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Public health spending, old-age productivity and economic growth: chaotic cycles under perfect foresight

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Abstract This paper analyses the dynamics of a double Cobb-Douglas economy with overlapping generations and public health investments that affect the supply of efficient labour of the old-aged. It is shown that the positive steady state of the economy is unique. Moreover, we provide necessary and sufficient conditions for the emergence of endogenous deterministic complex cycles when individuals are perfectly foresighted. Interestingly, (*i*) the equilibrium dynamics shows rather complicated phenomena such as a multiplicity of bubblings depending on the size of the public health system, and (*ii*) the higher the degree of thriftiness, the likelier an economy is exposed to endogenous fluctuations because the need to save when young to support consumption when old reduces.

Keywords Old-age labour productivity; OLG model; Perfect foresight; Public health expenditure

JEL Classification C62; I18; O41

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Journal of Economic Behavior and Organization – Research highlights

Title *Public health spending, old-age productivity and economic growth: chaotic cycles under perfect foresight*

Authors **Luciano Fanti*** (first author) and **Luca Gori**** (corresponding author) and the

- This paper presents an analysis of a very simple (one-dimensional) Cobb-Douglas economy with very complicated dynamics. Our model contributes to the literature on growth and cycles with endogenous health.

- Using the Diamond overlapping generations model of neoclassical growth, it is shown that the link between the public provision of health care services, which in turn affects the supply if efficient labour of the old-aged, and labour productivity causes complex dynamics depending on the size of the public health system, despite the assumption of perfectly foresighted individuals.

- Although the dynamic system is described by a bimodal map, the steady state if the economy is unique. Moreover, we provide necessary and sufficient conditions for the emergence of endogenous deterministic complex cycles when individuals are perfectly foresighted. Interestingly, (*i*) the equilibrium dynamics shows rather complicated phenomena such as a multiplicity of bubbling areas depending on the size of the public health system, and (*ii*) the higher the degree of thriftiness, the likelier an economy is exposed to endogenous fluctuations.

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1. Introduction

The macroeconomic literature has devoted a long lasting attention to the understanding of the fundamental causes that generate fluctuations in the economic activity (i.e., business cycles). Starting from the Keynesian tradition, business cycles are generated from changes in aggregate demand conditions through the sluggish adjustment of prices and wages (e.g., Hicks, 1937). While such a strand of literature was focused on short-run issues in explaining cycle, the emergence of the real business cycle literature (e.g., Long and Plosser, 1983), which aims to explain the existence of cycles in macroeconomic variables through the propagation of temporary fundamental shocks within the neoclassical growth model, has renewed the interest in long-run economic growth and fluctuations. However, such a literature is grounded on a stochastic origin of economic cycles.

Unlike this, another strand of literature argue that exogenous shocks are not necessary for the emergence of fluctuations. Indeed, non-monotonicities in the dynamic behaviour of economic variables can generate periodic as well as aperiodic, but deterministic, orbits that resemble random ones (e.g., Goodwin, 1967; Grandmont, 1985; Farmer, 1986; Reichlin, 1986; de Vilder, 1996). While the first author explains the occurrence of growth and cycles in a Marxian labour market context, which has been criticised for the lacking of sound micro-foundations, the other three ones find that business cycles can occur in the neoclassical overlapping generations (OLG) model with rational individuals, showing that endogenous deterministic, rather than exogenous stochastic, fluctuations can also be generated in orthodox models. However, in the neoclassical OLG growth model with production à la Diamond (1965) regular and complex cycles can emerge only either assuming not fully rational individuals, i.e. with either myopic or adaptive expectations about future factor prices (Benhabib and Day, 1982; Michel and de la Croix, 2000; de la Croix and Michel, 2002) or extending it with the assumptions of, e.g., endogenous labour supply (Medio and Negroni, 1996), production externality (Cazzavillan, 1996), market imperfections (Aloi et al., 2000), PAYG pensions depending on previous earnings (Wagener, 2003), or taking the accumulation of

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government debt into consideration (Yokoo, 2000), which result in higher dimensional systems than the one-dimensional Diamond's model and thus able to describe more complicated dynamical events such as the Neimark–Hopf bifurcations.[1] In fact, the economic literature has definitely shown, by resorting to various mathematical tools, that the Diamond's model with rational individuals (perfect foresight) can never possess an unstable equilibrium, so that business cycle is prevented in such a case (Galor and Ryder, 1989; Longo and Valori, 2001; Wendner, 2003; Chen et al., 2008). Therefore, extending the basic OLG growth model with production with hypotheses that, however, preserve the feature of the one-dimensional map, may hardly transform it in a model suited to explain fluctuations in macroeconomic variables if individuals are rational. An exception, however, is Bhattacharya and Qiao (2007) who assume individual lifetime as being dependent of the health status which is, in turn, augmented by private health investments accompanied by a complementary tax-financed public health program, and show that the economy may be exposed to aggregate endogenous fluctuations and even chaotic motions. However, and most important, in their model not only endogenous fluctuations can occur when the private and public inputs in the longevity function are fairly complementary, but savings are also independent from the future interest rate and, hence, the hypothesis of perfect foresight of agents is irrelevant.

As regards the health of people, an evident and important stylised fact is the widespread rise in health spending, which is mainly public provided in several countries in the world, especially in Europe. In fact, as noted by Leung and Wang (2010, p. 11) "With few exceptions, we have observed consistent and steady rises over time of health-care expenditure, both in absolute terms and as percentages of GDP, in almost all countries in the world. The total health-care expenditures

[1] Note that even in higher dimensional systems the emergence of business cycles in OLG models with perfectly foresighted individuals is an exception rather than the rule. For instance, the modified OLG model by Lines (2001), where the assumption that retirees are not endowed with the entire capital stock results in a two-dimensional system, is able to describe the initial phase of accumulation but unable to generate any type of cycles. For a survey on chaotic dynamics in economics see Rosser (1999).

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among European Economic Monetary Union countries, for example, reached an average 9% of GDP in the 1990s".

This paper aims to investigate the properties of the equilibrium dynamics in the two-period Diamond's (1965) OLG growth model under two slight assumptions: (*i*) individuals work not only when young but also in the second period of life in a measure tuned by an efficiency parameter, as in de la Croix and Michel (2007); (*ii*) the supply of efficient labour of the old-aged, whose value results from a combined effects of age experience, sick days, retirement time, disabilities and so on, depends on the individual health status,[2] which is, in turn, augmented by public investments on health. In particular, following a recent literature (Blackburn and Cipriani, 2002; de la Croix and Ponthiere, 2010), it is assumed that such an expenditure is transformed into better health according to a S-shaped relationship to capture the idea that health spending may have a more intense effect in promoting health only when a certain threshold is achieved (e.g., vaccine and immunization programs) while becoming scarcely effective when the health status is close to its saturating value.

Our main results are the following. First, we characterise the "bi-modal" shape of the phase map and show the existence and uniqueness of the positive equilibrium. Second, we analyse the local dynamics of the system and consider local stability and the emergence of a local bifurcation. We find necessary and sufficient conditions for the existence of oscillations around the steady state as well as necessary and sufficient conditions for such an equilibrium to be non-stationary. Moreover, numerical simulations also reveal that for *intermediate-sized* provision of tax-financed health services, chaotic cycles emerge and, in particular, multiple bubbling phenomena (see the discussion in Section 4 for details) can occur. We find that endogenous fluctuations are likelier if the degree of

[2] The link between health status and labour productivity has been early recognised by the pioneering Grossman (1972) who argues that: "… the level of ill-health measured by the rates of mortality and morbidity, influences the amount and productivity of labour supplied to an economy" (p. xiii). Moreover, some recent empirical works have found that health status is an important determinant of labour participation, especially for older workers (see, e.g., Cai, 2010; García-Gómez et al., 2010).

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thriftiness is fairly high, i.e. when individuals prefer to smooth consumption over their second period of life. The equilibrium dynamics in our model therefore may be cyclical, and most important, regular and chaotic business cycles appear to be the rule rather than the exception, while also showing a strong complexity.

This paper contributes to two strands of literature centred on the issues of: (*i*) endogenous, as an alternative to stochastically driven, business cycles, and (*ii*) endogenous lifetime. The value added of the paper is twofold. First, it is shown that deterministic chaos[3] is generated in a model in which it is generally prevented.[4] Second, while the effects of the introduction of endogenous health as the main determinant of the individual life span have recently been investigated assuming either private (Leung and Wang, 2010) or public (Chakraborty, 2004) systems or both (Bhattacharya and Qiao, 2007), less attention has been paid to the link between endogenous health and labour productivity, and thus this paper want to fill this gap in a simple OLG context à la Diamond.

Our findings (*i*) add another possible explanation, based on link between old-age labour productivity and the public health system, of the occurrence of complex cycles in the OLG growth model with perfectly foresighted individuals, and (*ii*) suggest possible policy consequences.

The remainder of the paper is organised as follows. Section 2 presents the model. Section 3 analyses the equilibrium dynamics of the economy and provides necessary and sufficient conditions for the existence of endogenous fluctuations. Section 4 illustrates the emergence deterministic chaos. Section 5 concludes.

[3] This result is of interest because it shows that the equilibrium dynamics may generate business cycles without the need of any exogenous shock. In fact, as Bhattacharya and Qiao (2007, p. 2528) claimed: "These fluctuations are interesting to economists because they represent stylized business cycles that are generated purely from within an economic system and not from exogenous stochastic shocks."

[4] We recall that in the absence of our two main assumptions, the equilibrium dynamics in the model would be globally stable.

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2. The model

Consider a general equilibrium OLG closed economy populated by identical individuals, identical firms and a government that finances a public health programme through labour income taxes.

2.1. Individuals

Each generation is composed by a continuum of agents of measure one. The typical agent lives for two periods. An agent born at time t draws utility from consumption when young-aged ($c_{1,t}$) and old-aged ($c_{2,t+1}$):

$$U_t\left(c_{1,t}, c_{2,t+1}\right), \tag{1}$$

We assume that $U(.)$ is increasing in its arguments and concave; it is homogeneous of degree one (homothetic preferences) and satisfies the Inada conditions.

Workers inelastically supply one unit of efficient labour when young, while receiving wage income at the rate w_t, and d_{t+1} units of efficient labour when old, and receive earnings equal to $d_{t+1}w^e{}_{t+1}$ in such a case, where $w^e{}_{t+1}$ is the wage individuals expect to earn at $t+1$. The coefficient d defines the endowment of efficient labour when old (e.g., the behaviour of the productivity over life). Therefore, the slope of earnings over the life cycle positively depends on: (*i*) the individual health level when old, which, in turn, determines the endowment of efficient labour (of course, also an early or late retirement age, through its effects on the labour supply of workers in their second period of life, is strictly related with the measure of health status); (*ii*) the effect of both age and experience on productivity.[5] Therefore, a priori the supply of efficient labour when old may be

[5] See de la Croix and Michel (2007, p. 511): "The parameter [d] defines the income growth ability over life, and is determined by different factors: health when old, determining the old-age endowment in efficient labor; retirement age; effect of experience on human capital."

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larger or smaller than that of the young, and this crucially depends on two counterbalancing forces: the negative effect of the reduction in health when old, and the positive effect due to the experience. In particular, we assume that the efficient labour supply of the old-aged at $t+1$, d_{t+1}, depends on the individual health status[6] in the same period, which is positively affected by health investments at t, h_t (namely, the higher the public health spending, the higher both the health status and labour efficiency when old, i.e. if an individual is not productive at work is exclusively for health reasons in this paper). In particular, it is assumed that the relationship between the old age endowment of efficient labour and health expenditure is described by the following *non-decreasing* – though bounded – function (see, e.g., Blackburn and Cipriani, 2002):[7]

$$d_{t+1} = d(h_t) = \frac{d_0 + d_1 \Delta (h_t)^{\delta}}{1 + \Delta (h_t)^{\delta}}, \tag{2}$$

where $\delta, \Delta > 0$, $0 \le d_0 < d_1$ $d(0) = d_0$, $\lim_{h \to \infty} d(h) = d_1$, $d'_h(h) = \dfrac{\delta \Delta h^{\delta - 1}(d_1 - d_0)}{(1 + \Delta h^{\delta})^2} > 0$, $d''_h(h) < 0$ if $\delta \le 1$ and $d''_{hh}(h) \gtreqless 0$ for any $h \lesseqgtr h_T := \left[\dfrac{\delta - 1}{(1+\delta)\Delta} \right]^{\frac{1}{\delta}}$ if $\delta > 1$.

Now, some clarification on Eq. (2) are in order. We define d_0 as the value of the "natural" or "biological" level of labour efficiency of the old-aged. The parameter d_1 captures the intensity of

[6] The age profile of labour productivity due to experience effects, whose behaviour is empirically rather controversial, however, is left exogenous in this paper because we concentrate on the health effect of it.

[7] Although Blackburn and Cipriani (2002) study the relationship between longevity and human capital instead of labour productivity and health investments, the line of reasoning to justify Eq. (2) may be the same. Realistically, in many cases health investments have a more intense effect in promoting individual health (and thus labour efficiency) when old when a certain threshold level of public health expenditure is reached, while becoming scarcely effective when efficient labour is close to its saturating value (e.g., the functional relationship between health investment and individual health (i.e. labour efficiency) when old may be S-shaped).

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the efficiency of public health investments on the supply of efficient labour.[8] Finally, from Eq. (2) it can readily be seen that a rise in δ (Δ) reduces (increases), *ceteris paribus*, the effectiveness of health spending as an inducement to higher efficient labour for different values of the health investment. In other words, it measures how labour productivity reacts to a change in h_t. In particular, when $\delta \leq 1$ ($\delta > 1$) threshold effects of health investments on the endowment of efficient labour when old are absent (exist) and, hence, the function $d(h_t)$ is concave (S-shaped), so that the labour supply of the old-aged increases less than proportionally (more [less] than proportionally until [once] the turning point h_T is achieved) from the starting point d_0 to the saturating value d_1 as h rises.

The government levies a wage tax at the constant rate $0 < \tau < 1$ on the young workers' labour income (see Chakraborty, 2004). Therefore, the budget constraint of an individual born at t simply reads as:

$$c_{1,t} + s_t = w_t(1-\tau), \tag{3.1}$$

i.e. the disposable income is divided into material consumption and savings (s_t).

Old individuals work with either a higher or lower efficiency than the young depending on their health status, and consume on the basis of the resources saved in the first period of life plus the expected interest accrued from time t to time $t+1$ at the rate r^e_{t+1}, as well as on the labour income they receive in such a period. Hence, the budget constraint of an old individual started working at t can be expressed as

$$c_{2,t+1} = (1 + r^e_{t+1})s_t + d_{t+1}w^e_{t+1}. \tag{3.2}$$

The representative individual at t chooses how much to save out of her disposable income to maximise the lifetime utility which, for simplicity, is assumed to be of the logarithmic type, that is

[8] A rise in d_1 may be interpreted as exogenous medical advances due, for instance, to scientific research (i.e., antibiotics, vaccination programmes, innovations in medical technologies and so on).

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$$U_t = \ln(c_{1,t}) + \beta_t \ln(c_{2,t+1}), \tag{4}$$

subject to Eqs. (3), where $0 < \beta_t < 1$ is the subjective discount factor. The constrained maximisation of Eq. (4), where actual and expected factor prices and the future supply of efficient labour are taken as given, yields the following saving rate:

$$s_t = \frac{\beta_t w_t (1-\tau)}{1+\beta_t} - \frac{d_{t+1} w^e{}_{t+1}}{(1+\beta_t)(1+r^e{}_{t+1})}. \tag{5}$$

Although the subjective discount factor might depend on health when old (for instance, as a "proxy" of the rate of longevity, see Chakraborty, 2004), we assume $\beta_t = \beta$ constant for any t, because this paper is essentially focused on the relationship between health and labour efficiency, rather than between health and longevity.

2.2. Firms

At time t firms produce a homogeneous good, Y_t, combining capital and labour, K_t and L_t, respectively, through the constant returns to scale Cobb-Douglas technology $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. Labour supply is $L_t = \bar{L}(1+d_t)$, where $\bar{L}$ is the constant number of workers in each cohort (young and old); then, without loss of generality, we set $\bar{L} = 1$. Therefore, output per efficient worker (y_t) as a function of capital per efficient worker (k_t) is

$$y_t = Ak_t^{\alpha}, \tag{6}$$

where $y_t := Y_t / L_t$ and $k_t := K_t / L_t$.

Firms maximise profits and perfect competition guarantees that factor inputs are paid their marginal products, that is

$$r_t = \alpha A k_t^{\alpha-1} - 1, \tag{7}$$

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$$w_t = (1-\alpha)Ak_t^{\alpha}, \tag{8}$$

where we have assumed that capital fully depreciates at the end of each period and output is sold at unit price.

2.3. Government

The government invests in public health (e.g. hospitals, vaccination programmes, new medical services, scientific research and so on), and the (per capita) budget constraint faced in period t reads as

$$h_t = \tau w_t, \tag{9}$$

the left-hand side being the health expenditure and the right-hand side the tax receipt.

2.4. Equilibrium

Market-clearing in goods and capital market leads to the equilibrium condition:

$$k_{t+1}(1+d_{t+1}) = s_t. \tag{10}$$

More in detail, using Eq. (5) to substitute out for s_t into Eq. (10), equilibrium implies:

$$k_{t+1} = \frac{\beta}{(1+\beta)[1+d(k_t)]} w_t(1-\tau) - \frac{d(k_t)}{(1+\beta)[1+d(k_t)]} \frac{w^e{}_{t+1}}{1+r^e{}_{t+1}}. \tag{11}$$

where the relationship between the productivity of the old-aged and the capital stock, $d(k_t)$, is obtained by Eq. (2) upon substitution of h_t from Eq. (9) and w_t from Eq. (8).

In the next section we analyse the dynamics of the economy under perfect foresight.

3. Dynamics under perfect foresight

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With perfect foresight, the expected interest and wage rates depend on the future value of the stock of capital per efficient worker, that is

$$\begin{cases} 1+r^{e}{}_{t+1} = \alpha A k_{t+1}{}^{\alpha-1} \\ w^{e}{}_{t+1} = (1-\alpha) A k_{t+1}{}^{\alpha} \end{cases}. \tag{12}$$

Therefore, combining Eqs. (8), (11), (12) and rearranging terms, the dynamic equilibrium sequence of capital can be written as

$$k_{t+1} = \frac{I k_t{}^{\alpha}}{\alpha(1+\beta)+(1+\alpha\beta)d(k_t)}, \tag{13}$$

where $I = \beta\alpha(1-\alpha)A(1-\tau) > 0$. Substituting out for $d(k_t)$ from Eqs. (2), (8) and (9) into (13), the dynamics of capital be expressed as:

$$k_{t+1} = \frac{I k_t{}^{\alpha}\left(1+B k_t{}^{\alpha\delta}\right)}{G+EB k_t{}^{\alpha\delta}}, \tag{14}$$

where $B := \Delta[\tau(1-\alpha)A]^{\delta} > 0$, $E := \alpha(1+\beta)+d_1(1+\alpha\beta) > 0$, $G := \alpha(1+\beta)+d_0(1+\alpha\beta) > 0$ and $E > G$ because $d_1 > d_0 \geq 0$.

Steady states of the time map Eq. (14) are determined as $k_{t+1} = k_t = k^*$. The following propositions show (*i*) the existence of a unique non-trivial time-invariant solution of Eq. (14), and (*ii*) despite the seemingly simplistic form implied by such a dynamic system, there exists a decreasing relationship between capital stocks at two successive dates (i.e. the law of motion in Eq. 14 may be non-monotonic) that might also generate complex cycles. The latter result depends on the mutual relationship between the intensity of the reaction of the supply of efficient labour when old to a change in the health spending, δ, and the health tax rate, τ.

Analysis of Eq. (14), therefore, gives the following propositions.

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Proposition 1. *(Existence and uniqueness of the steady state). (1) The zero equilibrium of the dynamic system described by Eq. (14) is unstable. (2) In addition, a unique non-trivial steady state* $k^* > 0$ *exists.*

Proof. Define the right-hand side of Eq. (14) as $J(k)$. Differentiating $J(k)$ with respect to k gives:

$$J'_k(k) = \frac{\alpha I\left(Mk^{2\alpha\delta} + Fk^{\alpha\delta} + G\right)}{k^{1-\alpha}\left(G + EBk^{\alpha\delta}\right)^2}, \tag{15}$$

where $M := EB^2 > 0$, $F := B[E + G - \delta(E - G)]$ and $E - G = (1+\alpha\beta)(d_1 - d_0) > 0$. Moreover, $F > 0$ ($F < 0$) if and only if $\delta < \bar{\delta}$ ($\delta > \bar{\delta}$), where

$$\bar{\delta} := \frac{E+G}{E-G} > 1. \tag{16}$$

Since $J(0) = 0$ and

$$\lim_{k\to 0^+} J'_k(k) = \alpha I \lim_{k\to 0^+} \frac{Mk^{2\alpha\delta} + Fk^{\alpha\delta} + G}{k^{1-\alpha}\left(G + EBk^{\alpha\delta}\right)^2} = +\infty, \tag{17}$$

then Point (1) of Proposition 1 holds.

Now, fixed points of Eq. (14) are determined as solutions to $k = J(k)$, that can also be rearranged as $Z_1(k) = Z_2(k)$, where $Z_1(k) := k^{1-\alpha}$ and $Z_2(k) := \dfrac{I\left(1 + Bk^{\alpha\delta}\right)}{G + EBk^{\alpha\delta}}$.

Therefore, since:

(*i*) $Z_1(0) = 0$, $Z'_{1,k}(k) = (1-\alpha)k^{-\alpha} > 0$ for any $k > 0$ and $\lim_{k\to+\infty} Z_1(k) = +\infty$, and

(*ii*) $Z_2(0) = \dfrac{I}{G} > 0$, $Z'_{2,k}(k) = \dfrac{-\alpha\delta IB(E-G)k^{\alpha\delta-1}}{\left(G + EBk^{\alpha\delta}\right)^2} < 0$ for any $k > 0$ and

$\lim_{k\to+\infty} Z_2(k) = \dfrac{I\left(\dfrac{1}{k^{\alpha\delta}} + B\right)}{\dfrac{G}{k^{\alpha\delta}} + EB} = \dfrac{I}{E} > 0$, with $\dfrac{I}{E} < \dfrac{I}{G}$, then

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for any $k > 0$ $Z_1(k) = Z_2(k)$ only once at k^*. This proves Point (2) of Proposition 1. **Q.E.D.**

Proposition 2. *(Stability and non-monotonic behaviour). (1) (Stability). Let* $0 < \delta \le \delta_2$ *hold. Then, the phase map* $J(k)$ *monotonically increases for any* $k > 0$ *and the unique non-trivial steady state* k^* *is locally asymptotically stable. (2) (Non monotonic behaviour). If* $\delta > \delta_2$ *and* $\tau_1(k^*) < \tau < \tau_2(k^*)$*, then the law of motion in Eq. (14) is non-monotonic, where*

$$\delta_2 := \frac{E + G + 2\sqrt{EG}}{E - G} > \bar{\delta}, \tag{18}$$

$$\tau_1(k^*) := \left\{ \frac{-[E + G - \delta(E - G)] - \sqrt{\Lambda}}{2(k^*)^{\alpha\delta} \Delta[(1-\alpha)A]^{\delta}} \right\}^{\frac{1}{\delta}}, \tag{19}$$

$$\tau_2(k^*) := \left\{ \frac{-[E + G - \delta(E - G)] + \sqrt{\Lambda}}{2(k^*)^{\alpha\delta} \Delta[(1-\alpha)A]^{\delta}} \right\}^{\frac{1}{\delta}}, \tag{20}$$

and $\Lambda := (E - G)^2 \delta^2 - 2(E - G)(E + G)\delta + (E - G)^2 > 0$ *for any* $\delta > \delta_2$.

Proof. Let first the proof of Point (1) be outlined. From Eq. (15) we find that:

$$\lim_{k \to +\infty} J_k'(k) = \lim_{k \to +\infty} \frac{\alpha I (Mk^{2\alpha\delta} + Fk^{\alpha\delta} + G)}{k^{1-\alpha}(G + EBk^{\alpha\delta})^2} = \alpha I \lim_{k \to +\infty} \frac{M + \frac{F}{k^{\alpha\delta}} + \frac{G}{k^{2\alpha\delta}}}{k^{1-\alpha}\left(\frac{G^2}{k^{2\alpha\delta}} + \frac{2GEB}{k^{\alpha\delta}} + E^2B^2\right)} = 0 . \tag{21}$$

Now, define $k^{\alpha\delta} := x$ as a new supporting variable. Then, Eq. (15) can be rearranged as

$$j(k, x) = \frac{\alpha I (Mx^2 + Fx + G)}{k^{1-\alpha}(G + EBx)^2}. \tag{22}$$

Solving Eq. (22) for x gives

$$\hat{x}_1 = \frac{-[E + G - \delta(E - G)] - \sqrt{\Lambda}}{2BE}, \tag{23}$$

$$\hat{x}_2 = \frac{-[E + G - \delta(E - G)] + \sqrt{\Lambda}}{2BE}. \tag{24}$$

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Then, it is easy to verify that $\Lambda > 0$ if and only if either $0 < \delta < \delta_1$ or $\delta > \delta_2$ holds, where $\delta_1 := \frac{E + G - 2\sqrt{EG}}{E - G} < 1$ and $\delta_2 > \bar{\delta}$ is defined by Eq. (18) above. Since $\delta_1 < 1$, it can be ruled out because $F < 0$ if and only if $\delta > \bar{\delta} > 1$ (see Eq. 16).

Now, using $k^{\alpha\delta} := x$, from (21) and (22) we obtain

$$\hat{k}_1 = \left\{ \frac{-[E + G - \delta(E - G)] - \sqrt{\Lambda}}{2BE} \right\}^{\frac{1}{\alpha\delta}}, \tag{23}$$

$$\hat{k}_2 = \left\{ \frac{-[E + G - \delta(E - G)] + \sqrt{\Lambda}}{2BE} \right\}^{\frac{1}{\alpha\delta}}, \tag{24}$$

which represent the two solutions of Eq. (15) for k. Therefore, if $0 < \delta < \bar{\delta}$, then $F > 0$ and, hence, no positive real roots of Eq. (20) can exist for x. If $\bar{\delta} < \delta < \delta_2$, then $F < 0$ and $\Lambda < 0$, and no positive real roots of Eq. (20) can exist for x. Therefore, through Eq. (15), for any $0 < \delta < \delta_2$, no positive real roots can exist for k and $J_k'(k) > 0$ for any $k > 0$. Hence, since Proposition 1 holds, $k^* > 0$ is the unique locally asymptotically stable steady state of the dynamic system Eq. (14) in that case. This proves Point (1).

We now proceed to prove Point (2). If $\delta > \delta_2$, then it is straightforward to verify that $F < 0$, $\Lambda > 0$ and, hence, two positive real roots of Eq. (20) exist in that case, namely $\hat{x}_1 > 0$ and $\hat{x}_2 > 0$ (see Eqs. 21 and 22). Therefore, through Eq. (15), two positive real roots exist for k as well, namely $\hat{k}_1 > 0$ and $\hat{k}_2 > 0$ (see Eqs. 23 and 24). Then $J_k'(k^*) > 0$ for any $0 < k^* < \hat{k}_1$, $J_k'(k^*) < 0$ for any $\hat{k}_1 < k^* < \hat{k}_2$, and $J_k'(k^*) > 0$ for any $k^* > \hat{k}_2$, where $\hat{k}_1 > 0$ and $\hat{k}_2 > 0$ represent the maximal and minimal points of the time map $J(k)$ for any $k > 0$, respectively. Hence, $\delta > \delta_2$ is a necessary condition for the law of motion in Eq. (14) to be oscillatory around the steady state k^*. Moreover, from the numerator of Eq. (15), $J_k'(k^*) \leq 0$ if and only if

$$T_1(k^*)\tau^{2\delta} + T_2(k^*)\tau^{\delta} + G \leq 0, \tag{25}$$

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where $T_1(k^*) := (k^*)^{2\alpha\delta}\Delta^2[(1-\alpha)A]^{2\delta}E > 0$ and $T_2(k^*) := (k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}[E+G-\delta(E-G)] < 0$ for any $\delta > \delta_2$. Defining now $\tau^{\delta} := p$ as a new supporting variable, (25) can be rearranged as

$$T_1(k^*)p^2 + T_2(k^*)p + G \leq 0, \tag{26}$$

Since $T_1(k^*) > 0$, $T_2(k^*) < 0$ for any $\delta > \delta_2$ and $G > 0$, then applying the Descartes' rule of sign we find the two positive real solutions of (26) for p when equality holds, that is:

$$p_1(k^*) := \frac{-[E+G-\delta(E-G)] - \sqrt{\Lambda}}{2(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}}, \tag{27}$$

$$p_2(k^*) := \frac{-[E+G-\delta(E-G)] + \sqrt{\Lambda}}{2(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}}, \tag{28}$$

where $p_2(k^*) > p_1(k^*)$ and $\Lambda > 0$ for any $\delta > \delta_2$. Therefore, inequality (26) is verified if and only if, for any $\delta > \delta_2$, $p_1(k^*) < p < p_2(k^*)$. Now, applying $\tau^{\delta} := p$, we get the two positive real solutions of (25) for τ when equality holds, namely Eqs. (19) and (20), where $\tau_2(k^*) > \tau_1(k^*)$. Therefore, inequality (25) is verified if and only if, for any $\delta > \delta_2$, $\tau_1(k^*) < \tau < \tau_2(k^*)$, which is sufficient to have $J_k'(k^*) < 0$ for any $\hat{k}_1 < k^* < \hat{k}_2$. This proves Point (2). **Q.E.D.**

Although Eq. (14) is *not* a unimodal function, Proposition 1 show the existence and uniqueness of the steady state. Moreover, Proposition 2 provid: (*i*) necessary and sufficient conditions for the unique steady state k^* (a closed-form solution for k^*, however, is prevented) to be locally asymptotically stable: i.e., when thresholds effects of public health investments on the level of individual health and thus on the supply of efficient labour when old are either absent or fairly low, and (*ii*) necessary and sufficient conditions for the law of motion in Eq. (14) to be oscillatory: i.e., when thresholds effects exist and the size of the public health spending is neither too small nor too large (i.e. *intermediate-sized*). The set of conditions stated in Point (2) of Proposition 2 is indeed *necessary* to generate endogenous fluctuations around the steady state.

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Therefore, we are now in a position to contribute to bring to light the *sufficient* conditions for the equilibrium to be non-stationary and, hence, for deterministic endogenous fluctuations to be generated. The following proposition deals with this argument and characterises, for the case when the law of motion in Eq. (14) is oscillatory, all the possible outcomes as regards stability and instability of the unique stationary state of real activity of the economy.

Proposition 3. *(Endogenous fluctuations). If Point (2) of Proposition 2 holds,* $\delta_2 < \delta < \delta(\tau, k^*)$ *and,*

(1) if $\tau_1(k^*) < \tau < \underline{\tau}(\tau, k^*)$*, then the law of motion in Eq. (14) is non-monotonic and convergent to* k^**;*

(2) if $\tau = \underline{\tau}(\tau, k^*)$*, then a flip bifurcation generically occurs;*

(3) if $\underline{\tau}(\tau, k^*) < \tau < \overline{\tau}(\tau, k^*)$*, then the law of motion in Eq. (12) is non-monotonic and divergent from* k^**;*

(4) if $\tau = \overline{\tau}(\tau, k^*)$*, then a reverse flip bifurcation generically occurs;*[9]

(5) if $\overline{\tau}(\tau, k^*) < \tau < \tau_2(k^*)$*, then the law of motion in Eq. (12) is non-monotonic and convergent to* k^**, where*

$$\delta(\tau, k^*) := \frac{\alpha I \Lambda}{4EG(E-G)(k^*)^{1-\alpha}}, \tag{29}$$

$$\underline{\tau}(\tau, k^*) := \left[\frac{-T_4(\tau, k^*) - \sqrt{\Omega}}{2T_3(\tau, k^*)}\right]^{\frac{1}{\delta}}, \tag{30}$$

$$\overline{\tau}(\tau, k^*) := \left[\frac{-T_4(\tau, k^*) + \sqrt{\Omega}}{2T_3(\tau, k^*)}\right]^{\frac{1}{\delta}}, \tag{31}$$

[9] Note that it can be shown numerically (see the next section) that both the flip bifurcation and reverse flip bifurcation are super-critical and, hence, attractive (i.e., the bifurcation points are symmetrical and stable).

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$T_3(\tau,k^*) := (k^*)^{2\alpha\delta}\Delta^2[(1-\alpha)A]^{2\delta}E\left[\alpha I + (k^*)^{1-\alpha}E\right] > 0,$

$T_4(\tau,k^*) := (k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}\left\{\alpha I[E+G-\delta(E-G)] + (k^*)^{1-\alpha}2EG\right\},$

$\Omega := \alpha^2 I^2\Lambda - 4\alpha IEG(E-G)(k^*)^{1-\alpha}\delta > 0$ *for any* $\delta_2 < \delta < \delta(\tau,k^*)$ *and, we recall* $I = I(\tau) := \beta\alpha(1-\alpha)A(1-\tau)$.

Proof. The proof of Proposition 3 is articulated as follows. From Eq. (15) we find that $J_k'(k^*) \le -1$ if and only if

$$T_3(\tau,k^*)\tau^{2\delta} + T_4(\tau,k^*)\tau^{\delta} + T_5(\tau,k^*) \le 0\,. \tag{32}$$

where $T_5(\tau,k^*) := G\left[\alpha I + (k^*)^{1-\alpha}G\right] > 0$. Using the definition $\tau^{\delta} := p$, (32) can be rearranged as

$$T_3(\tau,k^*)p^2 + T_4(\tau,k^*)p + T_5(\tau,k^*) \le 0\,, \tag{33}$$

Since $T_3(\tau,k^*) > 0$, $T_5(\tau,k^*) > 0$ and assuming $T_4(\tau,k^*) < 0$ for any $\delta_2 < \delta < \delta(\tau,k^*)$, then applying the Descartes' rule of sign there exists the two positive real solutions of (33) for p when equality holds, as given by:

$$\underline{p}(\tau,k^*) := \frac{-T_4(\tau,k^*) - \sqrt{\Omega}}{2T_3(\tau,k^*)}\,, \tag{34}$$

$$\overline{p}(\tau,k^*) := \frac{-T_4(\tau,k^*) + \sqrt{\Omega}}{2T_3(\tau,k^*)}\,, \tag{35}$$

where $\overline{p}(\tau,k^*) > \underline{p}(\tau,k^*)$. Therefore, inequality (33) is verified if and only if, for any $\delta_2 < \delta < \delta(\tau,k^*)$, $p_1(k^*) < p < p_2(k^*)$. Now, applying $\tau^{\delta} := p$, we get the two positive real solutions of (32) for τ when equality holds, namely Eqs. (30) and (31), where $\overline{\tau}(\tau,k^*) > \underline{\tau}(\tau,k^*)$. Therefore, for any $\delta_2 < \delta < \delta(\tau,k^*)$ we get, (*i*) $-1 < J_k'(k) < 0$ if $\tau_1(k^*) < \tau < \underline{\tau}(\tau,k^*)$, i.e. the steady state is stationary through oscillations, (*ii*) $J_k'(k^*) = -1$ if $\tau = \underline{\tau}(\tau,k^*)$, i.e. a flip bifurcation generically occurs, (*iii*) $J_k'(k^*) < -1$ if $\underline{\tau}(\tau,k^*) < \tau < \overline{\tau}(\tau,k^*)$, i.e. the steady state is non-stationary

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and deterministic endogenous fluctuations occurs, (*iv*) $J'_k(k^*) = -1$ if $\tau = \bar{\tau}(\tau, k^*)$, i.e. a reverse flip bifurcation generally occurs, and (*v*) $-1 < J'_k(k^*) < 0$ if $\bar{\tau}(\tau, k^*) < \tau < \tau_2(k^*)$, i.e. the steady state is stationary through oscillations. **Q.E.D.**

The phase map Eq. (14) is a first order non-linear difference equation in k and the law of motion of capital accumulation may be oscillatory, i.e. it may become a "backward-bending" function. In particular, this happens when: (*i*) threshold effects of public health investments exist as an inducement to higher efficient labour when old, and (*ii*) the provision of health services, as measured by the health tax rate τ, is *intermediate-sized* (see Point 2 of Proposition 2). In fact, when the public health expenditure is either small or large enough the law of motion is monotonic and the steady state is locally asymptotically stable regardless of the size of the parameter δ (see Point 1 of Proposition 2).

Indeed, two different (negative) effects emerge when the health tax rate raises, and it is set as neither too small nor large values. First, it reduces the disposable income of the young and thus it negatively affects savings and capital accumulation through this channel. Second, it increases health services and, hence, the individual health status which, in turn, implies a rise in the units of efficient labour of old age people; the rise in d, by reducing the need to save to sustain consumption, contributes to reduce capital accumulation and the steady state further.

The final result of this two negative forces, therefore, may be such that, when $\delta > \delta_2$, the phase map $J(k)$ is negatively sloped before the unique steady state is achieved, that is $k^* > \hat{k}_1$, at least for an intermediate-sized provision of health care services that affects the efficient labour supply of the old-aged.

Figure 1 shows in a stylised way the behaviour of the capital accumulation function Eq. (14) and the evolution of the unique steady state k^* for different sizes of the public health system, in the case

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threshold effects exist ($\delta > \delta_2$). The phase map Eq. (14) may increase monotonically or being of the bimodal type, and the unique equilibrium may be locally stable or unstable.

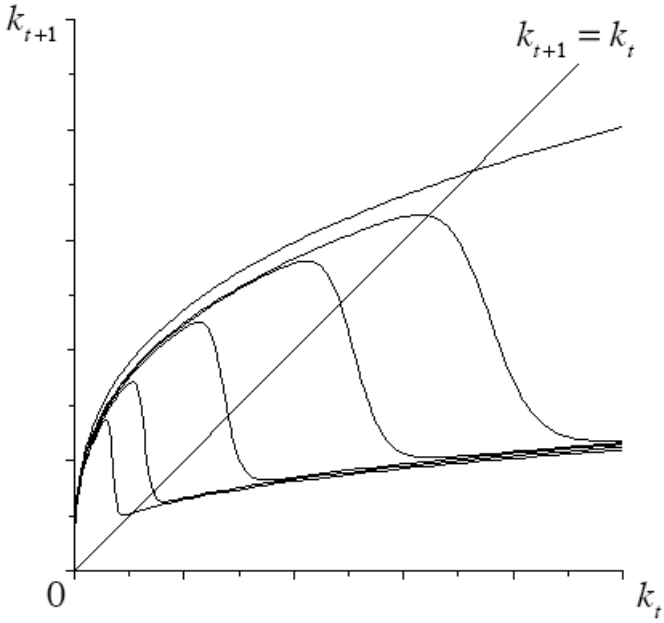

Figure 1. Pictorial view of the evolution of capital accumulation when τ varies.

3.1. Steady state effects of changes in the health tax rate

In order to clarify how the main steady-state macroeconomic variables react to a change in the health tax rate, τ, as well as to grasp the economic intuition behind the results, we now perform a sensitivity analysis by taking the following configuration of parameters:[10] $\alpha = 0.45$ (which is an average value between the output elasticity of capital usually referred to developed and developing countries, i.e. 0.36 and 0.50, respectively, see e.g., Kehoe and Perri, 2002 and Purdue University's

[10] It is worth noting that a calibration exercise for specific countries is beyond the scope of the present paper, whose aim is mainly concerned with the theoretical analysis of the dynamical features of an OLG economy through the relationship between (public) health system and health of mature workers. In this respect, this model may be considered a useful skeleton where more realistic components can also be added for calibrations analyses and policy exercises in both poor and rich countries.

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Global Trade Analysis Project 2005 database, GTAP), $\beta = 0.20$, $d_0 = 0$, $d_1 = 25$. This parameter set generates $\delta_2 = 1.3239$. Then we choose $A = 22$, $\delta = 14$[11] and $\Delta = 1$.

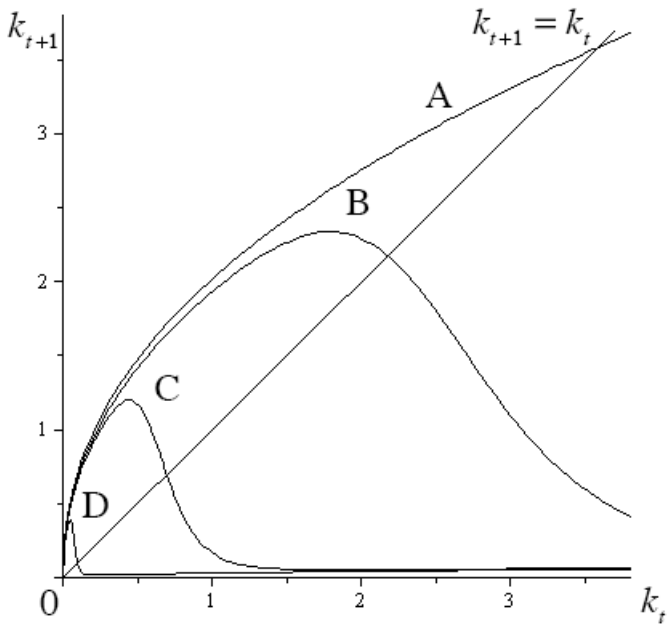

Figure 2. Pictorial view of the evolution of capital accumulation when τ varies for a given parameter set.

Table 1 below summarises Figure 2 from two point of views. The former is concerned with the evolution of the equilibrium stock of capital and other macroeconomic variables of interest when τ is increased. To this purpose, Table 1 illustrates the variations of k^* along with the per capita health expenditure, h^*, the supply of efficient labour when old, d^*, the level of per capita GDP, $Y^* = A(k^*)^\alpha (1 + d^*)$, the ratio of per capita health spending to per capita GDP, h^*/Y^*, and the steady state lifetime utility index of the representative agent, U^*.[12] The latter regards the slope of the phase

[11] We have used a value of δ close to that adopted by de la Croix and Ponthiere (2010) ($\delta = 10$).

[12] Note that Table 1 reports the different values of the variables of interest corresponding to Case A-Case D in Figure 2 (i.e., $0 < \tau < 0.2$) as well as for other several values of τ, which are not reported in the figure for reasons of clarity of the pictorial view.

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map evaluated at different equilibrium points (to this purpose Table 1 reports the values of the first derivative of Eq. 14 with respect to k_t evaluated at the steady state, namely $J'_k(k^*)$).

Table 1. Steady-state macroeconomic variables when τ varies.

τ	0 (A)	0.04 (B)	0.075 (C)	0.20 (D)	0.25	0.30	0.40
k^*	3.57	2.17	0.69	0.098	0.062	0.042	0.023
$J'_k(k^*)$	0.45	-0.85	-2.99	-4.26	-4.30	-4.29	-4.23
h^*	0	0.68	0.77	0.85	0.86	0.87	0.892
d^*	0	0.129	0.633	2.37	2.95	3.46	4.23
Y^*	39.05	35.27	30.5	26.08	24.94	23.77	21.24
h^*/Y^*	0	0.019	0.025	0.032	0.034	0.036	0.042
U^*	3.45	3.22	2.72	1.73	1.46	1.22	0.78

τ	0.60	0.80	0.92	0.94	0.96	0.98	0.99
k^*	0.0096	0.0047	0.003	0.0027	0.0023	0.0018	0.0008
$J'_k(k^*)$	-4.18	-4.30	-3.96	-3.64	-2.9	-0.98	0.43
h^*	0.899	0.87	0.81	0.79	0.76	0.69	0.49
d^*	4.63	3.28	1.44	1.04	0.60	0.14	0.001
Y^*	15.35	8.50	3.95	3.15	2.34	1.47	0.90
h^*/Y^*	0.058	0.102	0.207	0.253	0.32	0.46	0.54
U^*	-0.08	-1.21	-2.52	-2.92	-3.48	-4.47	-5.67

From Table 1, therefore, the following results hold.[13]

Result 1. *The steady-state stock of capital per efficient worker, k^*, monotonically reduces when τ raises.*

Result 2. *The steady-state per capita health expenditure, h^*, and the supply of efficient labour when old, d^*, are inverted U-Shaped functions of the health tax rate τ.*

Result 3. *The per capita GDP at the steady state , Y^*, monotonically reduces when τ raises.*

[13] Note that these results are robust to different parameter values, as extensive numerical simulations revealed.

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Result 4. *The steady state lifetime utility index of the representative individual, U^*, monotonically reduces when τ raises.*

Result 1 is not entirely surprising as the rise in the health tax rate negatively acts on k^* through two channels. First, the rise in τ reduces the disposable income of the young and, hence, the saving rate. Second, a higher health tax rate translates, for relatively low values of τ, into a higher health expenditure, which is eventually transformed in a higher supply of efficient labour, thus increasing the working income of the old-aged, *ceteris paribus* as regards the wage rate. The saving rate therefore shrinks because of the reduced need to support consumption in the second period of life.[14] This effect contributes to depress capital accumulation further on.

As regards Result 2, it is worth noting that the health expenditure is an inverted U-shaped function of the tax rate because of the existence of a sort of "Laffer curve" when τ raises. In fact, for increasing values of such a tax, the tax base (i.e. the wage rate) shrinks (see Result 1), meaning that the total health spending is reduced even if the tax rate is increased. As a consequence, the supply of efficient labour of the old, which solely depends on the government health spending, shrinks when the tax rate becomes fairly high. This causes a positive effect on capital accumulation which, nevertheless, is clearly of second order relative to the direct negative effect on k^* of the rise in τ. Therefore, as a final effect, we observe a strong reduction in the steady state stock of capital.

As regards Result 3, even if the effects of higher health taxes on per capita income may be a priori ambiguous, given the negative monotonic effect on capital accumulation and the possible increase in the supply of efficient labour when old, numerical simulations revealed that the negative effect of the reduced capital accumulation always prevails on the possible increase in the efficient

[14] Of course, as mentioned in the introduction, the public health spending can also positively affect the individual lifetime (see Chakraborty, 2004) and, through this channel, the saving rate. The analysis of this additional case is certainly interesting and left for further research.

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labour productivity when old and, hence, per capita GDP shrinks.[15] Finally, as a consequence of Result 3, also the lifetime utility index at the steady state always shrinks as well.

This paper, however, is mainly concerned with the effects of the public provision of health services on the stability of the economic system along with the possible emergence of deterministic endogenous cycles in an OLG model of neoclassical growth with production and rational expectations, which would be otherwise unable to generate interesting dynamical events. Therefore, we do not proceed further with a deeper investigation of the steady-state effects of the health tax rate τ, and concentrate on the study of the possible emergence of deterministic chaos.

Before starting with the numerical analysis of chaotic motion, it should be noted, however, that Table 1 clearly explains why stable or unstable non-monotonic behaviours occur only within a range of intermediate values of the health tax rate (see Point 2 of Proposition 2 and Proposition 3). In fact, for relatively small values of τ, the negative effect on savings and capital accumulation due to the reduction in the disposable income of the young is small and, hence, even if the supply of efficient labour when old increases in such a case, the final negative effect of a rise in τ (when it is fairly small) on the shape of the phase map Eq. (14) is not strong enough to generate a decreasing relationship between two different values of the capital stock at two successive dates. By contrast, when τ is high, even if the negative effect on savings and capital accumulation due to the strong reduction in the disposable income when young exists, it is mitigated by the reduction in health spending and, hence, in the supply of efficient labour when old (which, in turn, causes a positive effect on capital accumulation). As a consequence, the reduction caused by a rise in τ when the health tax rate is already high is not strong enough to continue to produce unstable oscillations around the steady state, i.e. the law of motion in Eq. (14) returns to be stable either through oscillations or monotonically.

[15] The ratio of per capita health spending to per capita GDP reported in Table 1 is consistent with the actual values observed in several countries around the world (see, e.g., the empirical work by Leung and Wang, 2010). For instance, for $0 < \tau < 0.45$, the ratio h^* / Y^* varies from 2.5 per cent when $\tau = 0.075$ to 5 per cent when $\tau = 0.45$.

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4. A numerical example of chaotic motions

In this section we wish to check for the possibility of chaotic motions that the phase map Eq. (14) may generate under the hypotheses of Proposition 3 above. To this purpose, an application of the theorem of Li and Yorke (1975) could be used.[16] However, given the economical rather than mathematical motivation of the paper, we only use a graphical tool (e.g., bifurcation diagrams) for a pictorial view of possible chaotic motions, without embarking in more sophisticated analyses for the detection of chaos (e.g. Lyapunov's exponents).

Therefore, below we again resort to numerical simulations to give an example of the chaotic behaviour generated by the financing of public health care services when $\delta > \delta_2$.

Figure 3-5 below (where Figures 4 and 5 represent an enlarged view of Figure 3) represent bifurcation for τ diagrams and show, for the same parameter values used in Section 3.1, the emergence of deterministic chaos depending on the values of τ. On the vertical axis we show the limit points of the equilibrium sequence of capital, and on the horizontal axis the value of τ ($0 < \tau < 1$). Figure 3 shows two bubbling phenomena linked by a stable two-period cycle (note that the scale of the figure prevents the vision of the second bubbling when τ is close to unity; such a case is portrayed in Figure 5 for reasons of expository clarity). When the health tax is low (i.e. $0 < \tau < 0.042$) a unique limit point exists. Then a flip bifurcation emerges followed by a period-doubling bifurcations cascade that leads to chaotic motions ($0.055 < \tau < 0.0985$) (see Figure 4), which is interwoven, as usually occurs, by small parametric windows (e.g., $0.091 < \tau < 0.915$) in which fluctuations become regular. At $\tau = 0.0986$ a reverse period-doubling bifurcations cascade

[16] Mitra (2001) also provided sufficient conditions to check for topological chaos when the Li–Yorke condition cannot be satisfied. However, since Proposition 2.3 by Mitra (2001, p. 143) holds only for unimodal maps, we cannot apply the result by such a result to ensure that the endogenous fluctuations generated in our model are chaotic, since the phase map Eq. (14) is bimodal.

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occurs, leading, for the large interval $0.125 < \tau < 0.924$, to a stable two-period cycle. Subsequently, the same qualitative change in the dynamics described above again holds (see Figure 5 in comparison with Figure 4): for $0.925 < \tau < 0.98$ a period-doubling route to chaos followed by a reverse period-doubling route to stability occurs.

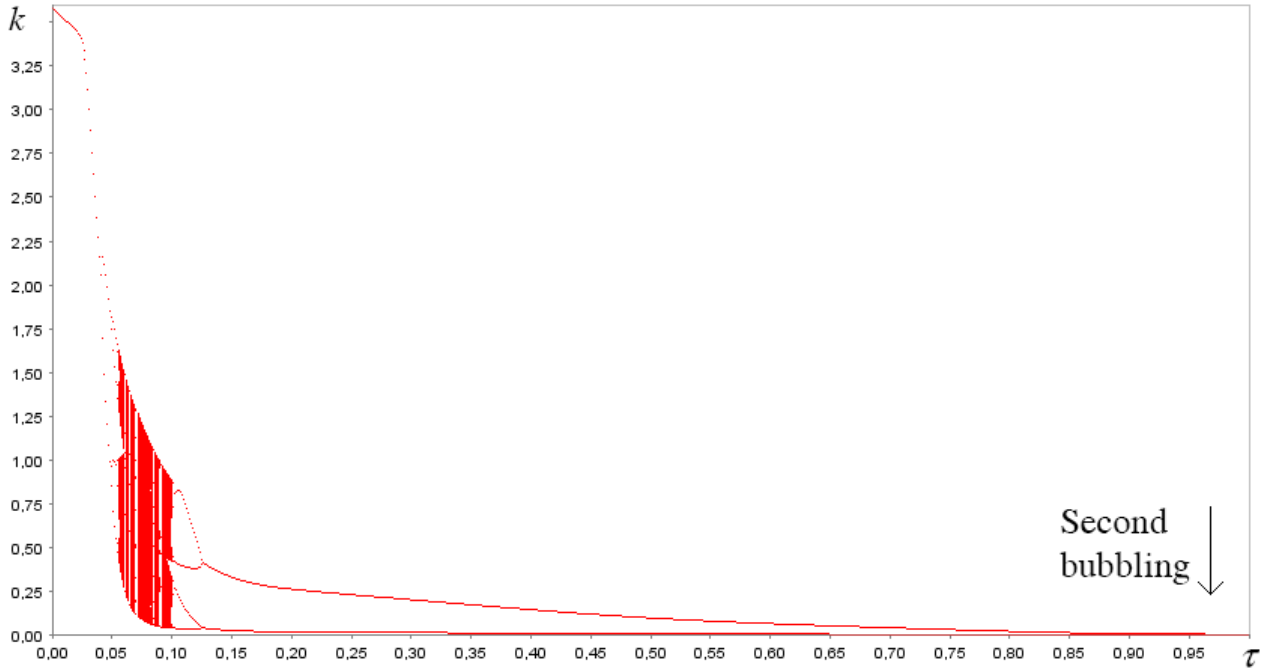

Figure 3. Bifurcation diagram for τ when $\beta = 0.2$ ($k_0 = 1$).

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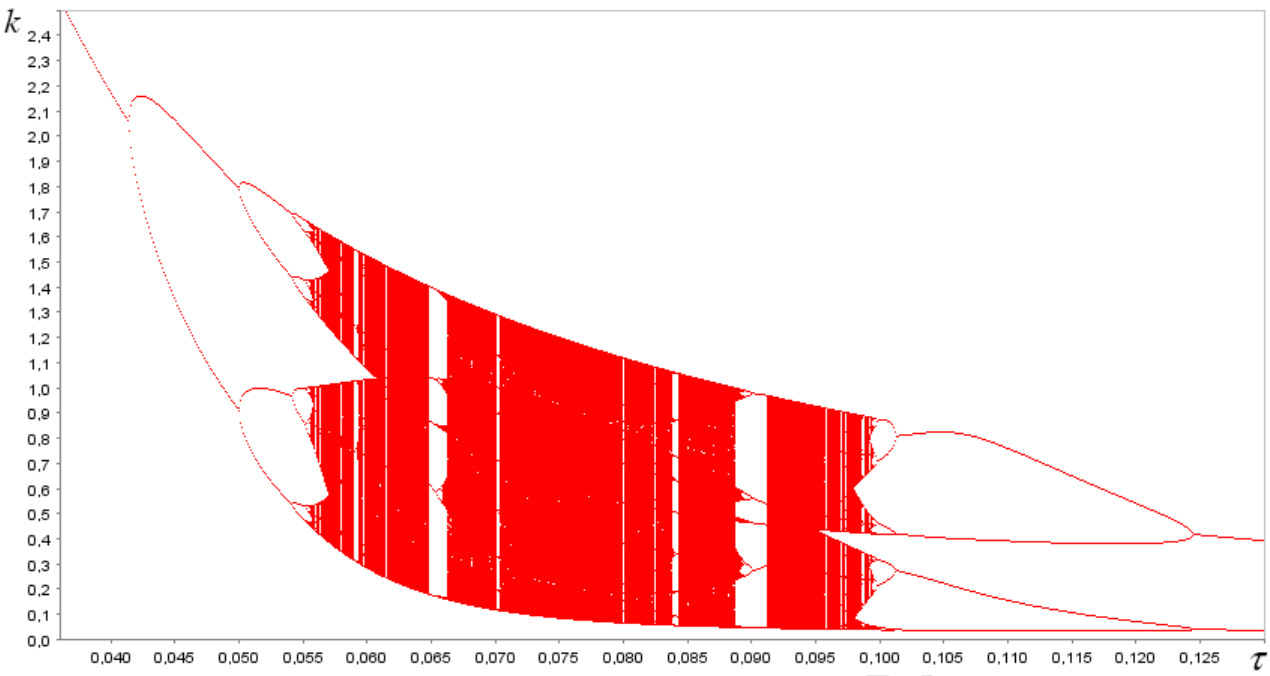

Figure 4. Enlarged view of the bifurcation diagram when $\beta = 0.2$ ($0.036 < \tau < 0.13$, $0 < k < 2.5$).

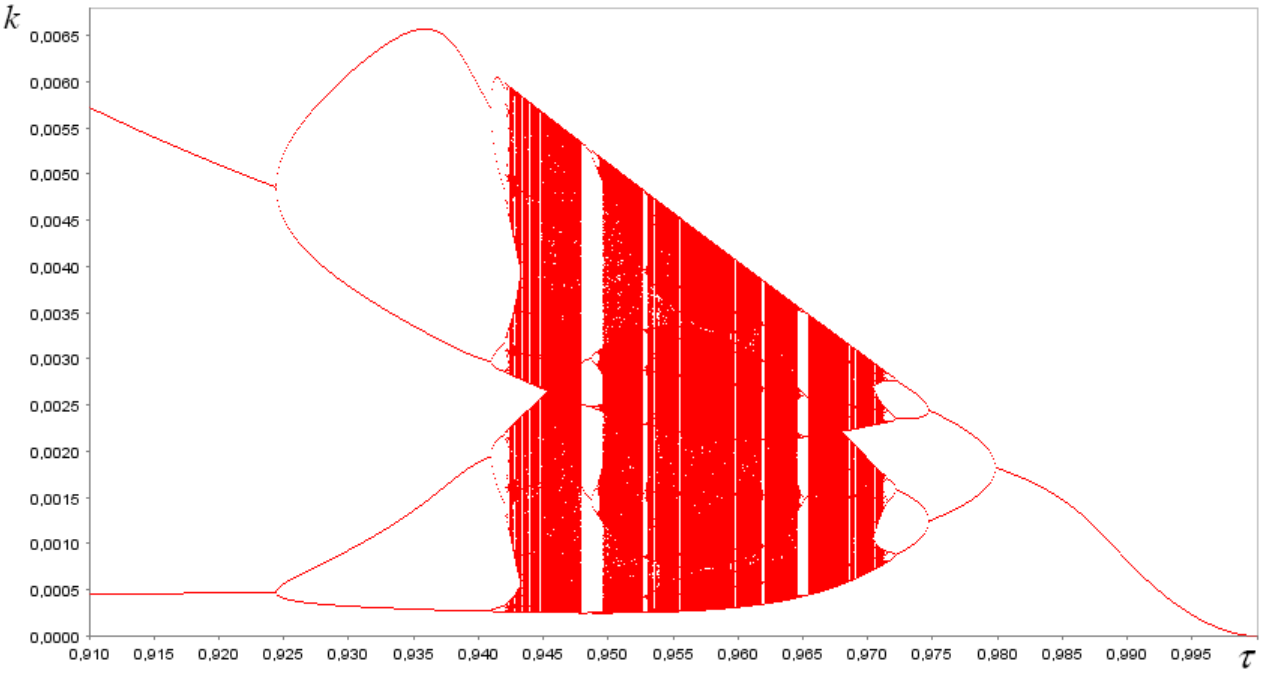

Figure 5. Enlarged view of the bifurcation diagram when $\beta = 0.2$ ($0.91 < \tau < 1$, $0 < k < 0.0068$).

Therefore, a schematic resume of the qualitative dynamics of this economy is the following. When the size of the public health system increases, *ceteris paribus*, the map Eq. (14) exhibits: (*i*) a

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period-doubling route to chaos when τ is fairly low; (*ii*) bubbling phenomena when τ rises.[17] As regards the latter result, we note that the regular bifurcation pattern reverses itself, undergoes period halving via flip bifurcations, and eventually returns to a unique steady state for larger parameter values. However, in this model we observe a return to a stable two-period cycle instead of a return to a steady state, followed, for further increases in the tax rate, by a second period-doubling route to chaos as well as a second bubbling in which the bifurcation pattern definitely returns to a stable steady state.

Now, in order to check for the robustness of the qualitative dynamics of the model, we now vary the preference parameter β from 0.2 to 0.6.[18] Using as a criterion to evaluate the stability of the unique steady state the measure of the parametric interval of τ for which the equilibrium is unstable, we may conclude that a rise in the subjective discount factor unexpectedly reduces stability. This holds because even if a rise in the degree of thriftiness increases the propensity to save and then, through this channel, works for stability, it also makes, *ceteris paribus*, the weight of old-age labour productivity higher and this instead tends to diminish capital accumulation because consumption smoothing beneficiates of old-aged labour and, hence, the need to save today to consume tomorrow is reduced. Therefore, a rise in β works against stability. In fact, while when $\beta = 0.2$ the economy is unstable for $0.042 < \tau < 0.98$, when $\beta = 0.6$ the parametric window of

[17] Pioneering discussions about this dynamical feature are Bier and Bountis (1984) and Stone (1993).

[18] In the economic literature several examples exists as regards different values of the subjective discount factor β used in numerical simulations. For instance, Strulik (2008) used $\beta = 0.09$, Strulik (2004) $\beta = 0.2$, de la Croix and Michel (2002) $\beta = 0.3$, while Blackburn and Cipriani (2002), Žamac (2007) and Leung and Wang (2010) considered $\beta = 0.6$. In this paper, therefore, we have chosen two typical values among those widely adopted in the economic literature.

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instability is enlarged to $0.021 < \tau < 0.9912$ (see Figure 6).[19] Moreover, and most important, with this new value of β the map Eq. (14) exhibits even three period-doubling routes to chaos followed by three bubbling phenomena, each of which is separated by two windows (for $0.058 < \tau < 0.217$ and $0.853 < \tau < 0.9695$) of two-period cycles (see Figures 7-9).

A rise in the subjective discount factor therefore implies that: (*i*) the portion of the entire domain of the health tax rate that guarantees stability of the steady state shrinks from 6.2 to 3 per cent; (*ii*) the portion of the domain of τ in which 2-period cycles occurs, reduces from 82.4 to 27.55 per cent, and finally (*iii*) the higher periodicity or chaotic regime occurs for the 69.5 per cent of the values of τ when $\beta = 0.6$ versus the 7.02 per cent when $\beta = 0.2$.

Therefore, it follows that when the individuals are more "parsimonious" and prefer to smooth consumption over their second period of life (i.e., β is high), the occurrence of business cycles is likelier, while also becoming more and more irregular.

[19] Similar to Figure 2, Figure 6 shows period-bubbling when τ is close to unity even for the case $\beta = 0.6$ (see Figure 9 for a clear pictorial view of it), thus giving rise to a triple, rather than double (as in the case of $\beta = 0.2$), period-bubbling phenomenon. Interestingly, we note that when $\alpha = 0.33$, $\beta = 0.2$ and $\delta = 30$ (with the other parameters being unchanged) the bifurcation diagram for τ allows to observe even quadruple bubbling areas.

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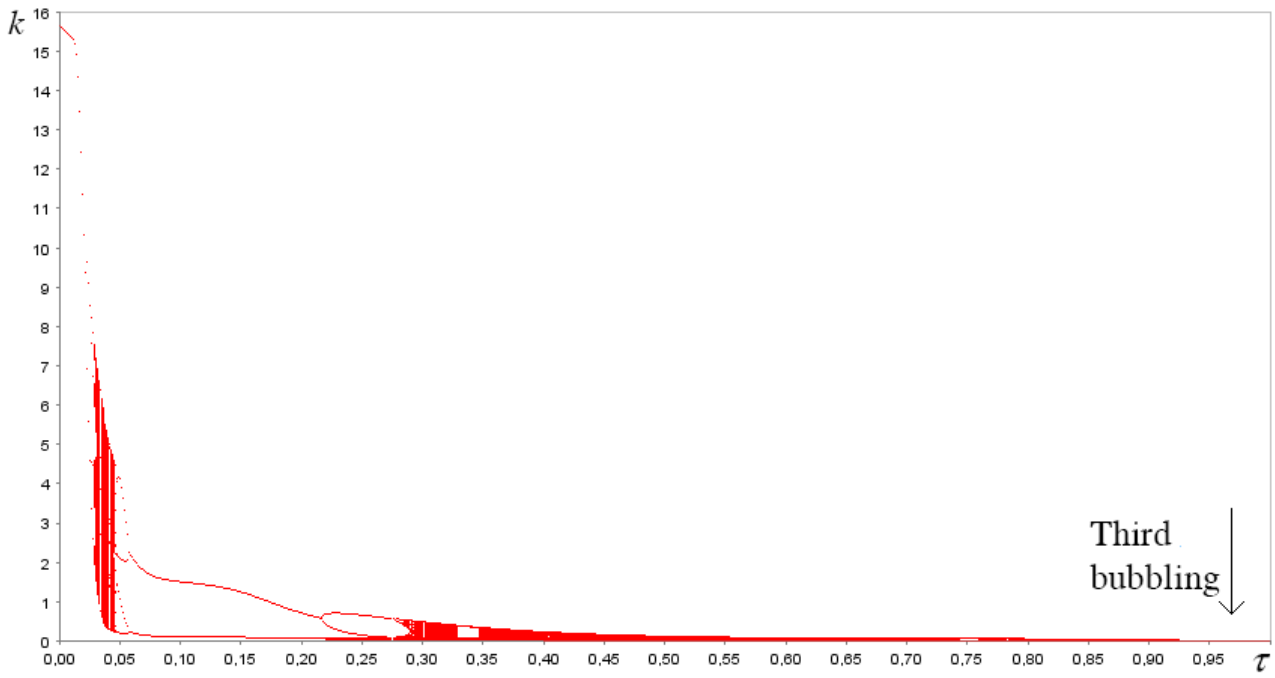

Figure 6. Bifurcation diagram for τ when $\beta = 0.6$ ($k_0 = 1$).

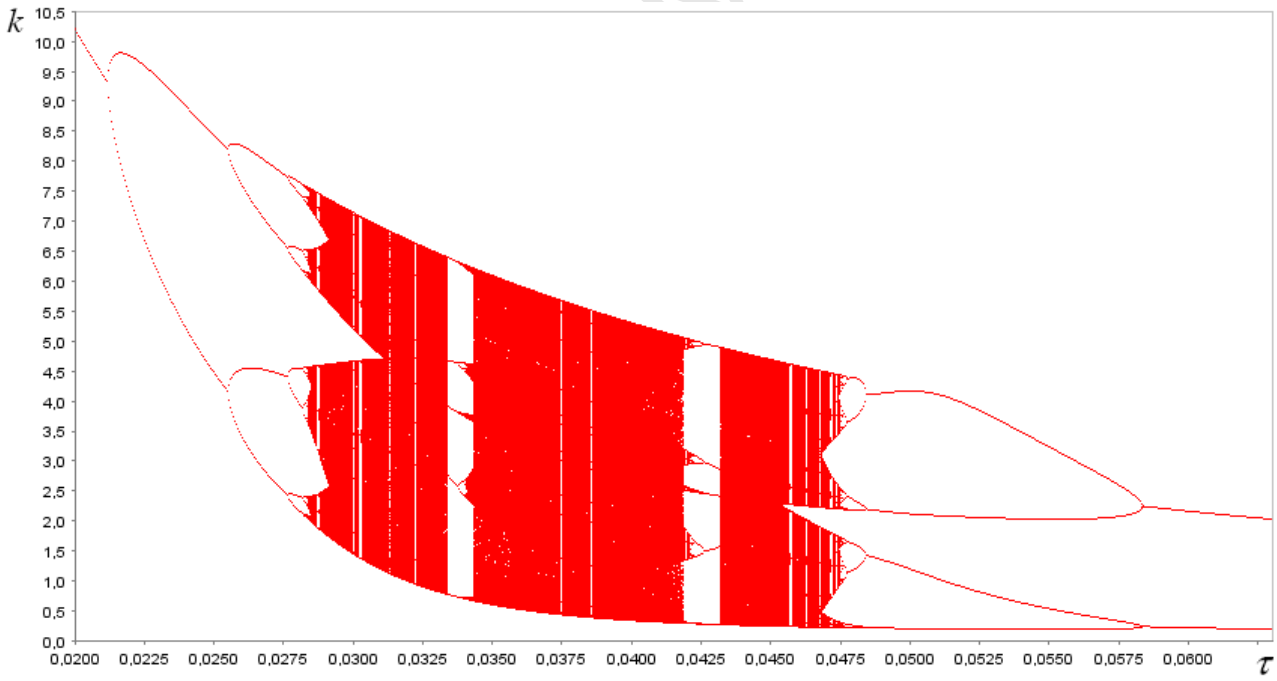

Figure 7. Enlarged view of the bifurcation diagram when $\beta = 0.6$ ($0.02 < \tau < 0.063$, $0 < k < 10.5$).

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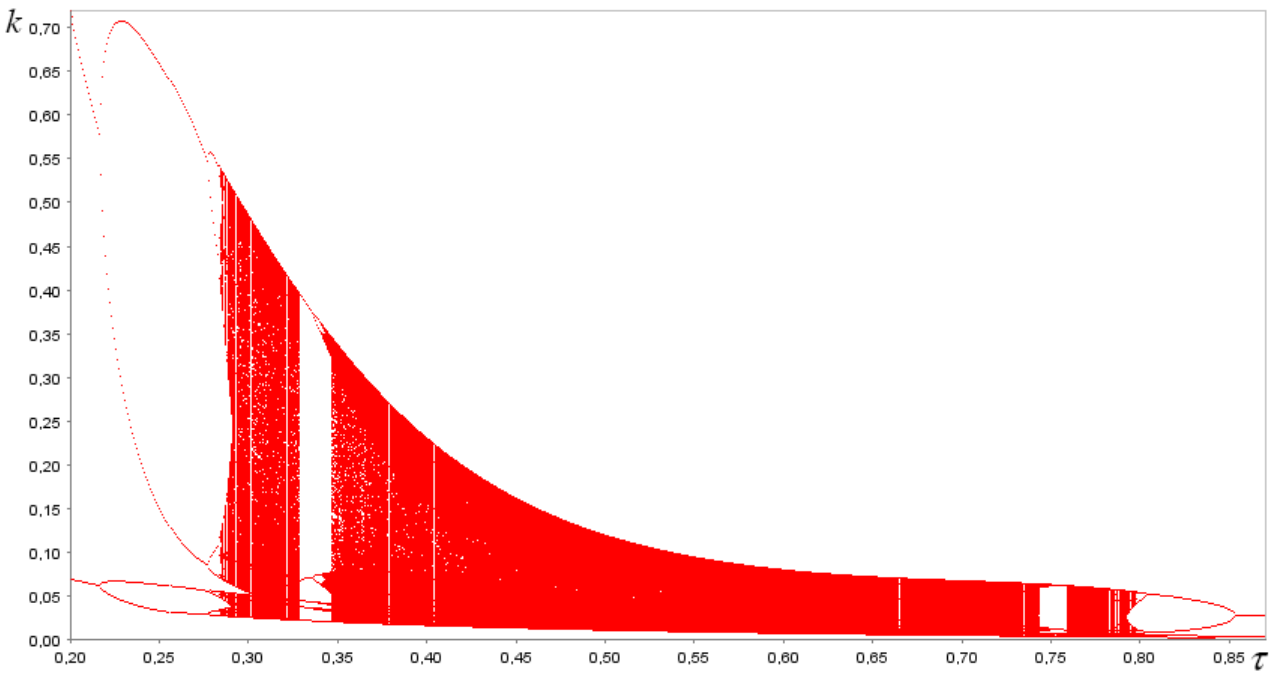

Figure 8. Enlarged view of the bifurcation diagram when $\beta = 0.6$ ($0.2 < \tau < 0.87$, $0 < k < 0.72$).

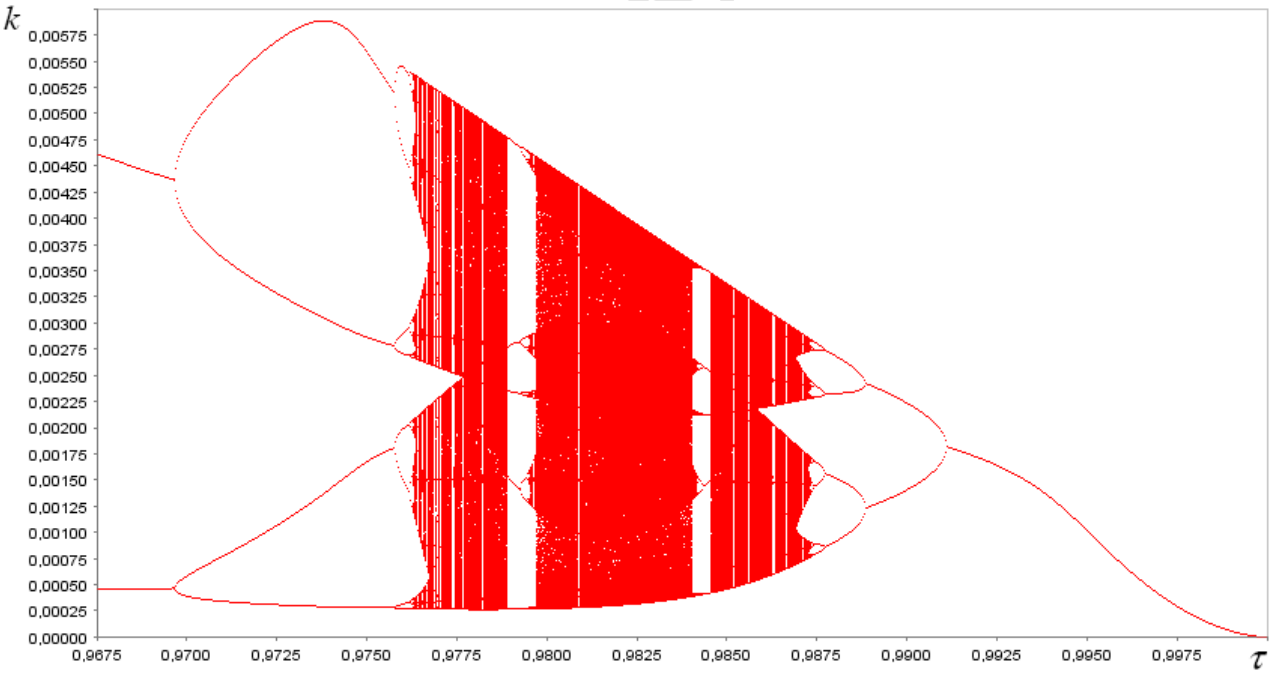

Figure 9. Enlarged view of the bifurcation diagram when $\beta = 0.6$ ($0.9675 < \tau < 1$, $0 < k < 0.006$).

We may therefore conclude that the existence of either a pair or triple bubbling areas linked each other by regular (i.e., two-period) cycles or irregular cycles instead of a stable steady state, seems to

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be the major and robust dynamical feature of this simple economy. Moreover, the complexity of the dynamical events described above – i.e., the multiplicity of bubblings – is notable for one-dimensional maps.[20]

Finally, we attempt to provide a "heuristic" interpretation of the complicated equilibrium dynamics of this economy. Although there is no simple closed-form solutions for the steady-state of Eq. (14), the above analysis has shown that the steady state equation has a unique interior fixed point and, as illustrated by Figure 1, the map (14) is bimodal when $\delta > \delta_2$. This means that even if the equilibrium occurs when the map is negatively sloped and, thus, it is locally unstable, the spiralling trajectories outward from such an unstable equilibrium remain entrapped due to the bimodal non-linearity feature of the dynamic system Eq. (14): in particular, given such a shape of the map, the more the steady state appears in the intermediate branch of the negatively inclined map (despite the slope of the map at the equilibrium point is fairly negative, and thus the large local instability of such an equilibrium), the likelier oscillations are regular. By contrast, the more the steady state appears in either the high or low branch of the negatively inclined map (despite the map at the equilibrium point is lesser inclined in such a case), the more likely we observe irregular fluctuations such as period doubling and period halving bifurcations.[21]

By looking at the case jointly portrayed in Figure 2 (changes in the shape of the phase map when τ raises) and Figures 3-5 (the corresponding bifurcation diagrams), it can be seen that the phase map at the equilibrium point is: (*i*) slightly sloped, (*ii*) highly sloped and (*iii*) slightly sloped as well.

[20] In fact, in an influential paper, Stone (1993) claimed: "The period-doubling route to chaos is a well known feature of a range of simple, nonlinear difference equations routinely used in modelling biological populations. It is not generally understood, however, that the process may easily break down and suddenly reverse, giving rise to distinctive period-halving bifurcations." (p. 617).

[21] A relevant mathematical reference on period doubling and period halving bifurcations is Nusse and Yorke (1988), while economic applications – in different contexts than ours – are, e.g., Hommes (1991; 1994), Zhang (1999) and Rosser (2001).

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Then we observe, respectively, irregular, regular (e.g., a two-period cycle) and irregular fluctuations in the economy (followed eventually by a definitive stabilisation for large enough values of the tax rate).

Moreover to better intuitively explain how a period-doubling cascade initiates and then reverses, it is again useful to examine the graph of the phase map and the bifurcation diagram for an increasing τ. In particular, we observe that the map has an increasing branch for small capital stocks and a plateau region that flattens out for large capital stocks. On the one hand, a rise in τ creates a peak (due to the effect of the non-linear dependence of the health production function on the policy variable), which is however translated towards left as the tax rate is continuously increased. This shrinks the increasing branch of the map and thus reduces the equilibrium point. On the other hand, the plateau region is always "horizontally" shifted towards left but its slope is substantially invariant for both increasing τ and sufficiently high capital stocks. However, the lower the equilibrium point the steeper the plateau region just beyond the equilibrium point (see Figure 1). Therefore two consequences emerge: (*i*) both the shrink of the increasing branch and the horizontal translation towards left of the plateau region of the map imply that, while the steady state capital stock is always lowered, the map at the equilibrium point changes its slope, which is in modulus relatively small for low values of τ, high for an increasing tax rate, and eventually small when τ is large enough; (*ii*) the interplay between the slope of the map at the equilibrium point and the slope of the plateau region beyond such an equilibrium determines whether the locally diverging trajectories will be entrapped in a regular cycle or in a chaotic area (i.e., highly diverging trajectories, arising form a high-sloped map at the equilibrium point, may approach "only" to a regular 2-period-cycle due to a "less" flattened plateau region, while slightly diverging trajectories,

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arising from a slight-sloped map at the equilibrium point, can emerge and approach to a highly complex region due to a "more" flattened plateau region). [22]

5. Conclusions

In this paper we studied the equilibrium dynamics of the Diamond overlapping generations model with perfect foresight extended with both endogenous health and old-age efficient labour. We showed that a unique positive equilibrium exists. However, while in the Diamond's model the equilibrium dynamics can ever display neither temporary nor permanent oscillations, the existence of a link between health and old-age labour productivity may be responsible of the appearance of robust endogenous fluctuations, which may be either regular or chaotic. In particular, in the tax-financed health services reduces capital per efficient worker, per capita GDP and welfare, because, by raising the health status, and thus the productivity of mature workers, the saving rate shrinks due to the reduced need to save to sustain old-age consumption.

Moreover, and most important, a rather low level of the health tax may trigger deterministic endogenous chaotic business cycles. Therefore, despite the simplicity of the one-dimensional map and the assumption of rational expectations, the qualitative features of the equilibrium dynamics are rich: indeed, a multiplicity of bubblings is shown.[23]

[22] As an example to illustrate this line of reasoning, we observe that, although the slope of the phase map Eq. (14) at the equilibrium point when $\tau = 0.82$ is, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} \cong -4.31$ – which corresponds to the maximal local instability – only a two-period cycle exists, while when $\tau = 0.945$, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} \cong -3.50$ – and thus the slope of the phase map is lower, i.e., the local instability is smaller than in the previous case – a chaotic behaviour exists.

[23] We note that the presence of period-doubling reversals may be economically interesting since they have the potential to curb dangerous chaotic fluctuations in macroeconomic variables.

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