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# Stochastic dynamics of determinantal processes by integration by parts

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## Abstract

We construct the interacting diffusion processes associated to determinantal processes on the whole space. Our construction is based on the notion of Dirichlet form and an integration by parts formula for functionals of determinantal processes on compact sets. We then prove the convergence of the Dirichlet forms on compact sets to a Dirichlet form defined on the whole space. We also prove the existence of diffusions associated with this limit Dirichlet form. Some examples of diffusions are also given.

*Keywords:* Dirichlet forms; infinite dimensional diffusion processes; closability; integration by parts; Malliavin calculus; determinantal processes.

*Mathematics Subject Classification:* 60G55, 60H07.

## 1 Introduction

Determinantal processes are point processes that exhibit a repulsion property, and were introduced to represent the configuration of fermions, cf. [14], [21] and [24]. They are known to be connected with the zeros of analytic functions, cf. [11] and references therein.

In this paper we construct Dirichlet forms related to determinantal processes, and we apply them to derive the existence of the associated interacting diffusion processes. For this we provide an integration by parts formula for functionals of determinantal processes which is based on a quasi-invariance result proved in [4]. In this paper, we provide another proof

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of the quasi-invariance result which makes some of the arguments more rigorous. This integration by parts formula on a compact set is extended to closed gradient and divergence operators thanks to the use of a set of test functionals different from the one considered in [4], cf. (3.1) and Theorem 3.9-(i) and (ii). Our approach follows the lines of [1] and our construction differs from the one considered in [25] which is based on sample-path identities. Such a construction can be applied to derive formulas for density estimation and sensitivity analysis for functionals of determinantal processes along the lines of [17].

Our main result, Theorem 4.1, provides the symmetric Dirichlet form associated to a determinantal process on a compact set. We construct the limiting Dirichlet form by making the compact sets tend to the whole space, and show that the limit is a proper Dirichlet form which is still closable in Theorem 4.4. We show that there exist some diffusions correctly associated to the determinantal point process on a compact set in Theorem 5.1. Afterwards, we show in Theorem 5.5 that there also exists a diffusion associated to the determinantal point process on the whole space. These diffusions can be explicitated on a compact set or when the number of points is finite, and we specifically give some examples in Section 6.

We proceed as follows. In Section 2 we recall the definitions of point processes and determinantal process, based on [5], [6], [15] and [26] for point processes, and on [11] for determinantal processes. We also refer to [2], [7] and [22] for the required background on functional analysis. In Section 3 we provide an integration by parts formula for functionals of determinantal processes and extend it by a closability argument. In Section 4 we consider the associated Dirichlet form, with application to the associated stochastic dynamics in Section 5. We also prove the non-collision property of the resulting diffusion. Finally, in Section 6 we give some examples of determinantal processes satisfying the hypotheses in the article.

## 2 Preliminaries

### Locally finite point processes

Let  $(S, d_S)$  be a locally compact Polish space, and denote by  $\mathcal{B}_S$  the associated Borel  $\sigma$ -algebra. For any subset  $B \subseteq S$ , let  $\sharp B$  denote the cardinality of  $B$ , setting  $\sharp B = \infty$  if  $B$  is not finite. We denote by  $N_{lf}$  the set of locally finite point configurations on  $S$ :

$$N_{lf} := \{B \subseteq S : \sharp(B \cap D) < \infty, \text{ for any compact } D \subseteq S\},$$

equipped with the  $\sigma$ -field

$$\mathcal{N}_{lf} := \sigma(\{B \in N_{lf} : \sharp(B \cap D) = m\}, m \geq 0, D \subseteq S \text{ compact}).$$

We define similarly  $N_f$  the set of finite point configurations on  $S$ :

$$N_f := \{B \subseteq S : \sharp B < \infty\},$$

$N_f$  is naturally equipped with the trace  $\sigma$ -field  $\mathcal{N}_f = \mathcal{N}_{lf}|_{N_f}$ . Lastly, for any bounded set  $D \subseteq S$ , let  $N_f^D$  be the space of finite configurations on  $D$ , and  $\mathcal{N}_f^D$  the associated  $\sigma$ -field. As

in [9], we define for any Radon measure  $\mu$  on  $(S, \mathcal{B}_S)$  the  $\mu$ -sample measure  $L^\mu$  on  $(\mathcal{N}_f, \mathcal{N}_f)$  by

$$\int_{\mathcal{N}_f} f(\alpha) L^\mu(d\alpha) := \sum_{n \geq 0} \frac{1}{n!} \int_{S^n} f(\{x_1, \dots, x_n\}) \mu(dx_1) \dots \mu(dx_n), \quad (2.1)$$

for any measurable  $f : \mathcal{N}_f \rightarrow \mathbb{R}$ .

Let  $\mathbf{X}$  be a locally finite point process on  $S$ , i.e. a measurable mapping defined on some probability space  $(\Omega, \mathcal{F}, P)$  and taking values on  $(\mathcal{N}_{lf}, \mathcal{N}_{lf})$ . We will scarcely denote by  $\mathbf{X}$  an element of  $\mathcal{N}_{lf}$ , omitting the fact that  $\mathbf{X}$  is in fact a mapping. Moreover, we assume that  $\mathbf{X}$  is *simple*, i.e.  $\mathbf{X}(\{x\}) \in \{0, 1\}$  almost surely (a.s.), for all  $x \in S$ , where  $\mathbf{X}(B)$  denotes the number of points of  $\mathbf{X}$  on  $B \subseteq S$ , i.e.  $\mathbf{X}(B) := \#\mathbf{X} \cap B$ . We also denote by

$$\mathbf{X}^D = \mathbf{X} \cap D = \{X_1, \dots, X_{\mathbf{X}(D)}\}$$

the restriction to  $D$  of the point process  $\mathbf{X} \equiv \{X_n\}_{1 \leq n \leq \mathbf{X}(S)}$ , where the previous notation includes the case  $\mathbf{X}(S) = \infty$ . In the following, we will denote by  $P$  the law of  $\mathbf{X}$  and by  $P_D$  the law of  $\mathbf{X}^D$ .

The correlation functions of  $\mathbf{X}$ , with respect to (*w.r.t.*) a Radon measure  $\mu$  on  $(S, \mathcal{B}_S)$ , are (if they exist) measurable functions  $\rho_k : S^k \rightarrow [0, \infty)$  such that

$$\mathbb{E} \left[ \prod_{i=1}^k \mathbf{X}(B_i) \right] = \int_{B_1 \times \dots \times B_k} \rho_k(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k).$$

for any family of mutually disjoint subsets  $B_1, \dots, B_k$  of  $S$ ,  $k \geq 1$ . We require in addition that  $\rho_k(x_1, \dots, x_k) = 0$  whenever  $x_i = x_j$  for some  $1 \leq i \neq j \leq k$ . When  $\rho_1$  exists, the measure  $\rho_1(x) \mu(dx)$  is known as the intensity measure of  $\mathbf{X}$ .

For any compact subset  $D \subseteq S$ , the Janossy densities of  $\mathbf{X}$ , *w.r.t.*  $\mu$  are (if they exist) measurable functions  $j_D^n : D^n \rightarrow \mathbb{R}$  satisfying, for all measurable functions  $f : \mathcal{N}_f^D \rightarrow \mathbb{R}$ ,

$$\mathbb{E} [f(\mathbf{X}^D)] = \sum_{n \geq 0} \frac{1}{n!} \int_{D^n} f(\{x_1, \dots, x_n\}) j_D^n(x_1, \dots, x_n) \mu(dx_1) \dots \mu(dx_n), \quad (2.2)$$

i.e.  $j_D$  is the density of  $P_D$  with respect to  $L_D^\mu$  (the restriction to  $\mathcal{N}_f^D$  of  $L^\mu$ ), when  $P_D \ll L_D^\mu$ . We remark that we shall sometimes use the simplified notation  $j_D(\mathbf{x}) := j_D^{\mathbf{x}(D)}(x_1, \dots, x_{\mathbf{x}(D)})$ .

### Kernels and integral operators

Let  $\mu$  be a Radon measure on  $(S, \mathcal{B}_S)$ . For any compact set  $D \subseteq S$ , we denote by  $L^2(D, \mu)$  the Hilbert space of complex-valued square integrable functions *w.r.t.* the restriction of the Radon measure  $\mu$  on  $D$ , equipped with the inner product

$$\langle f, g \rangle_{L^2(D, \mu)} := \int_D f(x) \overline{g(x)} \mu(dx), \quad f, g \in L^2(D, \mu)$$

where  $\bar{z}$  denotes the complex conjugate of  $z \in \mathbb{C}$ . By definition, a kernel  $K$  is a measurable function from  $S^2$  to  $\mathbb{C}$ . We say that  $K$  is locally square integrable if, for any compact  $D \subseteq S$ , we have

$$\int_{D^2} |K(x, y)|^2 \mu(dx) \mu(dy) < \infty. \quad (2.3)$$

To any locally square integrable kernel  $K$ , we associate the integral operator  $\mathcal{K}_D : L^2(D, \mu) \rightarrow L^2(D, \mu)$ , where  $D$  is a compact subset of  $S$ , defined for  $f \in L^2(D, \mu)$  by

$$\mathcal{K}_D f(x) := \int_D K(x, y) f(y) \mu(dy), \quad \text{for } \mu\text{-almost all } x \in D.$$

A straightforward application of the Cauchy-Schwarz inequality shows that the operator  $\mathcal{K}_D$  is bounded. In fact, it can be shown that  $\mathcal{K}_D$  is a compact operator.

To any locally square integrable kernel  $K$ , we associate the integral operator  $\mathcal{K}$  defined by

$$\mathcal{K}f(x) := \int_S K(x, y) f(y) \mu(dy), \quad \text{for } \mu\text{-almost all } x \in S$$

for functions  $f \in L^2(S, \mu)$  that vanish outside a compact subset of  $S$ . Letting  $\mathcal{P}_D$  denote the projection operator from  $L^2(S, \mu)$  onto  $L^2(D, \mu)$ , we have  $\mathcal{K}_D = \mathcal{P}_D \mathcal{K} \mathcal{P}_D$  and set  $K_D(x, y) := \mathbf{1}_D(x) K(x, y) \mathbf{1}_D(y)$ , for  $x, y \in S$ . The operator  $\mathcal{K}$  is said to be Hermitian or self-adjoint if

$$K(x, y) = \overline{K(y, x)}, \quad \text{for } \mu^{\otimes 2}\text{-almost all } (x, y) \in S^2. \quad (2.4)$$

Equivalently, this means that the integral operators  $\mathcal{K}_D$  are self-adjoint for any compact  $D \subseteq S$ . If  $\mathcal{K}_D$  is self-adjoint, by the spectral theorem for self-adjoint and compact operators we have that  $L^2(D, \mu)$  has an orthonormal basis  $\{\varphi_j^D\}_{j \geq 1}$  of eigenfunctions of  $\mathcal{K}_D$ . The corresponding eigenvalues  $\{\lambda_j^D\}_{j \geq 1}$  have finite multiplicity (except possibly the zero eigenvalue) and the only possible accumulation point of the eigenvalues is the zero eigenvalue. Then, the kernel  $K_D$  of  $\mathcal{K}_D$  can be written as

$$K_D(x, y) = \sum_{j \geq 1} \lambda_j^D \varphi_j^D(x) \overline{\varphi_j^D(y)}, \quad (2.5)$$

for  $x, y \in D$ . We say that an operator  $\mathcal{K}$  is positive (respectively non-negative) if its spectrum is included in  $(0, +\infty)$  (respectively  $[0, +\infty)$ ). For two operators  $\mathcal{K}$  and  $\mathcal{J}$ , we will say that  $\mathcal{K} > \mathcal{J}$  (respectively  $\mathcal{K} \geq \mathcal{J}$ ) in the operator ordering if  $\mathcal{K} - \mathcal{J}$  is a positive operator (respectively non-negative operator).

We say that a self-adjoint integral operator  $\mathcal{K}_D$  is of trace class if

$$\sum_{j \geq 1} |\lambda_j^D| < \infty.$$

We will then define the trace of the operator  $\mathcal{K}_D$  as  $\text{Tr } \mathcal{K}_D = \sum_{j \geq 1} \lambda_j^D$ . If  $\mathcal{K}_D$  is of trace class for every compact subset  $D \subseteq S$ , then we say that  $\mathcal{K}$  is locally of trace class. It is easily seen that  $\mathcal{K}^n$  is of trace class, for all  $n \geq 2$ , if  $\mathcal{K}$  is of trace class. We also have the

inequality  $\text{Tr}(\mathcal{K}^n) \leq \|\mathcal{K}\|^{n-1} \text{Tr}(\mathcal{K})$ , where  $\|\cdot\|$  denotes the operator norm. Finally, we define the Fredholm determinant of  $\mathbf{Id} + \mathcal{K}$ , where  $\|\mathcal{K}\| < 1$ , as

$$\text{Det}(\mathbf{Id} + \mathcal{K}) = \exp \left( \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \text{Tr}(\mathcal{K}^n) \right). \quad (2.6)$$

Here,  $\mathbf{Id}$  denotes the identity operator on  $L^2(S, \mu)$ .

## Determinantal point processes on $(S, d_S)$

Throughout this paper we shall work under the following hypothesis:

**(H1)**: *The operator  $\mathcal{K}$  is locally of trace class, satisfies (2.4), and its spectrum is contained in  $[0, 1)$ , i.e.  $0 \leq \mathcal{K} \leq \mathbf{Id}$  and  $\|\mathcal{K}\| < 1$ .*

Suppose that  $\mathcal{K}$  satisfies **(H1)**. A locally finite and simple point process  $\mathbf{X} \equiv \{X_n\}_{1 \leq n \leq \mathbf{X}(S)}$  on  $S$  is called a determinantal process if its correlation functions *w.r.t.* the Radon measure  $\mu$  on  $S$  exist and satisfy

$$\rho_k(x_1, \dots, x_k) = \det(K(x_i, x_j))_{1 \leq i, j \leq k},$$

for any  $k \geq 1$  and  $x_1, \dots, x_k \in S$ . It is worth noting that, under **(H1)**, we can choose a proper kernel for  $\mathcal{K}$ , in the sense of Lemma A.4 in [9], and we have  $\rho_k(x_1, \dots, x_k) \geq 0$  for  $\mu^{\otimes k}$ -a.e.  $(x_1, \dots, x_k) \in S^k$ .

Existence and uniqueness (in law) of determinantal processes is guaranteed under **(H1)** by the results in [14], [21] and [23]. See also Lemma 4.2.6 and Theorem 4.5.5 in [11]. More precisely, if a kernel  $K$  and its associated integral operator  $\mathcal{K}$  satisfy **(H1)**, then there exists a determinantal process  $\mathbf{X}$  on  $S$  with kernel  $K$ . Moreover, for any compact  $D \subseteq S$  there exist constants  $c_1(D), c_2(D) > 0$  such that  $P(\mathbf{X}(D) > k) \leq c_1(D)e^{-c_2(D)k}$  for all  $k \geq 1$ , and in this case the correlation functions  $\rho_k(x_1, \dots, x_k)$  uniquely determine the law of the process. This is because of the fact that for disjoint compacts  $D_1, \dots, D_k \subseteq S$ , the random vector  $(\mathbf{X}(D_1), \dots, \mathbf{X}(D_k))$  has a convergent Laplace transform in a neighborhood of zero, cf. [11] Remark 1.2.4.

Let  $\mathcal{K}$  be an operator satisfying the assumption **(H1)**. We define the operator  $\mathcal{J}[D]$  on  $L^2(D, \mu)$  by

$$\mathcal{J}[D] := (\mathbf{Id} - \mathcal{K}_D)^{-1} \mathcal{K}_D.$$

The operator  $\mathcal{J}[D]$  is called the local interaction operator and we emphasize the fact that unlike  $\mathcal{K}_D$ ,  $\mathcal{J}[D]$  is not a projection operator i.e., in general,  $\mathcal{J}[D] \neq \mathcal{P}_D(I - \mathcal{K})^{-1} \mathcal{K} \mathcal{P}_D$ . However,  $\mathcal{J}[D]$  has some notable properties which are summarized in [9]. Let us state the ones that are useful to our purposes. First,  $\mathcal{J}[D]$  is a bounded integral operator and, letting  $J[D]$  denote its kernel, as a consequence of (2.5), we have

$$J[D](x, y) = \sum_{j \geq 0} \frac{\lambda_j^D}{1 - \lambda_j^D} \varphi_j^D(x) \overline{\varphi_j^D(y)}, \quad (2.7)$$

for  $x, y \in D$ . Second, since  $\mathcal{J}[D] \leq (1 - \|\mathcal{K}\|)^{-1}\mathcal{K}_D$ , we have that  $\mathcal{J}[D]$  is a trace class operator. For  $\mathbf{x} = \{x_1, \dots, x_n\} \in \mathbb{N}_f^D$ , we denote by  $\det J[D](\mathbf{x}) = \det J[D](\{x_1, \dots, x_n\})$  the determinant  $\det (J[D](x_i, x_j))_{1 \leq i, j \leq n}$ . Note that the function

$$(x_1, \dots, x_n) \mapsto \det J[D](\{x_1, \dots, x_n\})$$

is  $\mu^{\otimes n}$ -a.e. non-negative (thanks to Lemma A.4 in [9] for example) and symmetric in  $x_1, \dots, x_n$  (see e.g. [9]), and we simply write  $\det J[D](\{x_1, \dots, x_n\}) = \det J[D](x_1, \dots, x_n)$ . The local interaction operator is related to the Janossy density of a determinantal process, due to the following proposition:

**Proposition 2.1** ([21]) *Assume that the kernel  $\mathcal{K}$  satisfies **(H1)**. Then, given a compact  $D \subseteq S$  and  $n \in \mathbb{N}^*$ , the determinantal process  $\mathbf{X}$  admits Janossy densities*

$$j_D^n(\mathbf{x}) = \text{Det}(\mathbf{Id} - \mathcal{K}_D) \det J[D](\mathbf{x}), \quad (2.8)$$

where  $\mathbf{x} = \{x_1, \dots, x_n\} \in \mathbb{N}_f^D$ . Moreover, it holds the following identity for the hole probability:  $j_D^0(\emptyset) = \text{Det}(\mathbf{Id} - \mathcal{K}_D)$ .

Let us now define the operator  $\mathcal{J} := (\mathbf{Id} - \mathcal{K})^{-1}\mathcal{K}$  which can be thought of as a global interaction operator. As proved in [9],  $\mathcal{J}$  satisfies the expected properties: it is a bounded integral operator, locally trace class and its kernel  $J(x, y)$  can be chosen as in Lemma A.4 of [9]. Furthermore, the determinantal process  $\mathbf{X}$  is stochastically dominated by a Poisson process  $\mathbf{Y}$  with intensity measure  $J(x, x) \mu(dx)$  (defined previously), i.e

$$\mathbb{E}[f(\mathbf{X})] \leq \mathbb{E}[f(\mathbf{Y})] \quad (2.9)$$

for any measurable function  $f : \mathbb{N}_f \rightarrow \mathbb{R}$  such that the expectations exist and  $f(\mathbf{x}) \leq f(\mathbf{y})$  whenever  $\mathbf{x} \subset \mathbf{y}$ .

### 3 Differential calculus and integration by parts

Hereafter we assume that  $S$  is a domain of  $\mathbb{R}^d$ ,  $d_S$  is the Euclidean distance,  $\mu$  is a Radon measure on  $S$  and  $D \subset S$  is a fixed bounded set. We denote by  $\|\cdot\|$  the Euclidean norm on  $\mathbb{R}^d$  and by  $x^{(i)}$  the  $i$ -th component of  $x \in \mathbb{R}^d$ .

#### Differential calculus

We denote by  $\mathcal{C}^\infty(D, \mathbb{R}^d)$  the set of all  $\mathcal{C}^\infty$ -vector fields  $v : D \rightarrow \mathbb{R}^d$  and by  $\mathcal{C}^\infty(D^k)$  the set of all  $\mathcal{C}^\infty$ -functions on  $D^k$ .

**Definition 1** *A random variable (r.v.)  $F(\mathbf{X}^D)$  is said to be in  $\mathcal{S}_D$  if*

$$F(\mathbf{X}^D) = f_0 \mathbf{1}_{\{\mathbf{x}(D)=0\}} + \sum_{k=1}^n \mathbf{1}_{\{\mathbf{x}(D)=k\}} f_k(X_1, \dots, X_k), \quad (3.1)$$

where  $n \geq 1$  is an integer, for any  $k = 1, \dots, n$ ,  $f_k \in \mathcal{C}^\infty(D^k)$  is a symmetric function and  $f_0 \in \mathbb{R}$  is a constant.

Then, the following lemma holds:

**Lemma 3.1**  $\mathcal{S}_D$  is dense in  $L_D^2 = L^2(\mathcal{N}_f^D, P_D)$ .

*Proof.* Let  $V \in L_D^2$  and assume that  $E[G(\mathbf{X}^D)V] = 0$  for all  $G(\mathbf{X}^D) \in \mathcal{S}_D$ . We need to show that  $V = 0$  a.s.. Since  $V$  is  $\mathcal{N}_f^D$ -measurable we have that  $V = u(\mathbf{X}(D), X_1, \dots, X_{\mathbf{X}(D)})$ , for some measurable function  $u$ . Therefore, for all integers  $n \geq 1$  and  $G(\mathbf{X}^D) \in \mathcal{S}_D$ , with a little abuse of notation, we get

$$G(\emptyset)u(0, \emptyset)P(\mathbf{X}(D) = 0) = 0 \quad \text{and} \quad E[G(X_1, \dots, X_n)u(n, X_1, \dots, X_n) \mid \mathbf{X}(D) = n] = 0. \quad (3.2)$$

The first equality above yields  $u(0, \emptyset) = 0$ . We now prove that, for any  $n \geq 1$ ,  $u(n, x_1, \dots, x_n) = 0$  a.s. *w.r.t.* the probability measure, say  $\pi_D^{(n)}$ , with density

$$j_D^{(n)}(x_1, \dots, x_n)\mu(dx_1) \dots \mu(dx_n)$$

(here,  $j_D^{(n)}$  are the Janossy densities of  $\mathbf{X}^D$  divided by  $P(\mathbf{X}(D) = n)$ .)

Denote by  $u_n^+$  and  $u_n^-$  the positive and the negative part of  $u(n, \cdot)$ , respectively. Clearly (take  $G \equiv 1$  in the second equality in (3.2)) we have

$$E_n := E[u_n^+(X_1, \dots, X_n) \mid \mathbf{X}(D) = n] = E[u_n^-(X_1, \dots, X_n) \mid \mathbf{X}(D) = n].$$

If  $E_n = 0$  then  $u_n^+ = u_n^- = 0$   $\pi_D^{(n)}$ -a.s., hence  $u_n = 0$   $\pi_D^{(n)}$ -a.s.. If  $E_n > 0$ , then consider the probability measures on  $D^n$ :

$$\pi_D^{(n)\pm}(dx_1 \dots dx_n) := \widehat{u}_n^\pm(x_1, \dots, x_n) \pi_D^{(n)}(dx_1, \dots, dx_n).$$

where

$$\widehat{u}_n^\pm(x_1, \dots, x_n) := \frac{1}{E_n} u_n^\pm(x_1, \dots, x_n).$$

Let  $R$  be a rectangular cell in  $\mathbb{R}^{dn}$  and take  $G(x_1, \dots, x_n) = \varphi_l(x_1, \dots, x_n)$  where  $\varphi_l$  is a sequence in  $\mathcal{C}_c^\infty(\mathbb{R}^{dn})$  such that  $\varphi_l(x_1, \dots, x_n)$  converges to  $\mathbf{1}_{\{(x_1, \dots, x_n) \in R\}}$  as  $l$  goes to infinity, for all  $x_1, \dots, x_n$ . Combining the second equality in (3.2) with the dominated convergence theorem, we have  $\pi_D^{(n)+}(R \cap D^n) = \pi_D^{(n)-}(R \cap D^n)$ . Therefore,  $\pi_D^{(n)+} \equiv \pi_D^{(n)-}$  on the Borel  $\sigma$ -field  $\mathcal{B}(D^n)$ . So  $u_n^+(x_1, \dots, x_n) = u_n^-(x_1, \dots, x_n)$   $\pi_D^{(n)}$ -a.s., and the claim follows.  $\square$

The gradient of  $F(\mathbf{X}^D) \in \mathcal{S}_D$  as in (3.1) is defined by

$$\nabla_x^{N_{lf}} F(\mathbf{X}^D) := \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \mathbf{1}_{\{X_i\}}(x) \nabla_x f_k(X_1, \dots, X_k), \quad x \in D, \quad (3.3)$$

where  $\nabla_x$  denotes the usual gradient on  $\mathbb{R}^d$  with respect to the variable  $x \in D$ . For  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ , we also let

$$\nabla_v^{N_{lf}} F(\mathbf{X}^D) := \sum_{k=1}^{\mathbf{X}(D)} \nabla_{X_k}^{N_{lf}} F(\mathbf{X}^D) \cdot v(X_k) = \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \nabla_{X_i} f_k(X_1, \dots, X_k) \cdot v(X_i), \quad (3.4)$$

where  $\cdot$  denotes the inner product on  $\mathbb{R}^d$ .



## Quasi-invariance

In this section, we will recall some results from [4] and make some of the proofs more precise. Let  $\text{Diff}_0(S)$  be the set of all diffeomorphisms from  $S$  into itself with compact support, i.e., for any  $\phi \in \text{Diff}_0(S)$ , there exists a compact outside of which  $\phi$  is the identity map. In particular, note that  $\text{Diff}_0(D)$  is the set of diffeomorphisms from  $D$  into itself. For any reference measure  $\mu$  on  $S$ ,  $\mu_\phi$  denotes the image measure of  $\mu$  by  $\phi$ . For  $\phi \in \text{Diff}_0(S)$  whose support is included in  $D$ , we introduce the isometry  $\mathcal{T}_\phi$ ,

$$\begin{aligned} \mathcal{T}_\phi : L^2(D, \mu_\phi) &\longrightarrow L^2(D, \mu) \\ f &\longmapsto f \circ \phi. \end{aligned}$$

Its inverse, which exists since  $\phi$  is a diffeomorphism, is trivially defined by  $f \mapsto f \circ \phi^{-1}$  and denoted by  $\mathcal{T}_\phi^{-1}$ . Note that  $\mathcal{T}_\phi$  and  $\mathcal{T}_\phi^{-1}$  are isometries, i.e.,

$$\langle \mathcal{T}_\phi \psi_1, \mathcal{T}_\phi \psi_2 \rangle_{L^2(D, \mu)} = \langle \psi_1, \psi_2 \rangle_{L^2(D, \mu_\phi)},$$

for any  $\psi_1$  and  $\psi_2$  belonging to  $L^2(D, \mu)$ . We also set:

$$\mathcal{K}_D^\phi = \mathcal{T}_\phi^{-1} \mathcal{K}_D \mathcal{T}_\phi \text{ and } \mathcal{J}^\phi[D] = \mathcal{T}_\phi^{-1} \mathcal{J}[D] \mathcal{T}_\phi.$$

Lastly, for any  $\mathbf{x} = \{\mathbf{x}_n\}_{1 \leq n \leq \mathbf{x}(S)} \in N_{lf}$ , we denote by  $\Phi$  the map:

$$\begin{aligned} \Phi : N_{lf} &\longrightarrow N_{lf} \\ \{\mathbf{x}_n\}_{1 \leq n \leq \mathbf{x}(S)} &\longmapsto \{\phi(\mathbf{x}_n)\}_{1 \leq n \leq \mathbf{x}(S)}. \end{aligned}$$

With the previous definitions in mind, we can give the following result from [4]:

**Lemma 3.2** *Let  $\mathcal{K}$  be an operator satisfying (H1). We have the following properties.*

- a)  $\mathcal{K}_D^\phi$  and  $\mathcal{J}^\phi[D]$  are continuous operators from  $L^2(D, \mu_\phi)$  into  $L^2(D, \mu_\phi)$ .
- b)  $\mathcal{K}_D^\phi$  is of trace class and  $\text{Tr}(\mathcal{K}_D^\phi) = \text{Tr}(\mathcal{K}_D)$ .
- c)  $\text{Det}(\mathbf{Id} - \mathcal{K}_D^\phi) = \text{Det}(\mathbf{Id} - \mathcal{K}_D)$ . This translates into the fact that  $P(\mathbf{X}(D) = 0) = P(\Phi(\mathbf{X})(D) = 0)$  which is expected since  $\phi$  is a diffeomorphism.
- d)  $\mathcal{K}_D^\phi$  is again an integral operator on  $L^2(D, \mu_\phi)$  whose kernel is given by  $(x, y) \mapsto K_D(\phi^{-1}(x), \phi^{-1}(y))$ . An analog formula also holds for the operator  $\mathcal{J}^\phi[D]$ , i.e. its kernel  $J[D]$  is given by  $(x, y) \mapsto J[D](\phi^{-1}(x), \phi^{-1}(y))$ .
- e)  $\mathcal{J}^\phi[D]$  is correctly defined as the local interaction operator associated with  $\mathcal{K}_D^\phi$ , i.e.  $(\mathbf{Id} - \mathcal{K}_D^\phi)^{-1} \mathcal{K}_D^\phi = \mathcal{J}^\phi[D]$ .

*Proof.* The proof can be found in [4], except for point e), which we now prove. We have by definition

$$\begin{aligned} (\mathbf{Id} - \mathcal{K}_D^\phi)^{-1} \mathcal{K}_D^\phi &= (\mathbf{Id} - \mathcal{T}_\phi^{-1} \mathcal{K}_D \mathcal{T}_\phi)^{-1} \mathcal{T}_\phi^{-1} \mathcal{K}_D \mathcal{T}_\phi \\ &= \mathcal{T}_\phi^{-1} (\mathbf{Id} - \mathcal{K}_D)^{-1} \mathcal{K}_D \mathcal{T}_\phi \\ &= \mathcal{J}^\phi[D], \end{aligned}$$

which is the aforementioned result. □

Then, it is known how the map  $\Phi$  transforms a determinantal process. More precisely,

**Lemma 3.3 (Theorem 7 in [4])** *Let  $\mathcal{K}_D$  be an operator satisfying **(H1)**. Then,  $\Phi(\mathbf{X})$  is again determinantal with kernel  $\mathcal{K}_D^\phi$  with respect to the measure  $\mu_\phi$  on  $(S, \mathcal{B}_S)$ .*

To prove the quasi-invariance of the determinantal measure restricted to a compact  $D \subseteq S$  with respect to the group of diffeomorphisms on  $D$ , we still need one last result:

**Lemma 3.4** *Let  $\mathcal{K}$  be an operator satisfying **(H1)**, and  $\mathcal{J}[D]$  the associated local interaction operator, with kernel  $J[D]$ . Then, we have that  $\det J[D](\mathbf{x}) > 0$ , for  $P_D$ -a.e.  $\mathbf{x} \in \mathbb{N}_f^D$ . However, we do not in general have  $\det J[D](\mathbf{x}) > 0$ , for  $L_D^\mu$ -a.e.  $\mathbf{x} \in \mathbb{N}_f^D$ , where  $L_D^\mu$  is the sample measure defined in (2.1).*

*Proof.* Recall that for a determinantal process with kernel  $\mathcal{K}$  satisfying **(H1)**, we have  $P_D \ll L_D^\mu$  and  $j_D = \frac{dP_D}{dL_D^\mu}$ . Moreover, it is known in the determinantal case that (2.8) holds, i.e.,

$$j_D(\mathbf{x}) = \text{Det}(\mathbf{Id} - \mathcal{K}_D) \det J[D](\mathbf{x}),$$

for  $\mathbf{x} \in \mathbb{N}_f^D$ . Since  $j_D$  is a density, we obviously have that  $j_D(\mathbf{x}) > 0$ , for  $P_D$ -a.e.  $\mathbf{x} \in \mathbb{N}_f^D$ . Hence, since  $\|\mathcal{K}_D\| < 1$ , we have  $\det J[D](\mathbf{x}) > 0$ , for  $P_D$ -a.e.  $\mathbf{x} \in \mathbb{N}_f^D$ . As to the concluding part of the lemma, we notice that in general, one does not have  $P_D \sim L_D^\mu$ . Indeed, consider for example the case where  $\text{Rank}(\mathcal{K}_D) \leq N \in \mathbb{N}^*$ . Then,  $j_D^{N+1}(x_1, \dots, x_{N+1}) = 0$ , for  $\mu^{\otimes(N+1)}$ -a.e.  $(x_1, \dots, x_{N+1}) \in D^{N+1}$  (since  $\mathbf{X}_D$  has less than  $N$  points almost surely, see [23] for details). It suffices to define the set

$$A := \{B \subseteq D : \#B = N + 1\},$$

which verifies  $P(A) = 0$  but  $L_D^\mu(A) = \frac{1}{n!}\mu(D)$ . □

**Remark 3.5** *If we suppose that, for any  $n \geq 1$ , the function*

$$(x_1, \dots, x_n) \longmapsto \det J[D](x_1, \dots, x_n)$$

*is strictly positive  $\mu^{\otimes n}$ -a.e. on  $D^n$ , then we have  $P_D \sim L_D^\mu$  and therefore  $\det J[D](\mathbf{x}) > 0$ , for  $L_D^\mu$ -a.a.  $\mathbf{x} \in \mathbb{N}_f^D$ .*

We are now ready to give the main result of this section. We emphasize that despite the result being the same as the one in [4], the proof given there implicitly uses the fact that  $\det J[D](\mathbf{x}) > 0$ , for  $L_D^\mu$ -a.a.  $\mathbf{x} \in \mathbb{N}_f^D$ , which is known not to be true in general. Henceforth, we shall also assume the following technical condition which ensures that there exists an integration by parts formula on the underlying space  $(S, d_S)$ .

**(H2)** : *The Radon measure  $\mu$  is absolutely continuous w.r.t. the Lebesgue measure  $\ell$  on  $S$ , with Radon-Nikodym derivative  $\rho = \frac{d\mu}{d\ell}$  which is strictly positive and continuously differentiable on  $S$ .*

Then for any  $\phi \in \text{Diff}_0(D)$ ,  $\mu_\phi$  is absolutely continuous with respect to  $\mu$  with density given by

$$p_\phi^\mu(x) = \frac{d\mu_\phi(x)}{d\mu(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)} \text{Jac}(\phi^{-1})(x), \quad (3.5)$$

where  $\text{Jac}(\phi^{-1})(x)$  is the Jacobian of  $\phi^{-1}$  at point  $x \in S$ . We draw attention to the fact that it is indeed  $\text{Jac}(\phi^{-1})(x)$  that appears in (3.5), which differs from equation (2.11) of [1]. We can now state the main result of this section:

**Proposition 3.6** *Let  $\mathcal{K}_D$  be a restriction operator satisfying (H1). Then, for any measurable non-negative  $f$  on  $D$ :*

$$\mathbb{E} \left[ e^{-\sum_{x \in \mathbf{X}^D} f \circ \phi(x)} \right] = \mathbb{E} \left[ e^{-\sum_{x \in \mathbf{X}^D} f(x)} e^{\sum_{x \in \mathbf{X}^D} \ln(p_\phi^\mu(x))} \frac{\det J^\phi[D](\mathbf{X}^D)}{\det J[D](\mathbf{X}^D)} \right]. \quad (3.6)$$

*Proof.* For any measurable non-negative  $f$  on  $D$ , we have

$$\begin{aligned} & \mathbb{E} \left[ e^{-\sum_{x \in \mathbf{X}^D} f(x)} e^{\sum_{x \in \mathbf{X}^D} \ln(p_\phi^\mu(x))} \frac{\det J^\phi[D](\mathbf{X}^D)}{\det J[D](\mathbf{X}^D)} \right] \\ &= \sum_{n \geq 0} \frac{1}{n!} \int_{D^n} e^{-\sum_{k=1}^n f(x_k)} \prod_{k=1}^n p_\phi^\mu(x_k) \\ & \quad \frac{\det J^\phi[D](x_1, \dots, x_n)}{\det J[D](x_1, \dots, x_n)} j_D(x_1, \dots, x_n) \mu(dx_1) \dots \mu(dx_n) \\ &= \sum_{n \geq 0} \frac{1}{n!} \int_{D^n} e^{-\sum_{k=1}^n f(x_k)} \prod_{k=1}^n p_\phi^\mu(x_k) \det J^\phi[D](x_1, \dots, x_n) \text{Det}(\mathbf{Id} - \mathcal{K}_D) \mu(dx_1) \dots \mu(dx_n) \\ &= \sum_{n \geq 0} \frac{1}{n!} \int_{D^n} e^{-\sum_{k=1}^n f(x_k)} \text{Det}(\mathbf{Id} - \mathcal{K}_D^\phi) \det J^\phi[D](x_1, \dots, x_n) \mu_\phi(dx_1) \dots \mu_\phi(dx_n), \end{aligned}$$

where we have used (2.8), (3.5) and Lemma 3.2, c). Then, we conclude by Lemma 3.3. Indeed,  $\text{Det}(\mathbf{Id} - \mathcal{K}_D^\phi) \det J^\phi[D](x_1, \dots, x_n)$  is the Janossy density of  $\Phi(\mathbf{X})$  with respect to  $L_D^{\mu_\phi}$  (see lemma 3.2, e)), which yields (3.6).  $\square$

### Integration by parts and closability

In this section we give an integration by parts formula for determinantal processes, based on closed gradient and divergence operators. The proof relies on an integration by parts formula on the set of test functionals  $\mathfrak{S}_D$  introduced in (3.1), extending and making more precise the arguments of Theorem 10 page 289 of [4] and its proof.

**(H3)** : *We suppose that, for any bounded set  $D$  and for any  $n \geq 1$ , the function*

$$(x_1, \dots, x_n) \longmapsto \det J[D](x_1, \dots, x_n)$$

*is continuously differentiable on the whole  $D^n$ .*

Assuming that **(H1)** and **(H3)** hold, the potential energy is the function  $U : \mathbb{N}_f^D \rightarrow \mathbb{R}$  defined by

$$U[D](\mathbf{x}) := -\log \det J[D](\mathbf{x}).$$

We insist that since  $\det J[D](\mathbf{x}) > 0$  for  $P$ -a.e.  $\mathbf{x} \in \mathbb{N}_f^D$ ,  $U$  is well defined for  $P$ -a.e.  $\mathbf{x} \in \mathbb{N}_f^D$ . We set

$$\begin{aligned} \nabla_v^{\mathbb{N}_{lf}} U[D](\mathbf{X}^D) &:= -\sum_{k=1}^{\infty} \mathbf{1}_{\{\mathbf{X}^{(D)}=k\}} \sum_{i=1}^k \frac{\nabla_{x_i} \det J[D](X_1, \dots, X_k)}{\det J[D](X_1, \dots, X_k)} \cdot v(X_i) \\ &= \sum_{k=1}^{\infty} \mathbf{1}_{\{\mathbf{X}^{(D)}=k\}} \sum_{i=1}^k U_{i,k}(X_1, \dots, X_k) \cdot v(X_i), \end{aligned} \quad (3.7)$$

for any vector field  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ .

Under Conditions **(H1)** and **(H2)** we define the vector field

$$\beta^\mu(x) := \frac{\nabla \rho(x)}{\rho(x)},$$

as well as the random variable

$$B_v^\mu(\mathbf{X}^D) := \sum_{k=1}^{\mathbf{X}^{(D)}} (-\beta^\mu(X_k) \cdot v(X_k) + \operatorname{div} v(X_k)), \quad v \in \mathcal{C}^\infty(D, \mathbb{R}^d),$$

where  $\operatorname{div}$  denotes the adjoint of the gradient  $\nabla$  on  $D$ , i.e.

$$\int_D g(x) \operatorname{div} \nabla f(x) \, dx = \int_D \nabla f(x) \cdot \nabla g(x) \, dx, \quad f, g \in \mathcal{C}^\infty(D).$$

**Lemma 3.7** *Assume that **(H1)**, **(H2)** and **(H3)** hold, and let  $D \subset S$  be a bounded set. Then, for any  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \mathcal{S}_D$  and vector field  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ , we have*

$$\mathbb{E}[G(\mathbf{X}^D) \nabla_v^{\mathbb{N}_{lf}} F(\mathbf{X}^D)] = \mathbb{E}[F(\mathbf{X}^D) \nabla_v^{\mathbb{N}_{lf}^*} G(\mathbf{X}^D)], \quad (3.8)$$

where

$$\nabla_v^{\mathbb{N}_{lf}^*} G(\mathbf{X}^D) := -\nabla_v^{\mathbb{N}_{lf}} G(\mathbf{X}^D) + G(\mathbf{X}^D) \left( -B_v^\mu(\mathbf{X}^D) + \nabla_v^{\mathbb{N}_{lf}} U[D](\mathbf{X}^D) \right).$$

*Proof.* For any vector field  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ , consider the flow  $\phi_t^v : D \rightarrow D$ ,  $t \in \mathbb{R}$ , where for a fixed  $x \in D$ , the curve  $t \mapsto \phi_t^v(x)$  is defined as the solution to the Cauchy problem

$$\frac{d}{dt} \phi_t^v(x) = v(\phi_t^v(x)), \quad \phi_0^v(x) = x.$$

For a given family of flows  $\{\phi_t^v : t \in \mathbb{R}, v \in \mathcal{C}^\infty(D, \mathbb{R}^d)\}$ , we define the mapping  $\Phi_t^v : \mathbb{N}_f^D \rightarrow \mathbb{N}_f^D$  by

$$\Phi_t^v(\mathbf{x}) := \{\phi_t^v(x) : x \in \mathbf{x}\}.$$

Following [1], for a functional  $R(\mathbf{X}^D)$  of the determinantal process, we define the gradient  $\nabla_v^{Nlf} R(\mathbf{X}^D)$  as the directional derivative along  $v$ , i.e.

$$\nabla_v^{Nlf} R(\mathbf{X}^D) := \left. \frac{d}{dt} R(\Phi_t^v(\mathbf{X}^D)) \right|_{t=0},$$

provided the derivative exists. It is easy to check that formulas (3.4) and (3.7) are consistent with this definition. Note that as observed previously, the image measure  $\mu \circ \phi_{-t}^v$  is absolutely continuous with respect to  $\mu$  on  $D$ , with Radon-Nikodym derivative

$$\frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \text{Jac}^{\phi_{-t}^v}(x),$$

where  $\text{Jac}^{\phi_{-t}^v}$  denotes the Jacobian of  $\phi_{-t}^v$ . Note also that

$$\text{Jac}^{\phi_t^v}(x) = \exp \left( - \int_0^t \text{div } v(\phi_z^v(x)) \, dz \right) \quad (3.9)$$

and therefore

$$\begin{aligned} \frac{d}{dt} \left( \frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \text{Jac}^{\phi_{-t}^v}(x) \right) &= - \exp \left( - \int_0^t \text{div } v(\phi_z^v(x)) \, dz \right) \left[ \frac{\nabla \rho(\phi_{-t}^v(x))}{\rho(x)} \cdot v(\phi_{-t}^v(x)) \right. \\ &\quad \left. + \frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \text{div } v(\phi_{-t}^v(x)) \right]. \end{aligned} \quad (3.10)$$

Using the quasi-invariance property of determinantal processes discussed in the previous section, for any  $t \in \mathbb{R}$  and  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \mathcal{S}_D$ , we have

$$\begin{aligned} &\mathbb{E}[F(\Phi_t^v(\mathbf{X}^D))G(\mathbf{X}^D)] \quad (3.11) \\ &= \mathbb{E} \left[ F(\mathbf{X}^D)G(\Phi_{-t}^v(\mathbf{X}^D)) \left( \prod_{k=1}^{\mathbf{x}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \text{Jac}^{\phi_{-t}^v}(X_k) \right) \frac{\det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \right]. \end{aligned}$$

We start by exchanging the derivative  $d/dt$  with the expectation sign  $\mathbb{E}$  in the above relation, for all  $t \in I_0$  a neighborhood of zero. This interchange will be justified by integrability after (3.12) below. In this case we have

$$\begin{aligned} &\mathbb{E} \left[ G(\mathbf{X}^D) \frac{d}{dt} F(\Phi_t^v(\mathbf{X}^D)) \right] \\ &= \mathbb{E} \left[ \left( \frac{d}{dt} G(\Phi_{-t}^v(\mathbf{X}^D)) \right) F(\mathbf{X}^D) \left( \prod_{k=1}^{\mathbf{x}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \text{Jac}^{\phi_{-t}^v}(X_k) \right) \frac{\det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \right] \\ &+ \mathbb{E} \left[ \left( \frac{d}{dt} \prod_{k=1}^{\mathbf{x}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \text{Jac}^{\phi_{-t}^v}(X_k) \right) F(\mathbf{X}^D)G(\Phi_{-t}^v(\mathbf{X}^D)) \frac{\det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \right] \\ &+ \mathbb{E} \left[ \left( \frac{d}{dt} \frac{\det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \right) F(\mathbf{X}^D)G(\Phi_{-t}^v(\mathbf{X}^D)) \prod_{k=1}^{\mathbf{x}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \text{Jac}^{\phi_{-t}^v}(X_k) \right]. \end{aligned}$$

The claimed integration by parts formula follows by evaluating the above relation at  $t = 0$ . In particular, we use (3.10) to evaluate the second term inside the expectation in the right-hand side of the above equality, and we use the relation

$$\begin{aligned} & \frac{d}{dt} \frac{\det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \\ &= - \sum_{i=1}^{\mathbf{x}(D)} \frac{\nabla_{x_i} \det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \cdot v(\phi_{-t}^v(X_i)) \end{aligned} \quad (3.12)$$

to evaluate the third term inside the expectation in the right-hand side of the above equality. Equality (3.12) holds  $P$ -a.e. which is enough for our purposes. Using the definition of functionals in  $\mathcal{S}_D$ , one checks that the r.v.

$$G(\mathbf{X}^D) \frac{d}{dt} F(\Phi_t^v(\mathbf{X}^D))$$

is uniformly bounded in  $t \in I_0$  by a positive constant. By the assumptions **(H2)** and **(H3)** and the form (3.1) of the functionals in  $\mathcal{S}_D$ , one may easily check that all the terms inside the expectations in the right-hand side of the above equality can be uniformly bounded in  $t \in I_0$  by integrable r.v.'s and this justifies the exchange of expectation and derivative in (3.11). We check this fact only for the latter term. Take

$$F(\mathbf{X}^D) = f_0 \mathbf{1}_{\{\mathbf{x}(D)=0\}} + \sum_{k=1}^n \mathbf{1}_{\{\mathbf{x}(D)=k\}} f_k(X_1, \dots, X_k)$$

of the form (3.1). By (3.12) we easily see that the modulus of the r.v.

$$\left( \frac{d}{dt} \frac{\det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_{\mathbf{x}(D)}))}{\det J[D](X_1, \dots, X_{\mathbf{x}(D)})} \right) F(\mathbf{X}^D) G(\Phi_{-t}^v(\mathbf{X}^D)) \prod_{k=1}^{\mathbf{x}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \text{Jac}^{\phi_{-t}^v}(X_k)$$

up to a positive constant is bounded above  $P$ -a.e. by

$$\sum_{k=1}^n \mathbf{1}_{\{\mathbf{x}(D)=k\}} \left( \prod_{i=1}^k \frac{\rho(\phi_{-t}^v(X_i))}{\rho(X_i)} \text{Jac}^{\phi_{-t}^v}(X_i) \right) \sum_{i=1}^k \left| \frac{\nabla_{x_i} \det J[D](\phi_{-t}^v(X_1), \dots, \phi_{-t}^v(X_k))}{\det J[D](X_1, \dots, X_k)} \cdot v(\phi_{-t}^v(X_i)) \right| \quad (3.13)$$

Since  $\rho$  is continuous on  $D$ , for any  $n \geq 1$ , the map

$$(x_1, \dots, x_n) \longmapsto \det J[D](x_1, \dots, x_n)$$

is continuously differentiable on  $D^n$ , and the Jacobian is given by (3.9), up to a positive constant. The term in (3.13) is bounded above  $P$ -a.e., uniformly in  $t \in I_0$ , by

$$\sum_{k=1}^n \mathbf{1}_{\{\mathbf{x}(D)=k\}} \frac{\rho(X_1)^{-1} \cdots \rho(X_k)^{-1}}{\det J[D](X_1, \dots, X_k)}.$$

To conclude the proof, we only need to check that the mean of this r.v. is finite. We have by definition of the Janossy densities, and since  $\det J[D](\mathbf{x}) > 0$ , for  $P_D$ -a.e.  $\mathbf{x} \in N_f^D$ :

$$\begin{aligned} \mathbb{E} \left[ \mathbf{1}_{\{\mathbf{x}^{(D)}=k\}} \frac{\rho(X_1)^{-1} \cdots \rho(X_k)^{-1}}{\det J[D](X_1, \dots, X_k)} \right] &= \frac{1}{k!} \int_{D^k} \frac{j_D^k(x_1, \dots, x_k)}{\det J[D](x_1, \dots, x_k)} \mathbf{1}_{\{j_D^k(x_1, \dots, x_k) > 0\}} dx_1 \cdots dx_k \\ &= \frac{\text{Det}(\mathbf{Id} - \mathcal{K}_D)}{k!} \ell(D^k) < \infty, \end{aligned}$$

where  $\ell$  denotes the Lebesgue measure. □

**Remark 3.8** *We remark that there is a sign change in (3.8), as compared to the results of [4], which is justified by the corrected formula for (3.5). This corrected version is also more in line with the corresponding integration by parts for the Poisson point process.*

Next, we extend the integration by parts formula by closability to a larger class of functionals. For  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ , we consider the closability of the linear operators  $\nabla_v^{N_{lf}} : \mathcal{S}_D \rightarrow L_D^2$  and  $\nabla_v^{N_{lf}^*} : \mathcal{S}_D \rightarrow L_D^2$  defined, respectively, by  $F(\mathbf{X}^D) \mapsto \nabla_v^{N_{lf}} F(\mathbf{X}^D)$  and  $F(\mathbf{X}^D) \mapsto \nabla_v^{N_{lf}^*} F(\mathbf{X}^D)$ . In addition we state our extension of the integration by parts formula (3.8) by closability. In the following, we denote by  $\overline{A}$  the minimal closed extension of a closable linear operator  $A$ , and by  $\text{Dom}(\overline{A})$  the domain of  $\overline{A}$ .

**Theorem 3.9** *Assume that (H1), (H2) and (H3) hold, and that for any bounded set  $D \subset S$ ,*

$$\int_{D^n} \left| \frac{\partial_{x_i^{(h)}} \det J[D](x_1, \dots, x_n) \partial_{x_j^{(k)}} \det J[D](x_1, \dots, x_n)}{\det J[D](x_1, \dots, x_n)} \right| \mathbf{1}_{\{\det J[D](x_1, \dots, x_n) > 0\}} \mu(dx_1) \cdots \mu(dx_n) < \infty \quad (3.14)$$

for any  $n \geq 1$ ,  $1 \leq i, j \leq n$  and  $1 \leq h, k \leq d$ . Then

- (i) *the linear operators  $\nabla_v^{N_{lf}}$  and  $\nabla_v^{N_{lf}^*}$  are well-defined and closable for any vector field  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ . In particular, we have*

$$\nabla_v^{N_{lf}}(\mathcal{S}_D) \subset L_D^2 \quad \text{and} \quad \nabla_v^{N_{lf}^*}(\mathcal{S}_D) \subset L_D^2.$$

- (ii) *for any vector field  $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$ , we have*

$$\mathbb{E} \left[ \overline{\mathbf{G} \nabla_v^{N_{lf}} \mathbf{F}} \right] = \mathbb{E} \left[ \overline{\mathbf{F} \nabla_v^{N_{lf}^*} \mathbf{G}} \right]$$

for all  $\mathbf{F} \in \text{Dom}(\overline{\nabla_v^{N_{lf}}})$ ,  $\mathbf{G} \in \text{Dom}(\overline{\nabla_v^{N_{lf}^*}})$  in the domains of the minimal closed extensions of  $\nabla_v^{N_{lf}}$  and  $\nabla_v^{N_{lf}^*}$ .

Note that under the assumptions (H1), (H2) and (H3), condition (3.14) is satisfied if, for any  $n \geq 1$  and any  $D \subset S$ , the function

$$(x_1, \dots, x_n) \mapsto \det J[D](x_1, \dots, x_n),$$

is strictly positive on the compact  $D^n$ .

*Proof of Theorem 3.9.* (i) Let  $v \in \mathcal{C}_c^\infty(D, \mathbb{R}^d)$  and  $F(\mathbf{X}^D) \in \mathcal{S}_D$ . For ease of notation, throughout this proof we write  $\nabla_v$  in place of  $\nabla_v^{N_{lf}}$  and  $\nabla_v^*$  in place of  $\nabla_v^{N_{lf}^*}$ . We clearly have

$$|\nabla_v F(\mathbf{X}^D)| \leq C$$

for some constant  $C > 0$ , almost surely, and therefore  $\nabla_v(\mathcal{S}_D) \subset L_D^2$ . The claim  $\nabla_v^*(\mathcal{S}_D) \subset L_D^2$  follows if we check that  $\|G(\mathbf{X}^D)\nabla_v U[D](\mathbf{X}^D)\|_{L_D^2} < \infty$  and  $\|G(\mathbf{X}^D)B_v^\mu(\mathbf{X}^D)\|_{L_D^2} < \infty$  for any  $G(\mathbf{X}^D) \in \mathcal{S}_D$ . The latter relation easily follows noticing that

$$|G(\mathbf{X}^D)B_v^\mu(\mathbf{X}^D)| \leq C$$

for some constant  $C > 0$ , almost surely. Taking

$$G(\mathbf{X}^D) = g_0 \mathbf{1}_{\{\mathbf{X}(D)=0\}} + \sum_{k=1}^m \mathbf{1}_{\{\mathbf{X}(D)=k\}} g_k(X_1, \dots, X_k)$$

of the form (3.1), by (3.7) we have

$$G(\mathbf{X}^D)\nabla_v U[D](\mathbf{X}^D) = - \sum_{k=1}^m \mathbf{1}_{\{\mathbf{X}(D)=k\}} g_k(X_1, \dots, X_k) \sum_{i=1}^k \frac{\nabla_{x_i} \det J[D](X_1, \dots, X_k)}{\det J[D](X_1, \dots, X_k)} \cdot v(X_i),$$

and for some positive constant  $C > 0$ ,

$$\begin{aligned} & \|G(\mathbf{X}^D)\nabla_v U[D](\mathbf{X}^D)\|_{L_D^2}^2 \\ &= \sum_{k=1}^m \frac{1}{k!} \int_{D^k} g_k^2(x_1, \dots, x_k) \\ & \quad \mathbf{1}_{\{\det J[D](x_1, \dots, x_k) > 0\}} \left( \sum_{i=1}^k \frac{\nabla_{x_i} \det J[D](x_1, \dots, x_k)}{\det J[D](x_1, \dots, x_k)} \cdot v(x_i) \right)^2 j_D^k(x_1, \dots, x_k) \mu(dx_1) \cdots \mu(dx_k) \\ &= \text{Det}(\mathbf{Id} - \mathcal{K}_D) \sum_{k=1}^m \frac{1}{k!} \int_{D^k} \frac{g_k^2(x_1, \dots, x_k)}{\det J[D](x_1, \dots, x_k)} \\ & \quad \mathbf{1}_{\{\det J[D](x_1, \dots, x_k) > 0\}} \left( \sum_{i=1}^k \nabla_{x_i} \det J[D](x_1, \dots, x_k) \cdot v(x_i) \right)^2 \mu(dx_1) \cdots \mu(dx_k) \\ &\leq C \text{Det}(\mathbf{Id} - \mathcal{K}_D) \sum_{k=1}^m \frac{1}{k!} \sum_{1 \leq i, j \leq k} \int_{D^k} \mathbf{1}_{\{\det J[D](x_1, \dots, x_k) > 0\}} \\ & \quad \frac{\nabla_{x_i} \det J[D](x_1, \dots, x_k) \cdot v(x_i) \nabla_{x_j} \det J[D](x_1, \dots, x_k) \cdot v(x_j)}{\det J[D](x_1, \dots, x_k)} \mu(dx_1) \cdots \mu(dx_k) \\ &< \infty, \end{aligned}$$

where the latter integral is finite by assumption (3.14).



To conclude, we only need to show that  $\nabla_v$  is closable (the closability of  $\nabla_v^*$  can be proved similarly). Let  $(F_n(\mathbf{X}^D))_{n \geq 1}$  be a sequence in  $\mathcal{S}_D$  converging to 0 in  $L_D^2$  and such that  $\nabla_v F_n(\mathbf{X}^D)$  converges to  $V$  in  $L_D^2$  as  $n$  goes to infinity. We need to show that  $V = 0$  a.s. We have

$$\begin{aligned} |\mathbb{E}[G(\mathbf{X}^D)V]| &= \lim_{n \rightarrow \infty} |\mathbb{E}[G(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)]| = \lim_{n \rightarrow \infty} |\mathbb{E}[F_n(\mathbf{X}^D)\nabla_v^* G(\mathbf{X}^D)]| \\ &\leq \|\nabla_v^* G(\mathbf{X}^D)\|_{L_D^2} \lim_{n \rightarrow \infty} \|F_n(\mathbf{X}^D)\|_{L_D^2} = 0, \quad G \in \mathcal{S}_D. \end{aligned} \quad (3.15)$$

Here, the second inequality in (3.15) follows by the integration by parts formula (3.8) of Lemma 3.7. The fact that  $\mathbb{E}[G(\mathbf{X}^D)V] = 0$  for all  $G(\mathbf{X}^D) \in \mathcal{S}_D$  implies  $V = 0$  a.s. is a consequence of the density Lemma 3.1.

(ii) By (i), both operators  $\nabla_v$  and  $\nabla_v^*$  are closable. Take  $\mathbf{F} \in \text{Dom}(\overline{\nabla_v})$ ,  $\mathbf{G} \in \text{Dom}(\overline{\nabla_v^*})$  and let  $(F_n(\mathbf{X}^D))_{n \geq 1}$ ,  $(G_n(\mathbf{X}^D))_{n \geq 1}$  be sequences in  $\mathcal{S}_D$  such that  $F_n(\mathbf{X}^D)$  converges to  $\mathbf{F}$ ,  $G_n(\mathbf{X}^D)$  converges to  $\mathbf{G}$ ,  $\nabla_v F_n(\mathbf{X}^D)$  converges to  $\overline{\nabla_v} \mathbf{F}$  and  $\nabla_v^* G_n(\mathbf{X}^D)$  converges to  $\overline{\nabla_v^*} \mathbf{G}$  in  $L_D^2$  as  $n$  goes to infinity. By Lemma 3.7 the integration by parts formula applies to r.v.'s in  $\mathcal{S}_D$ , therefore we have  $\mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] = \mathbb{E}[F_n(\mathbf{X}^D)\nabla_v^* G_n(\mathbf{X}^D)]$  for all  $n \geq 1$ . The claim follows if we prove that

$$\lim_{n \rightarrow \infty} \mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] = \mathbb{E}[\mathbf{G}\overline{\nabla_v} \mathbf{F}]$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}[F_n(\mathbf{X}^D)\nabla_v^* G_n(\mathbf{X}^D)] = \mathbb{E}[\mathbf{F}\overline{\nabla_v^*} \mathbf{G}].$$

We only show the first limit above; the second limit being proved similarly. We have

$$\begin{aligned} &|\mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] - \mathbb{E}[\mathbf{G}\overline{\nabla_v} \mathbf{F}]| \\ &= |\mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] - \mathbb{E}[G_n(\mathbf{X}^D)\overline{\nabla_v} \mathbf{F}] + \mathbb{E}[G_n(\mathbf{X}^D)\overline{\nabla_v} \mathbf{F}] - \mathbb{E}[\mathbf{G}\overline{\nabla_v} \mathbf{F}]| \\ &\leq |\mathbb{E}[G_n(\mathbf{X}^D)(\nabla_v F_n(\mathbf{X}^D) - \overline{\nabla_v} \mathbf{F})]| + |\mathbb{E}[(G_n(\mathbf{X}^D) - \mathbf{G})\overline{\nabla_v} \mathbf{F}]| \\ &\leq \|G_n(\mathbf{X}^D)\|_{L_D^2} \|\nabla_v F_n(\mathbf{X}^D) - \overline{\nabla_v} \mathbf{F}\|_{L_D^2} + \|G_n(\mathbf{X}^D) - \mathbf{G}\|_{L_D^2} \|\overline{\nabla_v} \mathbf{F}\|_{L_D^2}, \end{aligned}$$

which tends to 0 as  $n$  goes to infinity. □

## 4 Dirichlet forms

In this section we construct the symmetric Dirichlet form associated to a determinantal process (see Theorem 4.1 below.)

We start by recalling some definitions related to bilinear forms (see [18] for details). Let  $H$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and  $\mathcal{A} : \text{Dom}(\mathcal{A}) \times \text{Dom}(\mathcal{A}) \rightarrow \mathbb{R}$  a bilinear form defined on a dense subspace  $\text{Dom}(\mathcal{A})$  of  $H$ , the domain of  $\mathcal{A}$ . The form  $\mathcal{A}$  is said to be symmetric if  $\mathcal{A}(x, y) = \mathcal{A}(y, x)$ , for any  $x, y \in \text{Dom}(\mathcal{A})$ , and non-negative definite if  $\mathcal{A}(x, x) \geq 0$ , for any  $x \in \text{Dom}(\mathcal{A})$ . Let  $\mathcal{A}$  be symmetric and non-negative definite,  $\mathcal{A}$  is said closed if  $\text{Dom}(\mathcal{A})$  equipped with the norm

$$\|x\|_{\mathcal{A}} := \sqrt{\mathcal{A}(x, x) + \langle x, x \rangle}, \quad x \in \text{Dom}(\mathcal{A}),$$

is a Hilbert space. A symmetric and non-negative definite bilinear form  $\mathcal{A}$  is said closable if, for any sequence  $(x_n)_{n \geq 1} \subset \text{Dom}(\mathcal{A})$  such that  $x_n$  goes to 0 in  $H$  and  $(x_n)_{n \geq 1}$  is Cauchy *w.r.t.*  $\|\cdot\|_{\mathcal{A}}$  it holds that  $\mathcal{A}(x_n, x_n)$  converges to 0 in  $\mathbb{R}$  as  $n$  goes to infinity. Let  $\overline{\mathcal{A}}$  be closable and denote by  $\text{Dom}(\overline{\mathcal{A}})$  the completion of  $\text{Dom}(\mathcal{A})$  *w.r.t.* the norm  $\|\cdot\|_{\mathcal{A}}$ . It turns out that  $\mathcal{A}$  is uniquely extended to  $\text{Dom}(\overline{\mathcal{A}})$  by the closed, symmetric and non-negative definite bilinear form

$$\overline{\mathcal{A}}(x, y) = \lim_{n \rightarrow \infty} \mathcal{A}(x_n, y_n), \quad (x, y) \in \text{Dom}(\overline{\mathcal{A}}) \times \text{Dom}(\overline{\mathcal{A}}),$$

where  $\{(x_n, y_n)\}_{n \geq 1}$  is any sequence in  $\text{Dom}(\mathcal{A}) \times \text{Dom}(\mathcal{A})$  such that  $(x_n, y_n)$  converges to  $(x, y) \in \text{Dom}(\overline{\mathcal{A}}) \times \text{Dom}(\overline{\mathcal{A}})$  *w.r.t.* the norm  $\|\cdot\|_{\overline{\mathcal{A}}} + \|\cdot\|_{\overline{\mathcal{A}}}$ . A symmetric, non-negative definite and closed bilinear form  $\mathcal{A}$  is said symmetric coercive closed form if the following weak sector condition is satisfied:

$$\exists \text{ a constant } c > 0 \text{ such that } |\mathcal{A}^1(x, y)| \leq c \mathcal{A}^1(x, x)^{1/2} \mathcal{A}^1(y, y)^{1/2}, \quad x, y \in \text{Dom}(\mathcal{A}), \quad (4.1)$$

where

$$\mathcal{A}^1(x, y) := \mathcal{A}(x, y) + \langle x, y \rangle.$$

Suppose  $H = L^2(B, \mathcal{B}, \beta)$  where  $(B, \mathcal{B}, \beta)$  is a measure space. In such a case a symmetric coercive closed form  $\mathcal{A}$  is said to be a symmetric Dirichlet form if

$$\mathcal{A}(\min\{f^+, 1\}, \min\{f^+, 1\}) \leq \mathcal{A}(f, f), \mathcal{A}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{A}(f, f), \quad f \in \text{Dom}(\mathcal{A}),$$

where  $f^+$  denotes the positive part of  $f$ . Suppose that  $B$  is a Hausdorff topological space and let  $\mathcal{A}$  be a symmetric Dirichlet form. An  $\mathcal{A}$ -nest is an increasing sequence  $(C_n)_{n \geq 1}$  of closed subsets of  $B$  such that

$$\bigcup_{n \geq 1} \{f \in \text{Dom}(\mathcal{A}) : f = 0 \text{ } \beta\text{-a.e. on } B \setminus C_n\}$$

is dense in  $\text{Dom}(\mathcal{A})$  *w.r.t.* the norm  $\|\cdot\|_{\mathcal{A}}$ . We say that a subset  $B' \subset B$  is  $\mathcal{A}$ -exceptional if there exists an  $\mathcal{A}$ -nest  $(C_n)_{n \geq 1}$  with  $B' \subset B \setminus \bigcup_{n \geq 1} C_n$ . Throughout this paper we say that a property holds  $\mathcal{A}$ -almost everywhere ( $\mathcal{A}$ -a.e.) if it holds up to an  $\mathcal{A}$ -exceptional set. Moreover, a function  $f : B \rightarrow \mathbb{R}$  is called  $\mathcal{A}$ -almost continuous ( $\mathcal{A}$ -a.c.) if there exists an  $\mathcal{A}$ -nest  $(C_n)_{n \geq 1}$  such that the restriction  $f|_{C_n}$  of  $f$  to  $C_n$  is continuous for each  $n \geq 1$ .

Let  $B$  again a Hausdorff topological space. A symmetric Dirichlet form  $\mathcal{A}$  on the Hilbert space  $L^2(B, \mathcal{B}(B), \beta)$  is called quasi-regular if:

- (i) There exists an  $\mathcal{A}$ -nest  $(C_n)_{n \geq 1}$  consisting of compact sets.
- (ii) There exists a  $\|\cdot\|_{\mathcal{A}}$ -dense subset of  $\text{Dom}(\mathcal{A})$  whose elements have  $\mathcal{A}$ -a.c.  $\beta$ -versions.
- (iii) There exist  $f_k \in \text{Dom}(\mathcal{A})$ ,  $k \geq 1$ , having  $\mathcal{A}$ -a.c.  $\beta$ -versions  $\tilde{f}_k$ ,  $k \geq 1$ , such that  $(\tilde{f}_k)_{k \geq 1}$  is a separating set for  $B \setminus N$  (i.e. for any  $x, y \in B \setminus N$ ,  $x \neq y$ , there exists  $\tilde{f}_{k^*}$  such that  $\tilde{f}_{k^*}(x) \neq \tilde{f}_{k^*}(y)$ ), where  $N$  is a subset of  $B$  which is  $\mathcal{A}$ -exceptional.

We denote by  $\ddot{N}_f^D$  the set of non-negative integer-valued finite measures on  $D$ , equipped with the vague topology (recall that this topology is metrized by the so-called "d-hash" metric, see e.g. Appendix A2 pages 402-405 in [5]). For technical reasons, in this section and Section 5 we shall see  $N_f^D$  as a subspace of  $\ddot{N}_f^D$ , via the identification

$$\mathbf{x} \equiv \sum_{x \in \mathbf{x}} \delta_x, \quad \mathbf{x} \in N_f^D,$$

where  $\delta_x$  denotes the Dirac measure at  $x \in D$ . We shall denote by  $\mathcal{B}(\dot{\mathbb{N}}_f^D)$  the corresponding Borel  $\sigma$ -field. Letting  $\mathbf{X}$  denote a determinantal point process with kernel  $K$ , using obvious notation, we shall identify  $L_D^2$  with  $L^2(\dot{\mathbb{N}}_f^D, P_D)$ .

We consider the bilinear map  $\mathcal{E}_D$  defined on  $\mathcal{S}_D \times \mathcal{S}_D$  by

$$\mathcal{E}_D(F(\mathbf{X}^D), G(\mathbf{X}^D)) := \mathbb{E} \left[ \sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{N_{if}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{N_{if}} G(\mathbf{X}^D) \right].$$

For  $F(\mathbf{X}^D) \in \mathcal{S}_D$  of the form (3.1), i.e.

$$F(\mathbf{X}^D) = f_0 \mathbf{1}_{\{\mathbf{X}(D)=0\}} + \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} f_k(X_1, \dots, X_k),$$

we also define the linear Laplace operator  $\mathcal{H}_D$  by

$$\mathcal{H}_D F(\mathbf{X}^D) = \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k (-\beta^\mu(X_i) \cdot \nabla_{x_i} f_k(X_1, \dots, X_k) - \Delta_{x_i} f_k(X_1, \dots, X_k) + U_{i,k}(X_1, \dots, X_k) \cdot \nabla_{x_i} f_k(X_1, \dots, X_k)),$$

where  $\Delta = -\operatorname{div} \nabla$  denotes the Laplacian.

In the following, we consider the subspace  $\tilde{\mathcal{S}}_D$  of  $\mathcal{S}_D$  made of r.v.'s  $F(\mathbf{X}^D) \in \mathcal{S}_D$  of the form

$$F(\mathbf{X}^D) = f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_M(X_k) \right) \mathbf{1}_{\{\mathbf{X}(D) \leq K\}},$$

for some integers  $M, K \geq 1$ ,  $\varphi_1, \dots, \varphi_M \in \mathcal{C}^\infty(D)$ ,  $f \in \mathcal{C}_b^\infty(\mathbb{R}^M)$ . Note that  $\tilde{\mathcal{S}}_D$  is dense in  $L_D^2$  (see e.g. [18] p. 54).

The next Theorem 4.1 provides the Dirichlet form associated to a determinantal process.

**Theorem 4.1** *Let  $D \subset S$  be a bounded set. Under the assumptions of Theorem 3.9, we have:*

- (i) *The linear operator  $\mathcal{H}_D : \tilde{\mathcal{S}}_D \rightarrow L_D^2$  is symmetric, non-negative definite and well-defined, i.e.  $\mathcal{H}_D(\tilde{\mathcal{S}}_D) \subset L_D^2$ . In particular the operator square root  $\mathcal{H}_D^{1/2}$  of  $\mathcal{H}_D$  exists.*
- (ii) *The bilinear form  $\mathcal{E}_D : \tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D \rightarrow \mathbb{R}$  is symmetric, non-negative definite and well-defined, i.e.  $\mathcal{E}_D(\tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D) \subset \mathbb{R}$ .*
- (iii)  *$\mathcal{H}_D^{1/2}$  and  $\mathcal{E}_D$  are closable and the following relation holds:*

$$\overline{\mathcal{E}_D}(\mathbf{F}, \mathbf{G}) = \mathbb{E}[\overline{\mathcal{H}_D^{1/2} \mathbf{F}} \overline{\mathcal{H}_D^{1/2} \mathbf{G}}], \quad \forall \mathbf{F}, \mathbf{G} \in \operatorname{Dom}(\overline{\mathcal{H}_D^{1/2}}). \quad (4.2)$$

- (iv) *The bilinear form  $(\overline{\mathcal{E}_D}, \operatorname{Dom}(\overline{\mathcal{H}_D^{1/2}}))$  is a symmetric Dirichlet form.*

*Proof.* The proof of this theorem is based on Lemma 4.2 below.

(i) By Relation (4.4) in Lemma 4.2 we easily deduce that, for any  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$  we have

$$\mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] = \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D G(\mathbf{X}^D)] \quad \text{and} \quad \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] \geq 0.$$

Therefore,  $\mathcal{H}_D$  is symmetric and non-negative definite. It remains to check that, under the foregoing assumptions,  $\mathcal{H}_D$  is well-defined. Let  $F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$  be of the form

$$\begin{aligned} F(\mathbf{X}^D) &= \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}^{(D)}=k\}} h \left( \sum_{i=1}^k \varphi_1(X_i), \dots, \sum_{i=1}^k \varphi_m(X_i) \right) \\ &= \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}^{(D)}=k\}} h_k(X_1, \dots, X_k) \end{aligned}$$

for some integers  $m, n \geq 1$ ,  $\varphi_1, \dots, \varphi_m \in \mathcal{C}^\infty(D)$ ,  $h \in \mathcal{C}_b^\infty(\mathbb{R}^m)$ . By the definition of  $\mathcal{H}_D$ , for the well-definiteness of  $\mathcal{H}_D$  we only need to check that  $\|\mathbf{1}_{\{\mathbf{X}^{(D)}=k\}} B_{\nabla_{x_i} h_k}^\mu(\mathbf{X}^D)\|_{L_D^2} < \infty$  and  $\|\mathbf{1}_{\{\mathbf{X}^{(D)}=k\}} \nabla_{\nabla_{x_i} h_k}^{N_{if}} U[D](\mathbf{X}^D)\|_{L_D^2} < \infty$ . The assumptions guarantee these relations, cf. the proof of Theorem 3.9.

(ii) The symmetry and non-negative definiteness of  $\mathcal{E}_D$  follow from Lemma 4.2 below. It remains to check that, under the foregoing assumptions,  $\mathcal{E}_D$  is well-defined. By Step (i), for any  $F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ , we have  $\mathcal{H}_D F(\mathbf{X}^D) \in L_D^2$ . We conclude the proof by noting that, by Lemma 4.2, for any  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$  and some positive constant  $c > 0$ , we have

$$|\mathcal{E}_D(F(\mathbf{X}^D), G(\mathbf{X}^D))| = |\mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)]| \leq c \|\mathcal{H}_D F(\mathbf{X}^D)\|_{L_D^2} < \infty.$$

(iii) We first show that  $\mathcal{E}_D$  is closable. We apply Lemma 3.4 page 29 in [18]. We start checking that  $\mathcal{E}_D$  satisfies the weak sector condition (4.1). By Relation (4.5) in Lemma 4.2 we have, for any  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ ,

$$\begin{aligned} |\mathcal{E}_D^1(F(\mathbf{X}^D), G(\mathbf{X}^D))| &= |\mathbb{E}[\mathcal{H}_D^{1/2} F(\mathbf{X}^D)\mathcal{H}_D^{1/2} G(\mathbf{X}^D)] + \mathbb{E}[G(\mathbf{X}^D)F(\mathbf{X}^D)]| \\ &\leq \|\mathcal{H}_D^{1/2} F(\mathbf{X}^D)\|_{L_D^2} \|\mathcal{H}_D^{1/2} G(\mathbf{X}^D)\|_{L_D^2} + \|F(\mathbf{X}^D)\|_{L_D^2} \|G(\mathbf{X}^D)\|_{L_D^2} \\ &\leq 2\mathcal{E}_D^1(F(\mathbf{X}^D), F(\mathbf{X}^D))^{1/2} \mathcal{E}_D^1(G(\mathbf{X}^D), G(\mathbf{X}^D))^{1/2}. \end{aligned}$$

It remains to check that if  $(F_n(\mathbf{X}^D))_{n \geq 1} \subset \tilde{\mathcal{S}}_D$  is such that  $F_n(\mathbf{X}^D)$  converges to 0 in  $L_D^2$ , then  $\mathcal{E}_D(G(\mathbf{X}^D), F_n(\mathbf{X}^D))$  converges to 0, for any  $G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ . This easily follows by Lemma 4.2, the Cauchy-Schwarz inequality and the fact that  $\mathcal{H}_D G(\mathbf{X}^D)$  is square integrable (see the proof of Step (i)). The closability of  $\mathcal{H}_D^{1/2}$  follows by the closability of  $\mathcal{E}$ , Relation (4.5) in Lemma 4.2 and Remark 3.2 (i) page 29 in [18]. Finally, we prove Relation (4.2). Take  $\mathbf{F}, \mathbf{G} \in \text{Dom}(\overline{\mathcal{H}_D^{1/2}})$  and let  $(F_n(\mathbf{X}^D))_{n \geq 1}, (G_n(\mathbf{X}^D))_{n \geq 1}$  be sequences in  $\tilde{\mathcal{S}}_D$  such that  $F_n(\mathbf{X}^D)$  converges to  $\mathbf{F}$ ,  $G_n(\mathbf{X}^D)$  converges to  $\mathbf{G}$ ,  $\mathcal{H}_D^{1/2} F_n(\mathbf{X}^D)$  converges to  $\mathcal{H}_D^{1/2} \mathbf{F}$ , and  $\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D)$  converges to  $\mathcal{H}_D^{1/2} \mathbf{G}$  in  $L_D^2$  as  $n$  goes to infinity. By Lemma 4.2 we have

$$\mathcal{E}_D(F_n(\mathbf{X}^D), G_n(\mathbf{X}^D)) = \mathbb{E}[\mathcal{H}_D^{1/2} F_n(\mathbf{X}^D)\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D)], \quad \text{for all } n \geq 1.$$

The claim follows if we prove

$$\lim_{n \rightarrow \infty} \mathbb{E}[\mathcal{H}_D^{1/2} F_n(\mathbf{X}^D) \mathcal{H}_D^{1/2} G_n(\mathbf{X}^D)] = \mathbb{E}[\overline{\mathcal{H}_D^{1/2} \mathbf{F}} \overline{\mathcal{H}_D^{1/2} \mathbf{G}}].$$

We have:

$$\begin{aligned} & |\mathbb{E}[\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) \mathcal{H}_D^{1/2} F_n(\mathbf{X}^D)] - \mathbb{E}[\overline{\mathcal{H}_D^{1/2} \mathbf{G}} \overline{\mathcal{H}_D^{1/2} \mathbf{F}}]| \\ &= |\mathbb{E}[\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) \mathcal{H}_D^{1/2} F_n(\mathbf{X}^D)] - \mathbb{E}[\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) \overline{\mathcal{H}_D^{1/2} \mathbf{F}}]| \\ &\quad + |\mathbb{E}[\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) \overline{\mathcal{H}_D^{1/2} \mathbf{F}}] - \mathbb{E}[\overline{\mathcal{H}_D^{1/2} \mathbf{G}} \overline{\mathcal{H}_D^{1/2} \mathbf{F}}]| \\ &\leq |\mathbb{E}[\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) (\mathcal{H}_D^{1/2} F_n(\mathbf{X}^D) - \overline{\mathcal{H}_D^{1/2} \mathbf{F}})]| + |\mathbb{E}[(\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) - \overline{\mathcal{H}_D^{1/2} \mathbf{G}}) \overline{\mathcal{H}_D^{1/2} \mathbf{F}}]| \\ &\leq \|\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D)\|_{L_D^2} \|\mathcal{H}_D^{1/2} F_n(\mathbf{X}^D) - \overline{\mathcal{H}_D^{1/2} \mathbf{F}}\|_{L_D^2} \\ &\quad + \|\mathcal{H}_D^{1/2} G_n(\mathbf{X}^D) - \overline{\mathcal{H}_D^{1/2} \mathbf{G}}\|_{L_D^2} \|\overline{\mathcal{H}_D^{1/2} \mathbf{F}}\|_{L_D^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(iv) The bilinear form  $(\overline{\mathcal{E}_D}, \text{Dom}(\overline{\mathcal{H}_D^{1/2}}))$  defined by (4.2) is clearly symmetric, non-negative definite, and closed. Using the Cauchy-Schwarz inequality and equality (4.2) (i.e. reasoning similarly as the first part of Step (iii)) it is easily checked that the weak sector condition (4.1) holds. So by Definition 2.4 page 16 in [18] we have that  $(\overline{\mathcal{E}_D}, \text{Dom}(\overline{\mathcal{H}_D^{1/2}}))$  is a symmetric coercive closed form. We conclude the proof by applying Proposition 4.10 page 35 in [18]. First note that  $\tilde{\mathcal{S}}_D$  is dense in  $\text{Dom}(\overline{\mathcal{H}_D^{1/2}})$  (w.r.t. the norm  $(\overline{\mathcal{E}_D}^{1/2})^1$ ). By Exercise 2.7 page 47 in [18], for any  $\varepsilon > 0$  there exists an infinitely differentiable function  $\varphi_\varepsilon : \mathbb{R} \rightarrow [-\varepsilon, 1 + \varepsilon]$  (which has not to be confused with the functions  $\varphi_1, \dots, \varphi_M$  involved in the definition of the r.v.  $F(\mathbf{X}^D)$  below) such that  $\varphi_\varepsilon(t) = t$  for any  $t \in [0, 1]$ ,  $0 \leq \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \leq t - s$  for all  $t, s \in \mathbb{R}$ ,  $t \geq s$ ,  $\varphi_\varepsilon(t) = 1 + \varepsilon$  for  $t \in [1 + 2\varepsilon, \infty)$  and  $\varphi_\varepsilon(t) = -\varepsilon$  for  $t \in (-\infty, -2\varepsilon]$ . Note that  $|\varphi'_\varepsilon(t)|^2 \leq 1$  for any  $\varepsilon > 0$ ,  $t \in \mathbb{R}$  and  $\varphi_\varepsilon$  is in  $\mathcal{C}_b^\infty$ , for any  $\varepsilon > 0$ . Consider the r.v.

$$F(\mathbf{X}^D) = f \left( \sum_{k=1}^{\mathbf{x}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_M(X_k) \right) \mathbf{1}_{\{\mathbf{x}(D) \leq K\}},$$

for some integers  $M, K \geq 1$ ,  $\varphi_1, \dots, \varphi_M \in \mathcal{C}^\infty(D)$ ,  $f \in \mathcal{C}_b^\infty(\mathbb{R}^M)$ . Note that  $F(\mathbf{X}^D), \varphi_\varepsilon \circ F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ . Indeed

$$\begin{aligned} \varphi_\varepsilon \circ F(\mathbf{X}^D) &= \varphi_\varepsilon \left( f \left( \sum_{k=1}^{\mathbf{x}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_M(X_k) \right) \mathbf{1}_{\{\mathbf{x}(D) \leq K\}} \right) \\ &= \varphi_\varepsilon \left( f \left( \sum_{k=1}^{\mathbf{x}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_M(X_k) \right) \right) \mathbf{1}_{\{\mathbf{x}(D) \leq K\}}, \end{aligned}$$

because  $\varphi_\varepsilon(0) = 0$ . By Lemma 4.2 we have

$$\overline{\mathcal{E}_D}(\varphi_\varepsilon \circ F(\mathbf{X}^D), \varphi_\varepsilon \circ F(\mathbf{X}^D)) = \mathbb{E} \left[ \sum_{i=1}^{\mathbf{x}(D)} \nabla_{X_i}^{N_{if}} \varphi_\varepsilon \circ F(\mathbf{X}^D) \cdot \nabla_{X_i}^{N_{if}} \varphi_\varepsilon \circ F(\mathbf{X}^D) \right]$$

$$\begin{aligned}
&= \mathbb{E} \left[ \sum_{i=1}^{\mathbf{X}(D)} \|\nabla_{X_i}^{N_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D)\|^2 \right] \\
&= \mathbb{E} \left[ \sum_{k=1}^K \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \|\nabla_{X_i}^{N_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D)\|^2 \right] \\
&= \mathbb{E} \left[ \sum_{k=1}^K \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \left\| \sum_{m=1}^M \varphi'_\varepsilon \circ f \left( \sum_{l=1}^k \varphi_1(X_l), \dots, \sum_{l=1}^k \varphi_M(X_l) \right) \right. \right. \\
&\quad \left. \left. \times \partial_m f \left( \sum_{l=1}^k \varphi_1(X_l), \dots, \sum_{l=1}^k \varphi_M(X_l) \right) \nabla \varphi_m(X_i) \right\|^2 \right] \\
&\leq \mathbb{E} \left[ \sum_{k=1}^K \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^k \left\| \sum_{m=1}^M \partial_m f \left( \sum_{l=1}^k \varphi_1(X_l), \dots, \sum_{l=1}^k \varphi_M(X_l) \right) \nabla \varphi_m(X_i) \right\|^2 \right] \quad (4.3) \\
&= \overline{\mathcal{E}}_D(F(\mathbf{X}^D), F(\mathbf{X}^D)),
\end{aligned}$$

where in (4.3) we used the fact that  $|\varphi'_\varepsilon(t)|^2 \leq 1$ ,  $t \in \mathbb{R}$ . By this inequality we easily have, for any  $F(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ ,

$$\liminf_{\varepsilon \rightarrow 0} \overline{\mathcal{E}}_D(F(\mathbf{X}^D) \pm \varphi_\varepsilon \circ F(\mathbf{X}^D), F(\mathbf{X}^D) \mp \varphi_\varepsilon \circ F(\mathbf{X}^D)) \geq 0$$

and the proof is completed (since, as required by Proposition 4.10 page 35 in [18], we checked condition (4.6) page 34 in [18]). Indeed, for any  $\varepsilon > 0$ , by the above inequality and the symmetry of  $\mathcal{E}_D$  and  $\mathcal{H}_D$ , we have

$$\begin{aligned}
&\overline{\mathcal{E}}_D(F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D), F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)) \\
&= \overline{\mathcal{E}}_D(F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D), F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D)) \\
&= \mathbb{E}[(F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D))\mathcal{H}_D(F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D))] \\
&= \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D) + F(\mathbf{X}^D)\mathcal{H}_D \varphi_\varepsilon \circ F(\mathbf{X}^D) \\
&\quad - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}_D \varphi_\varepsilon \circ F(\mathbf{X}^D)] \\
&\geq \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D \varphi_\varepsilon \circ F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] \\
&= 0.
\end{aligned}$$

□

**Lemma 4.2** *Under the assumptions of Theorem 3.9, for any  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ , we have*

$$\mathbb{E} \left[ \sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{N_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{N_{lf}} G(\mathbf{X}^D) \right] = \mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] \quad (4.4)$$

$$= \mathbb{E}[\mathcal{H}_D^{1/2} F(\mathbf{X}^D)\mathcal{H}_D^{1/2} G(\mathbf{X}^D)]. \quad (4.5)$$

*Proof.* Let  $F(\mathbf{X}^D), G(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$  be, respectively, of the form

$$F(\mathbf{X}^D) = f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}},$$

$$G(\mathbf{X}^D) = g \left( \sum_{k=1}^{\mathbf{X}(D)} \gamma_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \gamma_{M_2}(X_k) \right) \mathbf{1}_{\{\mathbf{X}(D) \leq K_2\}},$$

for some integers  $M_1, M_2, K_1, K_2 \geq 1$ ,  $\varphi_1, \dots, \varphi_{M_1}, \gamma_1, \dots, \gamma_{M_2} \in \mathcal{C}^\infty(D)$ ,  $f \in \mathcal{C}_b^\infty(\mathbb{R}^{M_1})$ ,  $g \in \mathcal{C}_b^\infty(\mathbb{R}^{M_2})$ . Define

$$F_i(\mathbf{X}^D) = \partial_i f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}},$$

and

$$v_i(x) = \nabla \varphi_i(x), \quad x \in D.$$

By direct computation we find

$$\begin{aligned} \mathcal{H}_D F(\mathbf{X}^D) &= -\mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}} \sum_{i=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} \beta^\mu(X_k) \cdot v_i(X_k) \partial_i f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \\ &\quad - \mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}} \sum_{i,j=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} v_i(X_k) \sum_{l=1}^{\mathbf{X}(D)} v_j(X_l) \partial_i \partial_j f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \\ &\quad + \mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}} \sum_{i=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} \operatorname{div} v_i(X_k) \partial_i f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \\ &\quad + \mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}} \sum_{i=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} U_{k, \mathbf{X}(D)}(X_1, \dots, X_{\mathbf{X}(D)}) \cdot v_i(X_k) \partial_i f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \\ &= - \sum_{i=1}^{M_1} F_i(\mathbf{X}^D) \sum_{k=1}^{\mathbf{X}(D)} \beta^\mu(X_k) \cdot v_i(X_k) \\ &\quad - \mathbf{1}_{\{\mathbf{X}(D) \leq K_1\}} \sum_{i,j=1}^{M_1} \sum_{k=1}^{\mathbf{X}(D)} v_i(X_k) \sum_{l=1}^{\mathbf{X}(D)} v_j(X_l) \partial_i \partial_j f \left( \sum_{k=1}^{\mathbf{X}(D)} \varphi_1(X_k), \dots, \sum_{k=1}^{\mathbf{X}(D)} \varphi_{M_1}(X_k) \right) \\ &\quad + \sum_{i=1}^{M_1} F_i(\mathbf{X}^D) \sum_{k=1}^{\mathbf{X}(D)} \operatorname{div} v_i(X_k) \\ &\quad + \sum_{i=1}^{M_1} F_i(\mathbf{X}^D) \nabla_{v_i}^{N_{if}} U[D](\mathbf{X}^D), \end{aligned}$$

which yields

$$\begin{aligned} \mathcal{H}_D F(\mathbf{X}^D) &= \sum_{i=1}^{M_1} \left( -\nabla_{v_i}^{N_{if}} F_i(\mathbf{X}^D) + (B_{v_i}^\mu(\mathbf{X}^D) + \nabla_{v_i}^{N_{if}} U[D](\mathbf{X}^D)) F_i(\mathbf{X}^D) \right) \\ &= \sum_{i=1}^{M_1} \nabla_{v_i}^{N_{if}^*} F_i(\mathbf{X}^D). \end{aligned}$$

So, by Lemma 3.7 and since  $\tilde{\mathcal{S}}_D \subset \mathcal{S}_D$ , using obvious notation we have

$$\begin{aligned}
\mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] &= \sum_{i=1}^{M_1} \mathbb{E} \left[ G(\mathbf{X}^D) \nabla_{v_i}^{N_{lf}^*} F_i(\mathbf{X}^D) \right] = \sum_{i=1}^{M_1} \mathbb{E} \left[ F_i(\mathbf{X}^D) \nabla_{v_i}^{N_{lf}} G(\mathbf{X}^D) \right] \\
&= \sum_{i=1}^{M_1} \mathbb{E} \left[ F_i(\mathbf{X}^D) \sum_{l=1}^{\mathbf{X}^D} \sum_{j=1}^{M_2} \partial_j g \left( \sum_{m=1}^{\mathbf{X}^D} \gamma_1(X_m), \dots, \sum_{m=1}^{\mathbf{X}^D} \gamma_{M_2}(X_m) \right) \nabla \gamma_j(X_l) \cdot \nabla \varphi_i(X_l) \right] \\
&= \mathbb{E} \left[ \sum_{l=1}^{\mathbf{X}^D} \sum_{i=1}^{M_1} F_i(\mathbf{X}^D) \nabla \varphi_i(X_l) \cdot \sum_{j=1}^{M_2} G_j(\mathbf{X}^D) \nabla \gamma_j(X_l) \right] \\
&= \mathbb{E} \left[ \sum_{i=1}^{\mathbf{X}^D} \nabla_{X_i}^{N_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{N_{lf}} G(\mathbf{X}^D) \right].
\end{aligned}$$

Finally, since  $\mathcal{H}_D$  is symmetric and non-negative definite the square root operator  $\mathcal{H}_D^{1/2}$  is well-defined. Relation (4.5) follows by the properties of  $\mathcal{H}_D^{1/2}$ .  $\square$

We conclude this section with the following remark which provides the semigroup of the Dirichlet form  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$ . The connection between such semigroup and the transition semigroup of the diffusion associated to the determinantal process will be specified in Theorem 5.1 below.

**Remark 4.3** *Assume the hypotheses of Theorem 3.9. Then  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$  is a symmetric Dirichlet form. Its generator is by definition the linear operator  $\mathcal{H}_D^{\text{gen}} \mathbf{F} = \mathbf{G}$ , where  $\mathbf{G}$  is determined by the domain of the operator:*

$$\begin{aligned}
&\text{Dom}(\mathcal{H}_D^{\text{gen}}) \\
&:= \left\{ \mathbf{F} \in \text{Dom}(\overline{\mathcal{H}}_D^{1/2}) : \exists \mathbf{G} \in L_D^2 \forall \mathbf{Z} \in \text{Dom}(\overline{\mathcal{H}}_D^{1/2}) \overline{\mathcal{E}}(\mathbf{F}, \mathbf{Z}) = -\mathbb{E}[\mathbf{G}\mathbf{Z}] \right\}.
\end{aligned}$$

One may easily see that the operator  $(-\mathcal{H}_D^{\text{gen}}, \text{Dom}(\mathcal{H}_D^{\text{gen}}))$  is symmetric, non-negative definite and extends  $(\mathcal{H}_D, \tilde{\mathcal{S}}_D)$ . Moreover

$$\text{Dom}(\overline{\mathcal{H}}_D^{1/2}) = \text{Dom}((-\mathcal{H}_D^{\text{gen}})^{1/2})$$

and

$$\overline{\mathcal{E}}_D(\mathbf{F}, \mathbf{G}) = \mathbb{E}[(-\mathcal{H}_D^{\text{gen}})^{1/2} \mathbf{F} (-\mathcal{H}_D^{\text{gen}})^{1/2} \mathbf{G}], \quad \forall \mathbf{F}, \mathbf{G} \in \text{Dom}(\overline{\mathcal{H}}_D^{1/2}).$$

By definition, the symmetric semi-group of  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$  is the linear operator  $T_t \mathbf{F} := \exp(t\mathcal{H}_D^{\text{gen}}) \mathbf{F}$ ,  $t > 0$ ,  $\mathbf{F} \in L_D^2$  (which is defined by the spectral theory for symmetric operators on a Hilbert space).

In this section, we generalize Theorem 3.9 to the whole space. We begin by introducing an increasing series of compact sets  $(D_n)_{n \in \mathbb{N}}$ , such that  $\cup_{n \in \mathbb{N}} D_n = E$ . For such  $(D_n)_{n \in \mathbb{N}}$ , we



denote by

$$\mathcal{E}_n(F, G) := \mathcal{E}_{D_n}(F, G) = \mathbb{E} \left[ \sum_{i=1}^{\mathbf{X}(D_n)} \nabla_{X_i}^{N_{lf}} F(\mathbf{X}^{D_n}) \cdot \nabla_{X_i}^{N_{lf}} G(\mathbf{X}^{D_n}) \right],$$

which is well-defined for  $F, G \in \text{Dom}(\mathcal{E}_n) := \tilde{\mathfrak{D}}_{D_n}$ . We recall that we have showed in Theorem 4.1 that  $(\mathcal{E}_n, \text{Dom}(\mathcal{E}_n))$  is a closable Dirichlet form, which is associated with  $\mathcal{H}_n := \mathcal{H}_{D_n}$ . We show the following.

**Theorem 4.4** *Let  $\mu_{\mathcal{X}, \lambda}$  be a determinantal point process satisfying the hypotheses of Theorem 3.9. Assume moreover that the operator  $\mathcal{J}$  associated with the determinantal point process is continuous. Define  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  by*

$$\mathcal{E}(F, F) = \lim_{n \rightarrow \infty} \mathcal{E}_n(F, F) \quad (4.6)$$

with domain  $\text{Dom}(\mathcal{E}) := \bigcap_{n \in \mathbb{N}} \text{Dom}(\mathcal{E}_n) \subset L^2(N_{lf}, P)$ , and define  $\mathcal{E}(F, G)$  by polarization.

Then,  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is a closable Dirichlet form.

*Proof.* We begin by remarking that  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is a densely defined positive definite symmetric bilinear form. We have that  $(\mathcal{E}_n, \text{Dom}(\mathcal{E}_n))$  are increasing in the sense that

$$D_n \subset D_m \text{ and } \forall F \in D_n, \mathcal{E}_m(F) \leq \mathcal{E}_n(F).$$

when  $n > m$ . Then, recall the following lemma from [18]

**Lemma 4.5 (Prop I.3.7, (i) in [18])** *Suppose  $(\mathcal{E}_n, \mathcal{D}_n)_{n \in \mathbb{N}}$  is an increasing sequence of positive definite symmetric bilinear forms on  $L^2(N_{lf}, P)$ . Moreover, assume  $(\mathcal{E}_n, \mathcal{D}_n)_{n \in \mathbb{N}}$  are closable. Define*

$$\begin{cases} \mathcal{E}_\infty(F, F) = \lim_{n \rightarrow \infty} \mathcal{E}_n(F, F), \\ \text{Dom}(\mathcal{E}_\infty) = \{F \in \bigcap_{n \in \mathbb{N}} \mathcal{D}_n : \sup \mathcal{E}_n(F, F) < \infty\}. \end{cases}$$

Then,  $(\mathcal{E}_\infty, \text{Dom}(\mathcal{E}_\infty))$  is closable.

Therefore, by applying Lemma 4.5, we obtain that  $\mathcal{E}$  is closable as it is the increasing limit of  $(\mathcal{E}_n, \text{Dom}(\mathcal{E}_n))$ . Its domain is defined in Lemma 4.5 as  $\text{Dom}(\mathcal{E}) := \{f \in \bigcap_{n \in \mathbb{N}} \text{Dom}(\mathcal{E}_n) : \sup_n \mathcal{E}_n(f, f) < \infty\}$ . In the following, we write  $(\mathcal{E}, \overline{\text{Dom}(\mathcal{E})})$  for its closure. The Dirichlet property follows by using the same arguments as in Theorem 4.1.  $\square$

As a corollary, we obtain

**Corollary 4.6** *Assume that the hypotheses of Theorem 4.4 hold. Then, there exists a non-positive definite self-adjoint operator, denoted by  $-\mathcal{H}$  such that*

$$\mathcal{E}(F, G) = \mathbb{E}[(-\mathcal{H}F)G], \quad (4.7)$$

for  $F, G \in \overline{\text{Dom}(\mathcal{E})}$ .

*Proof.* This is a direct consequence of Theorem 4.4 and Theorem 1.3.1 in [8].  $\square$

## 5 Stochastic dynamics and quasi-regularity

In this section we establish the existence of diffusions corresponding to determinantal point processes associated to the Dirichlet form  $(\overline{\mathcal{E}}, \text{Dom}(\overline{\mathcal{H}}^{1/2}))$ , cf. Theorem 5.1 below.

### Associated diffusion process

We start recalling some notions, see Chapters IV and V in [18]. Given  $\pi$  in the set  $\mathcal{P}(\check{\mathbb{N}}_f^D)$  of the probability measures on  $(\check{\mathbb{N}}_f^D, \mathcal{B}(\check{\mathbb{N}}_f^D))$ , we call a  $\pi$ -stochastic process with state space  $\check{\mathbb{N}}_f^D$  the collection

$$\mathbf{M}_{D,\pi} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{M}_t)_{t \geq 0}, (\mathbf{P}_\mathbf{x})_{\mathbf{x} \in \check{\mathbb{N}}_f^D}, \mathbf{P}_\pi)$$

where  $\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$  is a  $\sigma$ -field on the set  $\Omega$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is the  $\mathbf{P}_\pi$ -completed filtration generated by the process  $\mathbf{M}_t : \Omega \rightarrow \check{\mathbb{N}}_f^D$  of  $(\mathcal{F}, \mathcal{B}(\check{\mathbb{N}}_f^D))$ -measurable mappings,  $\mathbf{P}_\mathbf{x}$  is a probability measure on  $(\Omega, \mathcal{F})$  for all  $\mathbf{x} \in \check{\mathbb{N}}_f^D$ , and  $\mathbf{P}_\pi$  is the probability measure on  $(\Omega, \mathcal{F})$  defined by

$$\mathbf{P}_\pi(A) := \int_{\check{\mathbb{N}}_f^D} \mathbf{P}_\mathbf{x}(A) \pi(d\mathbf{x}), \quad A \in \mathcal{F}.$$

A collection  $(\mathbf{M}_{D,\pi}, (\theta_t)_{t \geq 0})$  is called a  $\pi$ -time homogeneous Markov process with state space  $\check{\mathbb{N}}_f^D$  if  $\theta_t : \Omega \rightarrow \Omega$  is a shift operator, i.e.  $\mathbf{M}_s \circ \theta_t = \mathbf{M}_{s+t}$ ,  $s, t \geq 0$ , the map  $\mathbf{x} \mapsto \mathbf{P}_\mathbf{x}(A)$  is  $(\mathcal{B}(\check{\mathbb{N}}_f^D), \mathcal{B}(\mathbb{R}))$ -measurable for all  $A \in \mathcal{F}$ , and the time homogeneous Markov property

$$\mathbf{P}_\mathbf{x}(\mathbf{M}_t \in A | \mathcal{F}_s) = \mathbf{P}_{\mathbf{M}_s}(\mathbf{M}_{t-s} \in A), \quad \mathbf{P}_\mathbf{x} - a.s., \quad A \in \mathcal{F}, \quad 0 \leq s \leq t, \quad \mathbf{x} \in \check{\mathbb{N}}_f^D$$

holds. Recall that a  $\pi$ -time homogeneous Markov process  $(\mathbf{M}_{D,\pi}, (\theta_t)_{t \geq 0})$  with state space  $\check{\mathbb{N}}_f^D$  is said to be  $\pi$ -tight on  $\check{\mathbb{N}}_f^D$  if  $(\mathbf{M}_t)_{t \geq 0}$  is right-continuous with left limits  $\mathbf{P}_\pi$ -almost surely;  $\mathbf{P}_\mathbf{x}(\mathbf{M}_0 = \mathbf{x}) = 1 \forall \mathbf{x} \in \check{\mathbb{N}}_f^D$ ; the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is right continuous; the following strong Markov property holds:

$$\mathbf{P}_{\pi'}(\mathbf{M}_{t+\tau} \in A | \mathcal{F}_\tau) = \mathbf{P}_{\mathbf{M}_\tau}(\mathbf{M}_t \in A)$$

$\mathbf{P}_{\pi'}$ -almost surely for all  $\mathcal{F}_t$ -stopping time  $\tau$ ,  $\pi' \in \mathcal{P}(\check{\mathbb{N}}_f^D)$ ,  $A \in \mathcal{F}$  and  $t \geq 0$ , cf. Theorem IV.1.15 in [18]. In addition, a  $\pi$ -tight process on  $\check{\mathbb{N}}_f^D$  is said  $\pi$ -special standard process on  $\check{\mathbb{N}}_f^D$  if for any  $\pi' \in \mathcal{P}(\check{\mathbb{N}}_f^D)$  which is equivalent to  $\pi$  and all  $\mathcal{F}_t$ -stopping times  $\tau$ ,  $(\tau_n)_{n \geq 1}$  such that  $\tau_n \uparrow \tau$  then  $\mathbf{M}_{\tau_n}$  converges to  $\mathbf{M}_\tau$ ,  $\mathbf{P}_{\pi'}$ -almost surely.

The following theorem holds, in which  $E_\mathbf{x}$  denotes the expectation under  $\mathbf{P}_\mathbf{x}$ ,  $\mathbf{x} \in \check{\mathbb{N}}_f^D$ .

**Theorem 5.1** *Assume the hypotheses of Theorem 3.9. Then there exists a  $P_D$ -tight special standard process  $(\mathbf{M}_{D,P_D}, (\theta_t)_{t \geq 0})$  on  $\check{\mathbb{N}}_f^D$  with transition semigroup*

$$p_t f(\mathbf{x}) := E_\mathbf{x}[f(\mathbf{M}_t)], \quad \mathbf{x} \in \check{\mathbb{N}}_f^D, \quad f : \check{\mathbb{N}}_f^D \rightarrow \mathbb{R} \quad \text{square integrable.}$$

*In addition,  $(\mathbf{M}_{D,P_D}, (\theta_t)_{t \geq 0})$  is properly associated with the Dirichlet form  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$  in the sense that  $p_t f$  is an  $\overline{\mathcal{E}}_D$ -a.c.,  $P_D$ -version of  $\exp(t\mathcal{H}_D^{\text{gen}})f$  for all square integrable  $f : \check{\mathbb{N}}_f^D \rightarrow \mathbb{R}$  and  $t > 0$ , and such that*

$$\mathbf{P}_\mathbf{x}(\{\omega : t \mapsto \mathbf{M}_t(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \quad \overline{\mathcal{E}}_D\text{-a.e.}, \quad \mathbf{x} \in \check{\mathbb{N}}_f^D. \quad (5.1)$$

*Proof.* By Lemma 5.2 below the Dirichlet form  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$  is quasi-regular, hence by Theorem III.3.5 page 103 in [18], there exists a  $P_D$ -tight special standard process on  $\ddot{N}_f^D$ , say  $(\mathbf{M}_{D,P_D}, (\theta_t)_{t \geq 0})$ , whose transition semigroup  $(p_t)_{t \geq 0}$  is such that, for any square integrable function  $f : \ddot{N}_f^D \rightarrow \mathbb{R}$  and  $t > 0$ ,  $p_t f$  is a  $P_D$ -version of  $\exp(t\mathcal{H}_{\text{gen}})f$  and  $p_t f$  is  $\overline{\mathcal{E}}$ -a.c.. Since the form has clearly the local property (see the definition in [18]), by Theorem V.1.5 page 150 in [18] the tight special standard process is a  $P_D$ -diffusion associated to the form, i.e. relation (5.1) holds for  $\overline{\mathcal{E}}_D$ -a.e.  $\mathbf{x} \in \ddot{N}_f^D$ .  $\square$

For the sake of completeness we remark that by applying Theorem 6.4 page 141 in [18], one has that the  $P_D$ -diffusion on  $\ddot{N}_f^D$  properly associated with the Dirichlet form  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$  (defined in Theorem 5.1) is unique up to  $P_D$ -equivalence. We refer the reader to Definition 6.3 page 140 in [18] for the meaning of this notion.

### Quasi-regularity

The following lemma holds.

**Lemma 5.2** *Assume the hypotheses of Theorem 3.9. Then the form  $(\overline{\mathcal{E}}_D, \text{Dom}(\overline{\mathcal{H}}_D^{1/2}))$  is quasi-regular.*

*Proof.* As in [27] and [28], we apply Theorem 3.4 of [19]. For this purpose, we have to show that, for some countable dense set  $(\tilde{\mathbf{x}}_i)_{i \geq 1}$  in  $\ddot{N}_f^D$ , there exists a countable collection  $(F_{ij}(\mathbf{X}^D))_{i,j \geq 1}$  of random variables such that:

- (i) For some bounded metric  $\varrho$  on  $\ddot{N}_f^D$ , which is uniformly equivalent to the "d-hash" metric, we have  $\varrho(\mathbf{x}, \tilde{\mathbf{x}}_i) = \sup_j F_{ij}(\mathbf{x})$ ,  $\overline{\mathcal{E}}$ -a.e.  $\mathbf{x} \in \ddot{N}_f^D$ , for any  $i \geq 1$ .
- (ii)  $F_{ij} : N_{lf} \rightarrow \mathbb{R}$  is of the form

$$F_{ij}(\mathbf{x}) = f_{ij} \left( \sum_{x \in \mathbf{x}} h_1(x), \dots, \sum_{x \in \mathbf{x}} h_N(x) \right), \quad \mathbf{x} \in N_{lf}$$

for some integer  $N \geq 1$ ,  $h_1, \dots, h_N \in \mathcal{C}_c^\infty(S)$  and  $f_{ij}$  weakly differentiable;  $(F_{ij}(\mathbf{X}^D))_{i,j \geq 1} \subset \text{Dom}(\overline{\mathcal{H}}_D^{1/2})$  and, setting  $\mathbf{F}_{ij} = F_{ij}(\mathbf{X}^D)$ ,

$$\overline{\mathcal{E}}_D(\mathbf{F}_{ij}, \mathbf{F}_{ij}) = \mathbb{E} \left[ \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \right], \quad (5.2)$$

where  $\nabla_x^{N_{lf}} F_{ij}(\mathbf{x})$  is defined as in (3.3) with  $n = \infty$  and the symbol  $\nabla_x$  being the gradient w.r.t. the weak partial derivatives.

- (iii)  $\sup_{i,j} \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \in L_D^1$ .

*Proof of (i).* Such a metric  $\varrho$  is defined as the restriction on  $\ddot{N}_f^D$  of the following metric on  $\ddot{N}_{lf}$  (which we continue to denote by  $\varrho$ ) constructed in [28] as follows. Here  $\ddot{N}_{lf}$  denotes the set of non-negative integer-valued locally finite measures on  $S$ , equipped with the vague topology. Let  $\mathcal{T}_b$  be a countable open base in  $S$  and, for every  $O \in \mathcal{T}_b$ , take a sequence

$\ell_{O_1}, \ell_{O_2}, \dots \in \mathcal{C}_c^\infty(S)$  such that  $\ell_{O_n}(x) \uparrow \mathbf{1}_{\{x \in O\}}, \forall x \in S$ . For ease of notation set  $\ell_n = \ell_{O_n}$  and assume that this enumeration satisfies

$$\bigcup_{k=1}^j \text{supp}(\ell_k) \subseteq \{x \in S : \|x\| \leq j\}, \quad j = 1, 2, \dots$$

The desired metric is defined by

$$\varrho(\mathbf{x}, \mathbf{y}) := \sup_j 2^{-j} \left\{ 1 - \exp \left( - \frac{\left| \sum_{x \in \mathbf{x}} \ell_j(x) - \sum_{y \in \mathbf{y}} \ell_j(y) \right|}{\bar{\ell}_j} \right) \right\}, \quad \mathbf{x}, \mathbf{y} \in \check{N}_{lf}$$

where, for  $x = (x^{(1)}, \dots, x^{(d)})$ ,

$$\bar{\ell}_j := \sup \left\{ \left| \frac{\partial \ell_j(x)}{\partial x^{(q)}} \right| : x \in S, q = 1, \dots, d \right\} > 0.$$

Let  $(\tilde{\mathbf{x}}_i)_{i \geq 1}$  be a countable dense set in  $\check{N}_f^D$  and define

$$F_{ij}(\mathbf{x}) := 2^{-j} \left\{ 1 - \exp \left( - \frac{\left| \sum_{x \in \mathbf{x}} \ell_j(x) - \sum_{\tilde{\mathbf{x}}_i \in \tilde{\mathbf{x}}_i} \ell_j(\tilde{\mathbf{x}}_i) \right|}{\bar{\ell}_j} \right) \right\}, \quad \mathbf{x} \in \check{N}_{lf}$$

Then (i) follows.

*Proof of (ii).* Clearly  $F_{ij}$  has the claimed form. We now check that  $\mathbf{F}_{ij} \in \text{Dom}(\overline{\mathcal{H}_D^{1/2}})$ , i.e. there exists a sequence  $(F_{ij}^{(n)}(\mathbf{X}^D))_{n \geq 1} \subset \tilde{\mathcal{S}}_D$  such that, as  $n$  goes to infinity,  $F_{ij}^{(n)}(\mathbf{X}^D)$  converges to  $\mathbf{F}_{ij}$  in  $L_D^2$  and  $(\mathcal{H}_D^{1/2} F_{ij}^{(n)}(\mathbf{X}^D))_{n \geq 1}$  converges in  $L_D^2$ . Define the function

$$\bar{f}_{ij}(x) := \left| x - \sum_{\tilde{\mathbf{x}}_i \in \tilde{\mathbf{x}}_i} \ell_j(\tilde{\mathbf{x}}_i) \right|, \quad x \in \mathbb{R}$$

and consider the classical sequence of mollifiers defined by  $\eta_n(x) := Cn\eta(nx)$ ,  $n \geq 1$ , where

$$\eta(x) := \mathbf{1}_{\{|x| < 1\}} e^{-1/(1-x^2)}, \quad x \in \mathbb{R}$$

and  $C := \left( \int_{-1}^1 \eta(x) dx \right)^{-1}$ . Define the functions

$$f_{ij}^{(n)}(x) := 2^{-j} \left\{ 1 - \exp \left( - \frac{\eta_n * \bar{f}_{ij}(x)}{\bar{\ell}_j} \right) \right\}, \quad x \in \mathbb{R}$$

where  $*$  denotes the convolution product, and set  $F_{ij}^{(n)}(\mathbf{x}) := f_{ij}^{(n)}(\sum_{x \in \mathbf{x}} \ell_j(x)) \mathbf{1}_{\{\#\mathbf{x} \leq n\}}$ ,  $\mathbf{x} \in N_f^D$ , where the symbol  $\#\mathbf{x}$  denotes the number of points in the configuration  $\mathbf{x}$ . Since  $\eta_n \in \mathcal{C}_c^\infty(\mathbb{R})$  and  $\bar{f}_{ij}$  is locally integrable on  $\mathbb{R}$ , we have  $\eta_n * \bar{f}_{ij} \in \mathcal{C}^\infty(\mathbb{R})$  (see e.g. Proposition IV.20 in [2]). Therefore  $f_{ij}^{(n)} \in \mathcal{C}_b^\infty(\mathbb{R})$  and so  $F_{ij}^{(n)}(\mathbf{X}^D) \in \tilde{\mathcal{S}}_D$ . Since  $\bar{f}_{ij} \in \mathcal{C}(\mathbb{R})$ , we deduce that  $\eta_n * \bar{f}_{ij}$  converges to  $\bar{f}_{ij}$  uniformly on the compacts of  $\mathbb{R}$  (see e.g. Proposition IV.21 in

[2]), and so  $F_{ij}^{(n)}(\mathbf{X}^D)$  converges to  $\mathbf{F}_{ij}$  in  $L_D^2$ , as  $n$  goes to infinity. It remains to show that the sequence  $(\mathcal{H}_D^{1/2} F_{ij}^{(n)}(\mathbf{X}^D))_{n \geq 1}$  converges in  $L_D^2$ . For this we are going to prove that it is a Cauchy sequence. We start by noting that since  $\eta_n$  is integrable on  $\mathbb{R}$  and  $\bar{f}_{ij}$  is weakly differentiable we have (see e.g. Lemma VIII.4 in [2])

$$(\eta_n * \bar{f}_{ij})'(x) := \frac{d}{dx} \eta_n * \bar{f}_{ij}(x) = \eta_n * \frac{d}{dx} \bar{f}_{ij}(x)$$

where

$$\frac{d}{dx} \bar{f}_{ij}(x) = \mathbf{1}_{\{\sum_{\tilde{x}_i \in \tilde{x}_i} \ell_j(\tilde{x}_i) \in (x, \infty)\}} - \mathbf{1}_{\{\sum_{\tilde{x}_i \in \tilde{x}_i} \ell_j(\tilde{x}_i) \in (-\infty, x)\}}.$$

Thus  $|(\eta_n * \bar{f}_{ij})'| \leq 1$ , for any  $n \geq 1$ . Moreover,  $(\eta_n * \bar{f}_{ij})'(x)$  converges to  $\bar{f}'_{ij}(x)$  point-wise, as  $n$  goes to infinity. We also note that for any  $n \geq 1$  and  $x \in D$  we have

$$\begin{aligned} \nabla_x^{\mathbf{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D) &= \sum_{k=1}^n \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{h=1}^k \mathbf{1}_{\{X_h\}}(x) \nabla_{x_h} f_{ij}^{(n)} \left( \sum_{q=1}^k \ell_j(X_q) \right) \\ &= \mathbf{1}_{\{\mathbf{X}(D) \leq n\}} \sum_{h=1}^{\mathbf{X}(D)} \mathbf{1}_{\{X_h\}}(x) \nabla_{x_h} f_{ij}^{(n)} \left( \sum_{q=1}^{\mathbf{X}(D)} \ell_j(X_q) \right) \end{aligned}$$

where, for any  $\mathbf{x} \in \mathbf{N}_f^D$ ,

$$\nabla_x f_{ij}^{(n)} \left( \sum_{x \in \mathbf{x}} \ell_j(x) \right) = (2^{-j} / \bar{\ell}_j) (\eta_n * \bar{f}_{ij})' \left( \sum_{x \in \mathbf{x}} \ell_j(x) \right) \exp \left( -\eta_n * \bar{f}_{ij} \left( \sum_{x \in \mathbf{x}} \ell_j(x) \right) / \bar{\ell}_j \right) \nabla \ell_j(x).$$

So, for any  $n \geq 1$ ,  $i, j$  and  $k = 1, \dots, \mathbf{X}(D)$ , we have

$$\|\nabla_{X_k}^{\mathbf{N}_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \leq d4^{-j}, \quad \text{a.s..} \quad (5.3)$$

For any  $n, m \geq 1$ , we deduce

$$\begin{aligned} &\|\mathcal{H}_D^{1/2} F_{ij}^{(n)}(\mathbf{X}^D) - \mathcal{H}_D^{1/2} F_{ij}^{(m)}(\mathbf{X}^D)\|_{L_D^2}^2 = \mathbb{E}[\|\mathcal{H}_D^{1/2}(F_{ij}^{(n)}(\mathbf{X}^D) - F_{ij}^{(m)}(\mathbf{X}^D))\|^2] \\ &= \mathbb{E} \left[ \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{\mathbf{N}_{lf}} [F_{ij}^{(n)}(\mathbf{X}^D) - F_{ij}^{(m)}(\mathbf{X}^D)]\|^2 \right] \\ &\leq d4^{-j} \mathbb{E} \left[ \left( \mathbf{1}_{\{\mathbf{X}(D) \leq n\}} (\eta_n * \bar{f}_{ij})' \left( \sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) \exp \left( -\eta_n * \bar{f}_{ij} \left( \sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) / \bar{\ell}_j \right) + \right. \right. \\ &\quad \left. \left. - \mathbf{1}_{\{\mathbf{X}(D) \leq m\}} (\eta_m * \bar{f}_{ij})' \left( \sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) \exp \left( -\eta_m * \bar{f}_{ij} \left( \sum_{k=1}^{\mathbf{X}(D)} \ell_j(X_k) \right) / \bar{\ell}_j \right) \right)^2 \mathbf{X}(D) \right]. \end{aligned}$$

Due to the above remarks, applying Lebesgue's dominated convergence theorem (recall that  $\mathbf{X}(D)$  has a finite mean since it has exponential tails) one has that this quantity goes to zero

as  $n, m$  go to infinity, and the claim follows. It remains to check (5.2). By Lemma 4.2 we have

$$\mathbb{E} \left[ \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \right] = \mathbb{E}[(\mathcal{H}_D^{1/2} F_{ij}^{(n)}(\mathbf{X}^D))^2].$$

Clearly

$$\lim_{n \rightarrow \infty} \mathbb{E}[(\mathcal{H}_D^{1/2} F_{ij}^{(n)}(\mathbf{X}^D))^2] = \mathbb{E}[(\overline{\mathcal{H}_D^{1/2}} \mathbf{F}_{ij})^2] = \overline{\mathcal{E}_D}(\mathbf{F}_{ij}, \mathbf{F}_{ij}),$$

and so we only need to check

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \right].$$

This easily follows by Lebesgue's dominated convergence theorem. Indeed, by (5.3), for any  $i, j, n$ ,

$$\sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 \leq 4^{-j} d \mathbf{X}(D), \quad \text{a.s.} \quad (5.4)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}^{(n)}(\mathbf{X}^D)\|^2 = \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}(\mathbf{X}^D)\|^2, \quad \text{a.s..}$$

*Proof of (iii).* It suffices to note that by (5.4) we have

$$\sup_{i,j} \sum_{k=1}^{\mathbf{X}(D)} \|\nabla_{X_k}^{N_{lf}} F_{ij}(\mathbf{X}^D)\|^2 \leq \frac{d}{4} \mathbf{X}(D), \quad \text{a.s..}$$

□

## Extension to the whole space

We recall that in Section 4, we showed that we have proved that  $(\mathcal{E}_{D_n})_{n \in \mathbb{N}}$  converge to a Dirichlet form  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  which is closable, and is associated with the generator  $\mathcal{H}$ . To show that the limit of the previous Dirichlet forms is local and quasi-regular, we will use the following result which can be found in [16]. Here  $D = \mathcal{B}(0, r)$  and set  $j_{\mathcal{B}(0,r)}^i = j_r^i$  to denote the  $i$ -th Janossy density in  $\mathcal{B}(0, r)$ .

**Lemma 5.3 (Theorem 1 in [16])** *Suppose that  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is a given Dirichlet form, and assume that the following conditions hold:*

**(A1)**  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is closable on  $L^2(N_{lf}, P)$ ;

**(A2)**  $\forall i \in \mathbb{N}, \forall r \geq 0, j_r^i \in L^\infty(N_r^i, d\lambda) \quad \forall r \geq 0, \sum_{i=1}^\infty i P(\mathbf{X}(\mathcal{B}(0, r)) = i) < \infty$ . Here,  $N_r^i := \{\mathbf{x} \in N_{\mathcal{B}(0,r)}^f : \mathbf{x}(\mathcal{B}(0, r)) = i\}$ .

*Then  $(\mathcal{E}, \overline{\text{Dom}(\mathcal{E})})$  is a local, and quasi-regular form.*

We are then able to show the following:

**Theorem 5.4** *Let  $\mu_{\mathcal{X},\lambda}$  be a determinantal point process satisfying the hypotheses of Theorem 3.9. Assume moreover that the operator  $\mathcal{J}$  associated with the determinantal point process is continuous. Define  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  by (4.6). Then,  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is a closable, quasi-regular, and local Dirichlet form.*

*Proof.* We have already proved in Theorem 4.4 that  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is a closable Dirichlet form. In order to verify that  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is quasi-regular and local, it suffices to apply Lemma 5.3. Indeed, we have already proved **(A1)** and **(A2)** follows since by the results of [9],  $j_r^i(x_1, \dots, x_i) \leq J(x_1, x_1) \dots J(x_i, x_i)$  for  $\lambda$ -a.e.  $x_1, \dots, x_i \in \mathcal{B}(0, r)$  for any  $r > 0$ . We conclude that  $j_r^i(x_1, \dots, x_i) \in L^\infty(N_r^i, d\lambda)$  by continuity of  $J$ . The last part of **(A2)** follows directly from the fact that  $\mathcal{K}$  is locally trace-class. Hence,  $(\mathcal{E}, \text{Dom}(\mathcal{E}))$  is a closable quasi-regular local Dirichlet form.  $\square$

As a consequence, there exists a quasi-continuous diffusion correctly associated with the determinantal point process on  $N_{lf}$ , and satisfying the properties of Theorem 5.1.

**Theorem 5.5** *Let  $\mu_{\mathcal{X},\lambda}$  be a determinantal point process satisfying the conditions of Theorem 3.9. We assume moreover that  $d \geq 2$  and  $\rho(\cdot)J(\cdot, \cdot) \in L_{\text{loc}}^2$ . Then there exists a  $\mu_{\mathcal{X},\lambda}$ -tight special standard process  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$  on  $N_{lf}$  with transition semigroup*

$$p_t f(\xi) := \mathbb{E}_{\mathbf{x}}[f(\mathbf{M}_t)], \quad \mathbf{x} \in N_{lf}, \quad f : N_{lf} \longrightarrow \mathbb{R} \quad \text{square integrable.}$$

*In addition,  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$  is properly associated with the Dirichlet form  $(\mathcal{E}, \overline{\text{Dom}(\mathcal{E})})$  in the sense that  $p_t f$  is an  $\mathcal{E}$ -a.c.,  $\mu_{\mathcal{X},\lambda}$ -version of  $\exp(t\mathcal{H})f$  for all square integrable  $f : N_{lf} \longrightarrow \mathbb{R}$  and  $t > 0$ , and such that*

$$\mathbf{P}_{\mathbf{x}}(\{\omega : t \mapsto \mathbf{M}_t(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \quad \mathcal{E}\text{-a.e.}, \quad \mathbf{x} \in N_{lf}, \quad (5.5)$$

*i.e.  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$  is quasi-continuous. Moreover, such  $\mathbf{M}$  is up to  $\mu_{\mathcal{X},\lambda}$ -equivalence unique (in the sense of [18]) and hence  $\mu_{\mathcal{X},\lambda}$  is reversible with respect to  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$ .*

*Proof.* The first part is a consequence of Theorem 4.4. Lastly, reversibility of  $\mu_{\mathcal{X},\lambda}$  follows from Theorem 6.4 page 141 in [18].  $\square$

## Non-collision property of the corresponding diffusion

We start recalling the following lemma, which is borrowed from [20].

**Lemma 5.6** *Assume the hypotheses of Theorem 5.5 and let  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$  be the  $P$ -tight special standard process properly associated with the Dirichlet form  $(\mathcal{E}, \overline{\text{Dom}(\mathcal{E})})$ . Let  $u_n \in \text{Dom}(\mathcal{E})$ ,  $n \geq 1$ , be such that:  $u_n : N_{lf} \rightarrow \mathbb{R}$  is continuous,  $u_n \rightarrow u$  point-wise,  $u \in L^2$  and*

$$\sup_{n \geq 1} \mathcal{E}(u_n, u_n) < \infty.$$

*Then  $u$  is  $\overline{\mathcal{E}}$ -a.c. and, in particular,*

$$\mathbf{P}_{\mathbf{x}}(\{\omega : t \mapsto u(\mathbf{M}_t)(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \quad P\text{-a.e. } \mathbf{x} \in N_{lf}.$$

The next theorem provides the non-collision property of  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$ .

**Theorem 5.7** *Assume  $d \geq 2$ ,  $J$  continuous on  $S$ , and the hypotheses of Theorem 3.9. Then  $(\mathbf{M}_{\mathcal{X},\lambda}, (\theta_t)_{t \geq 0})$  takes values on  $N_{lf}$ , i.e.*

$$\mathbf{P}_{\mathbf{x}}(\{\omega : \mathbf{M}_t(\omega) \in N_{lf} \quad \forall 0 \leq t < \infty\}) = 1, \quad P\text{-a.e. } \mathbf{x} \in N_{lf}.$$

*Proof.* Since the proof is similar to the proof of Proposition 1 in [20] we skip some details. For every positive integer  $a$ , define  $u := \mathbf{1}_N$ , where

$$N := \{\mathbf{x} \in N_{lf} : \sup_{x \in [-a,a]^d} \mathbf{x}(\{x\}) \geq 2\}.$$

The claim follows if we prove that  $u$  is  $\mathcal{E}$ -a.c.. For this we are going to apply Lemma 5.6. Define

$$u_n(\mathbf{x}) = \Psi \left( \sup_{i \in A_n} \sum_{x \in \mathbf{x}} \phi_i(x) \right), \quad n \geq 1$$

where  $A_n := \mathbb{Z}^d \cap [-na, na]^d$  and  $\Psi$  and  $\phi$  are chosen as in the proof of Proposition 1 in [20]. Then  $u_n(\mathbf{X}) \in \text{Dom}(\mathcal{E})$ ,  $N_{lf} \rightarrow \mathbb{R}$  is continuous and  $u_n \rightarrow u$  point-wise. It remains to prove that  $\sup_{n \geq 1} \mathcal{E}(u_n(\mathbf{X}), u_n(\mathbf{X})) < \infty$ . For  $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ , we denote by  $I_i$  the function defined by

$$I_i(x) = \prod_{k=1}^d \mathbf{1}_{[-1/2, 3/2]}(nx_k - i_k), \quad x = (x_1, \dots, x_d) \in S.$$

The following upper bound holds:

$$\mathcal{E}(u_n(\mathbf{X}), u_n(\mathbf{X})) \leq Cn^2 \sum_{i \in A_n} \mathbb{E} \left[ \mathbf{1}_{\{\sum_{x \in \mathbf{x}} I_i(x) \geq 2\}} \sum_{x \in \mathbf{x}} I_i(x) \right], \quad \text{for some constant } C > 0.$$

Since  $\mathbf{X}$  is stochastically dominated by a Poisson process  $\mathbf{Y}$  with intensity measure  $J(x, x) \mu(dx)$  and, for any  $i \in A_n$ , the mapping  $\mathbf{x} \mapsto \mathbf{1}_{\{\sum_{x \in \mathbf{x}} I_i(x) \geq 2\}} \sum_{x \in \mathbf{x}} I_i(x)$  is "increasing", by (2.9) we have

$$\mathcal{E}(u_n(\mathbf{X}), u_n(\mathbf{X})) \leq Cn^2 \sum_{i \in A_n} \mathbb{E} \left[ \mathbf{1}_{\{\sum_{y \in \mathbf{Y}} I_i(y) \geq 2\}} \sum_{y \in \mathbf{Y}} I_i(y) \right]. \quad (5.6)$$

By the properties of the Poisson process, the right hand side of (5.6) is equal to

$$Cn^2 \sum_{i \in A_n} \left( 1 - e^{-\int_S I_i(x) J(x, x) \mu(dx)} \right) \int_S I_i(x) J(x, x) \mu(dx),$$

which is bounded above by

$$Cn^2 \sum_{i \in A_n} \left( \int_S I_i(x) J(x, x) \rho(x) dx \right)^2.$$

By using the Cauchy-Schwarz inequality, this term is further bounded by

$$Cn^2 \sum_{i \in A_n} \left( \int_S I_i(x) dx \right) \left( \int_S I_i(x) J(x, x)^2 \rho(x)^2 dx \right). \quad (5.7)$$



We have on the one hand that

$$\int_{\mathbb{R}^d} I_i(x) dx = (2/n)^d$$

and on the other hand,

$$\int_S I_i(x) J(x, x)^2 \rho(x)^2 dx \leq \left( \sup_{x \in D} J(x, x)^2 \rho(x)^2 \right) (2/n)^d,$$

since  $J$  and  $\rho$  are continuous on the support of  $I_i$  which is compact, and included in a compact  $D$ . Moreover,  $\sharp(A_n) \leq (2an)^d$ . Consequently, the quantity (5.7) is in turn bounded by

$$C_{d,a} n^{2-d}, \quad \text{for some constant } C_{d,a} > 0 \text{ which depends only on } d \text{ and } a.$$

The claim follows by the assumption  $d \geq 2$ .  $\square$

We note that in the proof we only need  $\rho(\cdot)J(\cdot, \cdot) \in L^2_{\text{loc}}$ , but in all common examples,  $J$  is in fact continuous.

## 6 Examples

### Bergman kernel

Let  $S = D = B(0, R) \subset \mathbb{R}^2$  be the ball centered at the origin with radius  $R \in (0, 1)$  and  $\{\varphi_k^{(R)}\}_{1 \leq k \leq N}$ ,  $N \geq 1$ , the orthonormal subset of  $L^2(B(0, R), \ell)$  defined by

$$\varphi_k^{(R)}(x) := \frac{1}{R} \sqrt{\frac{k+1}{\pi}} \left( \frac{x^{(1)}}{R} + i \frac{x^{(2)}}{R} \right)^k, \quad x = (x^{(1)}, x^{(2)}) \in B(0, R),$$

where  $\mu = \ell$  is the Lebesgue measure and  $i := \sqrt{-1}$  denotes the complex unit. We consider the modified Bergman operator  $\mathcal{K}_{\text{Be}} : B(0, R)^2 \rightarrow \mathbb{C}$  with associated kernel defined by

$$K_{\text{Be}}(x, y) := \sum_{k=1}^N R^{2(k+1)} \varphi_k^{(R)}(x) \overline{\varphi_k^{(R)}(y)}, \quad x, y \in B(0, R), \quad n \in \mathbb{N}^*$$

and denote by  $\mathcal{K}_{\text{Be}}$  the associated integral operator, which is easily seen to be Hermitian and locally of trace class with non-zero eigenvalues  $\kappa_k := R^{2(k+1)}$ ,  $k = 1, \dots, N$ . As a consequence, the spectrum of  $\mathcal{K}_{\text{Be}}$  is contained in  $[0, 1)$  and the triplet  $(\mathcal{K}_{\text{Be}}, K_{\text{Be}}, \mu)$  satisfies assumption **(H1)**. In addition, Condition **(H2)** is trivially satisfied since  $\mu = \ell$  is the Lebesgue measure. The Janossy densities of  $\mathbf{X} = \mathbf{X}^D$  defined in (2.2) are given by

$$j_D^k(x_1, \dots, x_k) = \text{Det}(\mathbf{Id} - \mathcal{K}_{\text{Be}}) \det J[D](x_1, \dots, x_k), \quad k = 1, \dots, N, \quad (x_1, \dots, x_k) \in D^k,$$

where the kernel  $J[D]$  is given by

$$J[D](x, y) := \sum_{h=1}^N \frac{R^{2(h+1)}}{1 - R^{2(h+1)}} \varphi_h^{(R)}(x) \overline{\varphi_h^{(R)}(y)},$$

cf. (2.7). Moreover,  $\mathbf{X}$  has at most  $N$  points, see e.g. [23], which means that  $j_D^k = 0$ , for  $k \geq N + 1$ . To prove condition **(H3)** it suffices to remark that the function

$$(x_1, \dots, x_k) \rightarrow \det(J[D](x_p, x_q))_{1 \leq p, q \leq k}$$

is continuously differentiable on  $D^k$ , for  $k \in \mathbb{N}^*$ . To show that Condition (3.14) is verified, we first consider the case  $n = N$ . Note that

$$(J[D](x_p, x_q))_{1 \leq p, q \leq N} = A^N(x_1, \dots, x_N)A^N(x_1, \dots, x_N)^*,$$

where the matrix  $A^N(x_1, \dots, x_N) := (A_{ph}^N)_{1 \leq p, h \leq N}$  is given by

$$A_{ph}^N := \frac{R^{h+1}}{\sqrt{1 - R^{2(h+1)}}} \varphi_h^{(R)}(x_p)$$

and  $A^N(x_1, \dots, x_N)^*$  denotes the transpose conjugate of  $A^N(x_1, \dots, x_N)$ . Hence,

$$\det J[D](x_1, \dots, x_N) = |\det A^N(x_1, \dots, x_N)|^2,$$

and since the previous determinant is a Vandermonde determinant, we have

$$\det A^N = \prod_{p=1}^N \sqrt{\frac{1+p}{\pi(1-R^{2(p+1)})}} \left( \prod_{p=1}^N (x_p^{(1)} + ix_p^{(2)}) \right) \prod_{1 \leq p < q \leq N} ((x_p^{(1)} - x_q^{(1)}) + i(x_p^{(2)} - x_q^{(2)})).$$

Note that Condition (3.14) with  $n = N$  is equivalent to

$$\int_{B(0,R)^N} \left| \frac{\partial_{x_i^{(p)}} |\det A^N(x_1, \dots, x_N)|^2 \partial_{x_j^{(q)}} |\det A^N(x_1, \dots, x_N)|^2}{|\det A^N(x_1, \dots, x_N)|^2} \right| dx_1 \dots dx_N < \infty,$$

for all  $1 \leq i, j \leq N$  and  $1 \leq p, q \leq 2$ , and so it is verified if we check

$$\int_{B(0,R)^N} \left| \frac{\partial_{x_1^{(1)}} |\det A^N(x_1, \dots, x_N)|^2}{|\det A^N(x_1, \dots, x_N)|^2} \right| dx_1 \dots dx_N < \infty.$$

This latter integral reduces to

$$\int_{B(0,R)^N} \left| \frac{2x_1^{(1)}}{(x_1^{(1)})^2 + (x_1^{(2)})^2} + 2 \sum_{j=2}^N \frac{x_1^{(1)} - x_j^{(1)}}{(x_1^{(1)} - x_j^{(1)})^2 + (x_1^{(2)} - x_j^{(2)})^2} \right| dx_1 \dots dx_N,$$

which is indeed a finite quantity. Consequently, we proved that Condition (3.14) is verified for  $n \geq N$  (indeed it is trivially satisfied for  $n > N$ .) Now, consider  $n < N$ . We have again

$$J[D](x_1, \dots, x_k) = A^N(x_1, \dots, x_k)A^N(x_1, \dots, x_k)^*,$$

where this time,  $A^N(x_1, \dots, x_k)$  is a rectangular  $k \times N$  matrix. Hence, by application of the Cauchy-Binet formula:

$$\det J[D](x_1, \dots, x_k) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N} |\det A^{i_1, \dots, i_k}(x_1, \dots, x_k)|^2,$$

where we have for  $1 \leq p, h \leq k$ ,

$$A_{ph}^{i_1, \dots, i_k} := \frac{R^{i_h+1}}{\sqrt{1 - R^{2(i_h+1)}}} \varphi_{i_h}^{(R)}(x_p),$$

which is a square matrix. We now consider fixed  $1 \leq i_1 < i_2 < \dots < i_k \leq N$  and wish to evaluate  $|\det A^{i_1, \dots, i_k}(x_1, \dots, x_k)|^2$ . In fact, we observe that

$$|\det A^{i_1, \dots, i_k}(x_1, \dots, x_k)|^2 = \prod_{p=1}^k \frac{1 + i_p}{\pi(1 - R^{2(i_p+1)})} |V_{i_1, \dots, i_k}(x_1, \dots, x_k)|^2,$$

where

$$V_{i_1, \dots, i_k}(x_1, \dots, x_k) := \det \left( \left( x_h^{i_p} \right)_{1 \leq p, h \leq k} \right)$$

is known in the literature as the generalized Vandermonde determinant. Here,  $V_{1, \dots, k}(x_1, \dots, x_k)$  is the classical Vandermonde determinant, and in the general case, we have deleted a certain number of rows from the matrix. The generalized Vandermonde determinant is known to factorize into the classical Vandermonde determinant and what is defined to be a Schur polynomial. To be more precise,

$$V_{i_1, \dots, i_k}(x_1, \dots, x_k) = V_{1, \dots, k}(x_1, \dots, x_k) s_{\lambda(i_1, \dots, i_k)}(x_1, \dots, x_k),$$

where  $\lambda(i_1, \dots, i_k) := (i_k - k + 1, \dots, i_2 - 1, i_1)$ , and  $s_{\lambda}$  is the Schur polynomial, which is known to be symmetric, and is a sum of monomials, see e.g. [10]. To sum up the previous calculations, we have

$$\det J[D](x_1, \dots, x_k) = |V_{1, \dots, k}(x_1, \dots, x_k)|^2 \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N} \left( \prod_{p=1}^k \frac{1 + i_p}{\pi(1 - R^{2(i_p+1)})} \right) |s_{\lambda(i_1, \dots, i_k)}(x_1, \dots, x_k)|^2. \quad (6.1)$$

The previous function of  $x_1, \dots, x_k$  is of class  $C^1$ , therefore condition **(H3)** is verified for  $k < N$ . Condition (3.14) is verified thanks to the form that  $s_{\lambda}$  takes, and in particular the fact that  $|s_{\lambda}(x_1, \dots, x_k)|^2 = 0$  if and only if  $x_1 = \dots = x_k = 0$ .

We now study the associated diffusion in a simple visual case. Assume that we have  $N = 2$ , i.e.

$$K_{\text{Be}}(x, y) = \frac{2}{\pi}(x\bar{y}) + \frac{3}{\pi}(x\bar{y})^2,$$

for  $x, y \in B(0, R)$ . Then, the associated determinantal point process has less than 2 points a.s. and:

$$\nabla_{x_1} U(\{x_1, x_2\}) = -2 \left[ \frac{x_1}{|x_1|^2} + \frac{x_1 - x_2}{|x_1 - x_2|^2} \right],$$

for  $x_1, x_2 \in \mathbb{R}^2$ . Therefore, the diffusion associated with the previous determinantal process starting from  $(x_1, x_2) \in \mathbb{C}^2$  is a (weak) solution of the following pair of S.D.E. on  $\mathbb{C}$ :

$$\begin{cases} dX_t^{(1)} = \sqrt{2} dB_t^1 + 2 \left[ \frac{X_t^{(1)}}{|X_t^{(1)}|^2} + \frac{X_t^{(1)} - X_t^{(2)}}{|X_t^{(1)} - X_t^{(2)}|^2} \right] dt, \\ dX_t^{(2)} = \sqrt{2} dB_t^2 + 2 \left[ \frac{X_t^{(2)}}{|X_t^{(2)}|^2} + \frac{X_t^{(2)} - X_t^{(1)}}{|X_t^{(2)} - X_t^{(1)}|^2} \right] dt, \\ X_0^{(1)} = x_1, \quad X_0^{(2)} = x_2, \end{cases}$$

where  $B^1$  and  $B^2$  are two independent complex brownian motions. Moreover, since  $s_{\lambda(i)}(x) = x^i$ , for  $i \in \mathbb{N}$  and  $x \in \mathbb{C}$ , we have by (6.1)

$$U(\{x\}) = -\ln \left( \frac{2}{\pi(1-R^4)} |x|^2 + \frac{3}{\pi(1-R^6)} |x|^4 \right).$$

Therefore, if we set  $C_1 := \frac{2}{\pi(1-R^4)}$  and  $C_2 := \frac{3}{\pi(1-R^6)}$ , then the diffusion starting from  $x_1 \in \mathbb{C}$  is a solution of the following S.D.E.:

$$\begin{cases} dX_t = \sqrt{2} dB_t + \frac{2C_1 X_t + 4C_2 (X_t)^3}{C_1 |X_t|^2 + C_2 |X_t|^4} dt, \\ X_0 = x_1, \end{cases}$$

where  $B$  is a complex brownian motion.

## Sine kernel on the circle

Fix  $N \in \mathbb{N}^*$  and define  $S = D = [-\frac{N}{2}, \frac{N}{2}] \subset \mathbb{R}$ . Let  $\{\varphi_k^{(N)}\}_{0 \leq k \leq N-1}$  denote the orthonormal subset of  $L^2([-\frac{N}{2}, \frac{N}{2}], \ell)$  defined by

$$\varphi_k^{(N)}(\theta) := \mathbf{1}_{[-\frac{N}{2}, \frac{N}{2}]}(\theta) \frac{1}{\sqrt{N}} e^{2i\pi k\theta/N},$$

where  $i := \sqrt{-1}$  denotes the complex unit. We consider the Dyson operator  $\mathcal{K}_{\text{Dy}}^N : [-\frac{N}{2}, \frac{N}{2}]^2 \rightarrow \mathbb{C}$  with associated kernel defined by

$$K_{\text{Dy}}^N(\theta_1, \theta_2) := \sum_{k=0}^{N-1} \varphi_k^{(N)}(\theta_1) \overline{\varphi_k^{(N)}(\theta_2)}, \quad \theta_1, \theta_2 \in \left[-\frac{N}{2}, \frac{N}{2}\right].$$

By a standard computation, one can rewrite  $K_{\text{Dy}}^N$  as

$$K_{\text{Dy}}^N(\theta_1, \theta_2) = \frac{\sin(\pi(\theta_1 - \theta_2))}{N \sin(\frac{\pi(\theta_1 - \theta_2)}{N})} \quad \theta_1, \theta_2 \in \left[-\frac{N}{2}, \frac{N}{2}\right].$$

Note that

$$K_{\text{Dy}}^N(\theta_1, \theta_2) \xrightarrow{N \rightarrow \infty} \frac{\sin(\pi(\theta_1 - \theta_2))}{\pi(\theta_1 - \theta_2)},$$

uniformly on compact sets. Therefore by Proposition 3.10 in [21], the determinantal point process with kernel  $K_{\text{Dy}}^N$  converges weakly to the sine kernel process. It is well-known that the operator  $\mathcal{K}_{\text{Dy}}^N$  satisfies Condition **(H1)** (see e.g. [23]). Since the measure  $\mu$  is the Lebesgue measure, Condition **(H2)** is trivially satisfied.

Again, when evaluating the determinant the matrix  $(K_{\text{Dy}}^N(\theta_i, \theta_j))_{1 \leq i, j \leq N}$ , for  $\theta_1, \dots, \theta_N \in [-\frac{N}{2}, \frac{N}{2}]$ , we notice that it is nothing but a Vandermonde determinant, which can be explicitly calculated:

$$\det (K_{\text{Dy}}(\theta_i, \theta_j))_{1 \leq i, j \leq N} = \frac{1}{N^N} \prod_{1 \leq i < j \leq N} |e^{2i\pi\theta_j/N} - e^{2i\pi\theta_i/N}|^2$$

$$= \frac{2^{N(N-1)}}{N^N} \prod_{1 \leq i < j \leq N} \sin\left(\pi \frac{\theta_j - \theta_i}{N}\right)^2. \quad (6.2)$$

Now, even though the operator  $\mathcal{J}[D]$  is not properly defined since the spectrum of  $\mathcal{K}$  contains  $N$  times the eigenvalues 1, the Janossy density is still well-defined, as was proved in [21, Lemma 3.4]. More precisely, in our case, if  $\mathbf{x} = \{\theta_1, \dots, \theta_N\}$ , the Janossy densities are equal to

$$j_{[-\frac{N}{2}, \frac{N}{2}] }^{(N)}(\mathbf{x}) = \det (K_{\text{Dy}}(\theta_i, \theta_j))_{1 \leq i, j \leq N},$$

and if  $|\mathbf{x}| \neq N$ , then  $j_{[-\frac{N}{2}, \frac{N}{2}] }(\mathbf{x}) = 0$ . As before, condition **(H3)** is satisfied as the previous function is continuously differentiable on  $[-\frac{N}{2}, \frac{N}{2}]$ . It remains to note that Condition (3.14) can be extended to include the case where  $1 \in \text{Spec}(\mathcal{K})$  by simply replacing the occurrence of  $\det J[D](x_1, \dots, x_k)$  with the  $k$ th-Janossy density  $j_{[-\frac{N}{2}, \frac{N}{2}] }^{(N)}(x_1, \dots, x_k)$ , which is well-defined in this case. Thanks to (6.2), we obtain

$$\left| \frac{\partial_{\theta_i} j_{[-\frac{N}{2}, \frac{N}{2}] }^{(N)}(\theta_1, \dots, \theta_n) \partial_{\theta_j} j_{[-\frac{N}{2}, \frac{N}{2}] }^{(N)}(\theta_1, \dots, \theta_n)}{j_{[-\frac{N}{2}, \frac{N}{2}] }^{(N)}(\theta_1, \dots, \theta_n)} \right| = C \prod_{k < k'} \sin\left(\pi \frac{\theta_k - \theta_{k'}}{N}\right)^2 \left| \sum_{k \neq i, k' \neq j} \cotan\left(\pi \frac{\theta_i - \theta_k}{N}\right) \cotan\left(\pi \frac{\theta_j - \theta_{k'}}{N}\right) \right|,$$

for  $i, j = 1, \dots, N$ , and  $\theta_1, \dots, \theta_N \in [-\frac{N}{2}, \frac{N}{2}]$ . The previous equation being continuous in  $\theta_1, \dots, \theta_N$ , it is integrable on a compact, and therefore Condition (3.14) is verified.

Therefore, the diffusion associated with the previous determinantal process starting from  $(\theta_1, \dots, \theta_N) \in [-\frac{N}{2}, \frac{N}{2}]^N$  is a (weak) solution of the following system of S.D.E. on  $\mathbb{R}$ :

$$\begin{cases} d\theta_t^{(1)} = \sqrt{2} dB_t^1 + \sum_{1 \leq j \leq N} \sum_{j \neq i} \frac{2\pi}{N} \cot\left(\frac{\pi(\theta_t^{(j)} - \theta_t^{(1)})}{N}\right) dt, \\ \vdots \\ d\theta_t^{(N)} = \sqrt{2} dB_t^N + \sum_{1 \leq j \leq N} \sum_{j \neq i} \frac{2\pi}{N} \cot\left(\frac{\pi(\theta_t^{(j)} - \theta_t^{(N)})}{N}\right) dt, \\ \theta_0^{(1)} = \theta_1, \dots, \theta_0^{(N)} = \theta_N, \end{cases} \quad (6.3)$$

where  $B^1, \dots, B^N$  are  $N$  independent brownian motions on  $\mathbb{R}$ . It can be noted that the S.D.E. (6.3) was solved in [3] in the strong sense.

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