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Approximate hedging with proportional transaction costs in stochastic volatility models with jumps

Thai Huu Nguyen · Serguei Pergamenschchikov

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Abstract We extend the results for the problem of option replication under proportional transaction costs in [23] to more general frameworks where stochastic volatility and jumps are combined to capture market's important features. In particular, we study the hedging error due to discrete readjustments by applying the Leland adjusting volatility principle to compensate transaction costs. In such contexts, jumps risk is approximately eliminated and the results established in [23] are recovered.

Keywords transaction costs · jump models · stochastic volatility · approximate hedging · theorem limit · super-hedging · quantile hedging

Mathematics Subject Classification (2010) 91G20 · 60G44

JEL Classification G11, G13

1 Introduction

Many suggested mathematical models for stock prices have been trying to capture important markets features, e.g. *leptokurtic feature*, *volatility clustering effect*, *implied volatility smile*. These market properties are tractable in stochastic volatility models. However, it is worth noticing that diffusion-based stochastic volatility models, where the market volatility can fluctuate autonomously, can not change suddenly and as a result, they could not take into account sudden and unpredictable market changes. Hence, for a realistic setting, the continuity assumption of stock price should be relaxed. In fact, as discussed in [27], the presence of jumps in the asset price can be recognized as the presence of participants in the option market. As an extension of the famous Black-Scholes framework, it is reasonable to suppose that good or bad news arrive according to a Poisson process. The changes of asset price are described by the jump-sizes and between two jump times, the asset price follows a geometric Brownian motion as in the classical Black-Scholes models. Such a combination is called a jump-diffusion model. As shown in [16], jump-diffusion models not only fit the data better than the classical geometric Brownian motion, but also well reproduce the leptokurtic feature of return distributions. See [27, 26, 16] and the references therein for detailed discussions.

It is well-known that in complete diffusion models, options can be completely replicated using the delta strategy adjusted continuously. However, it is not the case for models with jumps. In fact, jumps risk can not be released completely even using continuous time strategies and the only way to hedge perfectly an option against jumps is to buy and hold the underlying asset. In other words, the conception of replication does not indicate

T. Nguyen

Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, France E-mail: thaibopy@gmail.com; huu.nguyen@etu.univ-rouen.fr

S. Pergamenschchikov

Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS-Université de Rouen, France and National Research University - Higher School of Economics, Laboratory of Quantitative Finance, Moscow, Russia, E-mail: serge.pergamenschchikov@univ-rouen.fr

a right framework for risk management and hedging in the presence of jumps as in diffusion-based complete market models where Black-Scholes theory plays a central role. Therefore, discrete hedging for jump models is practically important and asymptotic properties of the hedging error are not a trivial task to show.

The situation becomes more challenging if one takes into account transaction costs which are needed for hedging activities in practice. Such a consideration is realistic and has been attractive to researchers for last years. In the absence of jumps, Leland [17] proposed a simple method to compensate trading costs, which is a modification of the well-known Black-Scholes PDE where volatility is artificially increased. Kabanov and Safarian [14] showed later that the Leland suggestion for constant transaction cost is not mathematically correct and hedging error in fact converges to a non-zero limit as the portfolio is frequently revised. The rate of convergence then investigated by Pergamenschikov [25] including a characterization of asymptotic distribution of the replication error. Many extensions have been made for different directions: studying the problem for option with general payoffs, using non-uniform rebalancing grid to accelerate the rate of convergence [18, 19, 5], considering the problem in more general models e.g. local volatility [19], trading costs based on the traded number of asset [19]. Recently, Nguyen and Pergamenschikov have studied the problem in stochastic volatility frameworks using a simpler form for adjusted volatility. It turns out that increasing volatility principle is still helpful for controlling losses caused by trading costs which are proportional to the trading volume (measured in dollar value or in physical number of traded asset). In fact, this has a simple connection to asset hedging in high frequency markets where the form of ask-bid price may be an essential factor for deciding laws of trading costs. We refer the reader to [14, 19, 25, 23] and the references therein for Leland's approach.

To the best of our knowledge, there is few study about the trading costs for models with jumps following the Leland spirit in discrete time setting. In fact, it is very intuitive to think that in the presence of transaction costs and even also jumps in the asset price, the option is more risky and should be evaluated at higher price than that in the absence of these risks. However, a more expensive option price would be equivalent to say that there has been some increase in its volatility values. That is the essential intuition behind the Leland algorithm.

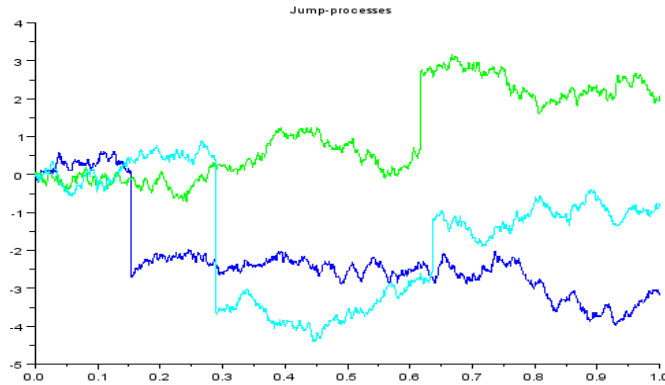


Fig. 1 Jump-diffusion paths with Gaussian jumps and intensity $\theta = 3$ in $[0, 1]$.

The aim of the present note is to build an extension of the achievement in [23] where a simple modification of Leland's algorithm [17] is suggested to take into account constant proportional transaction costs in stochastic volatility models. In fact, we try to capture not only the dependence structure (using stochastic volatility) but also short term behaviors of the stock price due to sudden market changes as well¹. This combination is expected to represent a market general enough for some practice purposes since stochastic volatility models well complement models with jumps [16]. In particular, we show that the impact of jumps can be partially negligible under some mild condition on jump sizes. It turns out that in the presence of both jumps in price and transaction costs, the

¹ This fact partially explains why jump-diffusion models are, in general, considered as a good choice, especially in short-term situations.

option becomes more expensive to hedge and super-hedge can be reached. In fact, asymptotic distribution of the hedging error is independent of jumps and consistent with those established in [23]. More interesting, the same thing is true when jumps are allowed in both asset price and its volatility. Such general frameworks provide the ability to explain large movements in volatility, which happen during crisis periods [7, 8].

The remainder of this paper is organized as follows. We shortly review the problem of quadratic hedging with jumps in Section 2. Section 3 is devoted to formulate the model and present our main results. General stochastic volatility with jumps will be discussed in Section 4. Proof of main results are reported in Section 5 and some useful Lemmas can be found in the Appendix.

2 Quadratic hedging with jumps: a short review

Estimating the hedging error in sense of the mean square means that for a given initial capital V_0 , we look for a self-financing strategy γ that minimizes the expectation value of the squared hedging error

$$\inf_{\gamma \in \mathcal{G}} \mathbf{E}(V_T - H)^2. \quad (2.1)$$

Let us suppose that the underlying asset follows a risk neutral dynamics $dS_t = S_t dZ_t$, where Z_t is a martingale Lévy process $dZ_t = \sigma dW_t + \int_{-\infty}^{\infty} z \tilde{J}_Z(dt \times dz)$ with diffusion coefficient $\sigma > 0$ and the compensated Poisson measure $\tilde{J}(dt \times dz)$. In this section, interest rate is assumed to be zero hence, option price is given by $C(t, S) = \mathbf{E}[h(S_T) | \mathcal{F}_t]$. Using Itô's formula one can determine the hedging error $\varepsilon(V_0, \gamma) = V_T(\gamma) - h(S_T)$ as the following

$$\int_0^T (\gamma - C_S(t, S_{t-})) \sigma dW_t + \int_0^T \int_{-\infty}^{\infty} \mathbf{B}(t, S_{t-}, z) \tilde{J}_Z(dt \times dz). \quad (2.2)$$

where $\mathbf{B}(t, x, z) = zx\gamma_t - C(t, x(1+z)) - C(t, x)$. Taking expectation and differentiating the right hand side of (2.2) one obtains the solution to (2.1)

$$\Delta(t, S_{t-}) = \tilde{\sigma}^{-2} \left(\sigma^2 C_S(t, S_{t-}) + \frac{1}{S_{t-}} \int_{-\infty}^{\infty} z (C(t, S_{t-}(1+z)) - C(t, S_{t-})) v_Z(dz) \right), \quad (2.3)$$

where $\tilde{\sigma}^2 = \sigma^2 + \int_{-\infty}^{\infty} z^2 v_Z(dz)$ and v_Z is the associated Lévy measure of Z . If $v_Z = 0$ we have the classical delta strategy as in Black-Scholes's model and the option is almost surely covered in this case. Nevertheless, when jumps are present it can be learned from (2.3) that jump risks can not be hedged out by simply taking the derivative of the option price as in [22].

Let us make a deeper view on strategy $\Delta(t, S_{t-})$. For simplicity, assume that there is a single jump size, i.e. $v = \delta_a$ for some constant a . Put $\alpha = a^2(\sigma^2 + a^2)^{-1}$ then, $\Delta(t, S_{t-}) = (1 - \alpha)C_S(t, S_{t-}) + \alpha \phi_t(t, S_{t-})$, where $\phi_t = (C(t, S_{t-}(1+a)) - C(t, S_{t-}))a^{-1}S_{t-}^{-1}$. In other words, the optimal quadratic hedging strategy is a linear combination of the well-known Merton strategy $C_S(t, S_{t-})$ and the jump-type strategy $\phi_t(t, S)$. If the payoff is not affine (in most of realistic cases) then $\phi_t \neq 0$. It is easy to see that $\Delta(t, S_{t-})$ converges to the Merton strategy as $\alpha \rightarrow 0$.

In summary, the Merton strategy is not the optimal in the presence of jumps. Jump risk can not be hedged completely even in continuous trading. Thus, quadratic hedging would be realistic in models with jumps but as we have seen above the variance $\mathbf{E}|\varepsilon(V_0, \gamma)|^2$ is computed under a risk-neutral measure. This measuring criterion is not very natural since the risk-neutral measure does not represent statistical description of market events and profit/loss of a portfolio may have a large variance while its risk-neutral variance can be small. Therefore, choosing a minimal martingale measure is important for such situations [27].

2.1 Discrete hedging and jumps

If jumps are allowed to appear in the stock price then one should distinguish two types of hedging errors: one is due to market incompleteness concerning jumps and the other one is due to the discrete nature of the hedging portfolio. These two types of hedging errors have different behaviors. In fact, as shown above that jump risks are not covered by simply using the classical delta strategy even with a continuous time policy. In other words, in the presence of jumps delta hedging is no longer optimal. Remark that the second one is generally related to the appearance of the diffusion term.

The literature of discrete hedging with jumps is vast and we only mention to [29, 30] for recent achievements. Seemingly, none of these mentioned papers discussed about trading costs. We conclude the section by noting that when both jumps and transaction costs are present, the problem of discrete hedging is of course more challenging to handle. We will see in Section 3 that these two risks can be partially controlled by following the usual discrete delta strategy obtained from the well-known Black-Scholes formula with an artificially increased volatility.

3 Model and main results

3.1 The market model

Let $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbf{P})$ be the standard filtered probability space with two standard independent $(\mathcal{F}_t)_{0 \leq t \leq 1}$ adapted Wiener processes $(W_t^{(1)})$ and $(W_t^{(2)})$ taking values in \mathbb{R} . Consider a financial market consisting of one non-risky asset set as the *numeraire* and the risky one (e.g. *stock*) S_t being assumed to take jumps at random times $(\tau_j)_{j \geq 1}$. Relative changes in value of the risky process at the jump time τ_j are characterized by the sequence of i.i.d. random variables $(\xi_j)_{j \geq 1}$. The jumps-size process is then defined by

$$\xi_t = \xi_0 1_{\{0\}} + \sum_{j \geq 1} \xi_{\tau_j} 1_{(\tau_j, \tau_{j+1}]}(t), \quad \xi_0 = 0. \quad (3.1)$$

We assume further that between the jumps times, S_t follows the classical geometric Brownian motion but with stochastic volatility. More precisely, we suppose that the risky asset dynamics is driven by the following equations

$$\begin{cases} dS_t = S_{t-} \left(b_t dt + \sigma(y_t) dW_t^{(1)} + d\zeta_t \right), \\ dy_t = \alpha_1(t, y_t) dt + \alpha_2(t, y_t) (\rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)}), \end{cases} \quad (3.2)$$

where $\zeta_t = \sum_{j=1}^{N_t} \xi_j$ is the jump part characterized by the Poisson process N_t with intensity parameter θ and $S_{t-} = \lim_{s \uparrow t} S_s$. We assume that the coefficients $\alpha_i, i = 1, 2$ are locally Lipschitz and linearly growth functions, which provide the existence of the unique strong solution y to the second equation [10].

In this paper, all sources of jump randomness mentioned above: two Brownian motions, the Poisson process N_t and the jumps sizes $(\xi_j)_{j \geq 1}$ are assumed to be independent. Note that limiting results of replication error will change only on its asymptotic distribution if the two Brownian motions are correlated, see [23].

Let us explain how the asset fluctuates. In fact, it can be observed that the risky asset price S_t changes continuously most of the time, but large jumps may occur from time to time. This important fact can not be adequately taken into account in the classical Black-Scholes context or other pure diffusion-type models. Moreover, it is easy to see that system (3.2) has the unique solution

$$S_t = S_0 \exp \left\{ \int_0^t b_s ds + \int_0^t \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma^2(y_s) ds + \sum_{j=1}^{N_t} \ln(1 + \xi_j) \right\}, \quad (3.3)$$

which means that the stock price is simply a product of the geometric Brownian motion with stochastic volatility $S_0 \exp \left\{ \int_0^t \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma^2(y_s) ds + \int_0^t b_s ds \right\}$ and the independent jump part $\prod_{j=1}^{N_t} (1 + \xi_j)$. The jump sizes should satisfy the natural condition

$$\xi_j > -1, \text{ for all } i = 1, 2, \dots \quad (3.4)$$

to guarantee the positivity of the stock price. We easily observe that at the jump time τ_j the relative change in value of S is given by $\Delta S_{\tau_j} = S_{\tau_j^-} (\xi_j + 1)$. Let $J(dt \times dz)$ be the random Poisson measure generated by the compound Poisson process ζ_t , that is

$$J([0, t] \times A) = \sum_{j \geq 1} \mathbf{1}_{\{\tau_j \leq t, \xi_j \in A\}}. \quad (3.5)$$

It is well known that $J(dt \times dz)$ is a σ -finite jump measure whose Lévy measure (intensity measure) is defined by $\nu = \theta F(dz)dt$, where $F(\cdot)$ is the common distribution of jump sizes (ξ_j) . The Lévy measure can be interpreted as the average number of jumps per unit of time. For convenience, we use the notation $\nu(dz) = \theta F(dz)$ for its density and denote the compensated stochastic Poisson measure by $\tilde{J}(dt \times dz) = J(dt \times dz) - \nu(dz)dt$. The jump measure $J(dt \times dz)$ permits to define the stochastic Poisson integral (integral with jumps)

$$\int_0^t \int_{\mathbb{R}} f(s, z) J(ds \times dz) = \sum_{j=1}^{N_t} f(\tau_j, Z_j), \quad (3.6)$$

for any predictable process $f(t, \xi_t)$. Adapting from the pricing principle we assume that the drift $b = -\theta \mathbf{E}\xi_1$ so that the stock price is now a local martingale given by

$$dS_t = S_{t-} \left(\sigma(y_t) dW_t^{(1)} + \int_{\mathbb{R}} z d\tilde{J}(dt \times dz) \right). \quad (3.7)$$

Remark 1 In this paper we do not discuss about the problem of change of measure and jump risk but accept the free-risk assumption of asset dynamics as the starting point. Clearly, jump-diffusion suggestion leads to an incomplete market, which is also an important feature of stochastic volatility settings. Hence, there are many ways to choose the pricing measure throughout the Girsanov technique. Such a procedure makes an essential change not only on the diffusion but also on the jump part of the asset dynamics [27, 22]. In [16], the rational expectations equilibrium is used to obtain a simple transform from the original physical probability to a risk-neutral probability so that many assets (bonds, stocks, derivatives on stocks) can be simultaneously priced in the same framework.

3.2 Assumptions and examples

The following condition on jump sizes is accepted in our consideration:

(C₁) *The common distribution F of jump sizes satisfies*

$$\int_{\mathbb{R}} z^2 F(dz) < \infty \quad \text{and} \quad \int_{-1}^0 \frac{1}{1+z} F(dz) < \infty.$$

The first integrability condition is nothing than the condition of finite variance for jump size distribution while the second one is equivalent to $\mathbf{E}(1 + \xi)^{-1} < \infty$. These are quite weak constraints and automatically fulfilled in the Merton jump-diffusion model [22] where jump size distribution is assumed to be log-normal. In [16], within an equilibrium-based setting, log-exponential distributions are suggested for jump sizes to obtain the convenient feature in analytical calculation. Again, this family of jump size distributions verifies condition (C₁).

Let us turn our attention to volatility assumption. As in [23], we assume that the volatility process satisfies the integrable condition.

(C₂) *The function σ is twice continuously differentiable such that*

$$0 < \sigma_{\min} \leq \inf_{y \in \mathbb{R}} \sigma(y) \quad \text{and} \quad \sup_{0 \leq t \leq 1} \mathbf{E} \max\{\sigma(y_t), |\sigma'(y_t)|\} < \infty.$$

In fact, condition (C₂) is fulfilled for almost of the famous stochastic volatility models, see [23] for more discussions.

Remark 2 It is important to note that in the present setting, the combination of stochastic volatility and jumps means that the asset price is not a Lévy process but a semi-martingale. As mentioned in [23], a general dynamics of volatility process do not insure the integrability of moments of the asset process [1,20]. This crucial feature prevent us from making a L^2 -based approximation as in deterministic volatility models [14,18,19]. Therefore, for approximation procedures, we follow the approach in [23,24] based on a truncation technique and convergence results obtained are guaranteed in probability.

We conclude this subsection with some well-known stochastic volatility models with jumps, see [27] and Section 4 for more examples.

The Bates models: The Bates models is a jump-diffusion stochastic volatility models obtained by adding proportional log-normal jumps to the Heston stochastic volatility model:

$$dS_t = S_t(\mu dt + \sqrt{y_t}dW_t^S + dZ_t); \quad dy_t = a(m - y_t)dt + b\sqrt{y_t}dW_t^y, \quad (3.8)$$

where W^S, W^y are Brownian motions with correlation ρ and Z is a compound Poisson process with intensity θ and log-normal distribution. Condition (C_1) is clearly verified since jumps follow the log-normal law. Bates's models exhibit some very nice properties from a practical point of view. Firstly, the characteristic function of the log-price is available in a closed-form, which is important for pricing purposes. Secondly, the implied volatility pattern for long term and short term options can be adjusted separately [27].

Ornstein-Uhlenbeck stochastic volatility models: It is possible to introduce a jump component in both price and volatility processes. Such models are suggested by Barndorff-Nielsen and Shephard to take into account leverage effect:

$$S_t = S_0 \exp X_t; \quad dX_t = (\mu + \beta \sigma_t^2)dt + \sigma_t dW_t + \rho dZ_t; \quad d\sigma_t^2 = -\theta \sigma_t^2 dt + dZ_t. \quad (3.9)$$

If $\rho = 0$ the volatility moves with jumps but the price process has continuous paths. The case $\rho \neq 0$, representing a strong correlation between volatility and price, provides the model flexibility but computation is now challenging. Remark that in this case σ is not the "true" volatility since the returns are also affected by changes of the Lévy process Z_t . If jumps still follow log-normal law then condition (C_1) is fulfilled.

3.3 Main Results

We study the problem of discrete hedging in friction contexts using the increasing volatility principle as in Leland's algorithm. A brief review on this literature can be found in [23]. See more in [25,15,19] for related results. In the present framework, we follow the setting in [23]. More precisely, we suppose that for each successful trade, traders are charged by a cost that is proportional to the trading volume with the cost coefficient κ . Here κ is a positive constant defined by market moderators. Let us suppose that the investor plans to revise his portfolio at dates (t_i) defined by

$$t_i = g(i/n), \quad g(t) = 1 - (1-t)^\mu. \quad (3.10)$$

To compensate transaction costs caused by hedging activities, the option seller is suggested to follow the Leland strategy defined by the piecewise process

$$\gamma_t^i = \sum_{i=1}^n \hat{C}_x(t_{i-1}, S_{t_{i-1}}^-) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad (3.11)$$

where \hat{C} is the solution to the following adjusted-volatility Black-Scholes PDE

$$C_t(t, x) + \frac{1}{2} \hat{\sigma}_t^2 x^2 C_{xx}(t, x) = 0, \quad 0 \leq t < 1; \quad C(1, x) = h(x) := (x - K)_+, \quad (3.12)$$

Here the adjusted volatility $\hat{\sigma}^2$ is given by

$$\hat{\sigma}^2(t) = \rho \sqrt{nf'(t)} \quad \text{with} \quad f = g^{-1}. \quad (3.13)$$

Motivation of this simple form is discussed in Remark 3, see more in [23]. Now, using strategy γ_t^n requires a cumulative trading volume measured in dollar value given by $\Gamma_n = \sum_{i=1}^n S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n|$. Thanks to Itô's lemma one represents the payoff as

$$\begin{aligned} h(S_1) &= \widehat{C}(0, S_0) + \int_0^1 \widehat{C}_x(t, S_{t-}) dS_t + \int_0^1 \left(\widehat{C}_t(t, S_t) + \frac{1}{2} \sigma^2(y_t) S_{t-}^2 \widehat{C}_{xx}(t, S_{t-}) \right) dt \\ &\quad + \sum_{0 \leq t \leq 1} \left(\widehat{C}(t, S_t) - \widehat{C}(t, S_{t-}) - \Delta S_t \widehat{C}_x(t, S_{t-}) \right), \end{aligned}$$

where $\Delta S_t = S_t - S_{t-}$ is the jump size of stock price at time t . The last sum of jumps can be represented as $\int_0^1 \int_{\mathbb{R}} \mathbf{B}(t, S_{t-}, z) J(dt \times dz)$, where

$$\mathbf{B}(t, x, z) = \widehat{C}(t, x(z+1)) - \widehat{C}(t, x) - zx \widehat{C}_x(t, x) \quad (3.14)$$

with the jump measure $J(dt \times dz)$ defined by (3.6). Define then $I_{3,n} = \int_0^1 \int_{\mathbb{R}} \mathbf{B}(t, S_{t-}, z) J(dt \times dz)$. Assuming that the initial capital (option price) is given by $V_0^n = \widehat{C}(0, S_0)$ and using (3.12), one represents² the hedging error as

$$V_1^n - h(S_1) = \frac{1}{2} I_{1,n} + I_{2,n} - I_{3,n} - \kappa \Gamma_n, \quad (3.15)$$

where $I_{1,n} = \int_0^1 (\widehat{\sigma}_t^2 - \sigma^2(y_t)) S_{t-}^2 \widehat{C}_{xx}(t, S_{t-}) dt$ and $I_{2,n} = \int_0^1 (\gamma_t^n - \widehat{C}_x(t, S_{t-})) dS_t$.

The goal now is study asymptotic property of the replication error $V_1^n - h(S_1)$. To describe asymptotic properties, let us introduce the following functions

$$\mathbf{v}(\lambda, x) = \frac{\ln(x/K)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2}, \quad q(\lambda, x) = \frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \quad \text{and} \quad \widetilde{\varphi}(\lambda, x) = \varphi(\mathbf{v}(\lambda, x)), \quad (3.16)$$

where φ is the standard normal density function. As shown below, the rate of convergence of our approximation will be controlled by the parameter β defined by

$$\frac{1}{4} \leq \beta := \frac{\mu}{2(\mu+1)} < \frac{1}{3}, \quad \text{for } 1 \leq \mu < 2. \quad (3.17)$$

Clearly, the closer to 2 this parameter is, the more rapidly the hedging error converges to its limit which is a non-zero value.

Before stating our result let us emphasize that using an enlarged volatility which diverges to infinity implies that asymptotic property of hedging error strongly depends on trading times near by the maturity. But remember that jumps are rare events and hence, jumps near by the expiry can be omitted with very small probability. Therefore, jumps in such contexts do not much affect asymptotic property of the hedging error as revisions become more shorter. This important fact proves that increasing volatility as in [23] is still helpful for models with jumps. The below theorems are in fact the achievement of [23] for continuous stochastic volatility models.

Theorem 1 *Under conditions (C₁) – (C₂), the sequence of $n^\beta (V_1^n - h(S_1) - \min(S_{1-}, K) + \kappa \Gamma(S_{1-}, y_1, \rho))$ converges to a centered-mixed Gaussian variable as n tends to infinity, where the positive function Γ is the limit of trading volume defined as*

$$\Gamma(x, y, \rho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \mathbf{E} |\sigma(y) \rho^{-1} Z + q(\lambda, x)| d\lambda, \quad (3.18)$$

in which Z is a standard normal variable independent of S_{1-}, y_1 .

The term $q(\lambda, x)$ in the limit of transaction costs can be removed using the modified Leland strategy, so-called Lépinette's strategy:

$$\widetilde{\gamma}_t^n = \sum_{i=1}^n \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}^-) - \int_0^{t_{i-1}} \widehat{C}_{xt}(t, S_{t-}) dt \right) \mathbf{1}_{(t_{i-1}, t_i]}(t). \quad (3.19)$$

² Recall that for Lebesgue and Itô's integrals one can replace S_{t-} by S_t .

Theorem 2 *Suppose that Lépinette's strategy is used for option replication. Then, under conditions $(C_1) - (C_2)$ the sequence of $n^\beta (\bar{V}_1^n - h(S_1) - \eta \min(S_1, K))$ converges to a centered-mixed Gaussian variable as n tends to infinity, where $\eta = 1 - \kappa \sigma(y_1) \rho^{-1} \sqrt{8/\pi}$.*

Remark 3 If σ is constant then a complete replication can be reached using the classical form for adjusted volatility $\sigma = \sigma^2 + \rho_0 \sqrt{n f'(t)}$, where $\rho_0 = \kappa \sigma \sqrt{8/\pi}$, even when jumps are allowed in the asset price process. This is consistent with [23]. However, the simple choice of $\hat{\sigma}$ defined in (3.13) has two-fold advantage. First, it allows to carry out a much more simple approximation procedure in comparison with the previous works [14, 18, 19, 5]. Second, strategy γ_t^n is always computed easily since it needs a simple modification in the well-known Black-Scholes framework while the one resulting from the use of classical form is not available for investors if volatility is driven by an extra random factor. In fact, by the Black-Scholes formula, it strongly depends on future realizations of the volatility process, which is impossible from a practical point of view, see [23].

3.4 Super-hedging and price reduction

As proved in [23], a suitable choice of ρ can lead to super-replication.

Proposition 1 *Let conditions $(C_1) - (C_2)$ hold and σ be a twice continuously differentiable and bounded function. Then there exists $\rho_* > 0$ such that $\lim_{n \rightarrow \infty} V_1^n \geq h(S_1)$ for any $\rho \geq \rho_*$. This property is true for both of Leland's strategy and Lépinette's strategy.*

From Black-Scholes's formula one observes that both strategy γ_t^n and $\tilde{\gamma}_t^n$ approach to the buy-and-hold one as $n \rightarrow \infty$. This means that option is now very expensive from the buyer's point of view. In fact it is close to the buy and hold price S_0 . In [25, 23] a simple method is suggested to lower the option price following the quantile hedging spirit. Let us adapt the main idea in these works for the present setting. Since $S_{1-} = S_1$ almost surely, we define

$$\delta_\varepsilon = \inf \{ a > 0 : Y(a) \geq 1 - \varepsilon \}, \quad (3.20)$$

where $Y(a) = \mathbf{P}((1 - \kappa) \min(S_1, K) > (1 - a)S_0)$. The quantity δ_ε is called quantile price of the option at level ε and the difference $(1 - \delta_\varepsilon)S_0$ is the reduction amount of option price (initial cost for quantile hedging). Clearly, the smaller value of δ_ε is, the cheaper the option is. We show that the option price is significantly reduced, compared with powers of parameter ε .

Proposition 2 *Let δ_ε be Leland price defined by (3.20) and assume that the jump sizes are almost surely non-negative, i.e.*

$$\xi_j \geq 0, \quad a.s. \quad \forall j \in \mathbb{N} \quad (3.21)$$

and $\sigma_{\max} = \sup_{y \in \mathbb{R}} \sigma(y) < \infty$. Then, for any $r > 0$,

$$\lim_{\varepsilon \rightarrow 0} (1 - \delta_\varepsilon) \varepsilon^{-r} = +\infty. \quad (3.22)$$

Proof Observe that $0 < \delta_\varepsilon \leq 1$ and δ_ε tends to 1 as $\varepsilon \rightarrow 0$. Set $b = 1 - \kappa$. Then for sufficiently small ε such that $\delta_\varepsilon > a > 1 - bK/S_0$ one has

$$1 - \varepsilon > \mathbf{P}(\min(S_1, K) > (1 - a)S_0) = 1 - \mathbf{P}(S_1/S_0 \leq (1 - a)).$$

Therefore, $\varepsilon < \mathbf{P}(S_1/S_0 \leq (1 - a)(1 - \kappa)^{-1}) = \mathbf{P}(P_t(\xi) \mathcal{E}_1(y) \leq z_a)$,

$$\varepsilon < \mathbf{P}(S_1/S_0 \leq (1 - a)(1 - \kappa)^{-1}) = \mathbf{P}(P_t(\xi) \mathcal{E}_1(y) \leq z_a), \quad (3.23)$$

where $z_a = (1 - a)(1 - \kappa)^{-1} e^{\lambda \mathbf{E} \xi_1}$ and

$$\mathcal{E}_t(\sigma) = \exp \left\{ \int_0^t \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma^2(y_s) ds \right\} \quad \text{and} \quad P_t(\xi) = \prod_{j=1}^{N_t} (1 + \xi_j). \quad (3.24)$$

By (3.21), $P_t(\xi) \geq 1$ for all $t \in [0, 1]$, which implies that the probability in the right side of (3.23) is bounded by $\mathbf{P}(\mathcal{E}_1(y) \leq z_a)$. Therefore, $\varepsilon < \mathbf{P}(\mathcal{E}_1(y) \leq z_a)$. At this point, the conclusion exactly follows from Proposition 4.2 in [23] and the proof is completed. \square

4 General stochastic volatility models with jumps

4.1 Introduction

Stochastic volatility with jumps (SVJ) models have been very popular in the option pricing literature since they provide flexibilities to capture important features of returns distribution. However, empirical studies [7, 8] show that they do not well reflex large movements in volatility assets during periods of market stress such as those in 1987, 1997, 2008. In other words, SVJ are misspecified for such purposes. The studies also suggest that it would be more reasonable to add an extra component into the volatility dynamics so that this new factor allows volatility to rapidly increase. Note that such expected effect can not be generated by only using jumps in returns (as in jump-diffusion models) nor by using diffusive stochastic volatility. In fact, jumps in returns can only create large movements but they do not have future impact on returns volatility. On the other hand, diffusive stochastic volatility driven by a Brownian motion only generates small increase via sequences of small normal increments. Empirical analyses in important works [7, 8] show that incorporating jumps in stochastic volatility can provide rapid changes in volatility.



Fig. 2 Implied volatility of EUR/USD, source from marketpulse.com.

It is important to note that introducing jumps in volatility does not mean an elimination of jumps in returns. Although jumps both in returns and volatility are rare, each of them plays an important part in generating crash-like movements. In crisis periods, jumps in returns and in volatility are more important factors than the diffusive stochastic volatility in producing large increases. We refer the reader to [7, 8] for more influential discussions about financial evidence for motivation of the use of jumps in volatility.

In this section, we study the problem of option replication under transaction costs in a general SV models with jumps in return as well as in volatility, which is clearly a generalization of the setting in Section 3. In such contexts, jumps in volatility can be also ignored as those in asset price, i.e. the results obtained in Section 3 are recovered when jumps are allowed in both asset price and volatility.

4.2 Specifications of SV models with jumps

Assume that under the objective probability measure, the dynamics of stock prices S are assumed to be given by

$$dS_t = S_{t-} \left(b_t(y_{t-}) dt + \sigma(y_{t-}) dW_t^{(1)} + d\zeta_t^S \right), \quad dy_t = \alpha_1(t, y_{t-}) dt + \alpha_2(t, y_{t-}) dW_t^{(2)} + d\zeta_t^y. \quad (4.1)$$

Here, ζ_t^S and ζ_t^y are two compound Poisson processes, in particular, for $r \in \{S, y\}$, $\zeta_t^r = \sum_{j=1}^{N_t^r} \xi_j^r$ with jump sizes (ξ^r). For a general setting, two Poisson processes N_t^r and two sequence of jump sizes (ξ_j^r), $r \in \{S, y\}$ can be correlated. For illustration, we give some possible specifications for jump components.

(i) *Stochastic volatility model (SV)*: Clearly, this corresponds to the case when there is no jump in both asset price and volatility, i.e. $\zeta_t^S = \zeta_t^y = 0, \forall t$. This basic SV model has been widely investigated in the literature. The problem of approximate hedging under proportional transaction costs is studied in [23,24]. Roughly speaking, adding some extra component generated by a diffusion to the returns distribution of a classic Black-Scholes setting gives a SV model.

(ii) *Stochastic volatility with jumps in volatility (SVJV)*: By allowing jumps in volatility process y one can obtain an extension of SV models, i.e. $\zeta_t^S = 0, \forall t$ but $\zeta_t^y \neq 0$. In such cases, option pricing implications are in fact inherited from SV models.

(iii) *Stochastic volatility with jumps in price (SVJP)*: Assume now that $\zeta_t^S \neq 0$ but $\zeta_t^y = 0$. This case is studied in Section 3.

(iv) *Stochastic volatility with common jumps in price and volatility (SVCJ)*: Suppose that both asset price and its volatility in a SV model are influenced by the same extra random factor modelled by a compound Poisson process. In other words, jumps in asset price and in volatility are driven by the same compound Poisson process $\zeta_t^S = \zeta_t^y$.

(v) *Stochastic volatility with state-dependent and correlated jumps (SVJJ)*: This is the most general case for the present setting (4.1).

4.3 Option replication with transaction costs in general SVJJ models

In this subsection we study the problem of option replication presented in Section 3 for general SVJJ models (4.1). We show that in the same hedging policy as in SVJ defined in Section 3, jump effects can be ignored in asset as well as in volatility. First, let us recall from Section 3 that the hedging error takes the form $V_1^n - h(S_1) = \frac{1}{2}I_{1,n} + I_{2,n} - I_{3,n} - \kappa\Gamma_n$, where $I_{i,n}$, $i = 1, 2, 3$ and Γ_n are defined as in (3.15). The following conditions on volatility dynamics are needed in this section.

(C₃) *The coefficient functions $\alpha_i, i = 1, 2$ are linearly bounded and locally Lipschitz. Furthermore, the common distribution of jump sizes in volatility F^y admits the integrability condition*

$$\int_{\mathbb{R}} z^2 F^y(dz) < \infty.$$

Condition (C₃) implies that $\sup_{0 \leq t \leq 1} \mathbf{E}y_t^2 < \infty$, which is necessary for approximation procedure.

Theorem 3 *Under conditions (C₁) – (C₂) – (C₃), the limit results in Theorems 1 and 2 still hold.*

5 Proofs

As usual, the main results established in Section 3 are just direct consequences of some specific types of limit theorem for martingales that we are searching for. For this aim, we construct a special approximation procedure following the one in [23]. Our main attempt is to prove that jump terms appearing in the approximation can be neglected at the desired rate n^β . For convenience, we recall in the first subsection the preliminary setup and refer to [23] for the motivation.

5.1 Preliminary

Define $m_1 = n - \left\lceil n(l^*/\lambda_0)^{2/(\mu+1)} \right\rceil$ and $m_2 = n - \left\lceil n(l_*/\lambda_0)^{2/(\mu+1)} \right\rceil$, where the notation $\lceil x \rceil$ stands for the integer part of a number x and $l_* = \ln^{-3} n$, $l^* = \ln^3 n$. Below we focus on the subsequence (t_j) of trading times and the

corresponding sequence (λ_j) defined as

$$t_j = 1 - (1 - j/n)^\mu \quad \text{and} \quad \lambda_j = \int_{t_j}^1 \widehat{\sigma}_u^2 du = \lambda_0 (1 - t_j)^{\frac{1}{4\beta}}, \quad m_1 \leq j \leq m_2. \quad (5.1)$$

Note that (t_j) is an increasing sequence with values in $[t^*, t_*]$, where $t_* = 1 - (l_*/\lambda_0)^{4\beta}$ and $t^* = 1 - (l^*/\lambda_0)^{4\beta}$, whereas (λ_j) is decreasing in $[l_*, l^*]$. Therefore, in the sequel we make use the notation $\Delta t_j = t_j - t_{j-1}$ whereas $\Delta \lambda_j = \lambda_{j-1} - \lambda_j$, for $m_1 \leq j \leq m_2$ to avoid the negative sign in discrete sums. Below, Itô integrals will be discretized throughout the following sequences of independent normal random variables

$$Z_{1,j} = \frac{W_{t_j}^{(1)} - W_{t_{j-1}}^{(1)}}{\sqrt{t_j - t_{j-1}}}, \quad Z_{2,j} = \frac{W_{t_j}^{(2)} - W_{t_{j-1}}^{(2)}}{\sqrt{t_j - t_{j-1}}}. \quad (5.2)$$

We set

$$p(\lambda, x, y) = \frac{\rho}{\sigma(y)} \left(\frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \right). \quad (5.3)$$

and write for short $p_{j-1} = p(\lambda_{j-1}, S_{t_{j-1}^-}, y_{t_{j-1}^-})$. This reduced notation is also frequently applied for functions appearing in the approximation procedure. With the sequence of revision times (t_j) in hand, we consider the centered sequences

$$\begin{cases} Z_{3,j} = |Z_{1,j} + p_{j-1}| - \mathbf{E} \left(|Z_{1,j} + p_{j-1}| \mid \mathcal{F}_{j-1} \right), \\ Z_{4,j} = |Z_{1,j}| - \mathbf{E} \left(|Z_{1,j}| \mid \mathcal{F}_{j-1} \right) = |Z_{1,j}| - \sqrt{2/\pi}. \end{cases} \quad (5.4)$$

The sequences $(Z_{3,j})$ and $(Z_{4,j})$ will serve in finding the Doob decomposition of considered terms. To represent the limit of transaction costs, we introduce the functions

$$\begin{cases} G(a) = \mathbf{E} (|Z + a|) = 2\varphi(a) + a(2\Phi(a) - 1), \\ \Lambda(a) = \mathbf{E} (|Z + a| - \mathbf{E}|Z + a|)^2 = 1 + a^2 - G^2(a), \end{cases} \quad (5.5)$$

for $a \in \mathbb{R}$ and $Z \sim \mathcal{N}(0, 1)$. We also write $o(n^{-r})$ for generic sequences of random variables $(X_n)_{n \geq 1}$ satisfying $\mathbf{P} - \lim_{n \rightarrow \infty} n^r X_n = 0$ while the notation $X_n = O(n^{-r})$ means that $n^r X_n$ is bounded in probability. For approximation analysis, we will make use of the functions

$$\phi(\lambda, x) = \exp \left\{ -\frac{x^2}{2\lambda} - \frac{\lambda}{8} \right\}, \quad \widehat{\phi}(\lambda, x) = \phi(\lambda, \eta(x)) \quad \text{with} \quad \eta(x) = |\ln(x/K)|. \quad (5.6)$$

5.2 Stopping time and technical condition

We first emphasize that in bounded volatility settings, it is possible to carry out an asymptotic analysis based on L^2 estimates as in the previous works [5, 18, 19]. For general stochastic volatility frameworks, this approach is no longer valid because k -th moments of the asset prices S may be infinite for $k > 1$ and S is not in general a martingale, see [1, 20]. We come over this difficulty by using a truncation technique. In particular, for any $L > 0$, we consider the stopping time

$$\tau^* = \tau_L^* = \inf \left\{ t \geq 0 : \mathbf{1}_{\{t \geq t^*\}} \eta_t^{-1} + \bar{\sigma}_t > L \right\} \wedge 1, \quad (5.7)$$

where $\eta_t = \eta(S_t)$ and $\bar{\sigma}_t = \max\{\sigma(y_t), |\sigma'(y_t)|\}$. Note that jumps may be not fully controlled for stopped process $S_{t \wedge \tau^*}$ as in [23]. Therefore, in the presence of jumps we consider its version defined by

$$S_t^* = S_0 \exp \left\{ \int_0^t b_s ds + \int_0^t \sigma_s^* dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma_s^{*2} ds + \sum_{j=1}^{N_t} \ln(1 + \xi_j) \right\}, \quad (5.8)$$

where $\sigma_t^* = \sigma(y_t) \mathbf{1}_{\{\sigma(y_t) \leq L\}}$. Here the dependency on L is dropped for simplicity. Then, it is clear that $S_t^* = S_t$ on the set $\{\tau^* = 1\}$. We easily observe that under condition (C_2) ,

$$\lim_{L \rightarrow \infty} \limsup_{t^* \rightarrow 1} \mathbf{P}(\tau^* < 1) = 0. \quad (5.9)$$

For simplicity, in the sequel we use the notation $\check{S}_u = (S_u, y_u)$. We carry out an approximation procedure for a class of continuously differentiable functions A from $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following technical condition, which is somehow more general than the one proposed in [23].

(H) *A is a continuously differentiable function having absolutely integrable derivative A' with respect to the first argument and*

$$\lim_{n \rightarrow \infty} n^\beta \left(\int_0^{t^*} |A(\lambda, \cdot, \cdot)| d\lambda + \int_{t^*}^{+\infty} |A(\lambda, \cdot, \cdot)| d\lambda \right) = 0.$$

Furthermore, there exist $\gamma > 0$ and a positive continuous function U such that

$$|A(\lambda, x, y)| \leq (\lambda^{-\gamma} + 1)U(x, y)\widehat{\phi}(\lambda, x), \quad (5.10)$$

where $\widehat{\phi}$ defined in (5.6) and

$$\sup_{0 \leq t \leq 1} \mathbf{E} U^4(\check{S}_t^*) < \infty. \quad (5.11)$$

Remark 4 In approximation of hedging error, the function $U(x, y)$ takes the form $\sqrt{x}\widetilde{\sigma}^m(y)$ (up to a multiple constant) where $m \geq 0$ with $\widetilde{\sigma}$ stands for σ or its derivatives σ' . Therefore, for any $L > 0$, condition (5.11) is fulfilled as long as $\sup_{0 \leq t \leq 1} \mathbf{E} S_t^{*2} < \infty$ but this is guaranteed by the condition of finite second moment of jump sizes (C_1) . See Appendix B.

For some positive constant L , we introduce the function

$$g^*(x) = g_L^*(x) = |x| \mathbf{1}_{\{|x| > L^{-1}\}} + L^{-1} \mathbf{1}_{\{|x| \leq L^{-1}\}}. \quad (5.12)$$

Putting $\eta_t^* = g^*(\eta_t)$, one observes that on the set $\{\tau^* = 1\}$,

$$\eta_t^* = L^{-1} \quad \text{and} \quad \widehat{\phi}(\lambda, S_t) = \phi(\lambda, \eta_t^*) = \phi(\lambda, L^{-1}) := \phi_L(\lambda), \quad \text{for all } t^* \leq u < 1. \quad (5.13)$$

5.3 Approximation for stochastic integrals

For the completeness of representation we recall here the asymptotic result established in [23], which serves the central role in the proof of the main results.

Proposition 3 *Let $A(\lambda, x, y)$ be a function such that A and its first partial derivatives $\partial_x A, \partial_y A$ satisfy **(H)**. Then, for $i = 1, 2$,*

$$\int_0^1 \widehat{\sigma}_t^2 \left(\int_t^1 A(\lambda_t, \check{S}_u) dW_u^{(i)} \right) dt = \rho^{-1} \sum_{j=m_1}^{m_2} \bar{A}_{j-1} Z_{i,j} \Delta \lambda_j + o(n^{-\beta}), \quad (5.14)$$

where $\bar{A}_j = \bar{A}(\lambda_j, \check{S}_{t_j^-})$ and $\bar{A}(\lambda, x, y) = \int_\lambda^\infty A(z, x, y) dz$.

Proof We follow the argument used in Proposition 7.1 in [23]. Although we are working under technical condition **(H)** which is slightly different from that in [23] but arguments are in fact the same. For reader's convenience let us give the proof in details since the approximation technique will be repeatedly used in our analysis. First, making use of the stochastic Fubini theorem one gets

$$\widehat{I}_n = \int_0^1 \widehat{\sigma}_t^2 \left(\int_t^1 A(\lambda_t, \check{S}_u) dW_u^{(i)} \right) dt = \int_0^1 \left(\int_0^u \widehat{\sigma}_t^2 A(\lambda_t, \check{S}_u) dt \right) dW_u^{(i)}.$$

Changing the variables $v = \lambda_t$ for the inner integral, we obtain

$$\int_0^u \widehat{\sigma}_t^2 A(\lambda_t, \check{S}_u) dt = \int_{\lambda_u}^{\lambda_0} A(v, \check{S}_u) dv = \bar{A}(\lambda_u, \check{S}_u) - \bar{A}(\lambda_0, \check{S}_u).$$

In other words, $\widehat{I}_n = \widehat{I}_{1,n} - \widehat{I}_{2,n}$, where $\widehat{I}_{1,n} = \int_0^1 \check{A}_u dW_u^{(i)}$, $\check{A}_u = \bar{A}(\lambda_u, \check{S}_u)$ and $\widehat{I}_{2,n} = \int_0^1 \bar{A}(\lambda_0, \check{S}_u) dW_u^{(i)}$. Moreover, we have

$$\widehat{I}_{1,n} = \int_0^{t^*} \check{A}_u dW_u^{(i)} + \int_{t^*}^{t^*} \check{A}_u dW_u^{(i)} + \int_{t^*}^1 \check{A}_u dW_u^{(i)} := R_{1,n} + R_{2,n} + R_{3,n}. \quad (5.15)$$

Let us first show that $\widehat{I}_{2,n} = o(n^{-\beta})$. For any $\varepsilon > 0$, one observes that

$$\mathbf{P}(n^\beta |\widehat{I}_{2,n}| > \varepsilon) \leq \mathbf{P}(n^\beta |\widehat{I}_{2,n}| > \varepsilon, \tau^* = 1) + \mathbf{P}(\tau^* < 1).$$

In view of (5.9), one needs to show that the first probability in the right side converges to 0. Indeed, by **(H)** one has

$$|\bar{A}(\lambda_0, x, y)| \leq CU(x, y) \int_{\lambda_0}^{\infty} e^{-\lambda/8} d\lambda \leq CU(x, y) e^{-\lambda_0/8}.$$

Putting $\check{A}_u^* = \check{A}_u \mathbf{1}_{\{\tau^*=1\}}$ and $\widehat{I}_{2,n}^* = \int_0^1 \check{A}_u^* dW_u^{(i)}$ and making use of the notation $\check{S}^* = (S^*, y)$ one has

$$\mathbf{P}(n^\beta |\widehat{I}_{2,n}| > \varepsilon, \tau_L^* = 1) = \mathbf{P}(n^\beta |\widehat{I}_{2,n}^*| > \varepsilon) \leq \varepsilon^{-2} n^{2\beta} \mathbf{E}(\widehat{I}_{2,n}^*)^2 \leq C \varepsilon^{-2} n^{2\beta} e^{-\lambda_0/8} \sup_{0 \leq t \leq 1} \mathbf{E}U^2(\check{S}_t^*),$$

which converges to zero by condition **(H)**. Hence, $\widehat{I}_{2,n} = o(n^{-\beta})$ as $n \rightarrow \infty$. Next, let us show that $R_{2,n}$ is the main part of $\widehat{I}_{1,n}$. For this aim, taking into account that $l^* \leq \lambda_u \leq \lambda_0$ for $0 \leq u \leq t^*$, we get $R_{1,n} = o(n^{-\beta})$.

Next, let us show the same property for the last term $R_{3,n}$ in (5.15). To this end, note again that

$$\mathbf{P}(n^\beta |R_{3,n}| > \varepsilon) \leq \mathbf{P}(n^\beta |R_{3,n}| > \varepsilon, \tau^* = 1) + \mathbf{P}(\tau^* < 1), \quad (5.16)$$

On the set $\{\tau^* = 1\}$ one has the estimate $|\check{A}_u| \leq U(\check{S}_u^*) \int_{\lambda_u}^{\infty} (1+z^{-\gamma}) \widehat{\phi}(z, S_u^*) dz = U(\check{S}_u^*) \check{f}_u^*$, where $\check{f}_u^* = \int_{\lambda_u}^{\infty} (1+z^{-\gamma}) \phi_L(z) dz$. Again one obtains by the Chebychev inequality

$$\mathbf{P}(n^\beta |R_{3,n}| > \varepsilon, \tau^* = 1) = \mathbf{P}(n^\beta |\widehat{R}_{3,n}| > \varepsilon) \leq n^{2\beta} \varepsilon^{-2} \int_{t^*}^1 \mathbf{E}(\widehat{A}_u^*)^2 du,$$

which is bounded by $n^{2\beta} \varepsilon^{-2} C(U) \int_{t^*}^1 (\check{f}_u^*)^2 du$ with $C(U) = \sup_{0 \leq u \leq 1} \mathbf{E}U^2(\check{S}_u^*) < \infty$. Taking into account that

$$\int_{t^*}^1 (\check{f}_u^*)^2 du = \lambda_0^{-4\beta} \int_0^{l^*} \left(\int_{\lambda}^{\infty} (1+z^{-\gamma}) \phi_L(z) dz \right)^2 d\lambda \leq C \lambda_0^{-4\beta} l_{*},$$

we conclude that $\lim_{n \rightarrow \infty} \mathbf{P}(n^\beta |R_{3,n}| > \varepsilon, \tau^* = 1) = 0$ and hence $R_{3,n} = o(n^{-\beta})$ in view of (5.9).

It remains to discretize the integral term $R_{2,n}$ using the sequence $(Z_{i,j})$. The key steps for this aim are the following. First, we represent $R_{2,n} = \int_{t^*}^{t^*} \check{A}_u dW_u^{(i)} = \sum_{j=m_1}^{m_2} \int_{t_{j-1}^j} \check{A}_u dW_u^{(i)}$, and replace the Itô integral in the last sum with $\bar{A}_{j-1} Z_{i,j} \sqrt{\Delta t_j}$. Next, Lemma 4 allows to substitute $\sqrt{\Delta t_j} = \rho^{-1} \Delta \lambda_j$ into the last sum to obtain the martingale \mathcal{M}_{m_2} defined by

$$\mathcal{M}_k = \rho^{-1} \sum_{j=m_1}^k \bar{A}_{j-1} Z_{i,j} \Delta \lambda_j, \quad m_1 \leq k \leq m_2.$$

We need to show that $\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta |R_{2,n} - \mathcal{M}_{m_2}| = 0$ or equivalently, $\sum_{j=m_1}^{m_2} B_{j,n} = o(n^{-\beta})$, where $B_{j,n} = \int_{t_{j-1}^j} \widetilde{A}_{u,j} dW_u^{(i)}$ and $\widetilde{A}_{u,j} = \bar{A}(\lambda_u, \check{S}_u) - \bar{A}(\lambda_{j-1}, \check{S}_{t_{j-1}^-})$. We show this without using the Itô's formula. For this aim, let $b > 0$ and introduce the set

$$\Omega_b = \left\{ \sup_{t^* \leq u \leq 1} \sup_{z \in \mathbb{R}} (|A(z, \check{S}_u)| + |\partial_x \bar{A}(z, \check{S}_u)| + |\partial_y \bar{A}(z, \check{S}_u)|) \leq b \right\}.$$

Then, for any $\varepsilon > 0$, $\mathbf{P}\left(n^\beta \left| \sum_{j=m_1}^{m_2} B_{j,n} \right| > \varepsilon\right)$ is bounded by

$$\mathbf{P}(\Omega_b^c) + \mathbf{P}(\tau^* < 1) + \mathbf{P}\left(n^\beta \left| \sum_{j=m_1}^{m_2} B_{j,n} \right| > \varepsilon, \Omega_b, \tau^* = 1\right).$$

Note that $\lim_{b \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \mathbf{P}(\Omega_b^c) = 0$ by Lemma 7. In view of (5.9), one needs to prove that the last probability converges to zero. To this end, put $\widehat{A}_{u,j} = \widetilde{A}_{u,j} \mathbf{1}_{\{\widetilde{A}_{u,j} \leq b \delta_{u,j}\}}$ and $\widehat{B}_{j,n} = \int_{t_{j-1}}^{t_j} \widehat{A}_{u,j} dW_u^{(i)}$, where $\delta_{u,j} = |\lambda_u - \lambda_{j-1}| + |S_{u^-}^* - S_{t_{j-1}^-}^*| + |y_{u^-} - y_{t_{j-1}^-}|$. Then, the last probability is equal to $\mathbf{P}\left(n^\beta \left| \sum_{j=m_1}^{m_2} \widehat{B}_{j,n} \right| > \varepsilon\right)$, which is smaller than $\varepsilon^{-2} n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E} \widehat{B}_{j,n}^2$ by the Chebychev inequality. Clearly, $\mathbf{E} \widehat{B}_{j,n}^2$ is bounded by

$$2b^2 \int_{t_{j-1}}^{t_j} ((\lambda_u - \lambda_{j-1})^2 + \mathbf{E}(S_{u^-}^* - S_{t_{j-1}^-}^*)^2 + \mathbf{E}(y_{u^-} - y_{t_{j-1}^-})^2) du \leq C((\Delta \lambda_j)^3 + (\Delta t_j)^2).$$

Consequently, $n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E} \widehat{B}_{j,n}^2 \leq C n^{2\beta} \sum_{j=m_1}^{m_2} (\Delta \lambda_j)^3 + (\Delta t_j)^2$. Taking into account Lemma 4 we conclude that the latter sum converges to 0 hence, the proof is completed. \square

Lemma 1 *Let $\iota(t) = \sup\{t_i : t_i \leq t\}$ and $A(\lambda, x, y)$ is a function satisfying condition **(H)**. Then,*

- (i). $\int_0^1 \left(\int_{\iota(t)}^t \widehat{\sigma}_u^2 A(\lambda_u, \check{S}_{u^-}) du \right) dW_t^{(i)} = o(n^{-\beta}), \quad i = 1, 2,$
- (ii). $\int_0^1 \left(\int_{\iota(t)}^t A(\lambda_u, \check{S}_{u^-}) dW_u^{(i)} \right) dW_t^{(j)} = o(n^{-\beta}), \quad i, j \in \{1, 2\}.$

Proof By assumption, $|A(\lambda, x, y)| \leq U(x, y) \widehat{\phi}(\lambda, x) (1 + \lambda^{-\gamma})$ for some constant γ and positive function $U(x, y)$ verifying (5.11). Denote by \mathbf{r}_n the considered double stochastic integral in (i). Put $\widetilde{A}_t = \int_{\iota(t)}^t \widehat{\sigma}_u^2 A(\lambda_u, \check{S}_{u^-}) du$, we represent \mathbf{r}_n as

$$\mathbf{r}_n = \int_0^{t^*} d\widetilde{A}_t W_t^{(i)} + \int_{t^*}^1 \widetilde{A}_t dW_t^{(i)} := \mathbf{r}_{1,n} + \mathbf{r}_{2,n},$$

We will prove that $\mathbf{r}_{i,n} = o(n^{-\beta}), i = 1, 2$. To this end, let $L > 0$ and consider $\tau^* = \tau_L^*$ defined as in (5.7). For $i = 1, 2$, by $\mathbf{r}_{i,n}^*$ we mean the "corrected" version of $\mathbf{r}_{i,n}$, i.e. S_u, y_u should be replaced by S_u^* and y_u^* respectively in A . Now, for any $\varepsilon > 0$,

$$\mathbf{P}\left(n^\beta |\mathbf{r}_n| > \varepsilon\right) \leq \mathbf{P}\left(n^\beta |\mathbf{r}_n| > \varepsilon, \tau^* = 1\right) + \mathbf{P}(\tau^* < 1), \quad (5.17)$$

Taking into account $\lambda_t \geq t^* \rightarrow \infty$ for $t \in [0, t^*]$ and using Chebychev's inequality, one bounds the first probability in the right side by

$$n^{2\beta} \varepsilon^{-2} \mathbf{E} \mathbf{r}_{1,n}^{*2} = n^{2\beta} \varepsilon^{-2} \int_0^{t^*} \mathbf{E} \widetilde{A}_t^{*2} dt \leq C n^{2\beta} \varepsilon^{-2} \mathbf{E} U^2(\check{S}_{t^-}^*) \int_0^{t^*} \mathbf{b}_t^2 dt,$$

where $\mathbf{b}_t = \int_{\iota(t)}^t \widehat{\sigma}_u^2 (1 + \lambda_u^{-\gamma}) e^{-\lambda_u/8} du$. Recall from (3.13) that

$$\widehat{\sigma}_u^2 = \rho \sqrt{n} (1 - u)^{\frac{1-\mu}{2\mu}} = \rho \sqrt{n} (\lambda_0 / \lambda_u)^{\widehat{\mu}}, \quad \text{with } \widehat{\mu} = (\mu - 1) / (1 + \mu). \quad (5.18)$$

Then, splitting the integral as the sum of integrals on the intervals $[t_{i-1}, t_i]$ and changing variable one gets

$$n^{2\beta} \int_0^{t^*} \mathbf{b}_t^2 dt \leq C n^{2\beta} n^{-2} \int_0^{t^*} \widehat{\sigma}_u^4 (1 + \lambda_u^{-\gamma})^2 e^{-\frac{\lambda_u}{4}} du \leq C n^{2\beta-3/2} \lambda_0^{\widehat{\mu}} \int_0^{t^*} \widehat{\sigma}_u^2 \lambda_u^{-\widehat{\mu}} (1 + \lambda_u^{-\gamma})^2 e^{-\frac{\lambda_u}{4}} du,$$

which is smaller than $C n^{2\beta-3/2} \lambda_0^{\widehat{\mu}} \int_{t^*}^\infty \lambda^{-\widehat{\mu}} (1 + \lambda^{-\gamma})^2 e^{-\frac{\lambda}{4}} d\lambda$. This implies the convergence to zero of the first probability in the right side of (5.17). In view of (5.9), one obtains $\mathbf{r}_{1,n} = o(n^{-\beta})$. Let us prove the same property for $\mathbf{r}_{2,n}$. In fact, the singularity at $t = 1$ requires a more delicate treatment. We make use of the stopping time τ^*

again. Put $\widehat{A}_u^* = A(\lambda_u, \check{S}_u^*) \mathbf{1}_{\{|A_u| \leq U(\check{S}_u^*, \widehat{f}_u^*)\}}$, $\widehat{f}_u^* = (1 + \lambda_u^{-\gamma}) \phi_L(\lambda)$ and $\widehat{\mathbf{r}}_{2,n} = \int_{t^*}^1 \left(\int_{\mathfrak{u}(t)}^t \widehat{\sigma}_u^2 \widehat{A}_u^* du \right) dW_t^{(i)}$. Then, by the Chebychev inequality one gets $\mathbf{P}(n^\beta |\mathbf{r}_{2,n}| > \varepsilon, \tau^* = 1) = \mathbf{P}(n^\beta |\widehat{\mathbf{r}}_{2,n}| > \varepsilon)$. The latter probability is bounded by

$$n^{2\beta} \varepsilon^{-2} \sup_{0 \leq u \leq 1} \mathbf{E} U^2(\check{S}_{u^-}^*) \int_{t^*}^1 \left(\int_{\mathfrak{u}(t)}^t \widehat{\sigma}_u^2 \widehat{f}_u^* du \right)^2 dt \leq C n^{2\beta} \varepsilon^{-2} \int_{t^*}^1 (1(t) - t) \int_{\mathfrak{u}(t)}^t \widehat{\sigma}_u^4 \widehat{f}_u^{*2} du dt := \mathbf{a}_n.$$

On the other hand, for some constant $C_{\varepsilon, \rho}$ independent of n ,

$$\mathbf{a}_n \leq C n^{2\beta - \frac{3}{2}} \varepsilon^{-2} \rho \lambda_0^{\widehat{\mu}} \int_{t^*}^1 \lambda_u^{-\widehat{\mu}} \widehat{\sigma}_u^2 \widehat{f}_u^{*2} du \leq C_{\varepsilon, \rho} n^{\frac{-2}{1+\widehat{\mu}}} \int_0^{t^*} \lambda^{-\widehat{\mu}} (1 + \lambda_u^{-\gamma})^2 \phi_L^2(\lambda) d\lambda,$$

which converges to 0 as $n \rightarrow \infty$. Hence, by taking into account (5.9) one concludes that $\mathbf{P}(n^\beta |\mathbf{r}_{2,n}| > \varepsilon)$ converges to 0. The second equality can be proved by the same way. \square

Lemma 2 *Suppose that $A = A(\lambda, x, y)$ verifies (H). Then, the following asymptotic properties hold in probability:*

- (i). $\int_0^1 \left(\int_0^t A(\lambda_r, \check{S}_r) dW_s^{(i)} \right) dW_t^{(j)} = O(n^{-2\beta})$, $i, j \in \{1, 2\}$.
- (ii). $\int_0^1 A(\lambda_r, \check{S}_{r^-}) dt = O(n^{-2\beta})$.
- (iii). $\int_0^1 \left(\int_r^1 A(\lambda_s, \check{S}_{s^-}) ds \right) dt = O(n^{-4\beta})$.

Proof The procedure used in the proof of Lemma 1 can be applied straightforwardly to obtain the first equality. Indeed, we can check directly that

$$\int_{[0, t^*] \cup [t^*, 1]} \left(\int_0^t A(\lambda_r, \check{S}_{r^-}) dW_s^{(i)} \right) dW_t^{(j)} = o(n^{-2\beta}).$$

Now, consider again the set $\{\tau^* = 1\}$ one can prove that $\int_{t^*}^1 \left(\int_0^t A(\lambda_r, \check{S}_{r^-}^*) dW_s^{(i)} \right) dW_t^{(j)} = O(n^{-2\beta})$ using again the truncation technique hence, (i) is verified. Next, let us prove (iii). By making use of the change of variable $\lambda_r = \lambda_0(1-t)^{1/(4\beta)}$, the double integral is written as

$$\widehat{\varepsilon}_n := \lambda_0^{-8\beta} 16\beta^2 \int_0^{\lambda_0} \lambda^{4\beta-1} \left(\int_\lambda^{\lambda_0} z^{4\beta-1} A(z, \check{S}_{v(z/\lambda_0)^-}) dz \right) d\lambda, \quad v(z) = 1 - z^{4\beta}.$$

By hypothesis, $A(\lambda, x, y)$ is bounded by $U(x, y)(1 + \lambda^{-\gamma}) \widehat{\phi}(\lambda, x)$ for some constant γ and some positive function U verifying (5.11). Hence, $\lambda_0^{8\beta} |\widehat{\varepsilon}_n|$ is bounded (up to a multiple constant) by the double integral

$$\int_0^{\lambda_0} \lambda^{4\beta-1} \left(\int_\lambda^{\lambda_0} z^{4\beta-1} (1 + z^{-\gamma}) U(\check{S}_{v(z/\lambda_0)^-}) \widehat{\phi}(z, S_{v(z/\lambda_0)^-}) dz \right) d\lambda.$$

Let ω outside the set $\{S_{1^-} = K\}$, which has zero probability by Lemma 5. It is clear that the integrand of the above integral is dominated by a continuous function depending on ω , which exponentially decreases to 0 at 0 and infinity hence, it is integrable on $[0, \infty)$. Therefore, the double integral converges to

$$\int_0^\infty \lambda^{4\beta-1} U(\check{S}_1) \left(\int_\lambda^\infty z^{4\beta-1} (1 + z^{-\gamma}) \widehat{\phi}(z, S_{1^-}) dz \right) d\lambda$$

by the dominated convergence theorem. Thus, $n^{4\beta} \widehat{\varepsilon}_n$ is bounded in probability. The equality (ii) is proved by the same way. \square

5.4 Eliminating jumps

In this subsection, we establish asymptotic results which will serve in eliminating jump effects in our approximation.

Lemma 3 *Suppose that*

$$|B(\lambda, x, y, z)| \leq \varpi(z) \psi(\lambda) U(x, y), \quad \text{for all } x > 0, z \in \mathbb{R}, \lambda > 0,$$

where U is a continuous function holding $\sup_{0 \leq t \leq 1} \mathbf{E} U^2(\check{S}_t^*) < \infty$ for any $L > 0$ in the definition of τ^* in (5.7). Suppose furthermore that

$$\int_{\mathbb{R}} \varpi^2(z) \nu(dz) < \infty \quad \text{and} \quad n^r \int_{t^*}^{\infty} \lambda^{4\beta-1} (\psi^2(\lambda) + \psi(\lambda)) d\lambda \rightarrow 0, \quad \text{for any } r > 0. \quad (5.19)$$

Then, for any $r > 0$,

$$\int_0^1 \int_{\mathbb{R}} A(\lambda_t, S_{t^-}, y_{t^-}, z) J(dt \times dz) = o(n^{-r}). \quad (5.20)$$

Proof For notation simplicity, one abbreviates $B(t, z) := A(\lambda_t, S_{t^-}, y_{t^-}, z)$. Let us decompose the integral in (5.20) as

$$\int_0^{t^*} \int_{\mathbb{R}} B(t, z) J(dt \times dz) + \int_{t^*}^1 \int_{\mathbb{R}} B(t, z) J(dt \times dz). \quad (5.21)$$

Clearly, for any $\delta > 0$ and $r > 0$, $\mathbf{P}\left(n^r \left| \int_{t^*}^1 \int_{\mathbb{R}} B(t, z) J(dt \times dz) \right| > \delta\right)$ is smaller than $\mathbf{P}(N_1 - N_{t^*} \geq 1) = 1 - e^{-\theta(1-t^*)}$, which converges to 0. Hence, it suffices to prove the same property for the first integral in (5.21). Indeed, this term can be represented as $\int_0^{t^*} \int_{\mathbb{R}} B(t, z) \tilde{J}(dt \times dz) + \int_0^{t^*} \int_{\mathbb{R}} B(t, z) \nu(dz) dt$. We prove that the compensator is almost surely exponentially negligible, i.e.

$$n^r \int_0^{t^*} \int_{\mathbb{R}} B(t, z) \nu(dz) dt \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty. \quad (5.22)$$

Indeed, by assumption and the change of variable defined in (5.1), it is estimated by

$$C n^r \lambda_0^{-4\beta} \int_{t^*}^{\infty} \lambda^{4\beta-1} \psi(\lambda) U(\check{S}_{t(\lambda)^-}) d\lambda \times \int_{\mathbb{R}} \varpi(z) \nu(dz),$$

which a.s. converges to zero due to (5.19) and the continuity of U , where $t(\lambda) = 1 - (\lambda/\lambda_0)^{4\beta}$. Hence, it remains to prove that for any $r > 0$, $\int_0^{t^*} \int_{\mathbb{R}} B(t, z) \tilde{J}(dt \times dz) = o(n^{-r})$ as $n \rightarrow \infty$. To this end, note that for any $\delta > 0$,

$$\mathbf{P}\left(\left|\int_0^{t^*} \int_{\mathbb{R}} B(t, z) \tilde{J}(dt \times dz)\right| > \delta\right) \leq \mathbf{P}(\tau^* < 1) + \mathbf{P}\left(\left|\int_0^{t^*} \int_{\mathbb{R}} B(t, z) \tilde{J}(dt \times dz)\right| > \delta, \tau^* = 1\right)$$

for any $L > 0$. Denote by ε_n the last probability. In view of (5.9), one needs to show that ε_n converges to 0. In fact, by assumption one also has the following estimate $|B(t, z)| \leq U(\check{S}_{t^-}^*) \psi(\lambda_t) \varpi(z) := \tilde{B}^*(t, z)$ on the set $\{\tau^* = 1\}$. Therefore, applying the isometry for jump integrals yields

$$\varepsilon_n = \mathbf{P}\left(n^r \left|\int_0^{t^*} \int_{\mathbb{R}} B(t, z) \mathbf{1}_{\{|B(t, z)| \leq \tilde{B}^*(t, z)\}} \tilde{J}(dt \times dz)\right| > \delta, \tau^* = 1\right),$$

which is bounded by

$$n^{2r} \delta^{-2} \mathbf{E} \int_0^{t^*} \int_{\mathbb{R}} \tilde{B}^{*2}(t, z) \nu(dz) dt \leq n^{2r} \delta^{-2} \sup_{0 \leq t \leq 1} \mathbf{E} U^2(\check{S}_{t^-}^*) \int_0^{t^*} \psi^2(\lambda_t) dt \times \int_{\mathbb{R}} \varpi^2(z) \nu(dz),$$

and the conclusion follows from (5.19). \square

5.5 Limit theorems for approximations

We first recall the following result in [12], which is extremely useful for studying asymptotic distribution of discrete martingales.

Theorem 4 [Theorem 3.2 and Corollary 3.1, p.58 in [12]] Let $\mathcal{M}_n = \sum_{i=1}^n X_i$ be a zero-mean, square integrable martingale and ζ be an a.s. finite random variable. Assume that the following convergences are satisfied in probability:

$$\sum_{i=1}^n \mathbf{E} \left(X_i^2 1_{\{|X_i| > \delta\}} | \mathcal{F}_{i-1} \right) \longrightarrow 0 \quad \text{for any } \delta > 0 \quad \text{and} \quad \sum_{i=1}^n \mathbf{E} (X_i^2 | \mathcal{F}_{i-1}) \longrightarrow \zeta^2.$$

Then, the sequence (\mathcal{M}_n) converges in law to X whose characteristic function is $\mathbf{E} \exp(-\frac{1}{2} \zeta^2 t^2)$, i.e. X has a Gaussian mixture distribution.

Below we will establish some special versions of Theorem 4. In particular, our aim is to study the asymptotic distribution of discrete martingales resulting from approximation (5.14) in Proposition 3.

Let $A_i = A_i(\lambda, x, y)$, $i \in I := \{1, 2, 3, 4\}$ be functions having property **(H)** and consider discrete martingales $(\mathcal{M}_k)_{m_1 \leq k \leq m_2}$ and $(\bar{\mathcal{M}}_k)_{m_1 \leq k \leq m_2}$ defined as

$$\mathcal{M}_k = \rho^{-1} \sum_{j=m_1}^k \sum_{i \in I \setminus \{4\}} A_{i,j-1} Z_{i,j} \Delta \lambda_j \quad \text{and} \quad \bar{\mathcal{M}}_k = \rho^{-1} \sum_{j=m_1}^k \sum_{i \in I \setminus \{3\}} A_{i,j-1} Z_{i,j} \Delta \lambda_j, \quad (5.23)$$

where $A_{i,j} = A_i(\lambda_j, \check{S}_{j-}^-)$ and $Z_{i,j}$ are defined as in (5.2) and (5.4). To describe the limit distributions let us introduce

$$\mathbf{L} = A_1^2 + 2A_1 A_3 (2\Phi(p) - 1) + A_3^2 \Lambda(p) + A_2^2, \quad \bar{\mathbf{L}} = A_1^2 + A_2^2 + (1 - 2/\pi) A_4^2, \quad (5.24)$$

where p is defined in (5.3). Define now

$$\zeta^2 = \check{\mu} \rho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\hat{\mu}} \mathbf{L}(\lambda, \check{S}_{1-}^-) d\lambda \quad \text{and} \quad \bar{\zeta}^2 = \check{\mu} \rho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\hat{\mu}} \bar{\mathbf{L}}(\lambda, \check{S}_{1-}^-) d\lambda \quad (5.25)$$

with

$$\check{\mu} = \frac{1}{2}(\mu + 1) \tilde{\mu}^{\frac{2}{\mu+1}} \quad \text{and} \quad \hat{\mu} = (\mu - 1)/(\mu + 1). \quad (5.26)$$

Proposition 4 Assume that $A_i = A_i(\lambda, x, y)$, $i = 1, 2, 3$ and their first partial derivatives $\partial_\lambda A_i$, $\partial_x A_i$, $\partial_y A_i$ are functions having property **(H)**. Then, for any fixed $\rho > 0$ the sequence $(n^\beta \mathcal{M}_{m_2})$ weakly converges to a mixed Gaussian variable with mean zero and variance ζ^2 defined as in (5.25). The same property still holds if some (or all) of the functions A_i are replaced by $\int_\lambda^\infty A_i(z, x, y) dz$.

Proof Note that the square integrability property is not guaranteed for the random variables (v_j) . To overcome this issue let us recall the stopping time $\tau^* = \tau_L^*$ defined in (5.7) and put $\tilde{A}_i(\lambda, x, y) = A_i(\lambda, x, y) \hat{\phi}^{-1}(\lambda, x) \phi_L(\lambda)$, where $\phi_L(\lambda)$ defined in (5.12). Let $v_j^* = \sum_{i=1}^3 \tilde{A}_i(\lambda_j, \check{S}_{j-}^*) Z_{i,j} \Delta \lambda_j$ and $\mathcal{M}_k^* = \sum_{j=m_1}^k v_j^*$.

Step 1: We will show throughout Theorem 4 that for any $L > 0$ the martingale $n^\beta \mathcal{M}_{m_2}^*$ weakly converges to a mixed Gaussian variable with mean zero and variance $\zeta^{*2}(L)$ defined as

$$\zeta^{*2}(L) = \check{\mu} \rho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\hat{\mu}} \tilde{\mathbf{L}}(\lambda, \check{S}_{1-}^-) d\lambda, \quad (5.27)$$

where $\tilde{\mathbf{L}}$ is obtained by replacing all A_i in the formula of \mathbf{L} in (5.24) by the corresponding modified functions \tilde{A}_i , $i = 1, 2, 3$. To this end, setting $\mathbf{a}_j^* = \mathbf{E}(v_j^{*2} 1_{\{|v_j^*| > \delta\}} | \mathcal{F}_{j-1})$, we first show that $\mathbf{P}(n^{2\beta} |\sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon)$ converges to 0. By hypothesis,

$$\max_{i=1,2,3} \left| \tilde{A}_i(\lambda_u, \check{S}_{u-}^*) \right| \leq U(\check{S}_{u-}^*) (1 + \lambda_u^{-\gamma}) \phi_L(\lambda) \leq U(\check{S}_{u-}^*) (1 + \lambda_u^{-\gamma}) \quad (5.28)$$

for some $\gamma > 0$ and positive function $U(\check{S})$ verifying (5.11). We observe that

$$\mathbf{P}\left(n^{2\beta} \left| \sum_{j=m_1}^{m_2} \mathbf{a}_j^* \right| > \varepsilon\right) = \mathbf{P}\left(n^{2\beta} \left| \sum_{j=m_1}^{m_2} \mathbf{a}_j^* \right| > \varepsilon\right) \leq \varepsilon^{-1} n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E} \mathbf{a}_j^*$$

by Markov's inequality. Using the Chebychev inequality and then again the Markov inequality one gets

$$\begin{aligned} \mathbf{E} \mathbf{a}_j^* &= \mathbf{E} \left(\mathbf{v}_j^{*2} 1_{\{|v_j^*| > \delta\}} \right) \leq \sqrt{\mathbf{E} v_j^{*4}} \sqrt{\mathbf{P}(|v_j^*| > \delta)} \leq \delta^{-2} \mathbf{E} v_j^{*4} \\ &\leq 9\delta^{-2} (1 + \lambda_u^{-\gamma})^4 (\Delta \lambda_j)^4 \mathbf{E} U^4(\check{S}_{u^-}^*) \sum_{i=1}^3 Z_{i,j}^4. \end{aligned}$$

Taking into account that all of $Z_{i,j}$ have bounded moments and using (5.28) we obtain

$$\varepsilon^{-1} n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E} \mathbf{a}_j^* \leq 9C\varepsilon^{-1} \delta^{-2} n^{2\beta} \sum_{j=m_1}^{m_2} (1 + \lambda_u^{-\gamma})^4 (\Delta \lambda_j)^4,$$

which converges to 0 by Lemma 4.

Let us verify the limit of the sum of conditional variances $\mathbf{E}(v_j^{*2} | \mathcal{F}_{j-1})$. Setting $v_{i,j}^* = \tilde{A}_{i,j-1}^* Z_{i,j} \Delta \lambda_j$, one obtains $\mathbf{E}(v_{1,j}^* v_{3,j}^* | \mathcal{F}_{j-1}) = \mathbf{E}(v_{2,j}^* v_{3,j}^* | \mathcal{F}_{j-1}) = 0$ since $Z_{1,j}$ and $Z_{2,j}$ are independent. It follows that

$$\mathbf{E}(v_j^{*2} | \mathcal{F}_{j-1}) = \mathbf{E}(v_{1,j}^{*2} | \mathcal{F}_{j-1}) + \mathbf{E}(v_{2,j}^{*2} | \mathcal{F}_{j-1}) + \mathbf{E}(v_{3,j}^{*2} | \mathcal{F}_{j-1}) + 2\mathbf{E}(v_{1,j}^* v_{2,j}^* | \mathcal{F}_{j-1}).$$

Observe that for $Z \sim N(0, 1)$ and some constant a , $\mathbf{E}(Z|Z+a) = 2\Phi(a) - 1$ and $\mathbf{E}(Z+a)^2 - (\mathbf{E}|Z+a|)^2 = \Lambda(a)$, where Φ is the standard normal distribution function and Λ is defined in (5.5). On the other hand, $\Delta \lambda_j = n^{-2\beta} (1 + o(1)) \check{\mu} \rho^{\frac{2}{\mu+1}} \lambda_{j-1}^{\hat{\mu}}$ by Lemma 4. So,

$$n^{2\beta} \mathbf{E}(v_j^{*2} | \mathcal{F}_{j-1}) = (1 + o(1)) \check{\mu} \rho^{\frac{2}{\mu+1}} \lambda_{j-1}^{\hat{\mu}} \tilde{\mathbf{L}}(\lambda_{j-1}, \check{S}_{i,j-1}^*) \Delta \lambda_j.$$

Therefore, by Lemma 8, the sum $n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E}(v_j^{*2} | \mathcal{F}_{j-1})$ converges in probability to $\zeta^{*2}(L)$ defined in (5.27). Thus, $n^\beta \mathcal{M}_{m_2}^*$ weakly converges to $\mathcal{N}(0, \zeta^{*2}(L))$ throughout Theorem 4.

Step 2: Let us show that $\sup_{\varepsilon > 0} \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}\left(n^\beta \mathcal{M}_{m_2}^* - n^\beta \mathcal{M}_{m_2} > \varepsilon\right) = 0$. To this end, recall that $\hat{\phi}(\lambda, S_t) = \phi_L(\lambda)$ and hence, $\tilde{A}_i = A_i$ for $i = 1, 2, 4$ on the set $\{\tau^* = 1\}$. Then, the conclusion directly follows from

$$\mathbf{P}\left(n^\beta |\mathcal{M}_{m_2}^* - \mathcal{M}_{m_2}| > \varepsilon\right) \leq \mathbf{P}\left(n^\beta |\mathcal{M}_{m_2}^* - \mathcal{M}_{m_2}| > \varepsilon, \tau^* = 1\right) + \mathbf{P}(\tau^* < 1)$$

and (5.9). Moreover, taking into account that $\zeta^{*2}(L)$ converges a.s. to ζ^2 as $L \rightarrow \infty$, we conclude that $n^\beta \mathcal{M}_{m_2}$ converges in law to $\mathcal{N}(0, \zeta^2)$, which completes the proof. \square

Let us consider martingales of the following form, resulting from the approximation for Lépinette's strategy, $\overline{\mathcal{M}}_k = \sum_{j=m_1}^k (A_{1,j-1} Z_{1,j} + A_{2,j-1} Z_{2,j} + A_{4,j-1} Z_{4,j}) \Delta \lambda_j$. Their limiting variance is defined throughout the function

$$\bar{\mathbf{L}}(\lambda, x, y) = A_1^2(\lambda, x, y) + A_2^2(\lambda, x, y) + (1 - 2/\pi) A_4^2(\lambda, x, y). \quad (5.29)$$

The following result is similar to Proposition 4.

Proposition 5 *Suppose that $A_i = A_i(\lambda, x, y)$, $i = 1, 2, 4$ and their first partial derivatives have property (H). Then, for any fixed $\rho > 0$ the sequence $(n^\beta \overline{\mathcal{M}}_{m_2})$ weakly converges to a mixed Gaussian variable with mean zero and variance $\bar{\zeta}^2$ given by (5.25). The same property still holds if some (or all) of the functions A_i are replaced by $\int_\lambda^\infty A_i^0(z, x, y) dz$.*

Proof The conclusion follows directly from the proof of Proposition 4 and the observation that $\mathbf{E}Z_{4,j}^2 = \mathbf{E}(|Z_{1,j}| - \sqrt{2/\pi})^2 = 1 - 2/\pi$, and $\mathbf{E}(Z_{i,j}Z_{4,j}) = 0$, for $i = 1, 2$ and $m_1 \leq j \leq m_2$. \square

The remaining part of the section is devoted to prove main results following the scheme of [23]. Our first step is establish the asymptotic representation at rate n^β for each term contributing in the hedging error. The approximation procedure also provides the residual parts as discrete martingales for which, Propositions 4 and 5 will be applied to obtain the limit distribution at the last step.

5.6 Approximation for $I_{1,n}$

The following approximation is obtained in [23].

Proposition 6 *Let $\check{H} = \int_{\lambda}^{\infty} (z^{-1/2}/2 - z^{-3/2} \ln(x/K)) \tilde{\varphi}(z, x) dz$ and define*

$$\mathcal{U}_{1,k} = \rho^{-1} \sum_{j=m_1}^k \sigma(y_{t_{j-1}}) S_{t_{j-1}^-} \check{H}_{j-1} Z_{1,j} \Delta \lambda_j, \quad m_1 \leq j \leq m_2.$$

Then, under (C₁) and (C₂), $\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta |I_{1,n} - 2 \min(S_1, K) - \mathcal{U}_{1,m_2}| = 0$.

Proof By (3.15), one represents $I_{1,n}$ as

$$I_{1,n} = \int_0^1 \hat{\sigma}_t^2 S_{t^-}^2 \hat{C}_{xx}(t, S_{t^-}) dt - \int_0^1 \sigma^2(y_t) S_{t^-}^2 \hat{C}_{xx}(t, S_{t^-}) dt.$$

The last term is n^β negligible by (ii) of Lemma 2. To study the first integral let us introduce the function $A(\lambda, x) = x^2 \hat{C}_{xx}(t, x)$ and split it as

$$\int_0^1 \hat{\sigma}_t^2 S_{t^-}^2 \hat{C}_{xx}(t, S_{t^-}) dt = \int_0^1 \hat{\sigma}_t^2 S_{1^-}^2 \hat{C}_{xx}(t, S_{1^-}) dt + \int_0^1 \hat{\sigma}_t^2 (A(\lambda_t, S_t) - A(\lambda_t, S_{1^-})) dt.$$

The first integral $\int_0^1 \hat{\sigma}_t^2 S_{1^-}^2 \hat{C}_{xx}(t, S_{1^-}) dt$ almost surely converges to $2 \min(S_{1^-}, K)$ faster than n^r for any $r > 0$, see [23]. Let us study the last term which describe jumps of A . Using the Itô Lemma for $A(\lambda_t, S_t) - A(\lambda_t, S_{1^-})$, we rewrite it as

$$\varepsilon_{1,n} + \varepsilon_{2,n} + \int_0^1 \int_t^1 \hat{\sigma}_t^2 \partial_x A(\lambda_t, S_{u^-}) \sigma(y_u) S_{u^-} dW^{(1)} dt, \quad (5.30)$$

where

$$\varepsilon_{1,n} := \int_0^1 \int_t^1 \hat{\sigma}_t^2 A_1(\lambda_t, S_{u^-}, y_{u^-}) du dt \quad \text{and} \quad \varepsilon_{2,n} := \int_0^1 \hat{\sigma}_t^2 \int_t^1 \int_{\mathbb{R}} \bar{A}(\lambda_t, S_{u^-}, z) J(du \times dz) dt$$

with

$$A_1(\lambda, x, y) = \partial_t A(\lambda, x) + \partial_{xx} A(\lambda, x) \sigma^2(y) x^2, \quad \bar{A}(\lambda, x, z) = A(\lambda, x(1+z)) - A(\lambda, x).$$

Then, the approximation procedure of Proposition 3 is used to get a discrete martingale approximation \mathcal{U}_{1,m_2} for the Itô's integral of (5.30).

Now, let us show that $\varepsilon_{i,n} = o(n^{-\beta})$, $i = 1, 2$. In fact, $\varepsilon_{1,n} = o(n^{-\beta})$ by (iii) of Lemma 2. The jump term $\varepsilon_{2,n}$ can be represented as $\varepsilon_{2,n} = \int_0^1 \int_{\mathbb{R}} (\int_0^u \hat{\sigma}_t^2 \bar{A}(\lambda_t, S_{u^-}, z) dt) J(du \times dz)$ by Fubini's theorem [2]. Changing variable $v = \int_u^1 \hat{\sigma}_t^2 dt$ as in (5.1), one gets

$$\int_0^u \hat{\sigma}_t^2 \bar{A}(\lambda_t, S_{u^-}, z) dt = \int_{\lambda_u}^{\lambda_0} \bar{A}(v, S_{u^-}, z) dv := D(\lambda_u, S_{u^-}, z)$$

and hence, $\varepsilon_{2,n} = \int_0^1 \int_{\mathbb{R}} D(\lambda_u, S_{u^-}, z) J(\mathbf{d}u \times \mathbf{d}z)$. On the other hand,

$$D(\lambda_u, S_{u^-}, z) = \int_{\lambda}^{\lambda_0} \bar{A}(\lambda_u, S_{u^-}, z) = \int_{S_{u^-}}^{S_{u^-}(1+z)} \int_{\lambda_u}^{\lambda_0} \partial_x A(v, \mathbf{x}) \mathbf{d}v \mathbf{d}\mathbf{x}.$$

Direct computation shows that $\partial_x A(v, \mathbf{x}) = 2\mathbf{x}\widehat{C}_{xx}(v, \mathbf{x}) + \mathbf{x}^2\widehat{C}_{xxx}(v, \mathbf{x})$ and

$$\widehat{C}_{xx}(v, \mathbf{x}) = \frac{1}{\mathbf{x}\sqrt{v}} \widetilde{\varphi}(v, \mathbf{x}), \quad \widehat{C}_{xxx}(v, \mathbf{x}) = -\frac{1}{\mathbf{x}^2 v} \widetilde{\varphi}(v, \mathbf{x}) \left(\frac{3}{2} \sqrt{v} + \frac{\ln(\mathbf{x}/K)}{\sqrt{v}} \right).$$

Denoting $\widetilde{\varphi}(v, \mathbf{x}) = \frac{1}{\sqrt{\mathbf{x}}} \phi_0(v) e^{-\frac{\ln^2(\mathbf{x}/K)}{2\lambda}}$ with $\phi_0(v) = \sqrt{\frac{K}{2\pi}} e^{-v/8}$ and using the fact that $v^k e^{-v^2/2}$ is uniformly bounded for all k , one has $|\partial_x A(v, \mathbf{x})| \leq C \frac{1}{\sqrt{\mathbf{x}}} (1+v^{-1}) \phi_0(v)$, for some positive constant C . This estimate implies that

$$|D(\lambda_u, S_{u^-}, z)| \leq C \left| \int_{S_{u^-}}^{S_{u^-}(1+z)} \frac{1}{\sqrt{\mathbf{x}}} \mathbf{d}\mathbf{x} \right| \times \int_v^{\lambda_0} (1+v^{-1}) \phi_0(v) \mathbf{d}v \leq C \overline{\omega}(z) \widetilde{\phi}_0(\lambda_u) S_{u^-}, \quad (5.31)$$

where

$$\overline{\omega}(z) = z \mathbf{1}_{\{z>0\}} + \frac{1}{(1-|z|)^{1/2}} \mathbf{1}_{\{-1<z\leq 0\}} + 1, \quad \widetilde{\phi}_0(\lambda) = \int_{\lambda}^{\infty} (1+v^{-1}) \phi_0(v) \mathbf{d}v. \quad (5.32)$$

Clearly, $\widetilde{\phi}_0$ and $\overline{\omega}$ satisfy condition (5.19) of Lemma 3 hence, $\varepsilon_{2,n} = o(n^{-r})$ for any $r > 0$. \square .

5.7 Approximation for $I_{2,n}$

Proposition 7 Under (\mathbf{C}_1) and (\mathbf{C}_2) , $n^\beta I_{2,n}$ converges to 0 in probability as $n \rightarrow \infty$.

Proof We represent $I_{2,n}$ as

$$\int_0^1 \sigma(y_t) S_{t-A}(t) \mathbf{d}W_t^{(1)} + \int_0^1 \int_{\mathbb{R}} z S_{t-A}(t^-) \widetilde{J}(\mathbf{d}t \times \mathbf{d}z) := \mathbf{b}_{1,n} + \mathbf{b}_{2,n}, \quad (5.33)$$

where $A(t) = \widehat{C}_x(t(t), S_{t(t)}) - \widehat{C}_x(t, S_t)$. We first claim that the Itô's integral of (5.33) can be omitted by Lemma 1. To see this, it suffices to apply the Itô's formula, one represents the difference A_t as

$$\begin{aligned} \int_{t(t)}^t \left(\widehat{C}_{xt}(u, S_{u^-}) + \sigma^2(y_u) S_{u^-}^2 \widehat{C}_{xxx}(u, S_{u^-}) \right) \mathbf{d}u + \int_{t(t)}^t \widehat{C}_{xx}(u, S_{u^-}) \sigma(y_u) S_{u^-} \mathbf{d}W_u^{(1)} \\ + \int_{t(t)}^t \int_{\mathbb{R}} (\widehat{C}_x(u, S_{u^-}(1+z)) - \widehat{C}_x(u, S_{u^-})) J(\mathbf{d}z \times \mathbf{d}u) \end{aligned}$$

In view of (3.12),

$$\widehat{C}_{xt}(u, x) = -\frac{1}{2} \widehat{\sigma}_u^2 \left(2x \widehat{C}_{xx}(u, x) + x^2 \widehat{C}_{xxx}(u, x) \right) := \widehat{\sigma}_u^2 \widetilde{A}(u, x). \quad (5.34)$$

Therefore, $\mathbf{b}_{1,n}$ equals the following sum

$$\begin{aligned} \int_0^1 \int_{t(t)}^t \widehat{\sigma}_u^2 \sigma(y_t) S_{t-A}(t) \widetilde{A}(u, S_{u^-}) \mathbf{d}u \mathbf{d}W_t^{(1)} + \int_0^1 \int_{t(t)}^t \sigma(y_t) S_{t-A}(t) \sigma^2(y_u) S_{u^-}^2 \widehat{C}_{xxx}(u, S_{u^-}) \mathbf{d}u \mathbf{d}W_t^{(1)} \\ + \int_0^1 \int_{t(t)}^t \int_{\mathbb{R}} \sigma(y_t) S_{t-A}(t) (\widehat{C}_x(u, S_{u^-}(1+z)) - \widehat{C}_x(u, S_{u^-})) J(\mathbf{d}z \times \mathbf{d}u) \mathbf{d}W_t^{(1)}. \end{aligned} \quad (5.35)$$

The first two integrals converge to 0 more rapidly than $n^{-\beta}$ by Lemma 1. Let us study the jump term in (5.35), which will be denoted by \mathbf{a}_n . Clearly, by the Fubini's theorem \mathbf{a}_n equals

$$\sum_{1 \leq i \leq n} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \Psi(u, S_{u^-}, z) \left(\int_u^{t_i} \sigma(y_t) S_{t-A}(t) \mathbf{d}W_t^{(1)} \right) J(\mathbf{d}z \times \mathbf{d}u), \quad (5.36)$$

where $\Psi(u, x, z) := \widehat{C}_x(u, x(1+z)) - \widehat{C}_x(u, x)$. We prove that $\mathbf{a}_n = o(n^{-r})$ for any $r > 0$ following the demonstration of Lemma 3 with some modification. In particular, we decompose the sum in (5.36) into two parts: $\mathbf{a}_{1,n}$, the first concerns the index i with $m_2 \leq i \leq n$ and the second one $\mathbf{a}_{2,n}$, which is the sum over the rest of index i , $1 \leq i \leq m_2$. Clearly, $\mathbf{P}(n^r |\mathbf{a}_{1,n}| < \delta) \leq \mathbf{P}(N_1 - N_{t^*} \leq 1) = 1 - e^{-\theta(1-t^*)}$, which converges to 0.

To study the second one $\mathbf{a}_{2,n}$, we run again the argument used to obtain the estimate (5.31). In particular, $|\Psi(u, x, z)|$ is bounded by $\phi_1(\lambda_u) \left| \int_x^{x(1+z)} \frac{dx}{x^{3/2}} \right| \leq \phi_1(\lambda_u) \sqrt{x} \overline{\omega}_0(z)$, where

$$\phi_1(\lambda) = \sqrt{K/(2\pi)} \lambda_t^{-1/2} e^{-\lambda/8} \quad \text{and} \quad \overline{\omega}_0(z) = 1 + \frac{1}{\sqrt{1+z}} \mathbf{1}_{\{-1 < z \leq 0\}} \quad (5.37)$$

Denote by $\mathbf{a}_{2,n}^c$ the compensator of $\mathbf{a}_{2,n}$. Then, it is clear that

$$\begin{aligned} |\mathbf{a}_{2,n}^c| &\leq \sum_{1 \leq i \leq m_2} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |\Psi(u, S_{u^-}, z)| \left| \int_u^{t_i} \sigma(y_t) S_{t^-} dW_t^{(1)} \right| \nu(dz) du \\ &\leq \sum_{1 \leq i \leq m_2} \int_{t_{i-1}}^{t_i} \phi_1(\lambda_u) \sqrt{S_{u^-}} \left| \int_u^{t_i} \sigma(y_t) S_{t^-} dW_t^{(1)} \right| du \times \int_{\mathbb{R}} \overline{\omega}_0(z) \nu(dz) \end{aligned} \quad (5.38)$$

It is important to highlight that $X_u := \sqrt{S_{u^-}} \left| \int_u^{t_i} \sigma(y_t) S_{t^-} dW_t^{(1)} \right|$ may not be squared integrable. To overcome this issue, consider the stopping time τ^* defined in (5.7) for some $L > 0$. On the set $\{\tau^* = 1\}$, one has for $u \in [t_{i-1}, t_i]$,

$$\mathbf{E} X_u^{2*} = \mathbf{E} \left(\sqrt{S_{u^-}^*} \int_u^{t_i} \sigma(y_t^*) S_{t^-}^* dW_t^{(1)} \right)^2 \leq \mathbf{E} S_{u^-}^* \int_u^{t_i} S_t^{*2} dt \leq CL^2 n^{-1}$$

and hence, $\mathbf{E} X_u^* \leq \sqrt{\mathbf{E} X_u^{2*}} \leq CLn^{-1/2}$ by Cauchy-Schwartz's inequality. Therefore,

$$\begin{aligned} \mathbf{P}(n^r |\mathbf{a}_{2,n}^c| > \delta, \tau^* = 1) &\leq n^r \delta^{-1} \sum_{1 \leq i \leq m_2} \int_{t_{i-1}}^{t_i} \phi_1(\lambda_u) \mathbf{E} X_u^* du \times \int_{\mathbb{R}} \overline{\omega}_0(z) \nu(dz) \\ &\leq n^r \delta^{-1} CLn^{-1/2} \sum_{1 \leq i \leq m_2} \int_{t_{i-1}}^{t_i} \phi_1(\lambda_u) du \times \int_{\mathbb{R}} \overline{\omega}_0(z) \nu(dz) \\ &\leq n^r \delta^{-1} CLn^{-1/2} \int_0^{t^*} \phi_1(\lambda_u) du \times \int_{\mathbb{R}} \overline{\omega}_0(z) \nu(dz) \end{aligned} \quad (5.39)$$

Now, taking into account $\int_{\mathbb{R}} \overline{\omega}_0(z) \nu(dz) < \infty$ and $\int_0^{t^*} \phi_1(\lambda_u) du$ rapidly converges to 0, one concludes that the right side of (5.38) converges to 0 for any $r > 0$ and so is $\mathbf{P}(n^r |\mathbf{a}_{2,n}^c| > \delta, \tau^* = 1)$. Noting that

$$\mathbf{P}(n^r |\mathbf{a}_{2,n}^c| > \delta) \leq \mathbf{P}(n^r |\mathbf{a}_{2,n}^c| > \delta, \tau^* = 1) + \mathbf{P}(\tau^* < 1)$$

and using (5.9) one obtains $n^r \mathbf{a}_{2,n}^c \rightarrow 0$ in probability for any $r > 0$.

Now, putting $\tilde{\alpha}_{2,n} = \alpha_{2,n} - \alpha_{2,n}^c$, we need to show that $\mathbf{P}(n^r |\tilde{\alpha}_{2,n}| > \delta) \rightarrow 0$. To this end, consider again the stopping time τ^* defined in (5.7) for some $L > 0$. On the set $\{\tau^* = 1\}$, one has $|\Psi(u S_{u^-}, z)| \leq \sqrt{S_{u^-}^*} \phi_1(\lambda_u) \overline{\omega}_0(z)$, where $S_{u^-}^*$ is the stopped version of S_{u^-} . Clearly, $\sup_{1 \leq i \leq n} \sup_{t_{i-1} \leq u \leq t_i} \mathbf{E} S_{u^-}^* |W_{t_{i-1}}^{(1)} - W_u^{(1)}|^2 \leq Cn^{-1}$ for some positive constant C . It then follows by the Chebychev inequality that $\mathbf{P}(n^r |\tilde{\alpha}_{2,n}| > \delta, \tau^* = 1) \leq n^{2r} \delta^{-2} \mathbf{E} \tilde{\alpha}_{2,n}^{*2}$, where $\tilde{\alpha}_{2,n}^{*2}$ is obtained by substituting S_{u^-} by $S_{u^-}^*$ in the function $\Psi(u, S_{u^-}, z)$. Now, the well-known isometry for jump integrals applying to $\tilde{\alpha}_{2,n} = \alpha_{2,n} - \alpha_{2,n}^c$ implies that $\mathbf{E} \tilde{\alpha}_{2,n}^{*2}$ is bounded by

$$\sum_{1 \leq i \leq m_2} \mathbf{E} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |\Psi(u, S_{u^-}^*, z)|^2 |W_{t_{i-1}}^{(1)} - W_u^{(1)}|^2 \nu(dz) du,$$

which is smaller than

$$\int_0^{t^*} \phi_1^2(\lambda_u) \mathbf{E} X_u^{*2} du \times \int_{\mathbb{R}} \varpi_0^2(z) \nu(dz) \leq C n^{-1} \int_0^{t^*} \phi_1^2(\lambda_u) du \times \int_{\mathbb{R}} \varpi_0^2(z) \nu(dz).$$

Again, $\int_{\mathbb{R}} \varpi_0^2(z) \nu(dz) < \infty$ by condition (C_1) . Therefore, for some constant depending on L ,

$$\mathbf{P}(n^r |\tilde{\alpha}_{2,n}| > \delta, \tau^* = 1) \leq C(L) n^{2r} \delta^{-2} n^{-1} \int_0^{t^*} \phi_1^2(\lambda_u) du \times \int_{\mathbb{R}} \varpi_0^2(z) \nu(dz),$$

which converges to 0 for any $r > 0$. Letting now $L \rightarrow \infty$ and using (5.9) we obtain that $|\tilde{\alpha}_{2,n}| = o(n^{-r})$ for any $r > 0$. By the same way, one can show that $n^r \mathbf{b}_{2,n} \rightarrow 0$ in probability for any $r > 0$ and the proof is completed. \square

5.8 Approximation for $I_{3,n}$

Proposition 8 *Suppose that (C_1) and (C_2) hold. Then, for any $r > 0$, $n^r |I_{3,n}| \rightarrow 0$ in probability as $n \rightarrow \infty$.*

Proof By (3.15), one has $\mathbf{B}(t, S_{t^-}, z) = \int_{S_{t^-}^{-(1+z)}}^{S_{t^-}^v} \widehat{C}_{xx}(t, u) du dv$. Recall that $\widehat{C}_{xx}(t, u) = u^{-1} \lambda_t^{-1/2} \tilde{\varphi}(\lambda_t, u) \leq u^{-3/2} \phi_1(\lambda_t)$, where $\phi_1(\lambda) = \sqrt{K/(2\pi)} \lambda^{-1/2} e^{-\lambda/8}$. Direct calculus leads to $|\mathbf{B}(t, S_{t^-}, z)| \leq C S_{t^-}^{1/2} \phi_1(\lambda_t) \varpi(z)$ where $\varpi(z)$ defined in (5.32). Therefore all assumptions in Lemma 3 are fulfilled and the conclusion follows. \square

5.9 Approximation for Γ_n

Let us study the trading volume J_n . It is easy to check that for $v \geq 0$, $1 - \Phi(v) \leq C v^{-1} \varphi(v)$ and $\int_0^{t^*} \tilde{\varphi}(\lambda_u, S_u) du + \int_{t^*}^1 \tilde{\varphi}(\lambda_u, S_u) du$ almost surely converges to 0 more rapidly than any power of n . Therefore, one can truncate the sum and keep only the part corresponding to index $m_1 \leq j \leq m_2$. Next, one can ignore jumps terms that may appear in approximations via Itô's formulas in the interval $[t^*, 1]$. For convenience, let us recall here the approximation result for Γ_n obtained in [23].

Proposition 9 *Under conditions $(C_1) - (C_2)$, the total trading volume J_n admits the following asymptotic form*

$$J_n = J(S_1, y_1, \rho) + (\mathcal{U}_{2,m_2} + \mathcal{U}_{3,m_2}) + o(n^{-\beta}).$$

5.10 Proof of Theorem 1

By Propositions 6-9, the hedging error is represented as $V_1^n - h(S_1) = \min(S_{1^-}, K) - \kappa \Gamma(S_{1^-}, y_1, \rho) + \mathcal{M}_{m_2}$, where the martingale part of the hedging error is given by $\mathcal{M}_k = \frac{1}{2} \mathcal{U}_{1,k} - \kappa (\mathcal{U}_{2,k} + \mathcal{U}_{3,k})$ and hence, the sequence $(n^\beta \mathcal{M}_{m_2})$ converges in law to a mixed Gaussian variable by Proposition 4 and Theorem 1 is proved. \square

5.11 Proof of Theorem 2

Suppose now that the Lépinette strategy $\tilde{\gamma}_t^n$ is applied for the replication problem. In the same principle one can represent the corresponding hedging error as $\tilde{V}_1^n - h(S_1) = \frac{1}{2} I_{1,n} + \bar{I}_{2,n} - I_{3,n} - \kappa \bar{\Gamma}_n$, where

$$\bar{I}_{2,n} = I_{2,n} + \sum_{i \geq 1} \Delta S_{t_i} \int_0^{t_i-1} \widehat{C}_{xt}(u, S_u) du$$

and $\bar{\Gamma}_n = \sum_{i=1}^n S_{t_i} |\tilde{\gamma}_{t_i}^n - \tilde{\gamma}_{t_{i-1}}^n|$ is the trading volume. Recall that $I_{2,n}$ est negligible by Proposition 7. Let us investigate the above sum. By (5.34), it can be represented as

$$\sum_{i \geq 1} \int_{\lambda_{i-1}}^{\lambda_0} \tilde{A}(u, S_u) dv \int_{t_{i-1}}^{t_i} \sigma(y_{t^-}) S_{t^-} dW_t^{(1)} + \sum_{i \geq 1} \int_{\lambda_{i-1}}^{\lambda_0} \tilde{A}(u, S_u) dv \int_{t_{i-1}}^{t_i} z dS_{t^-} \tilde{J}(dz \times dt)$$

using the usual change of variable, where \tilde{A} defined in (5.34). Now, the approximation technique of Proposition 3 can be applied to replace the first sum by martingale $\overline{\mathcal{U}}_{2,m_2}$ defined by

$$\overline{\mathcal{U}}_{2,k} = \rho^{-1} \sum_{j=m_1}^k \sigma(y_{t_{j-1}^-}) S_{t_{j-1}^-} Y_{j-1} Z_{1,j} \Delta \lambda_j, \quad m_1 \leq k \leq m_2$$

and $Y(\lambda, x) = \int_{\lambda}^{\infty} z^{-3/2} \ln(x/K) \tilde{\varphi}(z, x) dz$. On the other hand, one obtains the same estimate (5.31) for the integrand, which implies that the second sum can be omitted at order n^r for any $r > 0$ by Lemma 3.

Now, approximation representation for the trading volume $\overline{\Gamma}_n$ following the procedure in the approximation of Γ_n . The following is established in [23].

Proposition 10 *Under conditions (C₁) – (C₂),*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta |\overline{\Gamma}_n - \eta \min(S_1, K) - (\overline{\mathcal{U}}_{2,m_2} + \overline{\mathcal{U}}_{3,m_2})| = 0.$$

Hence, $\overline{\mathcal{M}}_{m_2} = \overline{\mathcal{U}}_{1,m_2} + \overline{\mathcal{U}}_{2,m_2} - \kappa_*(\overline{\mathcal{U}}_{2,m_2} + \overline{\mathcal{U}}_{3,m_2})$ is the martingale part of the hedging error for Lépinette's strategy, which can be represented in the form

$$\overline{\mathcal{M}}_k = \rho^{-1} \sum_{j=m_1}^k (A_{1,j-1} Z_{1,j} + A_{4,j} Z_{4,j-1} + A_{2,j-1} Z_{2,j}) \Delta \lambda_j$$

for explicit functions A_i holding the assumption of Proposition 5. Then, the convergence in law to a mixed Gaussian variable of the sequence $(n^\beta \overline{\mathcal{M}}_{m_2})$ is guaranteed by Proposition 5 and hence, Theorem 2 is proved. \square

5.12 Proof of Theorem 3

Note first that the approximation representation for the replication error is the same as in SVJ case, in particular, approximations of $I_{i,n}$, $i = 1, 2, 3$ are the same since martingale sums are resulted from Itô's formula in one dimension case. The only difference is that in finding the limit of the total transaction costs one has replaced $S_{t_{j-1}^-}$ and $y_{t_{j-1}^-}$ by terminal values S_{1^-} and y_{1^-} . Now, the two-dimension version of Itô's formula applied for the difference provides sums concerning to the dynamics of y_t . Using the elementary of Poisson process one can ignore the jump part of y_t in the time interval $[t^*, 1]$. Hence, the martingale approximation for this difference is the same as in SVJ case. However one needs to check the integrability of $\alpha_i(t, y_t)$, $i = 1, 2$. For this aim, condition $\sup_{0 \leq t \leq 1} \mathbf{E} y_t^2 < \infty$ is needed but this is fulfilled under condition (C₃) together with the linear growth and Lipschitz properties of these coefficients, see Appendix C. \square

6 Conclusion

Diffusion-based stochastic volatility models well account for volatility clustering, dependence in increments and long term smiles and skews but can not generate jumps nor realistic short-term implied volatility patterns. These shortcomings can be fixed by adding jumps into the model. There are two possible ways to emerge jumps into stochastic volatility models: adding an independent jump component to the return or in the volatility process itself. We showed that jumps in such frameworks do not affect asymptotic property of the replication error in approximate hedging with transaction costs. The results established in the present note is general enough for practical purposes.

It should be mentioned that in [29, 30], the authors studied the asymptotic property of hedging error resulting from discrete delta hedging in exponential Lévy models without transaction costs. More precisely, they showed that the normalized hedging error converges stably in finite-dimension laws to an explicit variable. This result was applied to a problem of hedging a discontinuous payoff option in Merton's jump-diffusion model. However, they left the jump residual as an unhedgeable term even in sense of approximate hedging.

For future directions, it would be interesting to investigate the problem of approximate hedging options which are written on multiple assets where jumps are allowed. In the absence of jumps in the price processes, a such study has appeared in [24] and hence, it is reasonable to believe that jumps influence can be also removed in the limiting property of the hedging error in multiple frameworks. Another interesting problem is to study asymptotic properties of jump risk in small transaction costs models. In fact, when $\kappa = \kappa_0 n^{-\alpha}$ for $0 < \alpha < 1/2$ the complete replication can be obtained with a better convergence for both Leland and Lépinette strategies [23]. For $\alpha = 1/2$, the classical form of volatility should be applied to get the complete replication. Such extensions are in progress research.

Appendix

A Auxiliary Lemmas

Lemma 4 *There exist two positive constants C_1, C_2 such that*

$$C_1 n^{-2\beta} \rho^{\frac{2}{\mu+1}} v_0(l_*) \leq \inf_{m_1 \leq j \leq m_2} |\Delta \lambda_j| \leq \sup_{m_1 \leq j \leq m_2} |\Delta \lambda_j| \leq C_2 n^{-2\beta} \rho^{\frac{2}{\mu+1}} v_0(l^*), \quad (\text{A.1})$$

where $v_0(x) = x^{(\mu-1)/(\mu+1)}$. Moreover,

$$\Delta \lambda_j = n^{-2\beta} \rho^{\frac{2}{\mu+1}} v_0(\lambda_{j-1})(1 + o(1)) \quad \text{and} \quad \Delta \lambda_j (\Delta t_j)^{-1/2} = \rho(1 + o(1)). \quad (\text{A.2})$$

Proof It follows directly from relation (5.1). \square

Lemma 5 *For any $K > 0$ and $0 < t \leq 1$, $\mathbf{P}(S_t = K) = 0$.*

Proof We prove that for $0 < t \leq 1$ and any real number a , $\mathbf{P}(\psi_t = a) = 0$, where $\psi_t = \int_0^t b_t dt + \int_0^t \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma^2(y_s) ds + \sum_{j=1}^{N_t} \ln(1 + \xi_j)$. Indeed, one can represent $W_t^{(1)} = \rho B_t + \sqrt{1 - \rho^2} Z_t$, where B_t is the Brownian driving y_t and Z is another Brownian independent of B . Now, conditionally on the Brownian B and jump terms $\sum_{j=1}^{N_t} \ln(1 + \xi_j)$, ψ_t is a Gaussian variable. \square

Lemma 6 *For any $\varepsilon > 0$ and $K > 0$, $\limsup_{v \rightarrow 1} \mathbf{P}(\inf_{v \leq t \leq 1} |\ln(S_t/K)| \leq \varepsilon) = 0$.*

Proof Let η be some positive number. Clearly, the above probability is bounded by

$$\begin{aligned} & \mathbf{P}(\inf_{v \leq u \leq 1} |\ln(S_t/K)| \leq \varepsilon, N_1 - N_v = 0, |\ln(S_1/K)| > \eta) \\ & \quad + \mathbf{P}(|\ln(S_1/K)| \leq \eta) + \mathbf{P}(N_1 - N_v \geq 1). \end{aligned} \quad (\text{A.3})$$

Let us show that the probability in (A.3) is equal to zero for v sufficiently close to 1. On the set $\{N_1 - N_v = 0\}$, we have $\ln(S_t/K) = \ln(S_1/K) - \psi_t$, where $\psi_t = -\frac{1}{2} \int_u^1 \sigma^2(s) ds + \int_u^1 \sigma(s) dW_r(s)$. We can check directly that $\psi_r^*(v) = \sup_{v \leq u \leq 1} |\psi_r(u)| \rightarrow 0$ a.s. as $v \rightarrow 1$. So, if $|\ln(S_1/K)| > \eta$ then for v sufficiently close to 1 and $v \leq u \leq 1$,

$$|\ln(S_t/K)| = |\ln(S_1/K) - \psi_t| \geq ||\ln(S_1/K)| - \psi_v| \geq \frac{1}{2} |\ln(S_1/K)| > \eta/2.$$

Therefore, for $\eta > 2\varepsilon$, one obtains $\inf_{v \leq u \leq 1} |\ln(S_t/K)| \geq \eta/2 > \varepsilon$ and so, the first probability in (A.3) is equal to 0. On the other hand, $\mathbf{P}(N_1 - N_v \geq 1) = 1 - e^{-\theta(1-v)} \rightarrow 0$ as $v \rightarrow 1$. Letting now $\eta \rightarrow 0$ we get $\mathbf{P}(|\ln(S_1/K)| \leq \eta) \rightarrow \mathbf{P}(S_1 = K) = 0$ in view of Lemma 5 and hence Lemma 6 is proved. \square

Lemma 7 *Suppose that $A = A(\lambda, x, y)$ and its partial derivatives $\partial_\lambda A, \partial_x A, \partial_y A$ verify condition (H). Set*

$$r_n = \sup_{(z,r,d) \in [l_*, l^*] \times \mathcal{B}} (|\partial_\lambda A(z, r, d)| + |\partial_x A(z, r, d)| + |\partial_y A(z, r, d)|),$$

where $\mathcal{B} = [S_{\min}^-, S_{\max}^-] \times [y_{\min}^-, y_{\max}^-]$ with

$$S_{\min}^- = \inf_{t^* \leq u \leq t_*} S_{u^-}, \quad S_{\max}^- = \sup_{t^* \leq u \leq t_*} S_{u^-}, \quad y_{\min}^- = \inf_{t^* \leq u \leq t_*} y_{u^-}, \quad y_{\max}^- = \sup_{t^* \leq u \leq t_*} y_{u^-}.$$

Then, $\lim_{b \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \mathbf{P}(r_n > b) = 0$.

Proof See Lemma A.4 in [23] with remark that the left continuity of S_{t-} and y_{t-} gives the same argument. \square

Lemma 8 *Suppose that $A = A(\lambda, x, y)$ and its first partial derivatives have property (H). Set $\bar{A}(\lambda, x, y) = \int_{\lambda} A(z, x, y) dz$ and $\tilde{A}(\lambda, x, y) = \bar{A}^2(\lambda, x, y)$. Then, for any $\gamma > 0$,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} \left| \sum_{j=m_1}^{m_2} \lambda_{j-1}^{\gamma} \tilde{A}(\lambda_{j-1}, \check{S}_{t_{j-1}^-}) \Delta \lambda_j - \int_0^{\infty} \lambda^{\gamma} \tilde{A}(\lambda, \check{S}_{1-}) d\lambda \right| = 0,$$

where $\check{S}_t = (S_t, y_t)$. The same property still holds if $\bar{A}(\lambda, x, y) = A(\lambda, x, y)$ or the product of these above kinds.

Proof See Lemme A.5 in [23]. \square

B Some moment estimates

Lemma 9 *Let y_t is some Itô's process and S_t be the asset process given by*

$$S_t = S_0 \exp \left\{ \int_0^t b_t dt + \int_0^t \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma^2(y_s) ds \right\} \prod_{j=1}^{N_t} (1 + \xi_j),$$

where N_t is a homogeneous Poisson process with intensity θ independent of $(\xi_j)_{j \geq 1}$, a sequence of i.i.d. variables. We assume that the jumps ingredient $(\xi_j)_{j \geq 1}$, N_t are independent of the Brownian motion W_t and of that of y_t . If b and σ are two bounded functions then, for any $m > 0$,

$$\mathbf{E} S_t^m \leq C(m) \exp\{\theta t (\mathbf{E}(1 + \xi_1)^m - 1)\}, \quad \text{for all } t \in [0, 1],$$

where $C(m)$ is some constant depending on m .

Proof Let us represent $S_t = \tilde{b}_t \mathcal{E}_t(\sigma) X_t$ with $X_t = \prod_{j=1}^{N_t} (1 + \xi_j)$ and

$$\tilde{b}_t = S_0 e^{\int_0^t b_s ds}, \quad \mathcal{E}_t(\sigma) = \exp \left\{ \int_0^t \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_0^t \sigma^2(y_s) ds \right\}.$$

By hypothesis, the stochastic exponential $\mathcal{E}_t(\sigma)$ is a martingale with expectation 1, independent of X_t . Therefore, $\mathbf{E} S_t^m \leq C \mathbf{E} \mathcal{E}_t^m(\sigma) \mathbf{E} X_t^m$ since $\sup_{0 \leq t \leq 1} \tilde{b}_t^m \leq C$. Because σ is bounded one has

$$\mathbf{E} \mathcal{E}_t^m(\sigma) = \mathbf{E} \mathcal{E}_t(m\sigma) e^{(m^2 - m)/2 \int_0^t \sigma^2(y_s) ds} \leq C \mathbf{E} \mathcal{E}_t(m\sigma) = C.$$

On the other hand, using the usual conditioning technique gives

$$\mathbf{E} X_t^m = \mathbf{E} \prod_{j=1}^{N_t} (1 + \xi_j)^m = \exp\{\theta t (\mathbf{E}(1 + \xi_1)^m - 1)\},$$

which implies the desired conclusion. \square

Lemma 10 *Under the assumptions of Lemma 9, for $0 \leq u \leq v \leq 1$,*

$$\mathbf{E} (S_u - S_v)^2 \leq C |u - v|,$$

for some constant C .

Proof For $0 \leq u < v \leq 1$, put $\tilde{b}_{v/u} = e^{\int_u^v b_s}$, $X_{u/v} = \prod_{j=N_u+1}^{N_v} (1 + \xi_j)$ and

$$\mathcal{E}_{v/u}(\sigma) = \exp \left\{ \int_u^v \sigma(y_s) dW_s^{(1)} - \frac{1}{2} \int_u^v \sigma^2(y_s) ds \right\}.$$

Then, $\mathcal{E}_{v/u}(\sigma)$ and $X_{u/v}$ are independent and

$$\sup_{0 \leq u \leq v \leq 1} (\mathbf{E} \mathcal{E}_{v/u}^2(\sigma) + \mathbf{E} e^{2 \int_u^v b_s}) < \infty \quad (\text{B.1})$$

since b and σ are bounded. Denote $\delta = \theta(v - u)$. It is easy to check that

$$\mathbf{E}(X_{u/v} - 1)^2 = e^{\delta(\mathbf{E}\xi_1 + \mathbf{E}\xi_1^2)} - 2e^{\delta\mathbf{E}\xi_1} + 1. \quad (\text{B.2})$$

Let us first show that $\mathbf{E}(X_{u/v} - 1)^2 \leq C\delta$, for some constant C . Obviously, for any finite interval $[a, b]$, $|e^x - 1| \leq Cx$ by Taylor's approximation. From condition (\mathbf{C}_3) , $\mathbf{E}\xi_1^2 < \infty$. Now, if $\mathbf{E}\xi_1 = 0$ then $\mathbf{E}(X_{u/v} - 1)^2 = e^{\delta\mathbf{E}\xi_1^2} - 1 \leq C\delta$. Similarly, in case $\mathbf{E}\xi_1 + \mathbf{E}\xi_1^2 = 0$ one has $\mathbf{E}\xi_1 \neq 0$ and hence $\mathbf{E}(X_{u/v} - 1)^2 = e^{\delta\mathbf{E}\xi_1} - 1 \leq C\delta$. Lastly, if both of $\mathbf{E}\xi_1$ and $\mathbf{E}\xi_1 + \mathbf{E}\xi_1^2$ are non zero one can estimate $\mathbf{E}(X_{u/v} - 1)^2$ by $|e^{\delta(\mathbf{E}\xi_1 + \mathbf{E}\xi_1^2)} - 1| + 2|e^{\delta\mathbf{E}\xi_1} - 1| \leq C\delta$. Using the same argument one can easily prove that

$$\mathbf{E}(\mathcal{E}_{u/v}(\sigma) - 1)^2 \leq C\delta \quad \text{and} \quad \mathbf{E}(\tilde{b}_{v/u} - 1)^2 \leq C\delta^2. \quad (\text{B.3})$$

Clearly, $\mathbf{E}(S_u - S_v)^2 = \mathbf{E}S_u^2 \mathbf{E} \left(\frac{S_v}{S_u} - 1 \right)^2$ and

$$\left(\frac{S_u}{S_v} - 1 \right)^2 \leq 2 \left(\tilde{b}_{v/u}^2 (\mathcal{E}_{u/v}(\sigma) - 1)^2 + (\tilde{b}_{v/u} - 1)^2 + \tilde{b}_{v/u}^2 \mathcal{E}_{u/v}^2(\sigma) (X_{u/v} - 1)^2 \right). \quad (\text{B.4})$$

By Lemma 9, $\sup_{0 \leq u \leq 1} \mathbf{E}S_u^2 < \infty$. Now, taking expectation in (B.4) and using (B.1), (B.2) and (B.3) one obtains the conclusion. \square

C Stochastic differential equations with jumps

In this section we recall the basic result in the theory of stochastic differential equations with jumps (SDEJ) of the form

$$dy_t = \alpha_1(t, y_t) dt + \alpha_2(t, y_t) dW_t + d\zeta_t, \quad (\text{C.1})$$

on the time interval $[0, T]$ with initial value y_0 , where $\zeta_t = \sum_{j=1}^{N_t} \xi_j$ is a compound Poisson process independent of the Brownian motion W and $\mathbf{E}y_0^2 < \infty$.

Theorem 5 *Suppose that α_i , $i = 1, 2$ are locally Lipschitz and linearly bounded functions and $\mathbf{E}y_0^2 < \infty$. Assume further that jump sizes of ζ_t have finite second moment. Then, there exists a unique solution y_t to (C.1) with initial value y_0 and*

$$\mathbf{E} \left(\sup_{0 \leq t \leq T} y_t \right)^2 < C(T)(1 + \mathbf{E}y_0^2) < \infty. \quad (\text{C.2})$$

Furthermore, for any $0 \leq s \leq t \leq T$, there exists a positive constant C such that

$$\mathbf{E}|y_t - y_s|^2 \leq C|t - s|. \quad (\text{C.3})$$

Proof The existence and uniqueness of the solution follows by adapting the classical method used for SDEs, see for instance Theorem 2.2 in [10]. To prove (C.3), we note that

$$\mathbf{E}|y_t - y_s|^2 \leq 3\mathbf{E} \left(\int_s^t \alpha_1(u, y_u) du \right)^2 + 3 \int_s^t \mathbf{E} \alpha_2^2(u, y_u) du + 3\mathbf{E} \left(\sum_{j=N_s+1}^{N_t} \xi_j \right)^2$$

By the linear boundedness of α_1, α_2 and (C.2) one gets

$$\mathbf{E} \left(\int_s^t \alpha_1(u, y_u) du \right)^2 \leq C|t-s| \int_s^t (1 + \mathbf{E}y_u^2) du \leq C|t-s|^2.$$

Similarly, $\mathbf{E} \alpha_2^2(u, y_u) du \leq C \int_s^t (1 + \mathbf{E}y_u^2) du \leq C|t-s|$. To compute $\mathbf{E} \left(\sum_{j=N_s+1}^{N_t} \xi_j \right)^2$ we apply the conditioning technique to get $\mathbf{E} \left(\sum_{j=N_s+1}^{N_t} \xi_j \right)^2 \leq \lambda|t-s| \text{Var} \xi_1 + (\mathbf{E} \xi_1)^2 (\lambda|t-s| + \lambda^2|t-s|^2)$ and the conclusion follows. \square

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