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Expansion of a singularly perturbed equation with a two-scale converging convection term

Alexandre MOUTON*

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Abstract

In many physical contexts, evolution convection equations may present some very large amplitude convective terms. As an example, in the context of magnetic confinement fusion, the distribution function which describes the plasma satisfies the Vlasov equation in which some terms are of the same order as ϵ^{-1} , $\epsilon \ll 1$ being the characteristic gyrokinetic period of the particles around the magnetic lines. In this paper, we aim to present a model hierarchy for modeling the distribution function for any value of ϵ by using some two-scale convergence tools. Following Frénod & Sonnendrücker's recent work, we choose the framework of a singularly perturbed convection equation where the convective terms admits a high amplitude part which periodically oscillates in time with high frequency $\epsilon^{-1} \gg 1$. In this abstract framework, we derive an expansion with respect to the small parameter ϵ and we recursively identify each term of this expansion. Finally, we apply this new model hierarchy to the context of a linear Vlasov equation in the presence of a high amplitude external magnetic field.

Key words. Vlasov equation, Two-scale convergence, Gyrokinetic approximations, convection equation.

AMS subject classifications. 35Q83, 76M40, 78A35, 82D10.

1 Introduction

For sixty years, Magnetic Confinement Fusion (MCF) is one of the most important technological challenge for producing domestic energy. Indeed, this worldwide project involves physicists, engineers and mathematicians in order to understand and reproduce on Earth the solar magnetic fusion reaction. One of the most famous examples of this work programme is the ITER project localized in Cadarache (France) which attempts to produce a fusion plasma in a tokamak reactor by confining it thanks to a strong external magnetic field. Besides the required technological aspects of MCF, it became necessary for thirty years to lead a rigorous study of the behaviour of such a plasma and this work takes the form of the derivation of mathematical models and of high precision numerical experiments.

In the present paper, we focus on the Vlasov equation in presence of a external magnetic field with an amplitude of the same order as $\epsilon^{-1} \gg 1$ and on its limit regime as $\epsilon \rightarrow 0$. Such an equation is the main subject of many previous works: indeed, many results about the mathematical justifications of Guiding-Center and Finite Larmor Radius limit regimes have been established by Bostan [5, 6], Frénod & Sonnendrücker [13, 15, 17], Frénod & Mouton [12, 28], Golse & Saint-Raymond [18, 19] and Han-Kwan [21, 22, 23]. Most of these results are based on the use of two-scale convergence and homogenization

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techniques (see Allaire [3] and Nguetseng [30]) or compactness methods. These mathematical studies allowed to validate and reinforce the tokamak plasma models presented by Littlejohn, Lee *et al.*, Dubin *et al.* or Brizard *et al.* (see [26], [24, 25], [9], [7, 8]).

The Vlasov equations we are focused on in the present paper are the following:

$$\begin{cases} \partial_t f_\epsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\epsilon + \left(\mathbf{E}_\epsilon + \mathbf{v} \times \mathbf{B}_\epsilon + \frac{\mathbf{v} \times \boldsymbol{\beta}_\epsilon}{\epsilon} \right) \cdot \nabla_{\mathbf{v}} f_\epsilon = 0, \\ f_\epsilon(t = 0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}), \end{cases} \quad (1.1)$$

$$\begin{cases} \partial_t f_\epsilon + \frac{\mathbf{v}_\perp}{\epsilon} \cdot \nabla_{\mathbf{x}_\perp} f_\epsilon + v_{\parallel} \partial_{x_{\parallel}} f_\epsilon + \left(\mathbf{E}_\epsilon + \mathbf{v} \times \mathbf{B}_\epsilon + \frac{\mathbf{v} \times \boldsymbol{\mathcal{M}}}{\epsilon} \right) \cdot \nabla_{\mathbf{v}} f_\epsilon = 0, \\ f_\epsilon(t = 0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}). \end{cases} \quad (1.2)$$

In both equations, $f_\epsilon = f_\epsilon(t, \mathbf{x}, \mathbf{v})$ is the distribution function which describes the evolution of the plasma in the phase space, t , \mathbf{x} and \mathbf{v} stand for the time, space and velocity variables, $\mathbf{E}_\epsilon = \mathbf{E}_\epsilon(t, \mathbf{x})$ and $\mathbf{B}_\epsilon = \mathbf{B}_\epsilon(t, \mathbf{x})$ are the external electric and magnetic fields, $\boldsymbol{\beta}_\epsilon = \boldsymbol{\beta}_\epsilon(t, \mathbf{x})$ is a given vector function assumed to oscillate in time with $\mathcal{O}(\epsilon^{-1})$ order frequency, $\boldsymbol{\mathcal{M}}$ is a fixed unit vector in \mathbb{R}^3 and, for any $\mathbf{v} \in \mathbb{R}^3$, $v_{\parallel} = \boldsymbol{\mathcal{M}} \cdot \mathbf{v}$ and $\mathbf{v}_\perp = \mathbf{v} - v_{\parallel} \boldsymbol{\mathcal{M}}$. These equations can be obtained by rescaling the linear Vlasov equation as follows. First, we start from the collisionless Vlasov equation

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0, \\ f(t = 0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}), \end{cases}$$

which describes the evolution of a cloud of particles with unit charge q and unit mass m submitted to external electric and magnetic fields $\mathbf{E} = \mathbf{E}(t, \mathbf{x})$ and $\mathbf{B} = \mathbf{B}(t, \mathbf{x})$.

We obtain (1.1) and (1.2) thanks to a rescaling procedure: indeed, for obtaining (1.1), we rescale the variables, the external fields and the distribution function by setting $\mathbf{x}' = \bar{x} \mathbf{x}'$, $\mathbf{v}' = \bar{v} \mathbf{v}'$, $t = \bar{t} t'$, $f(t, \mathbf{x}, \mathbf{v}) = \bar{f} f'(t', \mathbf{x}', \mathbf{v}')$, $\mathbf{E}(t, \mathbf{x}) = \bar{E} \mathbf{E}'(t', \mathbf{x}')$, $\mathbf{B}(t, \mathbf{x}) = \bar{B} \mathbf{B}'(t', \mathbf{x}')$, where $\bar{x}, \bar{v}, \bar{t}, \bar{f}, \bar{E}, \bar{B}$ are respectively the characteristic length, the characteristic velocity, the characteristic time, the characteristic distribution function, the characteristic electric field and the characteristic magnetic field. Introducing the characteristic gyrofrequency of the particles $\bar{\omega} = \frac{q\bar{B}}{m}$ and the characteristic Larmor radius $\bar{a}_L = \frac{\bar{v}}{\bar{\omega}}$, the rescaled Vlasov equation writes

$$\begin{cases} \partial_{t'} f' + \bar{t} \bar{\omega} \frac{\bar{a}_L}{\bar{x}} \mathbf{v}' \cdot \nabla_{\mathbf{x}'} f' + \left(\bar{t} \bar{\omega} \frac{\bar{E}}{\bar{v} \bar{B}} \mathbf{E}' + \bar{t} \bar{\omega} \mathbf{v}' \times \mathbf{B}' \right) \cdot \nabla_{\mathbf{v}'} f' = 0, \\ f'(t' = 0, \mathbf{x}', \mathbf{v}') = f^{0'}(\mathbf{x}', \mathbf{v}'), \end{cases}$$

Then, considering a strong external magnetic field induces the following ratios:

$$\bar{t} \bar{\omega} = \frac{1}{\epsilon}, \quad \frac{\bar{E}}{\bar{v} \bar{B}} = \epsilon, \quad \frac{\bar{a}_L}{\bar{x}} = \epsilon,$$

where ϵ is a dimensionless parameter close to 0. Hence, omitting the primed notations and adding ϵ in subscript for the distribution function and the electric and magnetic fields, we get equation (1.1). A similar procedure gives (1.2) by replacing the rescaling $\mathbf{x}' = \bar{x} \mathbf{x}'$ by $x'_{\parallel} = \bar{x}_{\parallel} x'_{\parallel}$, $\mathbf{x}'_{\perp} = \bar{x}_{\perp} \mathbf{x}'_{\perp}$ and by assuming that

$$\frac{\bar{a}_L}{\bar{x}_{\parallel}} = \epsilon, \quad \frac{\bar{a}_L}{\bar{x}_{\perp}} = 1,$$

instead of $\bar{a}_L = \epsilon \bar{x}$.

Both equations (1.1) and (1.2) can be viewed as particular formulations of a generic singularly perturbed convection equation of the form

$$\begin{cases} \partial_t u_\epsilon(t, \mathbf{x}) + \mathbf{A}_\epsilon(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} u_\epsilon(t, \mathbf{x}) + \frac{1}{\epsilon} \mathbf{L} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \cdot \nabla_{\mathbf{x}} u_\epsilon(t, \mathbf{x}) = 0, \\ u_\epsilon(t = 0, \mathbf{x}) = u^0(\mathbf{x}). \end{cases} \quad (1.3)$$

where $t \in [0, T]$ and $\mathbf{x} \in \mathbb{R}^n$ ($n \in \mathbb{N}^*$) are the variables ($T > 0$ is fixed), $\mathbf{A}_\epsilon : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\mathbf{L} : [0, T] \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are given vector functions and the solution quantity is $u_\epsilon : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$. We fix $\theta > 0$ and we assume that \mathbf{A}_ϵ and \mathbf{L} are divergence-free in \mathbf{x} -direction and that \mathbf{L} is θ -periodic in τ -direction, *i.e.*

$$\forall (t, \tau, \mathbf{x}) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n, \quad \nabla_{\mathbf{x}} \cdot \mathbf{A}_\epsilon(t, \mathbf{x}) = \nabla_{\mathbf{x}} \cdot \mathbf{L}(t, \tau, \mathbf{x}) = 0, \quad (1.4)$$

$$\forall (t, \tau, \mathbf{x}) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n, \quad \mathbf{L}(t, \tau + \theta, \mathbf{x}) = \mathbf{L}(t, \tau, \mathbf{x}). \quad (1.5)$$

We also assume that, for any fixed $\epsilon > 0$, the initial data u^0 and the vector functions \mathbf{A}_ϵ and \mathbf{L} satisfy the minimal required smoothness properties for insuring the existence and the uniqueness of the solution u_ϵ of (1.3).

The main goal of the present paper is to identify the required hypotheses for the sequence $(\mathbf{A}_\epsilon)_{\epsilon > 0}$ and for \mathbf{L} in order to develop u_ϵ as follows

$$u_\epsilon(t, \mathbf{x}) = \sum_{k=0}^{+\infty} \epsilon^k U_k \left(t, \frac{t}{\epsilon}, \mathbf{x} \right), \quad (1.6)$$

where each $U_k : [0, T] \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is θ -periodic in τ -direction and has to be identified.

A similar work has been lead by Frénod, Raviart and Sonnendrücker [13]: in this paper, the authors considered a simplified version by assuming that \mathbf{L} only depends on t and \mathbf{x} and that the vector function $\mathbf{A}_\epsilon = \mathbf{A}$ does not depend on ϵ . By assuming that $\mathbf{A}, \mathbf{L} \in L^\infty(0, T; W^{1, \infty}(\mathbb{R}^n))$, they proposed to write each U_k as

$$U_k(t, \tau, \mathbf{x}) = V_k(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) + W_k(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)), \quad k \in \mathbb{N},$$

and they identified each V_k and W_k thanks to a recurrence formula in k . Furthermore, thanks to additional assumptions on W_k , they established some two-scale convergence results of the form

$$u_{\epsilon, k}(t, \mathbf{x}) \longrightarrow U_k(t, \tau, \mathbf{x}) \quad \text{two-scale,}$$

where $(u_{\epsilon, k})_{k \in \mathbb{N}}$ is defined recursively by

$$\begin{cases} u_{\epsilon, k}(t, \mathbf{x}) := \frac{1}{\epsilon} \left(u_{\epsilon, k-1}(t, \mathbf{x}) - U_{k-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), & \text{if } k > 0, \\ u_{\epsilon, 0}(t, \mathbf{x}) := u_\epsilon(t, \mathbf{x}). \end{cases} \quad (1.7)$$

An alternative methodology has been proposed by Bostan [6] in the context of $\mathbf{L} = \mathbf{L}(\mathbf{x})$ and $\mathbf{A}_\epsilon = \mathbf{A}$ strongly depending on \mathbf{L} . This context is well suited for simplifying the expression of the 0-th order limit model established by Frénod, Raviart and Sonnendrücker in [13] and allows a simple justification of Guiding-Center and Finite Larmor Radius approximations of the linear Vlasov equations (1.1) and (1.2) where the distribution function f_ϵ strongly converges in $L^\infty(0, T; L^2(\mathbb{R}^6))$.

Another work from Frénod and Sonnendrücker [17] treats the equation (1.3) with a convection term \mathbf{A}_ϵ which strongly depends on ϵ and a convection term \mathbf{L} of the form

$$\mathbf{L}(t, \mathbf{x}) = \mathbb{M} \mathbf{x}.$$

Up to some well chosen hypotheses on the sequence $(\mathbf{A}_\epsilon)_{\epsilon > 0}$ and on $\mathbb{M} \in \mathcal{M}_n(\mathbb{R})$, the authors established the two-scale convergence of u_ϵ to 0-th order term U_0 and they provided a complete identification of this

limit function.

A first order expansion have been proposed by Frénod, Gutnic and Hirstoaga for the linear Vlasov equation in the context of axisymmetric charged particle beams: indeed, in [10], the authors proposed a first order approximation of the solution f_ϵ of the following Vlasov equation

$$\begin{cases} \partial_t f_\epsilon(t, r, v_r) + \frac{v_r}{\epsilon} \partial_r f_\epsilon(t, r, v_r) + \left(E_\epsilon(t, r) - \frac{r}{\epsilon} \right) \partial_{v_r} f_\epsilon(t, r, v_r) = 0, \\ f_\epsilon(t = 0, r, v_r) = f^0(r, v_r), \end{cases} \quad (1.8)$$

in which E_ϵ is given. This expansion has been written under the form

$$f_\epsilon(t, r, v_r) \approx F_0 \left(t, \frac{t}{\epsilon}, r, v_r \right) + \epsilon F_1 \left(t, \frac{t}{\epsilon}, r, v_r \right),$$

and it has been proved that, under well chosen hypotheses for $(E_\epsilon)_{\epsilon > 0}$ and f^0 , the sequence $(f_\epsilon)_{\epsilon > 0}$ two-scale converges to F_0 and the sequence $(f_{\epsilon,1})_{\epsilon > 0}$ defined by

$$f_{\epsilon,1}(t, r, v_r) = \frac{1}{\epsilon} \left(f_\epsilon(t, r, v_r) - F_0 \left(t, \frac{t}{\epsilon}, r, v_r \right) \right),$$

two-scale converges to F_1 , with F_0 and F_1 fully identified. As well as the problem treated in [17], the Vlasov equation (1.8) includes a 0-th order convection term which strongly depends on ϵ but a singularly perturbed convection term which linearly depends on the space variables r and v_r .

As a consequence, the aim of the present paper is to somehow generalize these results to the case where \mathbf{A}_ϵ strongly depends on ϵ , t and \mathbf{x} , and where \mathbf{L} depends on t , τ and \mathbf{x} . For this purpose, we follow the recursive methodology detailed in [13] and we first write some convection equation for each $u_{\epsilon,k}$ defined in (1.7), then we adapt the two-scale convergence results from [17] in order to prove that $u_{\epsilon,k}$ somehow two-scale converges to U_k and we propose a set of equations allowing to identify U_k . Such work is essentially motivated by applications to Vlasov equations like (1.1), (1.2) or (1.8) where the electric and magnetic fields may be self-consistent and obtained from Poisson or Maxwell equations which themselves depend on ϵ through the source terms computed thanks to the ϵ -dependent distribution function (see [12, 14, 17, 18, 19, 21, 22, 23]).

Thus, the present paper is organised as follows: in a first paragraph, we present the main convergence results along with the associated hypotheses which are required. The second part is devoted to the proof of these results. Finally, we present some applications to Vlasov equations (1.1), (1.2) and (1.8).

2 Two-scale convergence results

2.1 Notations and definitions

Before going further and presenting the main results, we introduce some notations and definitions. Considering a fixed $\theta > 0$, we define for any $p \in [1, +\infty]$ the space $L^p_{\#}(0, \theta)$ as the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which are θ -periodic and such that $f|_{[0, \theta]} \in L^p(0, \theta)$. In the same spirit, we define $\mathcal{C}_{\#}(0, \theta)$ stands for the subspace of $\mathcal{C}(\mathbb{R})$ constituted of θ -periodic functions and provided with the the norm induced by $\mathcal{C}(\mathbb{R})$. Having these notations in hands, we recall the definition of two-scale convergence as it has been introduced by Allaire [3] and Nguetseng [30] and a useful two-scale convergence criterion:

Definition 2.1. *Let X be a separable Banach space, X' its topological dual space, and $\langle \cdot, \cdot \rangle_{X, X'}$ the duality bracket associated to X and X' . Considering fixed $q \in [1, +\infty]$, $T > 0$, and q' such that $\frac{1}{q} + \frac{1}{q'} = 1$,*

a sequence $(u_\epsilon)_{\epsilon>0} \subset L^{q'}(0, T; X')$ two-scale converges to a function $U \in L^{q'}(0, T; L^q_\#(0, \theta; X'))$ if, for any test function $\psi \in L^q(0, T; \mathcal{C}_\#(0, \theta; X))$, we have

$$\lim_{\epsilon \rightarrow 0} \int_0^T \left\langle u_\epsilon(t), \psi \left(t, \frac{t}{\epsilon} \right) \right\rangle_{X, X'} dt = \int_0^T \int_0^\theta \langle U(t, \tau), \psi(t, \tau) \rangle_{X, X'} d\tau dt. \quad (2.1)$$

Theorem 2.2 (Allaire [3]). *If a sequence $(u_\epsilon)_{\epsilon>0} \subset L^{q'}(0, T; X')$ is bounded independently of ϵ , there exists a profile $U \in L^{q'}(0, T; L^q_\#(0, \theta; X'))$ such that, up to the extraction of a subsequence*

$$u_\epsilon \longrightarrow U \quad \text{two-scale in } L^{q'}(0, T; L^q_\#(0, \theta; X')). \quad (2.2)$$

Furthermore, the so-called two-scale limit U of u_ϵ is closely linked to the weak-* limit of $(u_\epsilon)_{\epsilon>0}$ in $L^{q'}(0, T; X')$. Indeed this function denoted by u satisfies

$$u(t) = \frac{1}{\theta} \int_0^\theta U(t, \tau) d\tau. \quad (2.3)$$

2.2 Two-scale convergence at 0-th order

For any $(t, \sigma, \mathbf{x}) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$ fixed, we consider the following differential system

$$\begin{cases} \partial_\tau \mathbf{X}(\tau) = \mathbf{L}(t, \tau, \mathbf{X}(\tau)), \\ \mathbf{X}(\sigma) = \mathbf{x}, \end{cases} \quad (2.4)$$

where the unknown is the vector function $\tau \mapsto \mathbf{X}(\tau)$. We assume from now that this system admits a unique solution in the class of θ -periodic functions in τ -direction and we denote this solution by $\tau \mapsto \mathbf{X}(\tau; \mathbf{x}, t; \sigma)$.

The first main result is the two-scale convergence of $(u_\epsilon)_{\epsilon>0}$ to a profile $U_0 = U_0(t, \tau, \mathbf{x})$. For this purpose, we consider some hypotheses derived from those which are required for proving Theorem 1.5 of [17]:

Hypothesis 2.3. *Fixing $p \in]1, +\infty[$, $q > 1$ and q' such that $\frac{1}{p} + \frac{1}{q'} < 1$ and $\frac{1}{q'} = \max(\frac{1}{q} - \frac{1}{n}, 0)$, we assume that*

- $u^0 \in L^p(\mathbb{R}^n)$,
- $(\mathbf{A}_\epsilon)_{\epsilon>0}$ is bounded independently of ϵ in $(L^\infty(0, T; (W^{1,q}(K))))^n$ for any compact subset $K \subset \mathbb{R}^n$,
- \mathbf{L} is smooth enough in order to insure that, for any compact subset $K \subset \mathbb{R}^n$,
 - \mathbf{L} is in $(L^\infty(0, T; L^\infty_\#(0, \theta; W^{1,q}(K))))^n$,
 - $(t, \tau, \mathbf{x}) \mapsto \partial_t \mathbf{X}(\tau; \mathbf{x}, t; 0)$ is in $(L^\infty(0, T; L^\infty_\#(0, \theta; W^{1,q}(K))))^n$,
 - $(t, \tau, \mathbf{x}) \mapsto \nabla_{\mathbf{x}} \mathbf{X}(\tau; \mathbf{x}, t; 0)$ is in $(L^\infty(0, T; L^\infty_\#(0, \theta; L^\infty(K))))^{n^2}$.

As a trivial consequence, we can write, up to a subsequence and for all compact $K \subset \mathbb{R}^n$,

$$\mathbf{A}_\epsilon \longrightarrow \mathcal{A}_0 = \mathcal{A}_0(t, \tau, \mathbf{x}) \quad \text{two-scale in } (L^\infty(0, T; L^\infty_\#(0, \theta; (W^{1,q}(K))))^n).$$

Assuming that the profile \mathcal{A}_0 is somehow known, we introduce α_0 and $\tilde{\mathbf{a}}_0$ as

$$\alpha_0(t, \tau, \mathbf{x}) = ((\nabla_{\mathbf{x}} \mathbf{X})(\tau; \mathbf{x}, t; 0))^{-1} (\mathcal{A}_0(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) - (\partial_t \mathbf{X})(\tau; \mathbf{x}, t; 0)), \quad (2.5)$$

and

$$\tilde{\mathbf{a}}_0(t, \mathbf{x}) = \frac{1}{\theta} \int_0^\theta \alpha_0(t, \tau, \mathbf{x}) d\tau. \quad (2.6)$$

With these hypotheses and definitions, we can identify the 0-th order term U_0 of the expansion (1.6):

Theorem 2.4. *Assume that Hypotheses 2.3 and that the sequence $(u_\epsilon)_{\epsilon > 0}$ is bounded in $L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$ independently of ϵ . Then, up to a subsequence, u_ϵ two-scale converges to the profile $U_0 = U_0(t, \tau, \mathbf{x})$ in $L^\infty(0, T; L^\infty_{\#}(0, \theta; L^p(\mathbb{R}^n)))$ defined by*

$$U_0(t, \tau, \mathbf{x}) = V_0(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)), \quad (2.7)$$

where $V_0 = V_0(t, \mathbf{x}) \in L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$ satisfies

$$\begin{cases} \partial_t V_0(t, \mathbf{x}) + \tilde{\mathbf{a}}_0(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} V_0(t, \mathbf{x}) = 0, \\ V_0(t = 0, \mathbf{x}) = u^0(\mathbf{x}). \end{cases} \quad (2.8)$$

Theorem 2.5. *U_0 satisfies the following equation:*

$$\partial_t U_0(t, \tau, \mathbf{x}) + \mathbf{a}_0(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} U_0(t, \tau, \mathbf{x}) = 0, \quad (2.9)$$

with \mathbf{a}_0 defined by

$$\begin{aligned} \mathbf{a}_0(t, \tau, \mathbf{x}) \\ = ((\nabla_{\mathbf{x}} \mathbf{X})(-\tau; \mathbf{x}, t; 0))^{-1} (\tilde{\mathbf{a}}_0(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) - (\partial_t \mathbf{X})(-\tau; \mathbf{x}, t; 0)). \end{aligned} \quad (2.10)$$

2.3 Two-scale convergence at k -th order

We fix $k \in \mathbb{N}^*$ and we aim to identify the k -th term of the expansion (1.6). Before stating the result, we need additional assumptions besides Hypotheses 2.3:

Hypothesis 2.6. *Defining the sequence $(\mathbf{A}_{\epsilon, i})_{\epsilon > 0}$ as*

$$\begin{cases} \mathbf{A}_{\epsilon, i}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(\mathbf{A}_{\epsilon, i-1}(t, \mathbf{x}) - \mathcal{A}_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), \quad \forall i = 1, \dots, k, \\ \mathbf{A}_{\epsilon, 0}(t, \mathbf{x}) = \mathbf{A}_\epsilon(t, \mathbf{x}), \end{cases}$$

we assume that, for all $i = 0, \dots, k$, $(\mathbf{A}_{\epsilon, i})_{\epsilon > 0}$ two-scale converges to the profile $\mathcal{A}_i = \mathcal{A}_i(t, \tau, \mathbf{x})$ in $(L^\infty(0, T; L^\infty_{\#}(0, \theta; W^{1, q}(K))))^n$ for any compact subset $K \subset \mathbb{R}^n$.

Hypothesis 2.7. *Defining the sequence $(u_{\epsilon, i})_{\epsilon > 0}$ as*

$$\begin{cases} u_{\epsilon, i}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(u_{\epsilon, i-1}(t, \mathbf{x}) - U_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), \quad \forall i = 1, \dots, k-1, \\ u_{\epsilon, 0}(t, \mathbf{x}) = u_\epsilon(t, \mathbf{x}), \end{cases}$$

we assume that, for all $i = 0, \dots, k-1$ and up to a subsequence, $(u_{\epsilon, i})_{\epsilon > 0}$ two-scale converges to the profile $U_i = U_i(t, \tau, \mathbf{x})$ in $L^\infty(0, T; L^\infty_{\#}(0, \theta; L^p(\mathbb{R}^n)))$.

Under these hypotheses and if $k > 1$, we define α_i , $\tilde{\mathbf{a}}_i$ and \mathbf{a}_i as

$$\alpha_i(t, \tau, \mathbf{x}) = ((\nabla_{\mathbf{x}} \mathbf{X})(\tau; \mathbf{x}, t; 0))^{-1} \mathcal{A}_i(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)), \quad (2.11)$$

$$\tilde{\mathbf{a}}_i(t, \mathbf{x}) = \frac{1}{\theta} \int_0^\theta \alpha_i(t, \tau, \mathbf{x}) d\tau, \quad (2.12)$$

$$\begin{aligned} \mathbf{a}_i(t, \tau, \mathbf{x}) &= \frac{1}{\theta} \int_0^\theta ((\nabla_{\mathbf{x}} \mathbf{X})(\sigma - \tau; \mathbf{x}, t; 0))^{-1} \mathcal{A}_i(t, \sigma, \mathbf{X}(\sigma - \tau; \mathbf{x}, t; 0)) d\sigma \\ &= \frac{1}{\theta} \int_0^\theta ((\nabla_{\mathbf{x}} \mathbf{X})(-\tau; \mathbf{x}, t; 0))^{-1} \alpha_i(t, \sigma, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) d\sigma, \end{aligned} \quad (2.13)$$

for all $i = 1, \dots, k-1$. We also define recursively the functions W_1, \dots, W_k and R_1, \dots, R_{k-1} as

$$W_i(t, \tau, \mathbf{x}) = \int_0^\tau \left(\sum_{j=0}^{i-1} (\mathbf{a}_j - \mathcal{A}_j) \cdot \nabla_{\mathbf{x}} U_{i-1-j} - R_{i-1} \right) (t, \sigma, \mathbf{X}(\sigma; \mathbf{x}, t; 0)) d\sigma, \quad (2.14)$$

$$\begin{aligned} R_i(t, \tau, \mathbf{x}) &= (\partial_t W_i)(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) - \frac{1}{\theta} \int_0^\theta (\partial_t W_i)(t, \sigma, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) d\sigma \\ &\quad + \sum_{j=0}^i (\tilde{\mathbf{a}}_j \cdot \nabla_{\mathbf{x}} W_{i-j})(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) \\ &\quad - \frac{1}{\theta} \sum_{j=0}^i \int_0^\theta (\alpha_j \cdot \nabla_{\mathbf{x}} W_{i-j})(t, \sigma, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) d\sigma, \end{aligned} \quad (2.15)$$

with the convention $W_0 = R_0 = 0$. Having these notations and assumptions in hands, we can now state a two-scale convergence result at k -th order by proceeding recursively:

Theorem 2.8. *We define $s' > 0$ such that $\frac{1}{s'} = 1 - \frac{1}{q} - \frac{1}{r}$ with $r \in [1, \frac{nq}{n-q}]$ and we define the functional space $X^{s'}(K) = (W^{1,q}(K))' \cup (W^{1,s'}(K))'$. We assume that Hypotheses 2.3-2.6-2.7 are satisfied and that, for any $K \subset \mathbb{R}^n$ compact,*

- W_k is in $L^\infty(0, T; L^\infty_\#(0, \theta; L^p(K)))$,
- $\partial_t W_k$ is in $L^\infty(0, T; L^\infty_\#(0, \theta; X^{s'}(K)))$,
- R_{k-1} is in $L^\infty(0, T; L^\infty_\#(0, \theta; X^{s'}(K)))$.

Then, if the sequence $(u_{\epsilon,k})_{\epsilon > 0}$ defined by

$$u_{\epsilon,k}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(u_{\epsilon,k-1}(t, \mathbf{x}) - U_{k-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), \quad (2.16)$$

is bounded independently of ϵ in $L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$, $u_{\epsilon,k}$ two-scale converges to the profile U_k in $(0, T; L^\infty_\#(0, \theta; L^p(\mathbb{R}^n)))$ characterized as follows:

$$U_k(t, \tau, \mathbf{x}) = V_k(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) + W_k(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)), \quad (2.17)$$

where $W_k = W_k(t, \tau, \mathbf{x})$ is defined in (2.14) and where $V_k = V_k(t, \mathbf{x})$ is the solution of

$$\left\{ \begin{array}{l} \partial_t V_k(t, \mathbf{x}) + \tilde{\mathbf{a}}_0(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} V_k(t, \mathbf{x}) \\ = -\frac{1}{\theta} \int_0^\theta (\partial_t W_k + \boldsymbol{\alpha}_0 \cdot \nabla_{\mathbf{x}} W_k)(t, \sigma, \mathbf{x}) d\sigma \\ - \sum_{i=1}^k \left[\frac{1}{\theta} \int_0^\theta \boldsymbol{\alpha}_i(t, \sigma, \mathbf{x}) \cdot [\nabla_{\mathbf{x}} V_{k-i}(t, \mathbf{x}) + \nabla_{\mathbf{x}} W_{k-i}(t, \sigma, \mathbf{x})] d\sigma \right] \\ V_k(t=0, \mathbf{x}) = 0. \end{array} \right. \quad (2.18)$$

Theorem 2.9. U_k satisfies the following equation:

$$\begin{aligned} \partial_t U_k(t, \tau, \mathbf{x}) + \mathbf{a}_0(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} U_k(t, \tau, \mathbf{x}) \\ = R_k(t, \tau, \mathbf{x}) - \sum_{i=1}^k \mathbf{a}_i(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} U_{k-i}(t, \tau, \mathbf{x}), \end{aligned} \quad (2.19)$$

where R_k is obtained from the definition (2.15) extended to the case $i = k$.

3 Characterization of U_k

In this section, we aim to prove the two-scale convergence results presented in Theorems 2.4, 2.5, 2.8 and 2.9. For this purpose, we choose to detail the proofs on the generic equation of the form

$$\left\{ \begin{array}{l} \partial_t g_\epsilon(t, \mathbf{x}) + \mathbf{A}_\epsilon(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} g_\epsilon(t, \mathbf{x}) + \frac{1}{\epsilon} \mathbf{L} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \cdot \nabla_{\mathbf{x}} g_\epsilon(t, \mathbf{x}) = \frac{1}{\epsilon} f_\epsilon(t, \mathbf{x}), \\ g_\epsilon(t=0, \mathbf{x}) = g^0(\mathbf{x}), \end{array} \right. \quad (3.1)$$

in which f_ϵ , \mathbf{A}_ϵ and \mathbf{L} are known and where g_ϵ is the unknown. The next lines are structured as follows: first, we detail some two-scale convergence results for the model (3.1) under some well-chosen hypotheses for \mathbf{A}_ϵ , \mathbf{L} and f_ϵ . Then we apply these results onto the equations satisfied by each $u_{\epsilon,i}$ recursively defined thanks to Hypothesis 2.7.

3.1 Two-scale convergence

We aim to establish some two-scale convergence results for the sequence $(g_\epsilon)_{\epsilon > 0}$ under some well-chosen hypotheses for \mathbf{A}_ϵ , \mathbf{L} and f_ϵ . These results are detailed in the following theorem:

Theorem 3.1. *We consider $s' > 0$ such that $\frac{1}{s'} = 1 - \frac{1}{q} - \frac{1}{r}$ with $r \in [1, \frac{nq}{n-q}[$ and, for all compact subset $K \subset \mathbb{R}^n$, we define $X^{s'}(K) = \left(W_0^{1,s'}(K) \right)' \cup \left(W_0^{1,q}(K) \right)'$. We assume that \mathbf{A}_ϵ and \mathbf{L} satisfy Hypotheses 2.3 and that g^0 and $(f_\epsilon)_{\epsilon > 0}$ have the following properties:*

- $g^0 \in L^p(\mathbb{R}^n)$,
- f_ϵ is bounded independently of ϵ in $W^{1,\infty} \left(0, T; X^{s'}(K) \right)$ and admits $F = F(t, \tau, \mathbf{x})$ as a two-scale limit in $L^\infty \left(0, T; L^\infty_{\#} \left(0, \theta; X^{s'}(K) \right) \right)$,
- F satisfies

$$\forall (t, \mathbf{x}), \quad \int_0^\theta F(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) d\tau = 0, \quad (3.2)$$

- The sequence $(f_{\epsilon,1})_{\epsilon > 0}$ defined by

$$f_{\epsilon,1}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(f_{\epsilon}(t, \mathbf{x}) - F \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), \quad (3.3)$$

is bounded independently in $L^{\infty} \left(0, T; X^{s'}(K) \right)$ and two-scale converges to the profile $F_1 = F_1(t, \tau, \mathbf{x})$ in $L^{\infty} \left(0, T; L^{\infty}_{\#} \left(0, \theta; X^{s'}(K) \right) \right)$,

- Defining the function $S = S(t, \tau, \mathbf{x})$ as

$$S(t, \tau, \mathbf{x}) = \int_0^{\tau} F(t, \sigma, \mathbf{X}(\sigma; \mathbf{x}, t; 0)) d\sigma, \quad (3.4)$$

we assume that S lies in $L^{\infty} \left(0, T; L^{\infty}_{\#} \left(0, \theta; L^p_{loc}(\mathbb{R}^n) \right) \right)$ and that $\partial_t S$ lies in $L^{\infty} \left(0, T; L^{\infty}_{\#} \left(0, \theta; X^{s'}(K) \right) \right)$.

If $(g_{\epsilon})_{\epsilon > 0}$ is bounded in $L^{\infty} \left(0, T; L^p_{loc}(\mathbb{R}^n) \right)$, then it admits a two-scale limit $G = G(t, \tau, \mathbf{x})$ in the space $L^{\infty} \left(0, T; L^{\infty}_{\#} \left(0, \theta; L^p(\mathbb{R}^n) \right) \right)$ and G is characterized thanks to the relation

$$G(t, \tau, \mathbf{x}) = H(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) + S(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)), \quad (3.5)$$

where $H = H(t, \mathbf{x}) \in L^{\infty} \left(0, T; L^p_{loc}(\mathbb{R}^n) \right)$ satisfies

$$\left\{ \begin{array}{l} \partial_t H(t, \mathbf{x}) + \tilde{\mathbf{a}}_0(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} H(t, \mathbf{x}) \\ \quad = \frac{1}{\theta} \int_0^{\theta} F_1(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) d\tau \\ \quad \quad - \frac{1}{\theta} \int_0^{\theta} [\partial_t S(t, \tau, \mathbf{x}) - \boldsymbol{\alpha}_0(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} S(t, \tau, \mathbf{x})] d\tau, \\ H(t = 0, \mathbf{x}) = g^0(\mathbf{x}). \end{array} \right. \quad (3.6)$$

Proof. Since $(g_{\epsilon})_{\epsilon > 0}$ is assumed to be bounded in $L^{\infty} \left(0, T; L^p_{loc}(\mathbb{R}^n) \right)$ independently of ϵ , it admits a two-scale limit $G = G(t, \tau, \mathbf{x})$ in the functional space $L^{\infty} \left(0, T; L^p_{\#} \left(0, \theta; L^p(\mathbb{R}^n) \right) \right)$. In the same spirit of [17], the next step of the proof consists in finding an equation which link the first order derivative of G in τ to the derivatives of G in \mathbf{x} directions. For this purpose, we consider a test function $\psi = \psi(t, \tau, \mathbf{x})$ defined on $[0, T] \times \mathbb{R} \times \mathbb{R}^n$ being θ -periodic in τ direction and with compact support $K \subset \mathbb{R}^n$ in \mathbf{x} direction. We multiply (3.1) by $\psi(t, \frac{t}{\epsilon}, \mathbf{x})$ and we integrate the result in t and \mathbf{x} . Some integrations by parts give

$$\begin{aligned} & \int_0^T \int_K g_{\epsilon}(t, \mathbf{x}) \left[(\partial_t \psi) \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) + \frac{1}{\epsilon} (\partial_{\tau} \psi) \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) + \mathbf{A}_{\epsilon}(t, \mathbf{x}) \cdot (\nabla_{\mathbf{x}} \psi) \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right. \\ & \quad \left. + \frac{1}{\epsilon} \mathbf{L} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \cdot (\nabla_{\mathbf{x}} \psi) \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right] d\mathbf{x} dt \\ & = -\frac{1}{\epsilon} \int_0^T \int_K f_{\epsilon}(t, \mathbf{x}) \psi \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) d\mathbf{x} dt + \int_K u^0(\mathbf{x}) \psi(0, 0, \mathbf{x}) d\mathbf{x}. \end{aligned}$$

Thanks to the considered assumptions for g^0 , \mathbf{A}_{ϵ} , \mathbf{L} and f_{ϵ} , we can multiply by ϵ and reach the limit $\epsilon \rightarrow 0$. This gives

$$\begin{aligned} & \int_0^{\theta} \int_0^T \int_K G(t, \tau, \mathbf{x}) \left[\partial_{\tau} \psi(t, \tau, \mathbf{x}) + \mathbf{L}(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \psi(t, \tau, \mathbf{x}) \right] d\mathbf{x} dt \\ & = - \int_0^{\theta} \int_0^T \int_K F(t, \tau, \mathbf{x}) \psi(t, \tau, \mathbf{x}) d\mathbf{x} dt d\tau. \end{aligned}$$

This means that G satisfies the following equation in $L^\infty\left(0, T; L^\infty_{\#}(0, \theta; L^p_{loc}(\mathbb{R}^n))\right)$:

$$\partial_\tau G(t, \tau, \mathbf{x}) + \mathbf{L}(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} G(t, \tau, \mathbf{x}) = F(t, \tau, \mathbf{x}).$$

According to Lemma 2.1 from [13] and thanks to the hypothesis (3.2), we can write G as follows

$$G(t, \tau, \mathbf{x}) = H(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) + \int_0^\tau F(t, \sigma, \mathbf{X}(\sigma - \tau; \mathbf{x}, t; 0)) d\sigma,$$

with $H = H(t, \mathbf{x}) \in L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$.

The next step consists in proving that H satisfies (3.6). For this purpose, we introduce the sequence $(h_\epsilon)_{\epsilon > 0}$ defined as

$$g_\epsilon(t, \mathbf{x}) = h_\epsilon\left(t, \mathbf{X}\left(-\frac{t}{\epsilon}; \mathbf{x}, t; 0\right)\right) + \int_0^{t/\epsilon} F\left(t, \sigma, \mathbf{X}\left(\sigma - \frac{t}{\epsilon}; \mathbf{x}, t; 0\right)\right) d\sigma.$$

Injecting this relation in (3.1) gives

$$\begin{cases} \partial_t h_\epsilon(t, \mathbf{x}) + \tilde{\mathbf{A}}_\epsilon(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} h_\epsilon(t, \mathbf{x}) \\ = f_{\epsilon,1}\left(t, \mathbf{X}\left(\frac{t}{\epsilon}; \mathbf{x}, t; 0\right)\right) - (\partial_t S)\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) - \tilde{\mathbf{A}}_\epsilon(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} S\left(t, \frac{t}{\epsilon}, \mathbf{x}\right), \\ h_\epsilon(t=0, \mathbf{x}) = g^0(\mathbf{x}), \end{cases} \quad (3.7)$$

where $\tilde{\mathbf{A}}_\epsilon$ is linked to \mathbf{A}_ϵ through the following relation:

$$\tilde{\mathbf{A}}_\epsilon(t, \mathbf{x}) = \left((\nabla_{\mathbf{x}} \mathbf{X})\left(\frac{t}{\epsilon}; \mathbf{x}, t; 0\right) \right)^{-1} \left(\mathbf{A}_\epsilon\left(t, \mathbf{X}\left(\frac{t}{\epsilon}; \mathbf{x}, t; 0\right)\right) - (\partial_t \mathbf{X})\left(\frac{t}{\epsilon}; \mathbf{x}, t; 0\right) \right).$$

From the definition of h_ϵ provided by (3.1) and the hypotheses made for F and $(g_\epsilon)_{\epsilon > 0}$, we can write

$$\forall t, \quad \|h_\epsilon(t, \cdot)\|_{L^p(K)} \leq \|g_\epsilon(t, \cdot)\|_{L^p(K)} + \theta \|F(t, \cdot, \cdot)\|_{L^\infty_{\#}(0, \theta; L^p(K))},$$

for all compact subset $K \subset \mathbb{R}^n$ so we deduce that the sequence $(h_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in $L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$ and, up to a subsequence, two-scale converges to H in $L^\infty\left(0, T; L^\infty_{\#}(0, \theta; L^p(\mathbb{R}^n))\right)$.

Indeed, if we consider a test function $\psi = \psi(t, \tau, \mathbf{x})$ on $[0, T] \times \mathbb{R} \times \mathbb{R}^n$ being θ -periodic in τ direction and with compact support $K \subset \mathbb{R}^n$ in \mathbf{x} direction, we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^n} h_\epsilon(t, \mathbf{x}) \psi\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) d\mathbf{x} dt \\ &= \lim_{\epsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^n} \left[g_\epsilon\left(t, \mathbf{X}\left(\frac{t}{\epsilon}; \mathbf{x}, t; 0\right)\right) - S\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) \right] \psi\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) d\mathbf{x} dt \\ &= \lim_{\epsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^n} \left[g_\epsilon(t, \mathbf{x}) \psi\left(t, \frac{t}{\epsilon}, \mathbf{X}\left(-\frac{t}{\epsilon}; \mathbf{x}, t; 0\right)\right) - S\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) \psi\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) \right] d\mathbf{x} dt \\ &= \frac{1}{\theta} \int_0^\theta \int_0^T \int_{\mathbb{R}^n} [G(t, \tau, \mathbf{x}) \psi(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) - S(t, \tau, \mathbf{x}) \psi(t, \tau, \mathbf{x})] d\mathbf{x} dt d\tau \\ &= \frac{1}{\theta} \int_0^\theta \int_0^T \int_{\mathbb{R}^n} [G(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) - S(t, \tau, \mathbf{x})] \psi(t, \tau, \mathbf{x}) d\mathbf{x} dt d\tau \\ &= \frac{1}{\theta} \int_0^\theta \int_0^T \int_{\mathbb{R}^n} H(t, \mathbf{x}) \psi(t, \tau, \mathbf{x}) d\mathbf{x} dt d\tau. \end{aligned}$$

Consequently, h_ϵ weakly-* converges to H in $L^\infty(0, T; L^p(\mathbb{R}^n))$ according to Theorem 2.2. However, as in [17], we are able to obtain a strong convergence result for h_ϵ in a well-chosen functional space:

Lemma 3.2. *For any compact subset $K \subset \mathbb{R}^n$, the sequence $(h_\epsilon)_{\epsilon > 0}$ strongly converges to H in $L^\infty\left(0, T; \left(W_0^{1,q}(K)\right)'\right)$.*

Proof. The procedure is almost similar to the proof of Lemma 4.1 of [17]. Indeed, from the assumptions made for the sequences $(g_\epsilon)_{\epsilon > 0}$, $(\mathbf{A}_\epsilon)_{\epsilon > 0}$, \mathbf{L} , $(f_\epsilon)_{\epsilon > 0}$ and $(f_{\epsilon,1})_{\epsilon > 0}$, we consider a compact subset K of \mathbb{R}^n and we successively prove that

- $(\tilde{\mathbf{A}}_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in $(L^\infty(0, T; W^{1,q}(K)))^n$ and is divergence-free in \mathbf{x} direction,
- $(\tilde{\mathbf{A}}_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in $(L^\infty(0, T; L^r(K)))^n$ for any $r \in [1, \frac{nq}{n-q}]$,
- $(\tilde{\mathbf{A}}_\epsilon h_\epsilon)_{\epsilon > 0}$ and $(\tilde{\mathbf{A}}_\epsilon S(\cdot, \frac{\cdot}{\epsilon}, \cdot))_{\epsilon > 0}$ are bounded independently of ϵ in the space $(L^\infty(0, T; L^s(K)))^n$ with s satisfying $\frac{1}{s} = \frac{1}{q} + \frac{1}{r}$,
- $(\nabla_{\mathbf{x}} \cdot (\tilde{\mathbf{A}}_\epsilon h_\epsilon))_{\epsilon > 0}$ and $(\nabla_{\mathbf{x}} \cdot (\tilde{\mathbf{A}}_\epsilon S(\cdot, \frac{\cdot}{\epsilon}, \cdot)))_{\epsilon > 0}$ are bounded in the space $(L^\infty(0, T; (W_0^{1,s'}(K))'))^n$ and consequently in $(L^\infty(0, T; X^{s'}(K)))^n$ independently of ϵ .

In addition of these results, we deduce from the hypotheses on F that $\partial_t S$ is in $L^\infty(0, T; L^\infty_\#(0, \theta; L^p(K)))$. At this point, we distinguish 2 different cases according to the considered value of s' in front of q :

1. Assume that $s' > q$. This leads to the continuous embedding $(L^q(K))' \subset (L^{s'}(K))'$ and, consequently, to the continuous embedding $(W_0^{1,q}(K))' \subset (W_0^{1,s'}(K))'$, so $X^{s'}(K) = (W_0^{1,s'}(K))'$. On another hand, Rellich's theorem gives the compact embedding $L^p(K) \subset (W_0^{1,q}(K))'$. Hence, $\partial_t S$ and $f_{\epsilon,1}$ lie in $L^\infty(0, T; (W_0^{1,s'}(K))')$, and the sequence $(f_{\epsilon,1})_{\epsilon > 0}$ is bounded independently of ϵ in this space. Finally, we can write that $(h_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in the following space:

$$\mathcal{U} = \left\{ h \in L^\infty(0, T; L^p(K)) : \partial_t h \in L^\infty\left(0, T; (W_0^{1,s'}(K))'\right) \right\}.$$

Aubin-Lions' lemma indicates that \mathcal{U} is compactly embedded in the space $L^\infty(0, T; (W_0^{1,q}(K))')$, so h_ϵ weakly-* converges to H in $L^\infty(0, T; L^p(K))$ and strongly converges to H in $L^\infty(0, T; (W_0^{1,q}(K))')$.

2. Assume that $s' \leq q$. As a consequence, we have the continuous embedding $(W_0^{1,s'}(K))' \subset (W_0^{1,q}(K))'$, $X^{s'}(K) = (W_0^{1,q}(K))'$ and the compact embedding $L^p(K) \subset (W_0^{1,q}(K))'$ so we are insured that $(\partial_t h_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in $L^\infty(0, T; (W_0^{1,q}(K))')$ and that $(h_\epsilon)_{\epsilon > 0}$ is bounded in the functional space \mathcal{U} defined by

$$\mathcal{U} = \left\{ h \in L^\infty(0, T; L^p(K)) : \partial_t h \in L^\infty\left(0, T; (W_0^{1,q}(K))'\right) \right\}.$$

Applying Aubin-Lions' lemma finally allows us to claim that the weak-* convergence of h_ϵ to H in $L^\infty(0, T; L^p(K))$ is a strong convergence in the space $L^\infty\left(0, T; \left(W_0^{1,q}(K)\right)'\right)$.

□

In order to conclude the proof of Theorem 3.1, we now consider a test function $\psi = \psi(t, \mathbf{x})$ on $[0, T] \times \mathbb{R}^n$ with compact support $K \subset \mathbb{R}^n$ in \mathbf{x} direction. If we multiply (3.7) by $\psi(t, \mathbf{x})$, integrate the result in t and \mathbf{x} , we obtain

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^n} h_\epsilon(t, \mathbf{x}) \left[\partial_t \psi(t, \mathbf{x}) + \tilde{\mathbf{A}}_\epsilon(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \psi(t, \mathbf{x}) \right] d\mathbf{x} dt - \int_{\mathbb{R}^n} g^0(\mathbf{x}) \psi(0, \mathbf{x}) d\mathbf{x} \\ & = \int_0^T \int_{\mathbb{R}^n} f_{\epsilon,1}(t, \mathbf{x}) \psi \left(t, \mathbf{X} \left(-\frac{t}{\epsilon}; \mathbf{x}, t; 0 \right) \right) d\mathbf{x} dt \\ & \quad - \int_0^T \int_{\mathbb{R}^n} \left[(\partial_t S) \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \psi(t, \mathbf{x}) - S \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \tilde{\mathbf{A}}_\epsilon(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \psi(t, \mathbf{x}) \right] d\mathbf{x} dt. \end{aligned}$$

Thanks to Lemma 3.2 and to the hypotheses we have formulated for \mathbf{A}_ϵ , $f_{\epsilon,1}$ and S , we can write the limit obtained when ϵ converges to 0: indeed, we can write

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} H(t, \mathbf{x}) \left[\partial_t \psi(t, \mathbf{x}) + \left[\frac{1}{\theta} \int_0^\theta \alpha_0(t, \tau, \mathbf{x}) d\tau \right] \cdot \nabla_{\mathbf{x}} \psi(t, \mathbf{x}) \right] d\mathbf{x} dt \\ & \quad + \int_{\mathbb{R}^n} g^0(\mathbf{x}) \psi(0, \mathbf{x}) d\mathbf{x} \\ & = -\frac{1}{\theta} \int_0^T \int_{\mathbb{R}^n} \int_0^\theta F_1(t, \tau, \mathbf{x}) \psi(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) d\tau d\mathbf{x} dt \\ & \quad + \int_0^T \int_{\mathbb{R}^n} \left[\frac{1}{\theta} \int_0^\theta \partial_t S(t, \tau, \mathbf{x}) d\tau \right] \psi(t, \mathbf{x}) d\mathbf{x} dt \\ & \quad - \int_0^T \int_{\mathbb{R}^n} \left[\frac{1}{\theta} \int_0^\theta S(t, \tau, \mathbf{x}) \alpha_0(t, \tau, \mathbf{x}) d\tau \right] \cdot \nabla_{\mathbf{x}} \psi(t, \mathbf{x}) d\mathbf{x} dt, \end{aligned}$$

which is exactly the variational formulation of (3.6) in $L^\infty(0, T; L^p(\mathbb{R}^n))$.

□

3.2 Identification of each U_k

Having Theorem 3.1 in hands, we can apply it for identifying each term U_k of the expansion (1.6). For obtaining some equations for U_0 , we simply use this theorem with the source term $f_\epsilon = 0$ on $[0, T] \times \mathbb{R}^n$ for each $\epsilon \geq 0$. As a consequence, assuming that $(u_\epsilon)_{\epsilon > 0}$ is bounded independently in ϵ in $L^\infty(0, T; L_{loc}^p(\mathbb{R}^n))$ in addition of Hypotheses 2.3 is sufficient to get the two-scale convergence of u_ϵ to the profile $U_0 = U_0(t, \tau, \mathbf{x})$ in $L^\infty\left(0, T; L_{\#}^\infty(0, \theta; L^p(\mathbb{R}^n))\right)$ entirely characterized by

$$U_0(t, \tau, \mathbf{x}) = V_0(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)),$$

where $V_0 = V_0(t, \mathbf{x}) \in L^\infty(0, T; L^p(\mathbb{R}^n))$ satisfies

$$\begin{cases} \partial_t V_0(t, \mathbf{x}) + \tilde{\mathbf{a}}_0(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} V_0(t, \mathbf{x}) = 0, \\ V_0(t = 0, \mathbf{x}) = u^0(\mathbf{x}). \end{cases}$$

This is the conclusion of Theorem 2.4. For reaching the results of Theorem 2.7, we derive in \mathbf{x} and t the relation (2.7) and we obtain

$$\nabla_{\mathbf{x}} U_0(t, \tau, \mathbf{x}) = ((\nabla_{\mathbf{x}} \mathbf{X})(-\tau; \mathbf{x}, t; 0))^T (\nabla_{\mathbf{x}} V_0)(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)),$$

and

$$\begin{aligned}
\partial_t U_0(t, \tau, \mathbf{x}) &= (\partial_t V_0)(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) + \partial_t \mathbf{X}(-\tau; \mathbf{x}, t; 0) \cdot (\nabla_{\mathbf{x}} V_0)(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) \\
&= [\partial_t \mathbf{X}(-\tau; \mathbf{x}, t; 0) - \tilde{\mathbf{a}}_0(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0))] \cdot (\nabla_{\mathbf{x}} V_0)(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) \\
&= \left[((\nabla_{\mathbf{x}} \mathbf{X})(-\tau; \mathbf{x}, t; 0))^{-1} [\partial_t \mathbf{X}(-\tau; \mathbf{x}, t; 0) - \tilde{\mathbf{a}}_0(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0))] \right] \\
&\quad \cdot \nabla_{\mathbf{x}} U_0(t, \tau, \mathbf{x}).
\end{aligned}$$

For identifying the higher order terms, more calculations are needed. First, we consider a fixed integer $k \in \mathbb{N}^*$ and we assume that Hypotheses 2.6 and 2.7 are satisfied at step k and that the results of Theorem 2.8 are true for $i = 0, \dots, k-1$, meaning that U_0, \dots, U_{k-1} are fully characterized. These assumptions authorize the definitions of $\boldsymbol{\alpha}_i, \tilde{\mathbf{a}}_i, \mathbf{a}_i, W_i$ and R_i for any $i = 0, \dots, k$ as it is suggested in paragraph 2.3. Then we can write an evolution equation for $u_{\epsilon, i}$ for any $i = 1, \dots, k$ thanks to a recurrence procedure: this equation writes

$$\left\{ \begin{aligned}
&\partial_t u_{\epsilon, i}(t, \mathbf{x}) + \mathbf{A}_{\epsilon}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} u_{\epsilon, i}(t, \mathbf{x}) + \frac{1}{\epsilon} \mathbf{L} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \cdot \nabla_{\mathbf{x}} u_{\epsilon, i}(t, \mathbf{x}) \\
&= \frac{1}{\epsilon} \sum_{j=0}^{i-1} \left(\mathbf{a}_j \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) - \mathbf{A}_{\epsilon, j}(t, \mathbf{x}) \right) \cdot \nabla_{\mathbf{x}} U_{i-1-j} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \\
&\quad - \frac{1}{\epsilon} R_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right), \\
&u_{\epsilon, i}(t=0, \mathbf{x}) = 0,
\end{aligned} \right. \quad (3.8)$$

for any $i > 0$.

As a consequence, we aim to apply Theorem 3.1 with f_{ϵ} defined by

$$f_{\epsilon}(t, \mathbf{x}) = \sum_{i=0}^{k-1} \left[\mathbf{a}_i \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) - \mathbf{A}_{\epsilon, i}(t, \mathbf{x}) \right] \cdot \nabla_{\mathbf{x}} U_{k-1-i} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) - R_{k-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right).$$

First, we have to verify if the sequence $(f_{\epsilon})_{\epsilon > 0}$ is bounded independently of ϵ in $L^{\infty} \left(0, T; X^{s'}(K) \right)$ for any compact subset $K \subset \mathbb{R}^n$. For this purpose, we first remark that Hypotheses 2.3 and 2.6 imply that there exists a constant $C = C(K) > 0$ such that

$$\left\| \mathbf{a}_i \left(t, \frac{t}{\epsilon}, \cdot \right) - \mathbf{A}_{\epsilon, i}(t, \cdot) \right\|_{W^{1, q}(K)} \leq C(K),$$

for any $t \in [0, T]$ and $\epsilon > 0$. Hence, following the same methodology as in the proof of Lemma 3.2 and assuming that R_{k-1} is in $L^{\infty} \left(0, T; L^{\infty}_{\#} \left(0, \theta; X^{s'}(K) \right) \right)$ leads to the existence of a constant $C' = C'(K) > 0$ such that

$$\|f_{\epsilon}(t, \cdot)\|_{X^{s'}(K)} \leq C'(K),$$

for any $\epsilon > 0$ and $t \in [0, T]$, where the norm $\|\cdot\|_{X^{s'}(K)}$ is either the usual norm on $(W_0^{1, q}(K))'$ or $(W_0^{1, s'}(K))'$ according to the sign of $s' - q$.

This result indicates that f_{ϵ} two-scale converges to the profile $F = F(t, \tau, \mathbf{x})$ in $L^{\infty} \left(0, T; L^{\infty}_{\#} \left(0, \theta; X^{s'}(K) \right) \right)$ characterized by

$$F(t, \tau, \mathbf{x}) = \sum_{i=0}^{k-1} [\mathbf{a}_i(t, \tau, \mathbf{x}) - \mathcal{A}_i(t, \tau, \mathbf{x})] \cdot \nabla_{\mathbf{x}} U_{k-1-i}(t, \tau, \mathbf{x}) - R_{k-1}(t, \tau, \mathbf{x}).$$

The next step consists in proving that W_k defined by

$$W_k(t, \tau, \mathbf{x}) = S(t, \tau, \mathbf{x}) = \int_0^\tau F(t, \sigma, \mathbf{X}(\sigma; \mathbf{x}, t; 0)) d\sigma,$$

is such that

$$\begin{aligned} W_k &\in L^\infty\left(0, T; L^\infty_{\#}(0, \theta; L^p(K))\right), \\ \partial_t W_k &\in L^\infty\left(0, T; L^\infty_{\#}(0, \theta; X^{s'}(K))\right), \\ W_k(t, \theta, \mathbf{x}) &= 0, \quad \forall t, \mathbf{x}. \end{aligned}$$

The first two points are handled thanks to the hypotheses for W_k which are added for claiming Theorem 2.8. The last point can be proved by using the definition of R_{k-1} . Indeed, we have

$$W_k(t, \theta, \mathbf{x}) = \int_0^\theta \left(\sum_{i=0}^{k-1} [\mathbf{a}_i - \mathcal{A}_i] \cdot \nabla_{\mathbf{x}} U_{k-1-i} - R_{k-1} \right) (t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) d\tau, \quad (3.9)$$

with

$$\begin{aligned} R_{k-1}(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) &= \partial_t W_{k-1}(t, \tau, \mathbf{x}) + \tilde{\mathbf{a}}_0(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} W_{k-1}(t, \tau, \mathbf{x}) \\ &\quad - \frac{1}{\theta} \int_0^\theta (\partial_t W_{k-1} + \boldsymbol{\alpha}_0 \cdot \nabla_{\mathbf{x}} W_{k-1})(t, \sigma, \mathbf{x}) d\sigma \\ &\quad + \sum_{i=1}^{k-1} \left[\frac{1}{\theta} \int_0^\theta \boldsymbol{\alpha}_i(t, \sigma, \mathbf{x}) \cdot [\nabla_{\mathbf{x}} W_{k-1-i}(t, \tau, \mathbf{x}) - \nabla_{\mathbf{x}} W_{k-1-i}(t, \sigma, \mathbf{x})] d\sigma \right], \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} &([\mathbf{a}_i - \mathcal{A}_i] \cdot \nabla_{\mathbf{x}} U_{k-1-j})(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) \\ &= [(\nabla_{\mathbf{x}} \mathbf{X})(\tau; \mathbf{x}, t; 0) [\tilde{\mathbf{a}}_i(t, \mathbf{x}) - \boldsymbol{\alpha}_i(t, \tau, \mathbf{x})]] \cdot (\nabla_{\mathbf{x}} U_{k-1-j})(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) \\ &= [\tilde{\mathbf{a}}_i(t, \mathbf{x}) - \boldsymbol{\alpha}_i(t, \tau, \mathbf{x})] \cdot (\nabla_{\mathbf{x}} V_{k-1-i}(t, \mathbf{x}) + \nabla_{\mathbf{x}} W_{k-1-i}(t, \tau, \mathbf{x})), \end{aligned} \quad (3.11)$$

for any $i = 0, \dots, k-1$. Hence, using the links between $\tilde{\mathbf{a}}_i$, $\boldsymbol{\alpha}_i$ and \mathcal{A}_i , we inject (3.10) and (3.11) in (3.9) for obtaining a new formulation for W_k :

$$\begin{aligned} W_k(t, \tau, \mathbf{x}) &= \sum_{j=0}^{k-1} \int_0^\tau [\tilde{\mathbf{a}}_j(t, \mathbf{x}) - \boldsymbol{\alpha}_j(t, \sigma, \mathbf{x})] \\ &\quad \cdot [\nabla_{\mathbf{x}} V_{k-1-j}(t, \mathbf{x}) + \nabla_{\mathbf{x}} W_{k-1-j}(t, \sigma, \mathbf{x})] d\sigma \\ &\quad - \int_0^\tau \left[\partial_t W_{k-1}(t, \sigma, \mathbf{x}) - \frac{1}{\theta} \int_0^\theta \partial_t W_{k-1}(t, \zeta, \mathbf{x}) d\zeta \right] d\sigma. \end{aligned} \quad (3.12)$$

Hence, it is straightforward that $W_k(t, \theta, \mathbf{x}) = 0$ for any t and for any \mathbf{x} .

The last property we have to satisfy for completing the proof of Theorem 2.8 consists in proving that the sequence $(f_{\epsilon,1})_{\epsilon > 0}$ defined by

$$\begin{aligned} f_{\epsilon,1}(t, \mathbf{x}) &= \frac{1}{\epsilon} \left(f_\epsilon(t, \mathbf{x}) - F\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) \right) \\ &= \frac{1}{\epsilon} \sum_{i=0}^{k-1} \left[\mathcal{A}_i\left(t, \frac{t}{\epsilon}, \mathbf{x}\right) - \mathbf{A}_{\epsilon,i}(t, \mathbf{x}) \right] \cdot \nabla_{\mathbf{x}} U_{k-1-i}\left(t, \frac{t}{\epsilon}, \mathbf{x}\right), \end{aligned}$$

is bounded independently of ϵ in $L^\infty\left(0, T; L^\infty_{\#}\left(0, \theta; X^{s'}(K)\right)\right)$ for any compact subset $K \subset \mathbb{R}^n$. To obtain this result, we remark that $f_{\epsilon,1}$ can write as

$$f_{\epsilon,1}(t, \mathbf{x}) = - \sum_{i=1}^k \mathbf{A}_{\epsilon,i}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} U_{k-i} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right),$$

and we use Hypothesis 2.6. Consequently, this sequence admits the profile $F_1 = F_1(t, \tau, \mathbf{x})$ defined as

$$F_1(t, \mathbf{x}) = - \sum_{i=1}^k \mathcal{A}_i(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} U_{k-i}(t, \tau, \mathbf{x}),$$

as a two-scale limit in $L^\infty\left(0, T; L^\infty_{\#}\left(0, \theta; X^{s'}(K)\right)\right)$.

We end the proof of Theorem 2.8 by assuming that $(u_{\epsilon,k})_{\epsilon > 0}$ is bounded independently of ϵ in $L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$: we deduce that $u_{\epsilon,k}$ two-scale converges to the profile $U_k = U_k(t, \tau, \mathbf{x})$ in $L^\infty\left(0, T; L^\infty_{\#}\left(0, \theta; L^p(\mathbb{R}^n)\right)\right)$ defined by

$$U_k(t, \tau, \mathbf{x}) = V_k(t, \mathbf{X}(-\tau; \mathbf{x}, t; 0)) + W_k(t, \tau, \mathbf{X}(-\tau; \mathbf{x}, t; 0)),$$

where $V_k = V_k(t, \mathbf{x}) \in L^\infty(0, T; L^p_{loc}(\mathbb{R}^n))$ satisfies

$$\left\{ \begin{array}{l} \partial_t V_k(t, \mathbf{x}) + \tilde{\mathbf{a}}_0(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} V_k(t, \mathbf{x}) \\ \quad = - \frac{1}{\theta} \int_0^\theta \left[\sum_{i=1}^k (\mathcal{A}_i \cdot \nabla_{\mathbf{x}} U_{k-i})(t, \tau, \mathbf{X}(\tau; \mathbf{x}, t; 0)) \right] d\tau \\ \quad \quad - \frac{1}{\theta} \int_0^\theta [\partial_t W_k(t, \tau, \mathbf{x}) + \boldsymbol{\alpha}_0(t, \tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}} W_k(t, \tau, \mathbf{x})] d\tau, \\ V_k(t=0, \mathbf{x}) = \begin{cases} u^0(\mathbf{x}), & \text{if } k=0, \\ 0 & \text{else.} \end{cases} \end{array} \right.$$

4 Application to Vlasov equation

In this last paragraph, we present some results which can be obtained when we apply Theorems 2.4 and 2.8 on linear Vlasov equations like (1.1), (1.2) or (1.8).

4.1 Guiding-Center regime

We focus on the following linear Vlasov equation:

$$\left\{ \begin{array}{l} \partial_t f_\epsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\epsilon + \left(\mathbf{E}_\epsilon + \mathbf{v} \times \mathbf{B}_\epsilon + \frac{\mathbf{v} \times \boldsymbol{\beta}_\epsilon}{\epsilon} \right) \cdot \nabla_{\mathbf{v}} f_\epsilon = 0, \\ f_\epsilon(t=0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}). \end{array} \right. \quad (4.1)$$

In this equation, $t \in [0, T]$ is the dimensionless time variable, $\mathbf{x} \in \mathbb{R}^3$ is the dimensionless space variable, $\mathbf{v} \in \mathbb{R}^3$ is the dimensionless velocity variable, $f_\epsilon = f_\epsilon(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}$ is the unknown distribution function, $\mathbf{E}_\epsilon = \mathbf{E}_\epsilon(t, \mathbf{x}) \in \mathbb{R}^3$ and $\mathbf{B}_\epsilon = \mathbf{B}_\epsilon(t, \mathbf{x}) \in \mathbb{R}^3$ are the given electric and magnetic fields, $f^0 = f^0(\mathbf{x}, \mathbf{v}) \in \mathbb{R}$ is the given initial distribution. We finally assume from now that the vector function $\boldsymbol{\beta}_\epsilon$ is of the form

$$\boldsymbol{\beta}_\epsilon(t, \mathbf{x}) = \boldsymbol{\beta} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right), \quad (4.2)$$

where $\beta : [0, T] \times \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a given function assumed to be θ -periodic and continuous in τ with $\theta > 0$ fixed.

Before going further, we introduce additional objects linked to β . First, let $\tilde{\beta}$ be defined such that $\partial_\tau \tilde{\beta} = \beta$ and $\tilde{\beta}(t, 0, \mathbf{x}) = 0$ for any t, \mathbf{x} . Second, we define the matrix $\mathfrak{B} = \mathfrak{B}(t, \tau, \mathbf{x})$ such that $\tilde{\mathfrak{B}}(t, \tau, \mathbf{x}) \mathbf{v} = \mathbf{v} \times \tilde{\beta}(t, \tau, \mathbf{x})$ for any $t, \tau, \mathbf{x}, \mathbf{v}$. Finally, we define $\mathcal{R} = \mathcal{R}(t, \tau, \mathbf{x}) = \exp\left(\tilde{\mathfrak{B}}(t, \tau, \mathbf{x})\right)$.

We fix $q > 3/2$ and we consider the following hypotheses:

Hypothesis 4.1. *We assume that the function \mathcal{R} satisfies*

- \mathcal{R} is θ -periodic in τ direction,
- $\mathcal{R} \in \left(L^\infty\left(0, T; L^\infty_\#(0, \theta; W^{1, \infty}(K))\right)\right)^{3 \times 3}$ for any compact subset $K \subset \mathbb{R}^3$,
- $\partial_t \mathcal{R} \in \left(L^\infty\left(0, T; L^\infty_\#(0, \theta; W^{1, q}(K))\right)\right)^{3 \times 3}$ for any compact subset $K \subset \mathbb{R}^3$.

Consequently, adding sufficient hypotheses on f^0 , $(\mathbf{E}_\epsilon)_{\epsilon > 0}$ and $(\mathbf{B}_\epsilon)_{\epsilon > 0}$ allows to establish a 0-th order two-scale convergence result:

Theorem 4.2. *We assume that Hypotheses 4.1 are satisfied and that $f^0 \in L^2(\mathbb{R}^6)$ and both sequences $(\mathbf{E}_\epsilon)_{\epsilon > 0}$ and $(\mathbf{B}_\epsilon)_{\epsilon > 0}$ are bounded in $(L^\infty(0, T; W^{1, q}(K)))^3$ independently of ϵ and for any $K \subset \mathbb{R}^3$ compact. We denote $\mathcal{E}_0 = \mathcal{E}_0(t, \tau, \mathbf{x})$ and $\mathcal{B}_0 = \mathcal{B}_0(t, \tau, \mathbf{x})$ as the respective two-scale limit of $(\mathbf{E}_\epsilon)_{\epsilon > 0}$ and $(\mathbf{B}_\epsilon)_{\epsilon > 0}$ in the space $(L^\infty(0, T; L^\infty_\#(0, \theta; W^{1, q}(K))))^3$ and we define \mathcal{L}_0 as*

$$\mathcal{L}_0(t, \tau, \mathbf{x}, \mathbf{v}) = \mathcal{E}_0(t, \tau, \mathbf{x}) + \mathbf{v} \times \mathcal{B}_0(t, \tau, \mathbf{x}). \quad (4.3)$$

Then, $(f_\epsilon)_{\epsilon > 0}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}^6))$ independently of ϵ and, up to the extraction of a subsequence, two-scale converges to the profile $F_0 = F_0(t, \tau, \mathbf{x}, \mathbf{v})$ in $L^\infty\left(0, T; L^\infty_\#(0, \theta; L^2(\mathbb{R}^6))\right)$. Furthermore, F_0 is characterized by

$$F_0(t, \tau, \mathbf{x}, \mathbf{v}) = G_0(t, \mathbf{x}, \mathcal{R}(t, -\tau, \mathbf{x}) \mathbf{v}), \quad (4.4)$$

with $G_0 = G_0(t, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$ solution of

$$\begin{cases} \partial_t G_0(t, \mathbf{x}, \mathbf{v}) + (\mathcal{J}_1(t, \mathbf{x}) \mathbf{v}) \cdot \nabla_{\mathbf{x}} G_0(t, \mathbf{x}, \mathbf{v}) \\ \quad + \mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} G_0(t, \mathbf{x}, \mathbf{v}) = 0, \\ G_0(t = 0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}), \end{cases} \quad (4.5)$$

where $\mathcal{J}_1(t, \mathbf{x})$ and $\mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v})$ are defined by

$$\mathcal{J}_1(t, \mathbf{x}) = \frac{1}{\theta} \int_0^\theta \mathcal{R}(t, \tau, \mathbf{x}) d\tau, \quad (4.6)$$

$$\mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\theta} \int_0^\theta \mathcal{J}_2(\mathcal{L}_0)(t, \tau, \mathbf{x}, \mathbf{v}) d\tau. \quad (4.7)$$

with \mathcal{J}_2 defined as

$$\begin{aligned} \mathcal{J}_2(\mathcal{L}_0)(t, \tau, \mathbf{x}, \mathbf{v}) = & \mathcal{R}(t, \tau, \mathbf{x})^{-1} \left[-\partial_t \mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v} - (\nabla_{\mathbf{x}} \mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v}) \mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v} \right. \\ & \left. + \mathcal{L}_0(t, \tau, \mathbf{x}, \mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v}) \right]. \end{aligned} \quad (4.8)$$

Proof. We first remark that the sequence $(f_\epsilon)_{\epsilon>0}$ is bounded independently of ϵ in $L^\infty(0, T; L^2(\mathbb{R}^6))$. Indeed, we have

$$\partial_t \left(\int_{\mathbb{R}^6} |f_\epsilon(t, \mathbf{x}, \mathbf{v})|^2 dx d\mathbf{v} \right) = 0.$$

Consequently, up to the extraction of a subsequence, f_ϵ two-scale converges to $F_0 = F_0(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty\left(0, T; L^\infty_{\#}(0, \theta; L^2(\mathbb{R}^6))\right)$, where F_0 has to be characterized.

Second, we remark that the vector function \mathbf{A}_ϵ defined by

$$\mathbf{A}_\epsilon(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}_\epsilon(t, \mathbf{x}) + \mathbf{v} \times \mathbf{B}_\epsilon(t, \mathbf{x}) \end{pmatrix},$$

is bounded independently of ϵ in $(L^\infty(0, T; W^{1,q}(K)))^6$ for any compact subset $K \subset \mathbb{R}^6$ and two-scale converges to \mathcal{A}_0 defined by

$$\mathcal{A}_0(t, \tau, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathcal{L}_0(t, \tau, \mathbf{x}, \mathbf{v}) \end{pmatrix}.$$

Under the hypotheses made on β , we find that F_0 satisfies

$$\partial_\tau F_0(t, \tau, \mathbf{x}, \mathbf{v}) + (\mathbf{v} \times \beta(t, \tau, \mathbf{x})) \cdot \nabla_{\mathbf{v}} F_0(t, \tau, \mathbf{x}, \mathbf{v}) = 0.$$

Invoking Hypotheses 4.1, we find that there exists a function $G_0 = G_0(t, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$ such that

$$F_0(t, \tau, \mathbf{x}, \mathbf{v}) = G_0(t, \mathbf{x}, \mathcal{R}(t, -\tau, \mathbf{x}) \mathbf{v}).$$

Hence, applying Theorem 2.4 gives the equation (4.5) for characterizing G_0 . □

We can remark here that the results of Theorem 4.2 are coherent with the Guiding-Center model presented in [15]. Indeed, taking $\mathbf{B}_\epsilon = 0$ and $\beta = \mathbf{e}_z$ leads to the matrix

$$\mathcal{R}(t, \tau, \mathbf{x}) = \begin{pmatrix} \cos \tau & \sin \tau & 0 \\ -\sin \tau & \cos \tau & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is 2π -periodic in τ . Consequently, assuming that \mathbf{E}_ϵ and \mathbf{B}_ϵ converge strongly in $(L^\infty(0, T; L^2_{loc}(\mathbb{R}^3)))^3$ to \mathbf{E} and \mathbf{B} , we have

$$\mathcal{J}_1(t, \mathbf{x}) \mathbf{v} = v_z \mathbf{e}_z,$$

and

$$\mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) = E_z(t, \mathbf{x}) \mathbf{e}_z + \mathbf{v} \times (B_z(t, \mathbf{x}) \mathbf{e}_z).$$

In order to characterize the higher order terms, it is necessary to add several assumptions. We fix an integer $k > 0$ and we consider the following hypotheses for $(f_\epsilon)_{\epsilon>0}$, $(\mathbf{E}_\epsilon)_{\epsilon>0}$ and $(\mathbf{B}_\epsilon)_{\epsilon>0}$:

Hypothesis 4.3. *Defining recursively the sequences $(\mathbf{E}_{\epsilon,i})_{\epsilon>0}$ and $(\mathbf{B}_{\epsilon,i})_{\epsilon>0}$ as*

$$\begin{cases} \mathbf{E}_{\epsilon,i}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(\mathbf{E}_{\epsilon,i-1}(t, \mathbf{x}) - \mathcal{E}_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), & \forall i = 1, \dots, k, \\ \mathbf{E}_{\epsilon,0}(t, \mathbf{x}) = \mathbf{E}_\epsilon(t, \mathbf{x}), \\ \mathbf{B}_{\epsilon,i}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(\mathbf{B}_{\epsilon,i-1}(t, \mathbf{x}) - \mathcal{B}_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), & \forall i = 1, \dots, k, \\ \mathbf{B}_{\epsilon,0}(t, \mathbf{x}) = \mathbf{B}_\epsilon(t, \mathbf{x}), \end{cases}$$

we assume that, for all $i = 0, \dots, k$, $(\mathbf{E}_{\epsilon,i})_{\epsilon>0}$ and $(\mathbf{B}_{\epsilon,i})_{\epsilon>0}$ two-scale converge to $\mathcal{E}_i = \mathcal{E}_i(t, \tau, \mathbf{x})$ and $\mathcal{B}_i = \mathcal{B}_i(t, \tau, \mathbf{x})$ respectively in $(L^\infty(0, T; L^\infty_{\#}(0, \theta; W^{1,q}(K))))^3$ for any compact subset $K \subset \mathbb{R}^3$.

Hypothesis 4.4. *Defining recursively the sequence $(f_{\epsilon,i})_{\epsilon>0}$ as*

$$\begin{cases} f_{\epsilon,i}(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\epsilon} \left(f_{\epsilon,i-1}(t, \mathbf{x}, \mathbf{v}) - F_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v} \right) \right), & \forall i = 1, \dots, k-1, \\ f_{\epsilon,0}(t, \mathbf{x}, \mathbf{v}) = f_{\epsilon}(t, \mathbf{x}, \mathbf{v}), \end{cases}$$

we assume that, up to a subsequence, the sequence $(f_{\epsilon,i})_{\epsilon>0}$ two-scale converges to the profile $F_i = F_i(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty \left(0, T; L^\infty_{\#} (0, \theta; L^2(\mathbb{R}^6)) \right)$ for all $i = 0, \dots, k-1$.

We now define \mathcal{L}_i for $i = 0, \dots, k$ such that

$$\mathcal{L}_i(t, \tau, \mathbf{x}, \mathbf{v}) = \mathcal{E}_i(t, \tau, \mathbf{x}) + \mathbf{v} \times \mathcal{B}_i(t, \tau, \mathbf{x}). \quad (4.9)$$

We define W_0, \dots, W_k such that $W_0 = 0$ and, for any $i = 1, \dots, k$,

$$\begin{aligned} W_i(t, \tau, \mathbf{x}, \mathbf{v}) &= \int_0^\tau \left(\begin{array}{c} (\mathcal{J}_1(t, \mathbf{x}) - \mathcal{R}(t, \sigma, \mathbf{x})) \mathbf{v} \\ \mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) - \mathcal{J}_2(\mathcal{L}_0)(t, \sigma, \mathbf{x}, \mathbf{v}) \end{array} \right) \\ &\quad \cdot \left(\begin{array}{c} \nabla_{\mathbf{x}} G_{i-1}(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} W_{i-1}(t, \sigma, \mathbf{x}, \mathbf{v}) \\ \nabla_{\mathbf{v}} G_{i-1}(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{v}} W_{i-1}(t, \sigma, \mathbf{x}, \mathbf{v}) \end{array} \right) d\sigma \\ &+ \sum_{j=1}^{i-1} \int_0^\tau \left[[\mathcal{J}_3(\mathcal{L}_j)(t, \mathbf{x}, \mathbf{v}) - \mathcal{R}(t, \sigma, \mathbf{x})^{-1} \mathcal{L}_j(t, \sigma, \mathbf{x}, \mathcal{R}(t, \sigma, \mathbf{x}) \mathbf{v})] \right. \\ &\quad \left. \cdot [\nabla_{\mathbf{v}} G_{i-j-1}(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{v}} W_{i-j-1}(t, \sigma, \mathbf{x}, \mathbf{v})] \right] d\sigma \\ &- \int_0^\tau \left[\partial_t W_{i-1}(t, \sigma, \mathbf{x}, \mathbf{v}) - \frac{1}{\theta} \int_0^\theta \partial_t W_{i-1}(t, \zeta, \mathbf{x}, \mathbf{v}) d\zeta \right] d\sigma, \end{aligned} \quad (4.10)$$

with $\mathcal{J}_3(\mathcal{L}_j)$ defined by

$$\mathcal{J}_3(\mathcal{L}_j)(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\theta} \int_0^\theta \mathcal{R}(t, \tau, \mathbf{x})^{-1} \mathcal{L}_j(t, \tau, \mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v}) d\tau, \quad (4.11)$$

and where G_0, \dots, G_{k-1} are linked with F_0, \dots, F_{k-1} and W_0, \dots, W_{k-1} thanks to the relation

$$F_i(t, \tau, \mathbf{x}, \mathbf{v}) = G_i(t, \mathbf{x}, \mathcal{R}(t, -\tau, \mathbf{x}) \mathbf{v}) + W_i(t, \tau, \mathbf{x}, \mathcal{R}(t, -\tau, \mathbf{x}) \mathbf{v}).$$

With these notations, we can establish a two-scale convergence result at the k -th order:

Theorem 4.5. *We assume that the hypotheses of Theorem 4.2 and that Hypotheses 4.3 and 4.4 are satisfied. We introduce R_{k-1} as follows*

$$\begin{aligned} R_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) &= \partial_t F_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) + (\mathcal{J}_1(t, \mathbf{x}) \mathbf{v}) \cdot \nabla_{\mathbf{x}} F_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) \\ &+ \left[\frac{1}{\theta} \int_0^\theta \mathcal{R}(t, \sigma, \mathbf{x})^{-1} \left[-\partial_t \mathcal{R}(t, \sigma, \mathbf{x}) \mathbf{v} - (\nabla_{\mathbf{x}} \mathcal{R}(t, \sigma, \mathbf{x}) \mathbf{v}) \mathcal{R}(t, \sigma, \mathbf{x}) \mathbf{v} \right] d\sigma \right] \\ &\quad \cdot \nabla_{\mathbf{v}} F_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) \\ &+ \sum_{i=0}^{k-1} \left[\frac{1}{\theta} \int_0^\theta \mathcal{R}(t, \sigma, \mathbf{x})^{-1} \mathcal{L}_i(t, \sigma + \tau, \mathbf{x}, \mathcal{R}(t, \sigma, \mathbf{x}) \mathbf{v}) d\sigma \right] \\ &\quad \cdot \nabla_{\mathbf{v}} F_{k-1-i}(t, \tau, \mathbf{x}, \mathbf{v}). \end{aligned} \quad (4.12)$$

In addition, taking s' such that $\frac{1}{s'} = 1 - \frac{1}{q} - \frac{1}{r}$ with $r \in [1, \frac{6q}{6-q}[$ and defining $X^{s'}(K) = (W^{1,q}(K))' \cup (W^{1,s'}(K))$, we assume that, for any compact subset $K \subset \mathbb{R}^6$,

- $W_k \in L^\infty \left(0, T; L^\infty_{\#} (0, \theta; L^2(K))\right)$,
- $\partial_t W_k, R_{k-1} \in L^\infty \left(0, T; L^\infty_{\#} \left(0, \theta; X^{s'}(K)\right)\right)$.

Then, if the sequence $(f_{\epsilon,k})_{\epsilon > 0}$ defined by

$$f_{\epsilon,k}(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\epsilon} \left(f_{\epsilon,k-1}(t, \mathbf{x}, \mathbf{v}) - F_{k-1} \left(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v} \right) \right), \quad (4.13)$$

is bounded independently of ϵ in $L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$, it two-scale converges to the profile $F_k = F_k(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty \left(0, T; L^\infty_{\#} (0, \theta; L^2(\mathbb{R}^6))\right)$ up to the extraction of a subsequence. Furthermore, F_k is fully characterized by

$$F_k(t, \tau, \mathbf{x}, \mathbf{v}) = G_k(t, \mathbf{x}, \mathcal{R}(t, -\tau, \mathbf{x}) \mathbf{v}) + W_k(t, \tau, \mathbf{x}, \mathcal{R}(t, -\tau, \mathbf{x}) \mathbf{v}), \quad (4.14)$$

where W_k is defined in (4.10) and where $G_k = G_k(t, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$ is the solution of

$$\left\{ \begin{array}{l} \partial_t G_k(t, \mathbf{x}, \mathbf{v}) + (\mathcal{J}_1(t, \mathbf{x}) \mathbf{v}) \cdot \nabla_{\mathbf{x}} G_k(t, \mathbf{x}, \mathbf{v}) \\ \quad + \mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} G_k(t, \mathbf{x}, \mathbf{v}) \\ = -\frac{1}{\theta} \int_0^\theta [\partial_t W_k(t, \tau, \mathbf{x}, \mathbf{v}) + (\mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v}) \cdot \nabla_{\mathbf{x}} W_k(t, \tau, \mathbf{x}, \mathbf{v})] d\tau \\ -\frac{1}{\theta} \int_0^\theta \mathcal{J}_2(\mathcal{L}_0)(t, \tau, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} W_k(t, \tau, \mathbf{x}, \mathbf{v}) d\tau \\ -\frac{1}{\theta} \sum_{i=0}^k \int_0^\theta [\mathcal{R}(t, \tau, \mathbf{x})^{-1} \mathcal{L}_i(t, \tau, \mathbf{x}, \mathcal{R}(t, \tau, \mathbf{x}) \mathbf{v})] \cdot \nabla_{\mathbf{v}} W_{k-i}(t, \tau, \mathbf{x}, \mathbf{v}) d\tau \\ -\sum_{i=1}^k \mathcal{J}_3(\mathcal{L}_i)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} G_{k-i}(t, \mathbf{x}, \mathbf{v}), \\ G_k(t=0, \mathbf{x}, \mathbf{v}) = 0. \end{array} \right. \quad (4.15)$$

4.2 Finite Larmor Radius regime

We focus now on the following linear equation:

$$\left\{ \begin{array}{l} \partial_t f_\epsilon + \frac{\mathbf{v}_\perp}{\epsilon} \cdot \nabla_{\mathbf{x}_\perp} f_\epsilon + v_{\parallel} \partial_{x_{\parallel}} f_\epsilon + \left(\mathbf{E}_\epsilon + \mathbf{v} \times \mathbf{B}_\epsilon + \frac{\mathbf{v} \times \mathcal{M}}{\epsilon} \right) \cdot \nabla_{\mathbf{v}} f_\epsilon = 0, \\ f_\epsilon(t=0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}), \end{array} \right. \quad (4.16)$$

in which $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $t \in [0, T]$, $f_\epsilon = f_\epsilon(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}$ is the unknown distribution function, $\mathbf{E}_\epsilon = \mathbf{E}_\epsilon(t, \mathbf{x}) \in \mathbb{R}^3$ is the external electric field, $f^0 = f^0(\mathbf{x}, \mathbf{v})$ is the initial distribution function, $\mathcal{M} = \mathbf{e}_z \in \mathbb{R}^3$ and $\mathbf{B}_\epsilon = \mathbf{B}_\epsilon(t, \mathbf{x}) \in \mathbb{R}^3$ constitute the external magnetic field.

Thanks to well-chosen hypotheses for f^0 and the sequences $(\mathbf{E}_\epsilon)_{\epsilon > 0}$ and $(\mathbf{B}_\epsilon)_{\epsilon > 0}$, it is possible to establish a 0-th order two-scale convergence result:

Theorem 4.6. *We assume that $f^0 \in L^2(\mathbb{R}^6)$ and that $(\mathbf{E}_\epsilon)_{\epsilon > 0}$ and $(\mathbf{B}_\epsilon)_{\epsilon > 0}$ are bounded independently of ϵ in $(L^\infty(0, T; W^{1,q}(K)))^3$ for $q > 3/2$ and for any compact subset $K \subset \mathbb{R}^3$.*

We denote with $\mathcal{E}_0 = \mathcal{E}_0(t, \tau, \mathbf{x})$ and $\mathcal{B}_0 = \mathcal{B}_0(t, \tau, \mathbf{x})$ the respective two-scale limits of $(\mathbf{E}_\epsilon)_{\epsilon > 0}$ and $(\mathbf{B}_\epsilon)_{\epsilon > 0}$ in $\left(L^\infty\left(0, T; L^\infty_{\#}(0, 2\pi; W^{1,q}(K))\right)\right)^3$ and we introduce the vector function \mathcal{L}_0 defined by

$$\mathcal{L}_0(t, \tau, \mathbf{x}, \mathbf{v}) = \mathcal{E}_0(t, \tau, \mathbf{x}) + \mathbf{v} \times \mathcal{B}_0(t, \tau, \mathbf{x}). \quad (4.17)$$

Up to a subsequence, f_ϵ two-scale converges to the profile $F_0 = F_0(t, \tau, \mathbf{x}, \mathbf{v})$ in $L^\infty\left(0, T; L^\infty_{\#}(0, 2\pi; L^2(\mathbb{R}^6))\right)$ and F_0 is fully characterized by

$$F_0(t, \tau, \mathbf{x}, \mathbf{v}) = G_0(t, \mathbf{x} + \mathcal{R}_1(-\tau)\mathbf{v}, \mathcal{R}_2(-\tau)\mathbf{v}), \quad (4.18)$$

where $\mathcal{R}_1, \mathcal{R}_2$ and $G_0 = G_0(t, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$ satisfy

$$\mathcal{R}_1(\tau) = \begin{pmatrix} \sin \tau & 1 - \cos \tau & 0 \\ \cos \tau - 1 & \sin \tau & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{R}_2(\tau) = \begin{pmatrix} \cos \tau & \sin \tau & 0 \\ -\sin \tau & \cos \tau & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.19)$$

$$\begin{cases} \partial_t G_0(t, \mathbf{x}, \mathbf{v}) + v_{\parallel} \partial_{x_{\parallel}} G_0(t, \mathbf{x}, \mathbf{v}) + \mathcal{J}_1(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{x}} G_0(t, \mathbf{x}, \mathbf{v}) \\ \qquad \qquad \qquad + \mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} G_0(t, \mathbf{x}, \mathbf{v}) = 0, \\ G_0(t = 0, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{x}, \mathbf{v}), \end{cases} \quad (4.20)$$

with $\mathcal{J}_1(\mathcal{L}_0)$ and $\mathcal{J}_2(\mathcal{L}_0)$ defined by

$$\mathcal{J}_i(\mathcal{L}_0) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}_i(-\tau) \mathcal{L}_0(t, \tau, \mathbf{x} + \mathcal{R}_1(\tau)\mathbf{v}, \mathcal{R}_2(\tau)\mathbf{v}) d\tau. \quad (4.21)$$

Proof. We easily remark that the sequence $(\mathbf{A}_\epsilon)_{\epsilon > 0}$ defined by

$$\mathbf{A}_\epsilon(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} 0 \\ 0 \\ v_{\parallel} \\ \mathbf{E}_\epsilon(t, \mathbf{x}) + \mathbf{v} \times \mathbf{B}_\epsilon(t, \mathbf{x}) \end{pmatrix},$$

is bounded independently of ϵ in $\left(L^\infty(0, T; W^{1,q}(K))\right)^6$ for any compact set $K \subset \mathbb{R}^6$ and for $q > 3/2$. Consequently, \mathbf{A}_ϵ two-scale converges to the vector function $\mathcal{A}_0 = \mathcal{A}_0(t, \tau, \mathbf{x}, \mathbf{v}) \in \left(L^\infty(0, T; L^\infty_{\#}(0, 2\pi; W^{1,q}(K))\right)^6$ defined by

$$\mathcal{A}_0(t, \tau, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} 0 \\ 0 \\ v_{\parallel} \\ \mathcal{E}_0(t, \tau, \mathbf{x}) + \mathbf{v} \times \mathcal{B}_0(t, \tau, \mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_{\parallel} \\ \mathcal{L}_0(t, \tau, \mathbf{x}, \mathbf{v}) \end{pmatrix}.$$

We also remark that the ODE system

$$\begin{cases} (\mathbf{X}, \mathbf{V})(\tau) = \mathbf{L}(t, \tau, \mathbf{X}(\tau), \mathbf{V}(\tau)), \\ (\mathbf{X}, \mathbf{V})(0) = (\mathbf{x}, \mathbf{v}), \end{cases}$$

with \mathbf{L} defined by

$$\mathbf{L}(t, \tau, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v}_{\perp} \\ \mathbf{v} \times \mathcal{M} \end{pmatrix},$$

admits the following 2π -periodic solution:

$$\begin{cases} \mathbf{X}(\tau; \mathbf{x}, \mathbf{v}, t; 0) = \mathbf{x} + \mathcal{R}_1(\tau)\mathbf{v}, \\ \mathbf{V}(\tau; \mathbf{x}, \mathbf{v}, t; 0) = \mathcal{R}_2(\tau)\mathbf{v}, \end{cases}$$

with \mathcal{R}_1 and \mathcal{R}_2 defined by (4.19).

Multiplying the Vlasov equation by f_ϵ and integrating the result in \mathbf{x} and \mathbf{v} , we obtain

$$\partial_t \left(\int_{\mathbb{R}^6} |f_\epsilon(t, \mathbf{x}, \mathbf{v})|^2 d\mathbf{x} d\mathbf{v} \right) = 0,$$

so we deduce $\|f_\epsilon(t, \cdot)\|_{L^2(\mathbb{R}^6)} = \|f^0\|_{L^2(\mathbb{R}^6)}$ for any ϵ and any t . Consequently, Theorem 2.4 allow us to claim that, up to the extraction of a subsequence, f_ϵ two-scale converges to the profile $F_0 = F_0(t, \tau, \mathbf{x}, \mathbf{v})$ in $L^\infty \left(0, T; L^\infty_{\#} \left(0, 2\pi; L^2(\mathbb{R}^6) \right) \right)$ characterized by

$$F_0(t, \tau, \mathbf{x}, \mathbf{v}) = G_0(t, \mathbf{x} + \mathcal{R}_1(-\tau) \mathbf{v}, \mathcal{R}_2(-\tau) \mathbf{v}),$$

where $G_0 = G_0(t, \mathbf{x}, \mathbf{v}) \in L^\infty \left(0, T; L^2_{loc}(\mathbb{R}^6) \right)$ satisfies (4.20). \square

For obtaining higher order two-scale convergence terms, we first consider a fixed $k \in \mathbb{N}^*$ and we assume that the electric and magnetic fields satisfy the following hypotheses:

Hypothesis 4.7. *Defining recursively the sequences $(\mathbf{E}_{\epsilon, i})_{\epsilon > 0}$ and $(\mathbf{B}_{\epsilon, i})_{\epsilon > 0}$ as*

$$\begin{cases} \mathbf{E}_{\epsilon, i}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(\mathbf{E}_{\epsilon, i-1}(t, \mathbf{x}) - \mathcal{E}_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), & \forall i = 1, \dots, k, \\ \mathbf{E}_{\epsilon, 0}(t, \mathbf{x}) = \mathbf{E}_\epsilon(t, \mathbf{x}), \\ \mathbf{B}_{\epsilon, i}(t, \mathbf{x}) = \frac{1}{\epsilon} \left(\mathbf{B}_{\epsilon, i-1}(t, \mathbf{x}) - \mathcal{B}_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x} \right) \right), & \forall i = 1, \dots, k, \\ \mathbf{B}_{\epsilon, 0}(t, \mathbf{x}) = \mathbf{B}_\epsilon(t, \mathbf{x}), \end{cases}$$

we assume that, for all $i = 0, \dots, k$ and up to the extraction of a subsequence, $(\mathbf{E}_{\epsilon, i})_{\epsilon > 0}$ and $(\mathbf{B}_{\epsilon, i})_{\epsilon > 0}$ two-scale converge to the profiles $\mathcal{E}_i = \mathcal{E}_i(t, \tau, \mathbf{x})$ and $\mathcal{B}_i = \mathcal{B}_i(t, \tau, \mathbf{x})$ respectively in $\left(L^\infty \left(0, T; L^\infty_{\#} \left(0, 2\pi; W^{1, q}(K) \right) \right) \right)^3$ for any compact subset $K \subset \mathbb{R}^3$.

Hypothesis 4.8. *Defining recursively the sequence $(f_{\epsilon, i})_{\epsilon > 0}$ as*

$$\begin{cases} f_{\epsilon, i}(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\epsilon} \left(f_{\epsilon, i-1}(t, \mathbf{x}, \mathbf{v}) - F_{i-1} \left(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v} \right) \right), & \forall i = 1, \dots, k-1, \\ f_{\epsilon, 0}(t, \mathbf{x}, \mathbf{v}) = f_\epsilon(t, \mathbf{x}, \mathbf{v}), \end{cases}$$

we assume that, up to a subsequence, the sequence $(f_{\epsilon, i})_{\epsilon > 0}$ two-scale converges to the profile $F_i = F_i(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty \left(0, T; L^\infty_{\#} \left(0, 2\pi; L^2(\mathbb{R}^6) \right) \right)$ for all $i = 0, \dots, k-1$.

Hence, defining \mathcal{L}_i as

$$\mathcal{L}_i(t, \tau, \mathbf{x}, \mathbf{v}) = \mathcal{E}_i(t, \tau, \mathbf{x}) + \mathbf{v} \times \mathcal{B}_i(t, \tau, \mathbf{x}), \quad (4.22)$$

for all $i = 0, \dots, k$, we define recursively the functions W_0, \dots, W_k such that $W_0 = 0$ and, for any $i > 0$,

$$\begin{aligned} & W_i(t, \tau, \mathbf{x}, \mathbf{v}) \\ &= \sum_{j=0}^{i-1} \int_0^\tau \left(\begin{aligned} & \mathcal{J}_1(\mathcal{L}_j)(t, \mathbf{x}, \mathbf{v}) - \mathcal{R}_1(-\sigma) \mathcal{L}_j(t, \sigma, \mathbf{x} + \mathcal{R}_1(\sigma) \mathbf{v}, \mathcal{R}_2(\sigma) \mathbf{v}) \\ & \mathcal{J}_2(\mathcal{L}_j)(t, \mathbf{x}, \mathbf{v}) - \mathcal{R}_2(-\sigma) \mathcal{L}_j(t, \sigma, \mathbf{x} + \mathcal{R}_1(\sigma) \mathbf{v}, \mathcal{R}_2(\sigma) \mathbf{v}) \end{aligned} \right) \\ & \quad \cdot \left(\begin{aligned} & \nabla_{\mathbf{x}} G_{i-1-j}(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} W_{i-1-j}(t, \sigma, \mathbf{x}, \mathbf{v}) \\ & \nabla_{\mathbf{v}} G_{i-1-j}(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{v}} W_{i-1-j}(t, \sigma, \mathbf{x}, \mathbf{v}) \end{aligned} \right) d\sigma \\ & - \int_0^\tau \left[\partial_t W_{i-1}(t, \sigma, \mathbf{x}, \mathbf{v}) - \frac{1}{2\pi} \int_0^{2\pi} \partial_t W_{i-1}(t, \zeta, \mathbf{x}, \mathbf{v}) d\zeta \right] d\sigma \end{aligned} \quad (4.23)$$

where, for $i = 0, \dots, k-1$, G_i is defined on $[0, T] \times \mathbb{R}^6$ thanks to the relation

$$F_i(t, \tau, \mathbf{x}, \mathbf{v}) = G_i(t, \mathbf{x} + \mathcal{R}_1(-\tau) \mathbf{v}, \mathcal{R}_2(-\tau) \mathbf{v}) \\ + W_i(t, \tau, \mathbf{x} + \mathcal{R}_1(-\tau) \mathbf{v}, \mathcal{R}_2(-\tau) \mathbf{v}).$$

Hence we have the following result for obtaining the k -th order term F_k :

Theorem 4.9. *We assume that the hypotheses of Theorem 4.6 and Hypotheses 4.7 and 4.8 are satisfied for a fixed $k \in \mathbb{N}^*$ and we introduce the function R_{k-1} defined by*

$$R_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) \\ = \partial_t F_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) + v_{||} \partial_{x_{||}} F_{k-1}(t, \tau, \mathbf{x}, \mathbf{v}) \\ + \frac{1}{2\pi} \sum_{i=0}^{k-1} \left[\int_0^{2\pi} \begin{pmatrix} \mathcal{R}_1(-\sigma) \mathcal{L}_i(t, \sigma + \tau, \mathbf{x} + \mathcal{R}_1(\sigma) \mathbf{v}, \mathcal{R}_2(\sigma) \mathbf{v}) \\ \mathcal{R}_2(-\sigma) \mathcal{L}_i(t, \sigma + \tau, \mathbf{x} + \mathcal{R}_1(\sigma) \mathbf{v}, \mathcal{R}_2(\sigma) \mathbf{v}) \end{pmatrix} d\sigma \right. \\ \left. \cdot \begin{pmatrix} \nabla_{\mathbf{x}} F_{k-1-i}(t, \tau, \mathbf{x}, \mathbf{v}) \\ \nabla_{\mathbf{v}} F_{k-1-i}(t, \tau, \mathbf{x}, \mathbf{v}) \end{pmatrix} \right]. \quad (4.24)$$

In addition, taking s' such that $\frac{1}{s'} = 1 - \frac{1}{q} - \frac{1}{r}$ with $r \in [1, \frac{6q}{6-q}[$ and defining $X^{s'}(K) = (W^{1,q}(K))' \cup (W^{1,s'}(K))$, we assume that, for any compact subset $K \subset \mathbb{R}^6$,

- $W_k \in L^\infty(0, T; L^\infty_{\#}(0, 2\pi; L^2(K)))$,
- $\partial_t W_k, R_{k-1} \in L^\infty(0, T; L^\infty_{\#}(0, 2\pi; X^{s'}(K)))$.

Then, if the sequence $(f_{\epsilon, k})_{\epsilon > 0}$ defined by

$$f_{\epsilon, k}(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\epsilon} \left(f_{\epsilon, k-1}(t, \mathbf{x}, \mathbf{v}) - F_{k-1} \left(t, \frac{t}{\epsilon}, \mathbf{x}, \mathbf{v} \right) \right), \quad (4.25)$$

is bounded independently of ϵ in $L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$, it two-scale converges to the profile $F_k = F_k(t, \tau, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^\infty_{\#}(0, 2\pi; L^2(\mathbb{R}^6)))$ up to the extraction of a subsequence. Furthermore, F_k is fully characterized by

$$F_k(t, \tau, \mathbf{x}, \mathbf{v}) = G_k(t, \mathbf{x} + \mathcal{R}_1(-\tau) \mathbf{v}, \mathcal{R}_2(-\tau) \mathbf{v}) \\ + W_k(t, \tau, \mathbf{x} + \mathcal{R}_1(-\tau) \mathbf{v}, \mathcal{R}_2(-\tau) \mathbf{v}), \quad (4.26)$$

where W_k is defined in (4.23) and where $G_k = G_k(t, \mathbf{x}, \mathbf{v}) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^6))$ is the solution of

$$\left\{ \begin{array}{l} \partial_t G_k(t, \mathbf{x}, \mathbf{v}) + v_{||} \partial_{x_{||}} G_k(t, \mathbf{x}, \mathbf{v}) \\ \quad + \mathcal{J}_1(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{x}} G_k(t, \mathbf{x}, \mathbf{v}) + \mathcal{J}_2(\mathcal{L}_0)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} G_k(t, \mathbf{x}, \mathbf{v}) \\ = -\frac{1}{2\pi} \int_0^{2\pi} [\partial_t W_k(t, \tau, \mathbf{x}, \mathbf{v}) + v_{||} \partial_{x_{||}} W_k(t, \tau, \mathbf{x}, \mathbf{v})] d\tau \\ \quad - \frac{1}{2\pi} \sum_{i=0}^k \int_0^{2\pi} \left[\begin{pmatrix} \mathcal{R}_1(-\tau) \mathcal{L}_i(t, \tau, \mathbf{x} + \mathcal{R}_1(\tau) \mathbf{v}, \mathcal{R}_2(\tau) \mathbf{v}) \\ \mathcal{R}_2(-\tau) \mathcal{L}_i(t, \tau, \mathbf{x} + \mathcal{R}_1(\tau) \mathbf{v}, \mathcal{R}_2(\tau) \mathbf{v}) \end{pmatrix} \right. \\ \quad \left. \cdot \begin{pmatrix} \nabla_{\mathbf{x}} W_{k-i}(t, \tau, \mathbf{x}, \mathbf{v}) \\ \nabla_{\mathbf{v}} W_{k-i}(t, \tau, \mathbf{x}, \mathbf{v}) \end{pmatrix} \right] d\tau \\ \quad - \sum_{i=1}^k \begin{pmatrix} \mathcal{J}_1(\mathcal{L}_i)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{x}} G_{k-i}(t, \mathbf{x}, \mathbf{v}) \\ \mathcal{J}_2(\mathcal{L}_i)(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} G_{k-i}(t, \mathbf{x}, \mathbf{v}) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{\mathbf{x}} G_{k-i}(t, \mathbf{x}, \mathbf{v}) \\ \nabla_{\mathbf{v}} G_{k-i}(t, \mathbf{x}, \mathbf{v}) \end{pmatrix}, \\ G_k(t=0, \mathbf{x}, \mathbf{v}) = 0. \end{array} \right. \quad (4.27)$$

4.3 Application to axisymmetric charged particle beams

In this last example, we focus on the following axisymmetric linear Vlasov equation:

$$\begin{cases} \partial_t f_\epsilon(t, r, v_r) + \frac{v_r}{\epsilon} \partial_r f_\epsilon(t, r, v_r) + \left(E_\epsilon(t, r) - \frac{r}{\epsilon} \right) \partial_{v_r} f_\epsilon(t, r, v_r) = 0, \\ f_\epsilon(t = 0, r, v_r) = f^0(r, v_r). \end{cases} \quad (4.28)$$

In this system, $f_\epsilon = f_\epsilon(t, r, v_r)$ is the unknown distribution function of the particles, $E_\epsilon = E_\epsilon(t, r)$ is the radial component of the external magnetic field, the variables $(t, r, v_r) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ stand for the time variable and the radial position and velocity variable, with the convention $f_\epsilon(t, r, v_r) = f_\epsilon(t, -r, -v_r)$, $E_\epsilon(t, r) = -E_\epsilon(t, -r)$ (see [10, 14, 29] for details).

The two-scale convergence of f_ϵ at 0-th order has been studied by Frénod, Sonnendrücker and Salvarani in [14] in a more rich context. We recall here this result and we adapt it in a straightforward way to the case of Vlasov equation (4.28):

Theorem 4.10 (Frénod, Sonnendrücker, Salvarani [14]). *We assume that the initial distribution f^0 is positive on \mathbb{R}^2 and that $f^0 \in L^1(\mathbb{R}^2; r dr dv_r) \cap L^2(\mathbb{R}^2; r dr dv_r)$. We also assume that the sequence $(E_\epsilon)_{\epsilon > 0}$ is bounded independently of ϵ in the space $L^\infty(0, T; W^{1,3/2}(K; r dr))$ for any $K \subset \mathbb{R}$ compact. Then, up to the extraction of a subsequence, f_ϵ two-scale converges to the profile $F_0 = F_0(t, \tau, r, v_r)$ in $L^\infty(0, T; L^\infty_\#(0, 2\pi; L^2(\mathbb{R}^2; r dr dv_r)))$ and E_ϵ two-scale converges to $\mathcal{E}_0 = \mathcal{E}_0(t, r, v_r)$ in $L^\infty(0, T; L^\infty_\#(0, 2\pi; W^{1,3/2}(K; r dr)))$ for any $K \subset \mathbb{R}$ compact, with F_0 defined by*

$$F_0(t, \tau, r, v_r) = G_0(t, r \cos \tau - v_r \sin \tau, r \sin \tau + v_r \cos \tau), \quad (4.29)$$

with $G_0 = G_0(t, r, v_r) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^2; r dr dv_r))$ solution of

$$\begin{cases} \partial_t G_0 + \mathcal{J}_1(\mathcal{E}_0) \partial_r G_0 + \mathcal{J}_2(\mathcal{E}_0) \partial_{v_r} G_0 = 0, \\ G_0(t = 0, r, v_r) = f^0(r, v_r), \end{cases} \quad (4.30)$$

where

$$\mathcal{J}_1(\mathcal{E}_0)(t, r, v_r) = -\frac{1}{2\pi} \int_0^{2\pi} \sin(\tau) \mathcal{E}_0(t, \tau, r \cos \tau + v_r \sin \tau) d\tau, \quad (4.31)$$

$$\mathcal{J}_2(\mathcal{E}_0)(t, r, v_r) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\tau) \mathcal{E}_0(t, \tau, r \cos \tau + v_r \sin \tau) d\tau. \quad (4.32)$$

In order to establish higher order two-scale convergence results, it is necessary to add some hypotheses on the external electric field E_ϵ . As in the previous paragraphs, we consider a fixed integer $k > 0$ and we formalize it as follows:

Hypothesis 4.11. *Defining recursively the sequence $(E_{\epsilon,i})_{\epsilon > 0}$ as*

$$\begin{cases} E_{\epsilon,i}(t, r) = \frac{1}{\epsilon} \left(E_{\epsilon,i-1}(t, r) - \mathcal{E}_{i-1} \left(t, \frac{t}{\epsilon}, r \right) \right), \quad \forall i = 1, \dots, k, \\ E_{\epsilon,0}(t, r) = E_\epsilon(t, r), \end{cases}$$

we assume that, for all $i = 0, \dots, k$, $(E_{\epsilon,i})_{\epsilon > 0}$ two-scale converges to the profile $\mathcal{E}_i = \mathcal{E}_i(t, \tau, r)$ in $L^\infty(0, T; L^\infty_\#(0, 2\pi; W^{1,3/2}(K; r dr)))$ for any $K \subset \mathbb{R}$ compact.

We also add some hypotheses about the two-scale convergence of f_ϵ at i -th order for $i = 0, \dots, k-1$:

Hypothesis 4.12. *Defining recursively the sequence $(f_{\epsilon,i})_{\epsilon>0}$ as*

$$\begin{cases} f_{\epsilon,i}(t, r, v_r) = \frac{1}{\epsilon} \left(f_{\epsilon,i-1}(t, r, v_r) - F_{i-1} \left(t, \frac{t}{\epsilon}, r, v_r \right) \right), & \forall i = 1, \dots, k-1, \\ f_{\epsilon,0}(t, r, v_r) = f_{\epsilon}(t, r, v_r), \end{cases}$$

we assume that, up to the extraction of a subsequence, $(f_{\epsilon,i})_{\epsilon>0}$ two-scale converges to $F_i = F_i(t, \tau, r, v_r)$ in $L^\infty \left(0, T; L^\infty_{\#} (0, 2\pi; L^2(\mathbb{R}^2; r dr dv_r)) \right)$ for $i = 0, \dots, k-1$.

Hence we can define recursively W_0, \dots, W_k as follows:

$$\begin{aligned} W_i(t, \tau, r, v_r) &= \int_0^\tau \left[\sum_{j=0}^{i-1} \begin{pmatrix} \mathcal{J}_1(\mathcal{E}_j)(t, r, v_r) + \sin(\sigma) \mathcal{E}_j(t, \sigma, r \cos \sigma + v_r \sin \sigma) \\ \mathcal{J}_2(\mathcal{E}_j)(t, r, v_r) - \cos(\sigma) \mathcal{E}_j(t, \sigma, r \cos \sigma + v_r \sin \sigma) \end{pmatrix} \right. \\ &\quad \cdot \begin{pmatrix} \partial_r G_{i-1-j}(t, r, v_r) + \partial_r W_{i-1-j}(t, \sigma, r, v_r) \\ \partial_{v_r} G_{i-1-j}(t, r, v_r) + \partial_{v_r} W_{i-1-j}(t, \sigma, r, v_r) \end{pmatrix} \\ &\quad \left. - \partial_t W_{k-1}(t, \sigma, r, v_r) + \frac{1}{2\pi} \int_0^{2\pi} \partial_t W_{k-1}(t, \zeta, r, v_r) d\zeta \right] d\sigma, \end{aligned} \quad (4.33)$$

where G_0, \dots, G_{k-1} are linked to F_0, \dots, F_{k-1} by the relations

$$\begin{aligned} F_i(t, \tau, r, v_r) &= G_i(t, r \cos \tau - v_r \sin \tau, r \sin \tau + v_r \cos \tau) \\ &\quad + W_i(t, \tau, r \cos \tau - v_r \sin \tau, r \sin \tau + v_r \cos \tau). \end{aligned}$$

We finally introduce the function R_{k-1} defined by

$$\begin{aligned} R_{k-1}(t, \tau, r, v_r) &= \partial_t F_{k-1}(t, \tau, r, v_r) \\ &\quad + \sum_{j=0}^{k-1} \left[\frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} \sin(\sigma) \mathcal{E}_j(t, \sigma + \tau, r \cos \sigma + v_r \sin \sigma) \\ \cos(\sigma) \mathcal{E}_j(t, \sigma + \tau, r \cos \sigma + v_r \sin \sigma) \end{pmatrix} d\sigma \right. \\ &\quad \left. \cdot \begin{pmatrix} \partial_r F_{k-1-j}(t, \tau, r, v_r) \\ \partial_{v_r} F_{k-1-j}(t, \tau, r, v_r) \end{pmatrix} \right]. \end{aligned} \quad (4.34)$$

Hence we can extend the main result of [10] to the k -th order:

Theorem 4.13. *We assume that the hypotheses of Theorem 4.10 and Hypotheses 4.11 and 4.12 are satisfied for a fixed $k \in \mathbb{N}^*$. In addition, taking s' such that $\frac{1}{s'} = 1 - \frac{1}{q} - \frac{1}{r}$ with $r \in [1, \frac{2q}{2-q}[$ and defining $X^{s'}(K; r dr dv_r) = (W^{1,q}(K; r dr dv_r))' \cup (W^{1,s'}(K; r dr dv_r))$, we assume that, for any compact subset $K \subset \mathbb{R}^2$,*

- $W_k \in L^\infty \left(0, T; L^\infty_{\#} (0, 2\pi; L^2(K; r dr dv_r)) \right)$,
- $\partial_t W_k, R_{k-1} \in L^\infty \left(0, T; L^\infty_{\#} \left(0, 2\pi; X^{s'}(K; r dr dv_r) \right) \right)$.

Then, if the sequence $(f_{\epsilon,k})_{\epsilon>0}$ defined by

$$f_{\epsilon,k}(t, r, v_r) = \frac{1}{\epsilon} \left(f_{\epsilon,k-1}(t, r, v_r) - F_{k-1} \left(t, \frac{t}{\epsilon}, r, v_r \right) \right), \quad (4.35)$$

is bounded independently of ϵ in $L^\infty(0, T; L^2_{loc}(\mathbb{R}^2; r dr dv_r))$, it two-scale converges to the profile $F_k = F_k(t, \tau, r, v_r)$ in $L^\infty(0, T; L^\infty_{\#}(0, 2\pi; L^2(\mathbb{R}^2; r dr dv_r)))$ with

$$F_k(t, \tau, r, v_r) = G_k(t, r \cos \tau - v_r \sin \tau, r \sin \tau + v_r \cos \tau) + W_k(t, \tau, r \cos \tau - v_r \sin \tau, r \sin \tau + v_r \cos \tau), \quad (4.36)$$

where W_k is defined in (4.33) and where $G_k = G_k(t, r, v_r) \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^2; r dr dv_r))$ is the solution of

$$\left\{ \begin{array}{l} \partial_t G_k(t, r, v_r) + \mathcal{J}_1(\mathcal{E}_0)(t, r, v_r) \partial_r G_k(t, r, v_r) + \mathcal{J}_2(t, r, v_r) \partial_{v_r} G_k(t, r, v_r) \\ = -\frac{1}{2\pi} \int_0^{2\pi} \partial_t W_k(t, \tau, r, v_r) d\tau \\ + \frac{1}{2\pi} \sum_{i=0}^k \int_0^{2\pi} \sin(\tau) \mathcal{E}_i(t, \tau, r \sin \tau + v_r \sin \tau) \partial_r W_{k-i}(t, \tau, r, v_r) d\tau \\ - \frac{1}{2\pi} \sum_{i=0}^k \int_0^{2\pi} \cos(\tau) \mathcal{E}_i(t, \tau, r \sin \tau + v_r \sin \tau) \partial_{v_r} W_{k-i}(t, \tau, r, v_r) d\tau \\ - \sum_{i=1}^k \mathcal{J}_1(\mathcal{E}_i)(t, r, v_r) \partial_r G_{k-i}(t, r, v_r) \\ - \sum_{i=1}^k \mathcal{J}_2(\mathcal{E}_i)(t, r, v_r) \partial_{v_r} G_{k-i}(t, r, v_r), \\ G_k(t=0, r, v_r) = 0. \end{array} \right. \quad (4.37)$$

5 Conclusions and perspectives

We have proposed some two-scale convergence results for a particular kind of convection equations in which a part of the convection term presents some high order frequency oscillations in time. These results can be viewed as an improvement of the calculations done by Frénod, Raviart and Sonnendrücker in [13] since the properties of the convection terms \mathbf{A}_ϵ and \mathbf{L} are less restrictive: indeed, in the present paper, the two-scale convergence can be proved with \mathbf{A}_ϵ which depends on ϵ and with \mathbf{L} which can also depend on ϵ in some particular sense. Along with these results, we have described the list of required hypotheses on $(\mathbf{A}_\epsilon)_{\epsilon>0}$ and \mathbf{L} for reaching the k -th order of two-scale convergence for $(u_\epsilon)_{\epsilon>0}$. Finally, we have applied these new results to three different rescaled linear Vlasov equations which are used in the context of MCF or charged particles beams. The limit systems which have been obtained consolidate the existing results and complete them by proposing a k -th order two-scale limit model.

From a numerical point of view, these new informations can be used for enriching the two-scale numerical methods which are currently based on the resolution of the 0-th order limit model: in particular, the limit model presented in Theorem 4.10 is discretized for approaching the solution of (4.28) but these numerical experiments are relevant for $\epsilon \ll 1$ (see [14, 29]). Combining this approach with the numerical resolution of higher order two-scale limit models like (4.37) may provide some relevant numerical results for values of ϵ which are less close to 0.

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