(\Sigma)$ be the set of probability measures on Σ . For $\mu \in \mathcal{M}_1(\Sigma)$, we denote by $T_*\mu$ its push-forward measure by T characterized by

$$\int_{\Sigma_{\mathcal{S}}} f(z) dT_*\mu(z) = \int_{\Sigma} f(T(x)) d\mu(x)$$

for every Borel function $f : \Sigma_{\mathcal{S}} \rightarrow \mathbb{R}$. Sometimes in the following proofs we define other stereographic or inverse stereographic projections \tilde{T} with point projection $\tilde{p} \in \mathcal{S}$. In this case, the projection is from \tilde{p} to the tangent plane of \mathcal{S} (or its inverse) at the opposite point of \tilde{p} on the sphere, corresponding to the whole plane \mathbb{R}^2 .

We need the following theorem of Hardy (see Corollary 2.7 of Hardy and Kuijlaars in [16] for a more general case) :

Theorem 3.1. (Hardy, [15]) *Let Δ be a unbounded closed subset of \mathbb{R}^2 and $V : \Delta \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous map such that there exists $\beta \geq 2$ verifying*

$$\liminf_{\substack{\|x\| \rightarrow \infty \\ x \in \Delta}} \{V(x) - \beta \log \|x\|\} > -\infty. \quad (3.2)$$

For $\mu \in \mathcal{M}_1(\Delta)$, let

$$I_V(\mu) = - \iint_{\Delta^2} \log \|x - y\| d\mu(x) d\mu(y) + \int_{\Delta} V(x) d\mu(x).$$

Then I_V admits a unique minimizer μ_V .

In the following we denote by Σ the support of μ_V . Hardy's proof is obtained by a compactification method on \mathcal{S} and a convexity argument. Moreover Σ can be unbounded hence it is not necessarily compact like in the classical theory (see [21]). In the following μ_V will denote this equilibrium measure, i.e.

$$I_V(\mu_V) = \inf\{I_V(\mu); \mu \in \mathcal{M}_1(\Delta)\}$$

and Δ denote \mathbb{R}^2 . We recall that the logarithmic capacity of a set Σ , denoted by $\text{cap}(\Sigma)$, is defined by

$$\text{cap}(\Sigma) = e^{-V_{\Sigma}}$$

where $V_{\Sigma} := \inf\{I_0(\mu) : \mu \in \mathcal{M}(\Sigma)\}$ and $I_0(\mu) = - \iint_{\Sigma^2} \log \|x - y\| d\mu(x) d\mu(y)$. We say that an assertion is true quasi-everywhere (q.e.) on Σ if it is the case except for a set of capacity zero, where the logarithmic potential of μ defined by

$$U^{\mu}(x) = - \int_{\Sigma} \log \|x - y\| d\mu(y)$$

is infinite and therefore its Lebesgue measure is zero.

3.2 Proof of the classical variational inequalities

Now we prove the classical inequalities connecting the logarithmic potential of the equilibrium measure μ_V to its associated external field V .

Proposition 3.2. *Let V and μ_V defined by Theorem 3.1 such that $\int_{\Sigma} \log(1 + \|x\|^2) d\mu_V(x) < +\infty$, then*

$$U^{\mu_V}(x) + \frac{V(x)}{2} \geq c_V \text{ q.e. on } \Sigma \quad (3.3)$$

$$U^{\mu_V}(x) + \frac{V(x)}{2} \leq c_V \text{ for any } x \in \Sigma \quad (3.4)$$

where $c_V := I_V(\mu_V) - \int_{\mathbb{R}^2} \frac{V(x)}{2} d\mu_V(x)$.

Proof. As in [15], let $\mathcal{V}(T(x)) = V(x) - \log(1 + \|x\|^2)$ be the external field on the sphere and, for $\mu \in \mathcal{M}_1(\Delta)$, $T_*\mu$ its push-forward by T . Hardy remarks in [15] that

$$I_V(\mu) = I_{\mathcal{V}}(T_*\mu) \text{ for any } \mu \in \mathcal{M}_1(\Delta)$$

where $I_{\mathcal{V}}(\nu) = - \int \int_{\Delta_{\mathbb{S}^2}} \log \|x - y\| d\nu(x) d\nu(y) + \int_{\Delta_{\mathbb{S}^2}} \mathcal{V}(x) d\nu(x)$ for $\nu \in \mathcal{M}_1(\Delta_{\mathbb{S}^2})$. Moreover Hardy and Kuijlaars notice in [16] that for any $\mu \in \mathcal{M}_1(\Delta)$,

$$\begin{aligned} U^{T_*\mu}(N) = +\infty &\iff - \int_{\mathbb{S}^2} \log \|p - N\| dT_*\mu(p) = +\infty \\ &\iff \frac{1}{2} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu(x) = +\infty \end{aligned}$$

by (3.1). Hence $U^{T_*\mu}(N)$ is finite. Moreover T_* is an homeomorphism from $\mathcal{M}_1(\Delta)$ to $\{\mu \in \mathcal{M}_1(\Delta_{\mathbb{S}^2}) : \mu(\{N\}) = 0\}$ (see [15], Lemma 2.1 for the proof). As a consequence the problem of minimization of I_V among probability measures on Δ is exactly the problem of minimization of $I_{\mathcal{V}}$ among probability measures on $\Delta_{\mathbb{S}^2}$.

This problem of finding the equilibrium measure on the sphere was studied by Saff and Dragnev in [12] and they proved that, for $\tilde{\mathcal{V}}$ an external field on the unit sphere, i.e. a non-negative lower semi-continuous function $\tilde{\mathcal{V}} : \mathbb{S}^2 \rightarrow \mathbb{R}_+$ such that $\tilde{\mathcal{V}}(x) < \infty$ on a set of positive Lebesgue measure, there exists a constant $c_{\tilde{\mathcal{V}}}$ such that

$$\begin{aligned} U^{\mu_{\tilde{\mathcal{V}}}}(x) + \frac{\tilde{\mathcal{V}}(x)}{2} &\geq c_{\tilde{\mathcal{V}}} \text{ q.e. on } \mathbb{S}^2 \\ U^{\mu_{\tilde{\mathcal{V}}}}(x) + \frac{\tilde{\mathcal{V}}(x)}{2} &\leq c_{\tilde{\mathcal{V}}} \text{ for any } x \in S_{\tilde{\mathcal{V}}} \end{aligned}$$

where $\mu_{\tilde{\mathcal{V}}}$ is the minimum of $I_{\tilde{\mathcal{V}}}$ among probability measures on \mathbb{S}^2 with support $S_{\tilde{\mathcal{V}}} \subset \mathbb{S}^2$ associated to the external field $\tilde{\mathcal{V}}$. Furthermore we have $c_{\tilde{\mathcal{V}}} = I_{\tilde{\mathcal{V}}}(\mu_{\tilde{\mathcal{V}}}) - \int_{\mathbb{S}^2} \frac{\tilde{\mathcal{V}}(x)}{2} d\mu_{\tilde{\mathcal{V}}}(x)$.

As V is lower semi-continuous, then it is the same for \mathcal{V} and, by (3.2) we get :

$$\mathcal{V}(N) = \liminf_{\substack{\|x\| \rightarrow \infty \\ x \in \Delta}} \{V(x) - \log(1 + \|x\|^2)\} > -\infty.$$

Therefore \mathcal{V} is bounded from below and we can use Saff and Dragnev's theorem on \mathbb{S}^2 . We have, for any $x \in \mathbb{R}^2$,

$$U^{T_*\mu_V}(T(x)) + \frac{\mathcal{V}(T(x))}{2} = - \int_{\Sigma_{\mathbb{S}^2}} \log \|T(x) - y\| dT_*\mu_V(y) + \frac{V(x)}{2} - \frac{\log(1 + \|x\|^2)}{2}$$

$$\begin{aligned}
&= - \int_{\Sigma} \log \|T(x) - T(y)\| d\mu_V(y) + \frac{V(x)}{2} - \frac{\log(1 + \|x\|^2)}{2} \\
&= - \int_{\Sigma} \log \left(\frac{\|x - y\|}{\sqrt{1 + \|x\|^2} \sqrt{1 + \|y\|^2}} \right) d\mu_V(y) + \frac{V(x)}{2} - \frac{\log(1 + \|x\|^2)}{2} \\
&= U^{\mu_V}(x) + \frac{1}{2} \int_{\Sigma} \log(1 + \|x\|^2) d\mu_V(y) + \frac{1}{2} \int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y) + \frac{V(x)}{2} \\
&\quad - \frac{\log(1 + \|x\|^2)}{2} \\
&= U^{\mu_V}(x) + \frac{V(x)}{2} + \frac{1}{2} \int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y). \tag{3.5}
\end{aligned}$$

We notice that the two notions of “quasi-everywhere” are the same because, for any $\mu \in \mathcal{M}_1(\mathbb{R}^2)$, we have

$$U^{T_*\mu}(T(x)) = U^{\mu}(x) + \frac{1}{2} \log(1 + \|x\|^2) + \frac{1}{2} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu(x).$$

With (3.5) and the fact that $\int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y)$ is finite, both inequalities (3.3) and (3.4) are clear because, quasi-everywhere on \mathbb{R}^2 , we get

$$\begin{aligned}
U^{\mu_V}(x) + \frac{V(x)}{2} &= U^{T_*\mu_V}(T(x)) + \frac{\mathcal{V}(T(x))}{2} - \frac{1}{2} \int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y) \\
&\geq c_V - \frac{1}{2} \int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y) \\
&\geq I_V(\mu_V) - \int_{\mathcal{S}} \frac{\mathcal{V}}{2} d\mu_V - \frac{1}{2} \int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y) \\
&\geq I_V(\mu_V) - \int_{\mathbb{R}^2} \frac{V(x)}{2} d\mu_V(x) + \frac{1}{2} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x) - \frac{1}{2} \int_{\Sigma} \log(1 + \|y\|^2) d\mu_V(y) \\
&\geq I_V(\mu_V) - \int_{\mathbb{R}^2} \frac{V(x)}{2} d\mu_V(x) = c_V
\end{aligned}$$

and the second inequality is obtained by the same way. \square

Remark 3.3. In particular, we have $U^{\mu_V}(x) + \frac{V(x)}{2} = c_V$ a.e. on Σ .

Example 3.4. If $V(x) = \log(1 + \|x\|^2)$, then $\liminf_{\|x\| \rightarrow +\infty} \{V(x) - 2 \log \|x\|\} = 0 > -\infty$. We have $\mathcal{V}(y) = 0$ for any $y \in \mathcal{S}$ and the unique minimizer of $I_V(T_*\mu) = - \iint_{\mathcal{S}^2} \log \|x - y\| d(T_*\mu)(x) d(T_*\mu)(y)$ is the uniform probability measure on \mathcal{S} , hence $\Sigma_{\mathcal{S}} = \mathcal{S}$ and $\Sigma = \mathbb{R}^2$. By the equality $U^{\mu_V}(x) + \frac{V(x)}{2} = c_V$ a.e. on Σ and the regularity of V we conclude that

$$d\mu_V(x) = \frac{\Delta V(x)}{4\pi} dx = \frac{dx}{\pi(1 + \|x\|^2)^2}.$$

This example is obviously essential for the link with the logarithmic energy on the sphere (see Section 7).

3.3 Convergence of the empirical measure

In the following we denote, for $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$, the counting measure of this set of points by

$$\nu_n = \sum_{i=1}^n \delta_{x_i}$$

and, for any $n \geq 2$, the Hamiltonian of the system (x_1, \dots, x_n) by

$$w_n(x_1, \dots, x_n) = - \sum_{i \neq j}^n \log \|x_i - x_j\| + n \sum_{i=1}^n V(x_i).$$

Proposition 3.5. *Let V and μ_V defined as in Theorem 3.1. If (x_1, \dots, x_n) minimizes w_n for any $n \geq 2$ then*

$$\lim_{n \rightarrow +\infty} I_V \left(\frac{\nu_n}{n} \right) = I_V(\mu_V) \text{ and } \lim_{n \rightarrow +\infty} \frac{\nu_n}{n} = \mu_V.$$

Proof. We consider the following discrete energy on \mathcal{S} with $\mathcal{V}(T(x)) = V(x) - \log(1 + \|x\|^2)$:

$$\begin{aligned} & E_{\log}^{\mathcal{V}}(T(x_1), \dots, T(x_n)) \\ & := - \sum_{i \neq j}^n \log \|T(x_i) - T(x_j)\| + (n-1) \sum_{i=1}^n \mathcal{V}(T(x_i)) \\ & = - \sum_{i \neq j}^n \log \left(\frac{\|x_i - x_j\|}{\sqrt{1 + \|x_i\|^2} \sqrt{1 + \|x_j\|^2}} \right) + (n-1) \sum_{i=1}^n V(x_i) - (n-1) \sum_{i=1}^n \log(1 + \|x_i\|^2) \\ & = - \sum_{i \neq j}^n \log \|x_i - x_j\| + (n-1) \sum_{i=1}^n V(x_i) =: \bar{w}_n(x_1, \dots, x_n). \end{aligned}$$

Let $(\bar{x}_1, \dots, \bar{x}_n)$ be a minimizer of \bar{w}_n for any $n \geq 2$, then $(T(\bar{x}_1), \dots, T(\bar{x}_n))$ minimizes $E_{\log}^{\mathcal{V}}$ for any $n \geq 2$. Brauchart, Dragnev and Saff (see Proposition 11 in [4]) proved that

$$\lim_{n \rightarrow +\infty} \frac{E_{\log}^{\mathcal{V}}(T(\bar{x}_1), \dots, T(\bar{x}_n))}{n^2} = I_{\mathcal{V}}(T_*\mu_V) \text{ and } \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \delta_{T(\bar{x}_i)} = T_*\mu_V$$

in the weak-star sense of measures. It is not difficult to see that if (x_1, \dots, x_n) minimizes w_n then

$$\lim_{n \rightarrow +\infty} I_V \left(\frac{\nu_n}{n} \right) = \lim_{n \rightarrow +\infty} \frac{w_n(x_1, \dots, x_n)}{n^2} = \lim_{n \rightarrow +\infty} \frac{\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n)}{n^2} = I_{\mathcal{V}}(T_*\mu_V) = I_V(\mu_V). \quad (3.6)$$

The weak-star convergence of $\frac{1}{n} \sum_{i=1}^n \delta_{T(\bar{x}_i)}$ to $T_*\mu_V$ means that for any continuous function $f : \mathcal{S} \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathcal{S}} f(p) dT_*\bar{\nu}_n(p) = \int_{\mathcal{S}} f(p) dT_*\mu_V(p)$$

where $\bar{\nu}_n = \sum_{i=1}^n \delta_{\bar{x}_i}$. Hence the previous equality is equivalent to

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} f(T(x)) d\bar{\nu}_n(x) = \int_{\Sigma} f(T(x)) d\mu_V(x).$$

Let g be a bounded continuous function on \mathbb{R}^2 , then there exists a continuous function $f = g \circ T^{-1}$ on \mathcal{S} such that $g(x) = f(T(x))$ because T^{-1} is continuous. Therefore

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} g(x) d\bar{\nu}_n(x) = \int_{\Sigma} g(x) d\mu_V(x)$$

and we obtain $\lim_{n \rightarrow +\infty} \frac{\bar{\nu}_n}{n} = \mu_V$ in the weak-star sense of measures. By (3.6) and the uniqueness of μ_V we conclude that $\frac{\nu_n}{n}$ converges to μ_V . \square

3.4 Assumptions on μ_V

Now we give assumptions on the equilibrium measure μ_V to apply our Gamma-Convergence method. At first we need μ_V to be absolutely continuous with respect to the Lebesgue measure :

$$(H1) : d\mu_V(x) = m_V(x)dx.$$

We want m_V to be positive and never equal to 0 :

$$(H2) : \forall R > 0, \forall x \in B_R \cap \Sigma, \exists M, \exists c_R \text{ such that } 0 < c_R < m_V(x) \leq M.$$

As we want to use the inequalities of Proposition 3.2, we assume that

$$(H3) : \int_{\Sigma} \log(1 + \|x\|^2) d\mu_V(x) < +\infty$$

in order to have the logarithmic potential of $T_*\mu_V$ bounded in N .

Finally we add a regularity assumption on the boundary of the support of μ_V in order to use the results of [24] for the upper bound of our Theorem 6.1

$$(H4) : \text{The support } \Sigma_{\mathcal{S}} \text{ of } T_*\mu_V \text{ is such that } \partial\Sigma_{\mathcal{S}} \text{ is } C^1.$$

Remark 3.6. A natural assumption on V such that (H4) holds implies that necessarily $V \in C^1(\mathbb{R}^2)$. In this case the inequality (3.3) is true everywhere in Σ .

4 Proof of the Splitting Formula

As in [24] we denote the blown-up quantities by primes :

$$x'_i = \sqrt{n}x_i, m'_V(x') = m_V(x), d\mu'_V(x') = m'_V(x')dx' = nm_V(x)dx.$$

Let $\zeta(x) := U^{\mu_V}(x) + \frac{V(x)}{2} - c_V$, then $\zeta(x) = 0$ q.e. in Σ and $\zeta(x) > 0$ q.e. in $\mathbb{R}^2 \setminus \Sigma$. We define $\nu_n = \sum_{i=1}^n \delta_{x_i}$

and $H_n := -2\pi\Delta^{-1}(\nu_n - n\mu_V)$ where Δ^{-1} is the convolution's operator with $\frac{1}{2\pi} \log \|\cdot\|$, hence such that $\Delta \circ \Delta^{-1} = I_2$ where Δ denote the usual laplacian.

Lemma 4.1. *Let μ_V verifying the assumptions (H1), (H2), (H3) then, for any $n \geq 2$ and for any configuration $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$, we have*

$$w_n(x_1, \dots, x_n) = n^2 I_V(\mu_V) - \frac{n}{2} \log n + \frac{1}{\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i). \quad (4.1)$$

Proof. Let $n \geq 2$ and $D = \{(x, x); x \in \mathbb{R}^2\}$. As $\nu_n = n\mu_V + \nu_n - n\mu_V$ and $(\mu_V \times \mu_V)(D) = 0$, we get, as in [24] :

$$\begin{aligned}
& w_n(x_1, \dots, x_n) \\
&= - \sum_{i \neq j}^n \log \|x_i - x_j\| + n \sum_{i=1}^n V(x_i) \\
&= - \iint_{D^c} \log \|x - y\| d\nu_n(x) d\nu_n(y) + n \int_{\mathbb{R}^2} V(x) d\nu_n(x) \\
&= - \iint_{D^c} \log \|x - y\| d(n\mu_V + \nu_n - n\mu_V)(x) d(n\mu_V + \nu_n - n\mu_V)(y) + n \int_{\mathbb{R}^2} V(x) d(n\mu_V + \nu_n - n\mu_V)(x) \\
&= n^2 I_V(\mu_V) + n \int_{\mathbb{R}^2} (2U^{\mu_V}(x) + V(x)) d(\nu_n - n\mu_V)(x) - \iint_{D^c} \log \|x - y\| d(\nu_n - n\mu_V)(x) d(\nu_n - n\mu_V)(y).
\end{aligned}$$

Now we use inequalities (3.3) and (3.4) to obtain

$$2U^{\mu_V}(x) + V(x) = \begin{cases} 2c_V & \text{a.e. on } \Sigma \\ 2\zeta(x) + 2c_V & \text{q.e. on } \mathbb{R}^2 \setminus \Sigma \end{cases}$$

and because the measures ν_n and $n\mu_V$ have same mass we have

$$w_n(x_1, \dots, x_n) = n^2 I_V(\mu_V) + 2n \sum_{i=1}^n \zeta(x_i) - \iint_{D^c} \log \|x - y\| d(\nu_n - n\mu_V)(x) d(\nu_n - n\mu_V)(y).$$

Now we want to prove the following equality :

$$- \iint_{D^c} \log \|x - y\| d(\nu_n - n\mu_V)(x) d(\nu_n - n\mu_V)(y) = \sum_{i=1}^n H_n^i(x_i) - n \int H_n d\mu_V.$$

We use the Green's formula, for any $R \geq 1$ and let \vec{u} be the outer unit normal vector :

$$\int_{B_R \setminus \cup_{i=1}^n B(x_i, \eta)} \|\nabla H_n\|^2 = \int_{\partial B_R} H_n \frac{\partial H_n}{\partial \vec{u}} + \sum_{i=1}^n \int_{\partial B(x_i, \eta)} H_n \frac{\partial H_n}{\partial \vec{u}} + 2\pi \int_{B_R \setminus \cup_{i=1}^n B(x_i, \eta)} H_n d(\nu_n - n\mu_V).$$

We notice that, for $p \in \mathcal{S} \setminus \{N\}$ and $T(x_i) = p_i \neq N$ for any $1 \leq i \leq n$:

$$\begin{aligned}
H_n(T^{-1}(p)) &= - \int_{\mathbb{R}^2} \log \|T^{-1}(p) - y\| d(\nu_n - n\mu_V)(y) \\
&= - \sum_{i=1}^n \log \|T^{-1}(p) - T^{-1}(p_i)\| - nU^{\mu_V}(T^{-1}(p)) \\
&= - \sum_{i=1}^n \log \|T^{-1}(p) - T^{-1}(p_i)\| + n \int_{\mathcal{S}} \log \|T^{-1}(p) - T^{-1}(u)\| dT_*\mu_V(u) \\
&= - \int_{\mathcal{S}} \log \|T^{-1}(p) - T^{-1}(u)\| d(T_*\nu_n - nT_*\mu_V)(u) \\
&= - \int_{\mathcal{S}} \log \left(\|p - u\| \sqrt{1 + \|T^{-1}(p)\|^2} \sqrt{1 + \|T^{-1}(u)\|^2} \right) d(T_*\nu_n - nT_*\mu_V)(u) \\
&= - \int_{\mathcal{S}} \log \|p - u\| d(T_*\nu_n - nT_*\mu_V)(u) - \frac{1}{2} \int_{\mathcal{S}} \log(1 + \|T^{-1}(p)\|^2) d(T_*\nu_n - nT_*\mu_V)(u)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \int_{\mathcal{S}} \log(1 + \|T^{-1}(u)\|^2) d(T_*\nu_n - nT_*\mu_V)(u) \\
& = -\int_{\mathcal{S}} \log \|p - u\| d(T_*\nu_n - nT_*\mu_V)(u) - \frac{1}{2} \int_{\mathcal{S}} \log(1 + \|T^{-1}(u)\|^2) d(T_*\nu_n - nT_*\mu_V)(u)
\end{aligned}$$

because $T_*\nu_n$ and $nT_*\mu_V$ have same mass and compact support. Hence on $\mathcal{S} \setminus \{N\}$, $H_n \circ T^{-1}$ verifies

$$\Delta(H_n \circ T^{-1}) = -2\pi(T_*\nu_n - nT_*\mu_V)$$

because T is conformal. As $H_n(T^{-1}(p)) = -\sum_{i=1}^n \log \|p - p_i\| - nU^{T_*\mu_V}(p) + C$ and $U^{T_*\mu_V}(p)$ is bounded as $p \rightarrow N \neq p_i$ for all i , then $H_n \circ T^{-1}$ is bounded at the neighborhood of N . Let $\tilde{p} \in \mathcal{S} \setminus \{N\}$, $\tilde{p} \neq p_i$ for any i , be a point projection and \tilde{T} be the corresponding inverse stereographic projection from \mathbb{R}^2 to \mathcal{S} , it follows, because \tilde{T} is conformal, that

$$\Delta(H_n \circ T^{-1} \circ \tilde{T}) = -2\pi(\nu_n - n\mu_V)$$

everywhere but not in $\tilde{T}^{-1}(N)$ where the function $H_n \circ T^{-1} \circ \tilde{T}$ is bounded. Now we use the following result: if $\Delta f = 0$ in the whole plane except for x_0 where f is bounded, then $\Delta f = 0$ in the whole plane (see [14], Theorem 1.28). Thus we get $\Delta(H_n \circ T^{-1} \circ \tilde{T}) = -2\pi(\nu_n - n\mu_V)$ in the whole plane and the following equality holds on \mathcal{S} :

$$\Delta(H_n \circ T^{-1}) = -2\pi(T_*\nu_n - nT_*\mu_V).$$

Now we notice that $\nabla(H_n \circ T^{-1})$ is bounded at the neighborhood of N , hence

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R} H_n \frac{\partial H_n}{\partial \vec{u}} = \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} (H_n \circ T^{-1}) \nabla(H_n \circ T^{-1}) \cdot \vec{u} = 0$$

where $T(\partial B_R) = \partial B_\varepsilon \cap \mathcal{S}$ is a circle on the sphere \mathcal{S} centred in N such that its radius ε goes to 0 as $R \rightarrow +\infty$. Now we set, for any $x \in \mathbb{R}^2$, $H_n^i(x) = H_n(x) + \log \|x - x_i\|$. As m_V is bounded in L^∞ we obtain, because H_n^i is C^1 in the neighborhood of x_i ,

$$\int_{\partial B(x_i, \eta)} H_n \frac{\partial H_n}{\partial \vec{u}} = 2\pi H_n^i(x_i) - 2\pi \log \eta + o(1).$$

Therefore by considering $\eta \rightarrow 0$ and $R \rightarrow +\infty$, we obtain

$$\int_{B_R \setminus \cup_{i=1}^n B(x_i, \eta)} \|\nabla H_n\|^2 = -2\pi n \log \eta + 2\pi \sum_{i=1}^n H_n^i(x_i) - 2\pi n \int H_n d\mu_V + o(1)$$

and by definition (2.3) of $W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2})$, it follows that

$$\begin{aligned}
W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2}) &= \lim_{\eta \rightarrow 0} \left(\frac{1}{2} \int_{B_R \setminus \cup_{i=1}^n B(x_i, \eta)} \|\nabla H_n\|^2 + \pi n \log \eta \right) \\
&= \lim_{\eta \rightarrow 0} \left(\frac{1}{2} \left(-2\pi n \log \eta + 2\pi \sum_{i=1}^n H_n^i(x_i) - 2\pi n \int H_n d\mu_V \right) + \pi n \log \eta \right) \\
&= \pi \sum_{i=1}^n H_n^i(x_i) - \pi n \int H_n d\mu_V < +\infty.
\end{aligned}$$

We notice, because $H_n^i = -\log *(\nu_n^i - n\mu_V)$ where $\nu_n^i = \nu_n - \delta_{x_i}$, that

$$H_n^i(x_i) = \int_{\mathbb{R}^2} -\log \|x_i - y\| d(\nu_n^i - n\mu_V)(y) = \int_{\mathbb{R}^2 \setminus \{x_i\}} -\log \|x_i - y\| d(\nu_n - n\mu_V)(y)$$

and if $x \notin \{x_1, \dots, x_n\}$ then

$$H_n(x) = \int_{\mathbb{R}^2 \setminus \{x\}} -\log \|x - y\| d(\nu_n - n\mu_V)(y).$$

Therefore we obtain

$$-\iint_{D^c} \log \|x - y\| d(\nu_n - n\mu_V)(x) d(\nu_n - n\mu_V)(y) = \sum_{i=1}^n H_n^i(x_i) - n \int H_n d\mu_V,$$

and finally we find our equality

$$-\iint_{D^c} \log \|x - y\| d(\nu_n - n\mu_V)(x) d(\nu_n - n\mu_V)(y) = \frac{1}{\pi} W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2}). \quad (4.2)$$

Thus we conclude that

$$w_n(x_1, \dots, x_n) = n^2 I_V(\mu_V) + \frac{1}{\pi} W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i).$$

Eventually, by equality $\int_{\mathbb{R}^2 \setminus \cup_{i=1}^n B(x_i, \eta)} \|\nabla H_n\|^2 = \int_{\mathbb{R}^2 \setminus \cup_{i=1}^n B(x'_i, \eta)} \|\nabla H'_n\|^2$, we get

$$W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2}) = W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) - \frac{\pi}{2} n \log n$$

and (4.1) is proved. \square

5 Useful Lemmas

Here we give essential results to prove the lower bound of our main theorem. Indeed our goal is to write $W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2})$ like an integral and to use an Ergodic Theorem to bound by below the next-order term in the previous splitting formula, as in [24].

5.1 Mass spreading result and modified density g

By a mass displacement method introduced in [22] we have the following result from [24] :

Lemma 5.1. (Sandier-Serfaty,[24]) *Assume $\Omega \subset \mathbb{R}^2$ is open and (ν, E) are such that $\nu = \sum_{p \in \Lambda} \delta_p$ for some*

finite subset Λ of $\hat{\Omega} := \{x \in \mathbb{R}^2; d(x, \Omega) < 1\}$ and $\operatorname{div} E = 2\pi(\nu - a(x)dx)$, $\operatorname{curl} E = 0$ in $\hat{\Omega}$ where $a \in L^\infty(\hat{\Omega})$.

Then, given any $\rho > 0$ there exists a measure g supported on $\hat{\Omega}$ and such that

- there exists a family \mathcal{B}_ρ of disjoint closed balls covering $\operatorname{Supp}(\nu)$, with the sum of the radii of the balls in \mathcal{B}_ρ intersecting with any ball of radius 1 bounded by ρ , and such that

$$g \geq -C(\|a\|_\infty + 1) + \frac{1}{4}|E|^2 \mathbf{1}_{\Omega \setminus \mathcal{B}_\rho} \text{ in } \hat{\Omega}$$

where C depends only on ρ ;

- we have

$$g = \frac{1}{2}|E|^2 \text{ outside } \bigcup_{p \in \Lambda} B(p, \lambda)$$

where λ depends only on ρ ;
- for any function χ compactly supported in Ω we have

$$\left| W(E, \chi) - \int \chi dg \right| \leq CN(\log N + \|a\|_\infty) \|\nabla \chi\|_\infty \quad (5.1)$$

where $N = \#\{p \in \Lambda; B(p, \lambda) \cap \text{Supp}(\nabla \chi) \neq \emptyset\}$ for some λ and C depending only on ρ ;
- for any $U \subset \Omega$

$$\#(\Lambda \cap U) \leq C(1 + \|a\|_\infty^2 |\hat{U}| + g(\hat{U})). \quad (5.2)$$

Remark 5.2. By assumptions (H1), (H2) and (H3), m_V decays sufficiently quickly at infinity, and by the fact that $\#\Lambda$ is finite we can apply the previous theorem in \mathbb{R}^2 with $\hat{\Omega} = \Omega = \mathbb{R}^2$ and $a = m_V$.

Definition 5.1. Assume $\nu_n = \sum_{i=1}^n \delta_{x_i}$. Letting $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ be the measure in blown-up coordinates and $E_{\nu_n} = \nabla H'_n$, we denote by g_{ν_n} the result of applying the previous proposition to (ν'_n, E_{ν_n}) .

Moreover we need the following result (Lemma 3.4 from [24]), which connects g and renormalized energy:

Lemma 5.3. (Sandier-Serfaty, [24]) For any $\nu_n = \sum_{i=1}^n \delta_{x_i}$, we have

$$W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) = \int_{\mathbb{R}^2} dg_{\nu_n}. \quad (5.3)$$

Proof. We apply the inequality (5.1) to χ_{K_R} defined in (2.4) for R sufficiently large in order to have $N = \#\{p \in \Lambda; B(p, \lambda) \cap \text{Supp}(\nabla \chi_{K_R}) \neq \emptyset\} = 0$. Hence we obtain

$$W(\nabla H'_n, \chi_{K_R}) = \int_{\mathbb{R}^2} \chi_{K_R} dg_{\nu_n}$$

and as $R \rightarrow +\infty$ we find $W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) = \int_{\mathbb{R}^2} dg_{\nu_n}$. □

5.2 Ergodic Theorem

We recall exactly the notations of Sandier and Serfaty from [23] and [24] for the two-parameter groups acting continuously on a metric Polish space X which is a space of functions :

- we define θ_λ acting on X by $\theta_\lambda u(x) = u(x + \lambda)$ for any $x, \lambda \in \mathbb{R}^2$;

- we also define T_λ^ε and T_λ acting on $\mathbb{R}^2 \times X$ by $T_\lambda^\varepsilon(x, u) := (x + \varepsilon\lambda, \theta_\lambda u)$ and $T_\lambda(x, u) := (x, \theta_\lambda u)$.

For a probability measure P on $\mathbb{R}^2 \times X$ we say that P is $T_{\lambda(x)}$ -invariant if for every function $\lambda(x)$ of class C^1 , it is invariant under the mapping $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$.

Let μ_V be a probability measure verifying (H1), (H2) and $\mu_V^R := \frac{\mu_V}{\mu_V(B_R)}$ the normalized measure on B_R of density m_V^R . Let $(f_\varepsilon)_\varepsilon$ and f be positive measurable functions on $\mathbb{R}^2 \times X$.

We assume that for any $\{x_\varepsilon, u_\varepsilon\}_\varepsilon$ such that for any $R > 0$, $\limsup_{\varepsilon \rightarrow 0} \int_{B_R} f_\varepsilon(x_\varepsilon + \varepsilon\lambda, \theta_\lambda u_\varepsilon) d\lambda < +\infty$, we have

the following properties :

- 1) (Coercivity) $\{u_\varepsilon\}_\varepsilon$ has a convergent subsequence,
- 2) (Γ -liminf) if $\{x_\varepsilon, u_\varepsilon\}_\varepsilon$ converge to $\{x, u\}$ then $\liminf_{\varepsilon \rightarrow 0} f_\varepsilon(x_\varepsilon, u_\varepsilon) \geq f(x, u)$.

Lemma 5.4. Let μ_V , X , $(f_\varepsilon)_\varepsilon$ and f be as above. We define $F_\varepsilon(u) := \int_{\mathbb{R}^2} f_\varepsilon(x, \theta_{\frac{x}{\varepsilon}} u) d\mu_V(x)$ and assume that there exists a sequence $(u_\varepsilon)_\varepsilon \in X$ such that $F_\varepsilon(u_\varepsilon) \leq C$ for any $\varepsilon > 0$. Let P_ε be the image of μ_V by $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$, then

i) $(P_\varepsilon)_\varepsilon$ admits a convergent's subsequence to a probability measure P ,

ii) the first marginal of P is μ_V ,

iii) P is $T_{\lambda(x)}$ -invariant,

iv) $(x, u) = \lim_{\varepsilon \rightarrow 0} (x_\varepsilon, \theta_{\frac{x_\varepsilon}{\varepsilon}} u_\varepsilon)$ P -a.e.,

v) $\liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} f_\varepsilon(x, u) dP_\varepsilon(x, u) \geq \int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u)$.

vi) Moreover we have

$$\int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u) = \int_{\mathbb{R}^2 \times X} \left(\lim_{R \rightarrow +\infty} \int_{B_R} f(x, \theta_\lambda u) d\lambda \right) dP(x, u). \quad (5.4)$$

Proof. **STEP 1 : Convergence of a subsequence of (P_ε) to a probability measure P**

For any $R > 0$ and $\varepsilon > 0$, we define $F_\varepsilon^R(u) = \int_{B_R} f_\varepsilon(x, \theta_{\frac{x}{\varepsilon}} u) d\mu_V^R(x)$. By assumption there exists a sequence $(u_\varepsilon)_\varepsilon \in X$ such that $F_\varepsilon^R(u_\varepsilon) \leq C_R$ for any $\varepsilon > 0$, where C_R is a constant depending on R .

We define $X_{K,R}^\varepsilon := \left\{ (\lambda, u) \in B_R \times X; \int_{B_R} f_\varepsilon(\lambda + \varepsilon x, \theta_x u) dx > K \right\}$ and for $\varepsilon < 1$, Y_R^ε the image of $B_{R(1-\varepsilon)}$ by $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$. Let P_ε^R be the image of μ_V^R by the same map. We have

$$\begin{aligned} \int_{Y_R^\varepsilon} \int_{B_R} f_\varepsilon(\lambda + \varepsilon x, \theta_x u) dx dP_\varepsilon^R(\lambda, u) &= \int_{B_{R(1-\varepsilon)}} \int_{B_R} f_\varepsilon(\lambda + \varepsilon x, \theta_{\frac{\lambda + \varepsilon x}{\varepsilon}} u_\varepsilon) dx d\mu_V^R(\lambda) \\ &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \mathbb{1}_{B_{R(1-\varepsilon)}}(\lambda) \mathbb{1}_{\varepsilon B_R}(y - \lambda) m_V^R(\lambda) d\lambda \right] f_\varepsilon(y, \theta_{\frac{y}{\varepsilon}} u_\varepsilon) dy \\ &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} [\mathbb{1}_{B_{R(1-\varepsilon)}} m_V^R * \mathbb{1}_{\varepsilon B_R}](y) f_\varepsilon(y, \theta_{\frac{y}{\varepsilon}} u_\varepsilon) dy \\ &\leq |B_R| \int_{B_R} \left(\sup_{\lambda \in B_{R(1-\varepsilon)}} m_V^R(y - \lambda) \right) f_\varepsilon(y, \theta_{\frac{y}{\varepsilon}} u_\varepsilon) dy \\ &= |B_R| \int_{B_R} \left(\sup_{\lambda \in B_{R(1-\varepsilon)}} \frac{m_V^R(y - \lambda)}{m_V^R(y)} \right) f_\varepsilon(y, \theta_{\frac{y}{\varepsilon}} u_\varepsilon) d\mu_V^R(y) \\ &\leq C_R |B_R| \sup_{y \in B_R} \sup_{\lambda \in B_{R(1-\varepsilon)}} \frac{m_V^R(y - \lambda)}{m_V^R(y)} \\ &\leq C_R \end{aligned}$$

by the assumption (H2). Therefore by the Markov inequality

$$P_\varepsilon^R(X_{K,R}^\varepsilon \cap Y_R^\varepsilon) \leq \frac{C_R}{K}$$

and by the following inequality for every set $A, B \subset B_R \times X$:

$$P_\varepsilon^R(A \cap B) + P_\varepsilon^R(B^c) = P_\varepsilon^R(A) + P_\varepsilon^R(B) - P_\varepsilon^R(A \cup B) + P_\varepsilon^R(B^c) = 1 - P_\varepsilon^R(A \cup B) + P_\varepsilon^R(A) \geq P_\varepsilon^R(A),$$

we get

$$P_\varepsilon^R(X_{K,R}^\varepsilon) \leq P_\varepsilon^R(X_{K,R}^\varepsilon \cap Y_R^\varepsilon) + P_\varepsilon^R((Y_R^\varepsilon)^c) \leq \frac{C_R}{K} + 1 - \mu_V^R(B_{R(1-\varepsilon)}).$$

Let $\delta > 0$ be a real and $(\varepsilon_n^R)_n$ be a subsequence which goes to 0 as $n \rightarrow +\infty$ and such that for any $n \in \mathbb{N}$, $1 - \mu_V^R(B_{R(1-\varepsilon_n^R)}) < \delta 2^{-n}$, we get

$$P_\varepsilon^R \left(\bigcup_{k=1}^n X_{2^k/\delta, R}^{\varepsilon_k^R} \right) \leq \sum_{k=1}^n P_\varepsilon^R \left(X_{2^k/\delta, R}^{\varepsilon_k^R} \right) \leq \sum_{k=1}^n \left[\frac{C_R \delta}{2^k} + \frac{\delta}{2^k} \right] \leq \tilde{C}_R \delta.$$

We define $K_n^R := \left(\bigcup_{k=1}^n X_{2^k/\delta, R}^{\varepsilon_k^R} \right)^c$, then we have $P_\varepsilon^R(K_n^R) \geq 1 - \tilde{C}_R \delta$. If $\{\lambda_n, u_n\} \in K_n^R$ for every n , then

it follows, for any $R > 0$ and any $n > R$, that $\int_{B_R} f_\varepsilon(\lambda_n + \varepsilon x, \theta_x u_n) dx \leq \frac{2^R}{\delta} < +\infty$ and by the property 1) of coercivity, $(u_n)_n$ has a convergent's subsequence. Moreover, as $\lambda_n \in B_R$ for any n , $(\lambda_n)_n$ admits a convergent's subsequence. Now we use the following simple lemma (Lemma 2.1 in [23]) :

Lemma 5.5. (E. Lesigne, [23]) *Assume $(P_n)_n$ are Borel probability measures on a Polish metric space X and that for any $\delta > 0$ there exists $(K_n)_n$ such that $P_n(K_n) \geq 1 - \delta$ for every n and such that if $(x_n)_n$ satisfies for every n that $x_n \in K_n$, then any subsequence of $(x_n)_n$ admits a convergent subsequence. Then $(P_n)_n$ admits a subsequence which converges weakly to a probability measure P .*

Therefore $\left(P_{\varepsilon_n^R}^R \right)_n$ admits a subsequence which converges weakly to a probability measure P^R . By diagonal extraction we construct a subsequence $\left(P_{\varepsilon_n^R}^R \right)_n$ which converges weakly to a probability measure P .

STEP 2 : P is $T_{\lambda(x)}$ -invariant. Let λ be a C^1 function on \mathbb{R}^2 , Φ a bounded continuous function on $\mathbb{R}^2 \times X$ and P_λ the image of P by $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$. By the change of variables $y = \varepsilon \lambda(x) + x = (\varepsilon \lambda + I_2)(x)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) &= \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP_\varepsilon(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi(x, \theta_{\lambda(x) + \frac{x}{\varepsilon}} u_\varepsilon) d\mu_V(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi(x, \theta_{\frac{\varepsilon \lambda(x) + x}{\varepsilon}} u_\varepsilon) d\mu_V(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{\Phi \left((\varepsilon \lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon \right) m_V((\varepsilon \lambda + I_2)^{-1}(y)) dy}{|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon \lambda)^{-1}(y)))|} \end{aligned}$$

where $D\lambda$ is the differential of λ . We define

$$f(y, \varepsilon \lambda + I_2) := \frac{\Phi \left((\varepsilon \lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon \right) m_V((\varepsilon \lambda + I_2)^{-1}(y))}{|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon \lambda)^{-1}(y)))|}.$$

For $R > 0$ and ε sufficiently small such that $|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon \lambda)^{-1}(y)))| \geq 1/2$, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} f(y, \varepsilon \lambda + I_2) dy &= \lim_{\varepsilon \rightarrow 0} \int_{B_R} f(y, \varepsilon \lambda + I_2) dy + \lim_{\varepsilon \rightarrow 0} \int_{B_R^c} f(y, \varepsilon \lambda + I_2) dy \\ &= \int_{B_R} \lim_{\varepsilon \rightarrow 0} f(y, \varepsilon \lambda + I_2) dy + \lim_{\varepsilon \rightarrow 0} \int_{B_R^c} f(y, \varepsilon \lambda + I_2) dy \end{aligned}$$

because Φ and m_V are bounded. We get

$$\lim_{\varepsilon \rightarrow 0} \int_{B_R^c} f(y, \varepsilon \lambda + I_2) dy \leq \lim_{\varepsilon \rightarrow 0} K \int_{B_R^c} m_V((\varepsilon \lambda + I_2)^{-1}(y)) dy \longrightarrow 0 \text{ as } R \rightarrow +\infty.$$

and it follows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} f(y, \varepsilon \lambda + I_2) dy = \int_{B_R} \lim_{\varepsilon \rightarrow 0} f(y, \varepsilon \lambda + I_2) dy + \lim_{\varepsilon \rightarrow 0} o_R(1).$$

Thus we can exchange the limit and the integral and we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) &= \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi((\varepsilon \lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon) d\mu_V(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi((\varepsilon \lambda + I_2)^{-1}(y), u) dP_\varepsilon(y, u). \end{aligned}$$

The first step gives that $(P_\varepsilon)_\varepsilon$ is tight hence for any $\delta > 0$ there exists a compact set $K_\delta \subset \mathbb{R}^2 \times X$ such that $P_\varepsilon(K_\delta^c) < \delta$. Φ is continuous and bounded on K_δ therefore Φ is uniformly continuous on this compact set and $\Phi((\varepsilon \lambda + I_2)^{-1}(y), u)$ converges uniformly to $\Phi(y, u)$ on K_δ . Now, as $\delta \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi((\varepsilon \lambda + I_2)^{-1}(y), u) dP_\varepsilon(y, u) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi(y, u) dP_\varepsilon(y, u) = \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP(x, u).$$

We proved that $\int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) = \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP(x, u)$, i.e. P is $T_{\lambda(x)}$ -invariant.

STEP 3 : $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u)$. It is clear with the Γ -liminf property and the following result (see Lemma 2.2 in [23] for the proof) :

Lemma 5.6. (Sandier-Serfaty,[23]) *Assume that X is a Polish metric space, that $\{P_\varepsilon\}_{\varepsilon > 0}$, P are Borel probability measures on X such that $P_\varepsilon \rightarrow P$ as $\varepsilon \rightarrow 0$, and that $\{f_\varepsilon\}_{\varepsilon > 0}$ and f are positive measurable functions on X such that $\liminf_{\varepsilon \rightarrow 0} f_\varepsilon(x_\varepsilon) \geq f(x)$ whenever $x_\varepsilon \rightarrow x$. Then $\liminf_{\varepsilon \rightarrow 0} \int f_\varepsilon dP_\varepsilon \geq \int f dP$.*

The results *ii*) and *iv*) are obvious by construction of P and the point *vi*) follows from the Wiener's Ergodic Theorem proved by Becker in [2] (see Theorem 3) as in [23] and [24] and because P is $T_{\lambda(x)}$ -invariant. \square

6 Asymptotic Expansion of the Hamiltonian

We define $\alpha := \frac{1}{\pi} \int_{\mathbb{R}^2} \min_{\mathcal{A}_{m_V(x)}} W dx = \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx$ and

$$F_n(\nu) = \begin{cases} \frac{1}{n} \left(\frac{1}{\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \int \zeta d\nu \right) & \text{if } \nu \text{ is of the form } \sum_{i=1}^n \delta_{x_i} \\ +\infty & \text{otherwise.} \end{cases}$$

For $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ let E_{ν_n} be a solution of $\operatorname{div} E_{\nu_n} = 2\pi(\nu'_n - m'_V)$, $\operatorname{curl} E_{\nu_n} = 0$ and

$$P_{\nu_n} := \int_{\mathbb{R}^2} \delta_{(x, E_{\nu_n}(x\sqrt{n}+))} d\mu_V(x).$$

With the following result we generalize Theorem 2 of [24] for a broader class of equilibrium measures (for example, with unbounded support). We use a Gamma-Convergence method (see [3] or [19] for details) as in [24] hence we show a lower bound, by Ergodic Theorem, and an upper bound by the compact case seen in [24]. These two bounds give the convergence of $\frac{1}{n} \left[w_n(x_1, \dots, x_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right]$ to α for a minimizer (x_1, \dots, x_n) of w_n .

6.1 Main result

Theorem 6.1. *Let $1 < p < 2$ and $X = \mathbb{R}^2 \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$. Let μ_V be the probability measure verifying (H1), (H2), (H3), (H4) associated to the equilibrium problem in the plane with external field V verifying the growth assumption (3.2).*

A. Lower bound : *Let $(\nu_n)_n$ such that $\frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) \leq C$, then*

- 1) P_{ν_n} is a probability measure on X and admits a subsequence which converges to a probability measure P on X ,
- 2) the first marginal of P is μ_V ,
- 3) P is $T_{\lambda(x)}$ -invariant,
- 4) $E \in \mathcal{A}_{m_V(x)}$ P -a.e.,
- 5) we have the lower bound

$$\liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E) \geq \alpha. \quad (6.1)$$

B. Upper bound. *Conversely, assume P is a $T_{\lambda(x)}$ -invariant probability measure on X whose first marginal is μ_V and such that for P -almost every (x, E) we have $E \in \mathcal{A}_{m_V(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of measures on \mathbb{R}^2 and a sequence $\{E_n\}_n$ in $L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{div} E_n = 2\pi(\nu'_n - m'_V)$ and such that defining $P_n = \int_{\mathbb{R}^2} \delta_{(x, E_n(x\sqrt{n}+))} d\mu_V(x)$, we have $P_n \rightarrow P$ as $n \rightarrow +\infty$ and*

$$\limsup_{n \rightarrow +\infty} F_n(\nu_n) \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E). \quad (6.2)$$

C. Consequences for minimizers. *If (x_1, \dots, x_n) minimizes w_n for every n and $\nu_n = \sum_{i=1}^n \delta_{x_i}$, then :*

- 1) for P -almost every (x, E) , E minimizes W over $\mathcal{A}_{m_V(x)}$,
- 2) we have

$$\lim_{n \rightarrow +\infty} F_n(\nu_n) = \lim_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E) = \alpha \quad (6.3)$$

hence we obtain the asymptotic expansion of the energy w_n , for its minimizer, as $n \rightarrow +\infty$:

$$w_n(x_1, \dots, x_n) = I_V(\mu_V)n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx \right) n + o(n). \quad (6.4)$$

Remarks 6.2. - The lower bound (6.1) is valid for any configurations.

- The following equality from (6.3)

$$\lim_{n \rightarrow +\infty} \frac{1}{n} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) = \int_{\mathbb{R}^2} \min_{\mathcal{A}_{m_V}(x)} W dx$$

is explained by Serfaty in [25] as follows : “minimizers of w_n provide configurations of points in the plane whose associated vector fields E minimize, after blow-up and taking the limit $n \rightarrow +\infty$, the renormalized energy (heuristically) for almost every blow-up center”.

6.2 Proof of the lower bound with Ergodic Theorem

We follow the same lines as in [24]. Let χ be a C^∞ cutoff function with support the unit ball B_1 and integral equal to 1. We define

$$\mathbf{f}_n(x, \nu, E, g) := \begin{cases} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} dg(y) & \text{if } (\nu, E, g) = \theta_{\sqrt{n}x}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n) \\ +\infty & \text{else} \end{cases}$$

where $\bar{\nu}_n$ is a subsequence such that $F_n(\bar{\nu}_n) \leq C$, which we assume to exist. Hence $\bar{\nu}'_n, \bar{E}_n, \bar{g}_n$ are respectively its expression in blow-up coordinates, its associated vector field and the corresponding Radon measure by Definition 5.1. We define and bound by above the following expression

$$\begin{aligned} \mathbf{F}_n(\nu, E, g) &:= \int_{\mathbb{R}^2} \mathbf{f}_n(x, \theta_{x\sqrt{n}}(\nu, E, g)) d\mu_V(x) \\ &= \int_{\mathbb{R}^2} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} d(\theta_{x\sqrt{n}} \# g) d\mu_V(x) \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi(y - x\sqrt{n}) dx d\bar{g}_n(y) \\ &= \frac{1}{n\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi(y - u) du d\bar{g}_n(y) \\ &\leq \frac{1}{n\pi} (\bar{g}_n(\mathbb{R}^2) + \bar{g}_n^-(U^c)) \\ &= \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + \frac{\bar{g}_n^-(U^c)}{n\pi} \end{aligned}$$

by (5.3), where $U = \{x' : d(x', \mathbb{R}^2 \setminus \Sigma) \geq 1\}$. We know that \bar{g}_n^- is supported in the union of balls $B(x', C)$ for $x' \in \text{Supp}(\nu'_n)$ and bounded above by a constant. Thus, as in [24], $\bar{g}_n^-(U^c)$ is bounded by a constant times the number of balls intersecting U^c , hence

$$\bar{g}_n^-(U^c) \leq C \bar{\nu}'_n \{x' : d(x', U^c) \leq C\} \leq Cn |\{x; d(x; \partial\Sigma) \leq C/\sqrt{n}\}| + o(n)$$

because m_V is bounded. Using standard estimates on the volumes of tubular neighborhoods, since $\partial\Sigma$ is C^1 by assumption, we conclude that $\bar{g}_n^-(U^c) = o(n)$. Hence, if $(\nu, E, g) = (\bar{\nu}'_n, \bar{E}_n, \bar{g}_n)$, as $n \rightarrow +\infty$:

$$\mathbf{F}_n(\nu, E, g) \leq \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + o(1)$$

and $\mathbf{F}_n(\nu, E, g) = +\infty$ otherwise.

Now we want to use Ergodic Theorem 5.4 with $\varepsilon = \frac{1}{\sqrt{n}}$ and $X = \mathcal{M}_+ \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2) \times \mathcal{M}$ where $p \in]1, 2[$, \mathcal{M}_+ is the set of nonnegative Radon measures on \mathbb{R}^2 and \mathcal{M} the set of Radon measures bounded below by

$-C_V := -C(\|m_V\|_\infty^2 + 1)$. Let Q_n be the image of μ_V by $x \mapsto (x, \theta_{x\sqrt{n}}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n))$. We have :
1) \mathbf{f}_n is coercive by Lemma 6.3 of [24]. Indeed, if $(x_n, \nu_n, E_n, g_n)_n$ is so that, for any $R > 0$,

$$\limsup_{n \rightarrow +\infty} \int_{B_R} \mathbf{f}_n \left(x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, E_n, g_n) \right) d\lambda < +\infty$$

then the integrand is bounded for a.e. λ . By assumption on \mathbf{f}_n , $\theta_\lambda(\nu_n, E_n, g_n) = \theta_{x_n\sqrt{n}+\lambda}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n)$, hence it follows that

$$(\nu_n, E_n, g_n) = \theta_{x_n\sqrt{n}}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n),$$

and for any $R > 0$ there exists $C_R > 0$ such that for any $n > 0$

$$\begin{aligned} \int_{B_R} \mathbf{f}_n \left(x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, E_n, g_n) \right) d\lambda &= \int_{B_R} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V \left(x_n + \frac{\lambda}{\sqrt{n}} \right)} d(\theta_{\lambda+x_n\sqrt{n}} \# \bar{g}_n(y)) d\lambda \\ &= \frac{1}{\pi} \int_{B_R} \int_{\mathbb{R}^2} \frac{\chi(y - x_n\sqrt{n} - \lambda)}{m_V \left(x_n + \frac{\lambda}{\sqrt{n}} \right)} d\bar{g}_n(y) d\lambda \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \chi * \left(\mathbb{1}_{B_R(x_n\sqrt{n})} \frac{1}{m_V(\cdot/\sqrt{n})} \right) d\bar{g}_n(y) < C_R. \end{aligned}$$

As $m_V(x) \leq M$, we get

$$\frac{1}{M\pi} \int_{\mathbb{R}^2} \chi * \mathbb{1}_{B_R(x_n\sqrt{n})} d\bar{g}_n(y) < C_R.$$

This and the fact that \bar{g}_n is bounded below implies that $\bar{g}_n(B_R(x_n\sqrt{n}))$ is bounded independently of n . Hence by the same argument as in Lemma 6.3 from [24] we have the convergence of a subsequence of (ν_n, E_n, g_n) . Remark that we don't want the convergence of a subsequence of $(x_n)_n$.

2) We have the Γ -liminf property : if $(x_n, \nu_n, E_n, g_n) \rightarrow (x, \nu, E, g)$ as $n \rightarrow +\infty$ then

$$\liminf_{n \rightarrow +\infty} \mathbf{f}_n(x_n, \nu_n, E_n, g_n) \geq f(x, \nu, E, g) := \frac{1}{\pi} \int \frac{\chi(y)}{m_V(x)} dg(y)$$

obviously if the left-hand side is finite, by Fatou Lemma. Therefore Ergodic Theorem 5.4 implies that:

i) Q_n admits a subsequence which converges to Q which has μ_V for first marginal,

ii) $(x, \nu, E, g) = \lim_{n \rightarrow +\infty} (x_n, \theta_{x_n\sqrt{n}}(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n))$ Q -a.e.,

iii) Q is $T_{\lambda(x)}$ -invariant,

iv) $\liminf_{n \rightarrow +\infty} \mathbf{F}_n(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} dg(y) \right) dQ(x, \nu, E, g)$.

v) $\frac{1}{\pi} \int \int \frac{\chi(y)}{m_V(x)} dg(y) dQ(x, \nu, E, g) = \int \left(\lim_{R \rightarrow +\infty} \int_{B_R} \int \frac{\chi(y - \lambda)}{m_V(x)} dg(y) d\lambda \right) dQ(x, \nu, E, g)$.

We obtain by iv) the following inequality

$$\liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) \geq \liminf_{n \rightarrow +\infty} \mathbf{F}_n(\bar{\nu}'_n, \bar{E}_n, \bar{g}_n) \geq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \chi(y) dg(y) \right) \frac{dQ(x, \nu, E, g)}{m_V(x)}.$$

By v) it follows that

$$\begin{aligned} \int \left(\int \chi(y) dg(y) \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} &= \int \lim_{R \rightarrow +\infty} \left[\int_{B_R} \left(\int \chi(y - \lambda) dg(y) \right) d\lambda \right] \frac{dQ(x, \nu, E, g)}{m_V(x)} \\ &= \int \lim_{R \rightarrow +\infty} \left(\frac{1}{\pi R^2} \int \chi * \mathbb{1}_{B_R} dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} \end{aligned}$$

and we get $\lim_{R \rightarrow +\infty} \frac{1}{\pi R^2} \int \chi * \mathbb{1}_{B_R} dg < +\infty$, therefore there exists $C > 0$ such that $\forall R > 1$, $\int_{B_R} dg \leq CR^2$. Now we obtain $E \in \mathcal{A}_{m_V(x)}$. Indeed by (5.2) with $\Lambda = (x_1, \dots, x_n)$ we have

$$\nu(B_R) = \#(\Lambda \cap B_R) \leq C(1 + \|m_V\|_\infty^2 |\widehat{B}_R| + g(B_R)) \leq CR^2 + C \int_{B_R} dg \leq CR^2.$$

As $\operatorname{div} \bar{E}_n = 2\pi(\bar{\nu}_n - m'_V)$ and $\operatorname{curl} \bar{E}_n = 0$, we deduce from *ii*) by pass to limit that $\operatorname{div} E = 2\pi(\nu - m_V)$ and $\operatorname{curl} E = 0$. We use the same argument that in the step 3 of the proof of Theorem 2, part A, in [24], which use (5.1) to prove

$$\lim_{R \rightarrow +\infty} \frac{1}{R^2} \left| W(E, \chi * \mathbb{1}_{B_R}) - \int \chi * \mathbb{1}_{B_R} dg \right| = 0.$$

Now we notice that P_n is the marginal of Q_n corresponding to the variables (x, E) . P_n converges to a probability measure P which is $T_{\lambda(x)}$ -invariant and we have the following lower bound

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbb{1}_{\mathbb{R}^2}) &\geq \int \left(\int \chi dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} = \int \lim_{R \rightarrow +\infty} \left(\frac{1}{\pi R^2} \int \chi * \mathbb{1}_{B_R} dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} \\ &\geq \frac{1}{\pi} \int W(E) \frac{dQ(x, \nu, E, g)}{m_V(x)} = \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP(x, E). \end{aligned}$$

Thus the lower bound (6.1) is proved. The fact that the right-hand side is larger than α is obvious because the first marginal of $\frac{dP}{m_V}$ is the Lebesgue measure.

6.3 Proof of the upper bound by compactification and conclusion

We use Theorem 2.B. from [24] when K is a compact set of \mathbb{R}^2 and ∂K is C^1 :

Theorem 6.3. (Sandier-Serfaty,[24]) *Let P be a $T_{\lambda(x)}$ -invariant probability measure on $X = K \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ with first marginal $dx_{|K}/|K|$ and such that for P almost every (x, E) we have $E \in \mathcal{A}_{m(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of empirical measures on K and a sequence $\{E_n\}_n$ in $L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{div} E_n = 2\pi(\nu'_n - m')$, $P_n := \int_K \delta_{(x, E_n(\sqrt{n}x+))} dx \rightarrow P$ as $n \rightarrow +\infty$ and*

$$\limsup_{n \rightarrow +\infty} F_n(\nu_n) \leq \frac{|K|}{\pi} \int W(E) dP(x, E).$$

Proof. We have the following two cases :

First case : If $\Sigma_{\mathcal{S}} \neq \mathcal{S}$, then there exists a projection point $\tilde{p} \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$ and a corresponding stereographic projection $\tilde{T} : \mathcal{S} \rightarrow \mathbb{R}^2$ such that $\tilde{T}(\Sigma_{\mathcal{S}}) = K$ is a compact of \mathbb{R}^2 .

Let P be a $T_{\lambda(x)}$ -invariant probability measure on $\Sigma \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$, then if we denote by T_*P the push-forward of P by T for its first marginal and $\tilde{T}_*(T_*P)$ the push-forward of T_*P by \tilde{T} , always for its first marginal, then $\tilde{T}_*(T_*P)$ is a $T_{\lambda(x)}$ -invariant probability measure on $K \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ such that $E \in \mathcal{A}_{\tilde{m}_V(x)}$ for $\tilde{T}_*(T_*P)$ -a.e. (x, E) with $\tilde{m}_V(x) := \left| DT^{-1}(\tilde{T}^{-1}(x)) \right| \left| D\tilde{T}^{-1}(x) \right| m_V(T^{-1} \circ \tilde{T}^{-1}(x))$ where $|DT^{-1}(p)|$ is the determinant of the Jacobian of T^{-1} on p . Therefore we have

$$d(\tilde{T}_* T_* P)(x, E) = |K| \tilde{m}_V(x) dP_K(x, E)$$

where P_K is a $T_{\lambda(x)}$ -invariant probability measure on $K \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ with first marginal $dx_{|K}/|K|$ and such that $E \in \mathcal{A}_{|K|^{-1}} P_K$ -a.e.. Hence we use the previous Theorem, because $\partial \Sigma_{\mathcal{S}}$ is C^1 and \tilde{T} and T

are conformal homeomorphisms, in order to obtain the existence of sequences $(\nu_n)_n$ and $(E_n)_n$ such that $\operatorname{div} E_n = 2\pi(\nu'_n - |K|^{-1})$, $\operatorname{curl} E_n = 0$ and

$$\limsup_{n \rightarrow +\infty} F_n(\nu_n) \leq \frac{|K|}{\pi} \int W(E) dP_K(x, E) = \frac{1}{\pi} \int W(E) \frac{d(\tilde{T}_* T_* P)(x, E)}{\tilde{m}_V(x)} = \frac{1}{\pi} \int W(E) \frac{dP(x, E)}{m_V(x)}.$$

Furthermore, by the scaling formula for vector fields, there exist $(\nu_n)_n$ and $(E_n)_n$ such that $\operatorname{div} E_n = 2\pi(\nu'_n - m'_V)$ and $\operatorname{curl} E_n = 0$.

The convergence of P_n is obvious by the convergence given in the previous Theorem 6.3 and the change of variable.

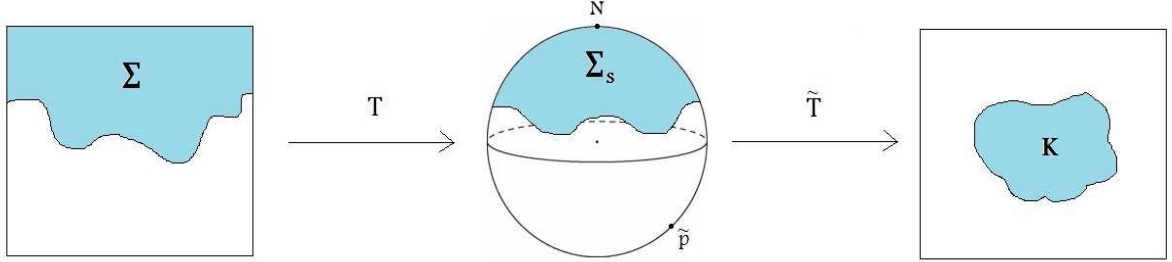


Illustration of the projections when $\Sigma_S \neq \mathcal{S}$

Second case : If $\Sigma_S = \mathcal{S}$ then we cut Σ_S in two compact hemispheres H_N (north) and H_S (south). For H_S we use the stereographic projection T and the previous theorem in $T(H_S)$. For H_N we use a stereographic projection \tilde{T} with point projection $\tilde{p} \notin H_N$. We apply the previous theorem to the compact $\tilde{T}(H_N)$. Because $\operatorname{curl} E_n = 0$ and $\mu_V(\partial B_R) = 0$ for any ball of radius R , we can pick the two constructions and obtain the upper bound with the same technical details as above. \square

Part *C* follows from *A* and *B* : inequalities (6.1) and (6.2) become equalities. By Gamma-Convergence we minimize $\frac{1}{\pi} \int \frac{W}{m_V} dP$ over vector fields of \mathcal{A}_{m_V} and we find α .

7 Consequence : the Logarithmic Energy on the Sphere

7.1 Asymptotic of the optimal logarithmic energy on the unit sphere

An important case is the equilibrium measure associated to the potential $V(x) = \log(1 + \|x\|^2)$ corresponding to the external field $\mathcal{V} \equiv 0$ on \mathcal{S} and where $T_* \mu_V$ is the uniform probability measure on \mathcal{S} . Hence (see Example 3.4) we find $d\mu_V(x) = \frac{dx}{\pi(1 + \|x\|^2)^2}$, $\Sigma = \mathbb{R}^2$ hence $\zeta(x) = 0$ for any $x \in \mathbb{R}^2$. We define

$$\bar{w}_n(x_1, \dots, x_n) := - \sum_{i \neq j}^n \log \|x_i - x_j\| + (n-1) \sum_{i=1}^n V(x_i)$$

and we recall that the logarithmic energy of the spherical configuration (y_1, \dots, y_n) is

$$E_{\log}(y_1, \dots, y_n) = - \sum_{i \neq j}^n \log \|y_i - y_j\|.$$

Lemma 7.1. *We have the following equalities*

$$\min_{(x_i) \in \mathbb{R}^{2n+2}} \bar{w}_{n+1}(x_1, \dots, x_{n+1}) = \min_{(x_i) \in \mathbb{R}^{2n}} w_n(x_1, \dots, x_n) = \min_{(y_i) \in \mathcal{S}^n} \mathbb{E}_{\log}(y_1, \dots, y_n)$$

Proof. These equalities are clear by the following identity, for any $(x_1, \dots, x_n) \in \mathbb{R}^{2n}$, $y_i = T(x_i) \in \mathcal{S}$ and $y_i \neq N$ for every $1 \leq i \leq n$:

$$\begin{aligned} \mathbb{E}_{\log}(y_1, \dots, y_n) &:= - \sum_{i \neq j}^n \log \|y_i - y_j\| \\ &= - \sum_{i \neq j}^n \log \|T(x_i) - T(x_j)\| \\ &= - \sum_{i \neq j}^n \log \left(\frac{\|x_i - x_j\|}{\sqrt{1 + \|x_i\|^2} \sqrt{1 + \|x_j\|^2}} \right) \\ &= - \sum_{i \neq j}^n \log \|x_i - x_j\| + (n-1) \sum_{i=1}^n \log(1 + \|x_i\|^2) \\ &= \bar{w}_n(x_1, \dots, x_n). \end{aligned}$$

Hence we can write, with $\nu_n = \sum_{i=1}^n \delta_{x_i}$ and thanks to equality (3.1):

$$\begin{aligned} w_n(x_1, \dots, x_n) &= \bar{w}_n(x_1, \dots, x_n) + \sum_{i=1}^n \log(1 + \|x_i\|^2) \\ &= \mathbb{E}_{\log}(y_1, \dots, y_n) - 2 \int_{\mathcal{S}} \log \|y - N\| dT_* \nu_n(y) \\ &= - \sum_{i \neq j} \log \|y_i - y_j\| - 2 \sum_{i=1}^n \log \|y_i - N\| = \mathbb{E}_{\log}(y_1, \dots, y_n, N). \end{aligned}$$

Now, let $y_{n+1} = N$ be the point projection, then (y_1, \dots, y_{n+1}) is a minimizer of \mathbb{E}_{\log} if and only if (x_1, \dots, x_{n+1}) is a minimizer of \bar{w}_{n+1} if and only if (x_1, \dots, x_n) is a minimizer of w_n . \square

Lemma 7.2. *If $(\bar{x}_1, \dots, \bar{x}_n)$ is a minimizer of \bar{w}_n and $\nu_n = \sum_{i=1}^n \delta_{\bar{x}_i}$ then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) = \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x)$$

Proof. Let $(\bar{x}_1, \dots, \bar{x}_n)$ be a minimizer of \bar{w}_n . We notice that for every $n \geq 2$ such that $y_i = T(\bar{x}_i) \neq N$ for any $1 \leq i \leq n$,

$$\frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) = -\frac{2}{n} \int_{\mathbb{R}^2} \log \left(\frac{1}{\sqrt{1 + \|x\|^2}} \right) d\nu_n(x) = -\frac{2}{n} \int_{\mathcal{S}} \log \|y - N\| dT_* \nu_n(y)$$

and by the previous lemma, (y_1, \dots, y_n) is a minimizer of \mathbb{E}_{\log} on \mathcal{S} . Now we use Theorem 15 of [4] about the optimal point separation which yields the existence of constants C and n_0 such that for any $n \geq n_0$ and

any minimizer $\{y_1, \dots, y_n\} \in \mathcal{S}^n$ of the logarithmic energy on the sphere, we have

$$\min_{i \neq j} \|y_i - y_j\| > \frac{C}{\sqrt{n-1}}.$$

Therefore it is clear that there exists a sequence of points $(N_n)_{n \geq n_0}$ on the sphere such that for any $n \geq n_0$ and any $1 \leq i \leq n$

$$\|y_i - N_n\| > \frac{C}{\sqrt{n-1}}. \quad (7.1)$$

For every $n \geq 2$, let T_n be the inverse stereographic projection with north pole N_n , then we get, for any $n \geq n_0$:

$$\frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) = -\frac{2}{n} \int_{\mathcal{S}} \log \|y - N_n\| d(T_n)_* \nu_n(y).$$

For $n \geq n_0$ and $\delta > 0$ sufficiently small we define, for any $0 < r \leq \delta$,

$$n(r) := \#\{y_i; y_i \in B(N_n, r) \cap \mathcal{S}\}$$

and $r_i = \|y_i - N_n\|$ where $y_i \in B(N_n, \delta) \cap \mathcal{S}$. We notice that there exists a constant C such that $n(r) \leq Cr^2 n$ for any r . Hence we have, by integration by parts and the separation (7.1) :

$$\begin{aligned} -\sum_{i=1}^{n_\delta} \log r_i &= -\int_{1/\sqrt{n-1}}^{\delta} \log rn'(r) dr \\ &= -\log \delta n(\delta) + \int_{1/\sqrt{n-1}}^{\delta} \frac{n(r)}{r} dr \\ &\leq -C\delta^2 \log \delta n + Cn \int_{1/\sqrt{n-1}}^{\delta} r dr \\ &\leq C\delta^2 |\log \delta| n. \end{aligned}$$

Thus for $(y_1, \dots, y_{n_\delta}) \in B(N_n, \delta) \cap \mathcal{S}$, it follows that

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B(N_n, \delta) \cap \mathcal{S}} \log \|y - N\| d(T_n)_* \nu_n(y) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} -\sum_{i=1}^{n_\delta} \frac{1}{n} \log \|y_i - N\| = 0 \quad (7.2)$$

and by the fact that $\frac{\nu_n}{n}$ goes weakly to the measure μ_V on B_R for any R , we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) &= \lim_{n \rightarrow +\infty} \frac{1}{n} \left(\int_{B_R} \log(1 + \|x\|^2) d\nu_n(x) + \int_{B_R^c} \log(1 + \|x\|^2) d\nu_n(x) \right) \\ &= \int_{B_R} \log(1 + \|x\|^2) d\mu_V(x) + \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B_R^c} \log(1 + \|x\|^2) d\nu_n(x). \end{aligned}$$

Hence it follows from (7.2) that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) &= \lim_{R \rightarrow +\infty} \left(\int_{B_R} \log(1 + \|x\|^2) d\mu_V(x) + \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B_R^c} \log(1 + \|x\|^2) d\nu_n(x) \right) \\ &= \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x). \end{aligned}$$

□

Thus we obtain, by the both previous lemmas :

Corollary 7.3. *If (x_1, \dots, x_n) is a minimizer of w_n and $\nu_n = \sum_{i=1}^n \delta_{x_i}$ then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\nu_n(x) = \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x).$$

The following result proves the existence of the constant C in the Conjecture 1 of Rakhmanov, Saff and Zhou.

Theorem 7.4. *If $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$ is a minimizer of E_{\log} then, as $n \rightarrow +\infty$*

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 \right) n + o(n).$$

Proof. Let $(y_1, \dots, y_n) \in \mathbb{S}^2$ be a minimizer of E_{\log} . Without loss of generality, we can choose this configuration such that $y_i \neq N$ for any $1 \leq i \leq n$, hence there exists (x_1, \dots, x_n) such that $\frac{y_i}{2} = T(x_i)$ for any i and we get

$$\begin{aligned} E_{\log}(y_1, \dots, y_n) &= - \sum_{i \neq j}^n \log \|y_i - y_j\| \\ &= - \sum_{i \neq j}^n \log \|T(x_i) - T(x_j)\| - n(n-1) \log 2 \\ &= \bar{w}_n(x_1, \dots, x_n) - n(n-1) \log 2. \end{aligned}$$

and by Lemma 7.1, (y_1, \dots, y_n) is a minimizer of E_{\log} if and only if (x_1, \dots, x_n) is a minimizer of \bar{w}_n . By the lower bound (6.1) and the convergence of Lemma 7.2 we have, for a minimizer $(\bar{x}_1, \dots, \bar{x}_n)$ of \bar{w}_n :

$$\begin{aligned} &\liminf_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &= \liminf_{n \rightarrow +\infty} \frac{1}{n} \left[w_n(\bar{x}_1, \dots, \bar{x}_n) - \sum_{i=1}^n \log(1 + \|\bar{x}_i\|^2) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &\geq \alpha - \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x). \end{aligned}$$

Upper bound (6.2) and Corollary 7.3 yield, for $(\bar{x}_1, \dots, \bar{x}_n)$ a minimizer of \bar{w}_n and (x_1, \dots, x_n) a minimizer of w_n :

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(x_1, \dots, x_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &= \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[w_n(x_1, \dots, x_n) - \sum_{i=1}^n \log(1 + \|x_i\|^2) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &= \alpha - \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x). \end{aligned}$$

Thus we get

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] = \alpha - \int_{\mathbb{R}^2} \log(1 + \|x\|^2) d\mu_V(x)$$

and we have the following asymptotic expansion, as $n \rightarrow +\infty$, when $(\bar{x}_1, \dots, \bar{x}_n)$ is a minimizer of \bar{w}_n :

$$\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) = n^2 I_V(\mu_V) - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx - \int_{\mathbb{R}^2} V(x) d\mu_V(x) \right) n + o(n).$$

Now we prove that, for any $x \in \mathbb{R}^2$, $U^{\mu_V}(x) + \frac{V(x)}{2} = 0$. We have to use the identity

$$\int_0^{2\pi} -\log \|x - re^{i\theta}\| d\theta = \begin{cases} -2\pi \log r & \text{if } r \geq \|x\| \\ -2\pi \log \|x\| & \text{if } r < \|x\|. \end{cases}$$

Thus it follows, for any $x \in \mathbb{R}^2$:

$$\begin{aligned} U^{\mu_V}(x) &= - \int \frac{\log \|x - y\|}{\pi(1 + \|y\|^2)^2} dy = \int_{r=0}^{+\infty} \frac{r}{\pi(1 + r^2)^2} \left(\int_{\theta=0}^{2\pi} -\log \|x - re^{i\theta}\| d\theta \right) dr \\ &= -2 \int_{\|x\|}^{+\infty} \frac{r \log r}{(1 + r^2)^2} dr - 2 \left(\int_0^{\|x\|} \frac{r}{(1 + r^2)^2} dr \right) \log \|x\| \\ &= - \left(\left[\frac{-\log r}{(1 + r^2)} \right]_{\|x\|}^{+\infty} + \int_{\|x\|}^{+\infty} \frac{dr}{r(1 + r^2)} \right) + \left[\frac{1}{1 + r^2} \right]_0^{\|x\|} \log \|x\| \\ &= - \frac{\log \|x\|}{1 + \|x\|^2} - \frac{1}{2} \left[\log \left(\frac{r^2}{1 + r^2} \right) \right]_{\|x\|}^{+\infty} + \frac{\log \|x\|}{1 + \|x\|^2} - \log \|x\| \\ &= -\frac{1}{2} \log(1 + \|x\|^2) = -\frac{V(x)}{2}. \end{aligned}$$

Therefore we obtain, by a simple integration by parts as in the previous computation :

$$\begin{aligned} I_V(\mu_V) &= \int_{\mathbb{R}^2} U^{\mu_V}(x) d\mu_V(x) + \int_{\mathbb{R}^2} V(x) d\mu_V(x) \\ &= \frac{1}{2} \int_{\mathbb{R}^2} V(x) d\mu_V(x) \\ &= \int_0^{+\infty} \frac{r \log(1 + r^2)}{(1 + r^2)^2} dr = \frac{1}{2}. \end{aligned}$$

Hence we get, as $n \rightarrow +\infty$,

$$\begin{aligned} \bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) &= \frac{n^2}{2} - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{1}{2} \int \log(\pi(1 + \|x\|^2)^2) d\mu_V(x) - 1 \right) n + o(n) \\ &= \frac{n^2}{2} - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \int \log(1 + \|x\|^2) d\mu_V(x) - 1 \right) n + o(n) \\ &= \frac{n^2}{2} - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} \right) n + o(n) \end{aligned}$$

and the asymptotic expansion of E_{\log} , for its minimizer (y_1, \dots, y_n) , is, as $n \rightarrow +\infty$:

$$E_{\log}(y_1, \dots, y_n) = \left(\frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + \left(\frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log(2) \right) n + o(n).$$

□

7.2 Computation of renormalized energy for the triangular lattice and upper bound for the term of order n

Sandier and Serfaty proved in [23] that

$$W(\Lambda_{1/2\pi}) = -\frac{1}{2} \log \left(\sqrt{2\pi b} |\eta(\tau)|^2 \right)$$

where $\Lambda_{1/2\pi}$ is the triangular lattice corresponding to the density $m = 1/2\pi$, $\tau = a + ib = 1/2 + i\frac{\sqrt{3}}{2}$ and η is the Dedekind eta function defined, with $q = e^{2i\pi\tau}$, by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

We recall Chowla-Selberg formula (see [7] or [6] for details) :

$$\prod_{m=1}^{|D|} \Gamma \left(\frac{m}{|D|} \right)^{\frac{w}{2} \left(\frac{D}{m} \right)} = 4\pi \sqrt{-D} b |\eta(\tau)|^4$$

for τ a root of the integral quadratic equation $\alpha z^2 + \beta z + \gamma = 0$ where $D = b^2 - 4ac < 0$, $\left(\frac{D}{m} \right)$ is the Kronecker symbol, w the number of roots of unity in $\mathbb{Q}(i\sqrt{-D})$ and when the class number of $\mathbb{Q}(i\sqrt{-D})$ is equal to 1. In our case $b = \sqrt{3}/2$, $w = 6$, $\alpha = \beta = \gamma = 1$ because τ is a root of unity, hence $D = -3$, $\left(\frac{-3}{1} \right) = 1$ and $\left(\frac{-3}{2} \right) = -1$ by the Gauss Lemma. Therefore we obtain

$$\Gamma(1/3)^3 \Gamma(2/3)^{-3} = 4\pi \sqrt{3} \frac{\sqrt{3}}{2} |\eta(\tau)|^4$$

and by Euler's reflection formula $\Gamma(1 - 1/3)\Gamma(1/3) = \frac{\pi}{\sin(\pi/3)}$ we get

$$\frac{\Gamma(1/3)^6 3\sqrt{3}}{8\pi^3} = \frac{4\pi \sqrt{3} \times \sqrt{3} |\eta(\tau)|^4}{2}$$

and finally we have

$$|\eta(\tau)|^4 = \frac{\Gamma(1/3)^6 \sqrt{3}}{16\pi^4}.$$

Now it is possible to find the exact value of the renormalized energy of the triangular lattice Λ_1 of density $m = 1$:

$$\begin{aligned} W(\Lambda_1) &= 2\pi W(\Lambda_{1/2\pi}) - \pi \frac{\log(2\pi)}{2} \\ &= -\pi \log \left(\sqrt{2\pi b} |\eta(\tau)|^2 \right) - \pi \frac{\log(2\pi)}{2} \\ &= \pi \log \pi - \frac{\pi}{2} \log 3 - 3\pi \log(\Gamma(1/3)) + \frac{3}{2} \pi \log 2 \\ &= \pi \log \left(\frac{2\sqrt{2}\pi}{\sqrt{3}\Gamma(1/3)^3} \right) \end{aligned}$$

and $W(\Lambda_1) \approx -11.342962807137$. Thus we get

$$\begin{aligned} & \frac{1}{\pi}W(\Lambda_1) + \frac{\log \pi}{2} + \log 2 \\ &= \frac{1}{\pi} \left(\pi \log \pi - \frac{\pi}{2} \log 3 - 3\pi \log(\Gamma(1/3)) + \frac{3}{2}\pi \log 2 \right) + \frac{\log \pi}{2} + \log 2 \\ &= 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} \approx -0.055605304943 \end{aligned}$$

and we find exactly the value conjectured by Brauchart, Hardin and Saff in [5]. Therefore Conjecture 2 is true if and only if the triangular lattice Λ_1 is a global minimizer of W among vector fields in \mathcal{A}_1 , i.e.

$$\min_{\mathcal{A}_1} W = W(\Lambda_1) = \pi \log \left(\frac{2\sqrt{2}\pi}{\sqrt{3}\Gamma(1/3)^3} \right).$$

Thus we obtain the following result

Theorem 7.5. (1) *If (y_1, \dots, y_n) is a minimizer of E_{\log} , then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[E_{\log}(y_1, \dots, y_n) - \left(\frac{1}{2} - \log 2 \right) n^2 + \frac{n}{2} \log n \right] \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}$$

(2) *CONJECTURES 2 and 3 are equivalent.*

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