HAL open science

Optimal Values of Multidimensional Mean-Payoff Games

Romain Brenguier, Jean-François Raskin

▶ To cite this version:

Romain Brenguier, Jean-François Raskin. Optimal Values of Multidimensional Mean-Payoff Games. [Research Report] Université Libre de Bruxelles (U.L.B.), Belgium. 2014. hal-00977352v6

HAL Id: hal-00977352

https://hal.science/hal-00977352v6

Submitted on 22 May 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Pareto Curves of Multidimensional Mean-Payoff Games*

Romain Brenguier, Jean-François Raskin

Université Libre de Bruxelles (U.L.B.), Belgium

Abstract In this paper, we study the set of thresholds that the protagonist can force in a zero-sum two-player multidimensional mean-payoff game. The set of maximal elements of such a set is called the *Pareto curve*, a classical tool to analyze *trade-offs*. As thresholds are vectors of real numbers in multiple dimensions, there exist usually an infinite number of such maximal elements. Our main results are as follow. First, we study the geometry of this set and show that it is definable as a finite union of convex sets given by linear inequations. Second, we provide a $\Sigma_2\mathsf{P}$ algorithm to decide if this set intersects a convex set defined by linear inequations, and we prove the optimality of our algorithm by providing a matching complexity lower bound for the problem. Furthermore, we show that, under natural assumptions, i.e. fixed number of dimensions and polynomially bounded weights in the game, the problem can be solved in deterministic polynomial time. Finally, we show that the Pareto curve can be effectively constructed, and under the former natural assumptions, this construction can be done in deterministic polynomial time.

1 Introduction

Two-player zero-sum games played on graphs are adequate models for open reactive systems [14], i.e systems maintaining a continuous interaction with their environment. In such model, `Eve` (the protagonist) models the system, `Adam` (the antagonist) models the environment, and a winning strategy for `Eve` in this game represents a controller that enforces a good property (modeled as the winning condition in the game) against all possible behaviors of the environment. Recently, there has been a large effort to study quantitative extensions of those graph games, see e.g. [6]. Those extensions are useful to model quantitative aspects of reactive systems such as mean energy or peak energy consumption, mean response time, etc. In practice, a system is most often exhibiting several such quantitative aspects, and they may be conflicting, e.g. one may need to consume more energy in order to ensure of a lower mean response time. This is why there is a clear need to study multi-dimensional quantitative games.

In [17], the *threshold problem* for multi-dimensional mean-payoff games is studied, i.e. given a d-dimensional value vector $v \in \mathbb{R}^d$, does `Eve` have a strategy

* Work supported by ERC Starting Grant inVEST (279499).

against all strategies of Adam to enforce values larger or equal to v. As weights in the game are given as vectors in multiple dimensions, there are usually an infinite number of incomparable thresholds that Eve is able to enforce. The set of maximal thresholds that Eve can enforce is called the *Pareto curve*, it is the classical tool to analyze *trade-offs*. Another application of the Pareto curve is the study of multiplayer games. For instance to compute Nash equilibria, a multiplayer game with mean-payoff objectives is transformed into a multidimensional mean-payoff two-player game [2], and the Pareto curve of this multidimensional game allows us to compute the equilibria of the original multiplayer game. In this paper, we study the Pareto curve and the set of thresholds that Eve can enforce in a multidimensional mean-payoff games.

Contributions To effectively analyze the trade-offs in systems formalized by multidimensional mean-payoff games, we need algorithms to answer queries about Pareto curves or to compute an effective representation of them. This is the subject of this paper. Our main contributions are as follows.

First, we characterize the geometry of the set of thresholds that Eve can force: we show that this infinite set can be effectively represented as a (finite) union of convex sets defined by linear inequations. We obtain this result both for games where the mean-payoff is given dimension by dimension using $\liminf$ (Theorem 4), and for a mixture of $\liminf$ and $\limsup$ (Theorem 10). Using this symbolic representation as a finite union of convex sets, it is now possible for instance to optimize linear functions by calls to linear programming.

Second, we study the computational complexity of natural associated decision problems. We provide a $\Sigma_2\mathsf{P}$ algorithm to decide if this set of thresholds intersects a convex set defined by linear inequations, and we prove the optimality of our algorithm by providing a matching complexity lower bound for the problem (Theorem 6). To obtain this result and several others in our paper, we extensively use techniques from discrete geometry [10] but we also need to establish new non-trivial results. In particular, we provide new results on the complexity of manipulating and querying linear sets defined by sets of linear inequations (Theorem 3). We believe that those results are of interest on their own. Equipped with those new results, we show that, even if the Pareto curve is represented by an exponential number of convex sets, each of them being defined by an exponential number of linear inequations, they are well behaved. Indeed, all the inequations that are needed to represent the Pareto curve and its downward closure (the set of thresholds that can be forced by Eve), have encoding that are bounded by polynomial functions in the size of the game.

Third, we show that it is possible to answer queries on the set thresholds that Eve can force (Theorem 7) and to construct the Pareto curve (Theorem 8) in *deterministic polynomial time* for fixed number of dimensions and polynomially bounded weights. Those results are of practical relevance as the number of dimension while multiple is often quite low in practice, and polynomially bounded weight is also a reasonable assumption, see [18,3,8] for papers where those two properties are exploited.

Related works In [1], Alur et al. consider languages of infinite words definable by Boolean queries over multidimensional mean-payoff automata. They study the accumulation points of infinite runs as a way to define an acceptance condition. They do not consider the construction of the Pareto curves associated to languages. Here, we show how to construct the Pareto curves in the more general and challenging setting of multidimensional mean-payoff games.

In [13], Papadimitriou et al. define a general procedure to construct *approximations* of Pareto curves. For models with *fixed* number of dimensions, they identify conditions that are sufficient to ensure that this approximation can be constructed in polynomial time. This technique has been used e.g. to provide approximate constructions of the Pareto curves for discounted sum Markov decision processes [7]. With the technique of [13], we can obtain approximations of the Pareto curves of multidimensional mean-payoff games in polynomial time for fixed number of dimensions. Here we provide a stronger result as we show how to construct *exact* representations of the Pareto curves (and not only approximations!) of multidimensional mean-payoff games in deterministic polynomial time for fixed number of dimensions.

Structure of the paper Section 2 defines the problems that we solve. Section 3 establishes general complexity results on the geometric objects that we use in the core of our paper. Section 4 solves the $\liminf$ case. We concentrate on this case first as it exhibits all the difficulties of the general case with simpler notations. Section 5 deals with the construction of a concrete representation of the Pareto curve. In Section 6 we solve the general problem in which $\liminf$ and $\limsup$ are mixed.

2 Preliminaries

Arenas We define *arenas* for two players that we call `Eve` and `Adam`. An *arena* $\mathcal{A}$ is a tuple $\langle \mathsf{States}_\exists, \mathsf{States}_\forall, \mathsf{Edges} \rangle$, where:

- $\mathsf{States} = \mathsf{States}_\exists \uplus \mathsf{States}_\forall$ is a finite set of *states* partitioned between the states of `Eve` and those of `Adam`;[1]
- $\mathsf{Edges} \subseteq \mathsf{States} \times \mathsf{States}$ is the set of *edges*. W.l.o.g. we assume that for all $s \in \mathsf{States}$, there exists $s' \in \mathsf{States}$ such that $(s, s') \in \mathsf{Edges}$.

A play proceeds as follows. Whenever we arrive at a state s: if $s \in \mathsf{States}_\exists$, then `Eve` selects a state s' such that $(s, s') \in \mathsf{Edges}$; if $s \in \mathsf{States}_\forall$, then `Adam` selects a state s' such that $(s, s') \in \mathsf{Edges}$. The game then continues from s' and this is repeated to form an infinite sequence of states. Formally, a *play* in the arena $\mathcal{A}$ is an infinite sequence of states $\rho = \rho_0\rho_1\cdots$ such that for all $i \geq 0$, $(\rho_i, \rho_{i+1}) \in \mathsf{Edges}$. We write $\rho_{\leq n}$ for the prefix $\rho_0 \cdots \rho_n$. A *history* h of the arena $\mathcal{A}$ is a (finite and non-empty) prefix of a play, i.e. an element of $\mathsf{States}^* \cdot \mathsf{States}$.

[1] We will write $|\mathsf{States}|$ for the cardinal of States.

Strategies Let $\mathcal{A}$ be an arena, a *strategy* for `Eve` maps histories ending in a state of $\mathsf{States}_\exists$ to a successor of that state. Formally, it is a function $\sigma_\exists \colon \mathsf{States}^* \cdot \mathsf{States}_\exists \to \mathsf{States}$, such that for all histories h and states s, $(s, \sigma_\exists(h \cdot s)) \in \mathsf{Edges}$. Similarly, a *strategy* for `Adam` is a function $\sigma_\forall \colon \mathsf{States}^* \cdot \mathsf{States}_\forall \to \mathsf{Actions}$, such that for all for all histories h and states s, $(s, \sigma_\forall(h \cdot s)) \in \mathsf{Edges}$. A strategy $\sigma_\forall$ is *memoryless* if for all histories h and h', and all states s, $\sigma_\forall(h \cdot s) = \sigma_\forall(h' \cdot s)$. We write $\mathbb{M}$ for the (finite) set of memoryless strategies of `Adam`. Let $\sigma_\exists$ be a strategy for `Eve`, a play ρ is *compatible* with the strategy $\sigma_\exists$ if, for all $k \geq 0$, if $\rho_k \in \mathsf{States}_\exists$ then $\rho_{k+1} = \sigma_\exists(\rho_{\leq k})$. We write $\mathsf{Outcome}_\mathcal{A}(s, \sigma_\exists)$ for the set of plays in $\mathcal{A}$ that are compatible with strategy $\sigma_\exists$ and have initial state s (i.e. ρ such that $\rho_0 = s$). These plays are called *outcomes* of $\sigma_\exists$ from s. We simply write $\mathsf{Outcome}(s, \sigma_\exists)$ when $\mathcal{A}$ is clear from context. The set of outcomes $\mathsf{Outcome}_\mathcal{A}(s, \sigma_\forall)$ of a strategy of `Adam` is defined symmetrically.

Weighted game A *weighted game* $\mathcal{G} = \langle \mathcal{A}, w, I, J \rangle$ is an arena $\mathcal{A}$ equipped with a *weight function* $w \colon \mathsf{Edges} \mapsto \mathbb{Z}^d$, and a partition of the set of dimensions $[\![1, d]\!] = \{1, 2, \ldots, d\}$ into $I \uplus J = [\![1, d]\!]$. We call d the dimension of $\mathcal{G}$. Given a weight function w, we write w_i for the projection to the i-th dimension of the function w. We write $W_\mathcal{G}$ for the maximal absolute value appearing in the weights: $W_\mathcal{G} = \max\{|w_i(e)| \mid i \in [\![1, d]\!], e \in \mathsf{Edges}\}$. The *mean-payoff inferior* and *mean-payoff superior* over dimension i of a play ρ are given by:

$$\underline{\mathsf{MP}}_i(\rho) = \liminf_{n \to \infty} \frac{1}{n} \sum_{0 \leq k < n} w_i(\rho_k, \rho_{k+1}),$$

$$\overline{\mathsf{MP}}_i(\rho) = \limsup_{n \to \infty} \frac{1}{n} \sum_{0 \leq k < n} w_i(\rho_k, \rho_{k+1}).$$

The goal of `Eve` is to maximize the mean-payoff inferior for the dimensions in I, and the mean-payoff superior for the dimensions in J. Let $\mathcal{G}$ be a weighted game, s a state of $\mathcal{G}$, and $v \in \mathbb{R}^d$, we say that a strategy $\sigma_\exists$ *ensures thresholds* v from state s if for all outcomes $\rho \in \mathsf{Outcome}_\mathcal{A}(s, \sigma_\exists)$, for all dimensions $i \in I$, $\underline{\mathsf{MP}}_i(\rho) \geq v_i$, and for all dimensions $j \in J$, $\overline{\mathsf{MP}}_j(\rho) \geq v_j$.

Pareto optimality We are interested in strategies of `Eve` that ensure thresholds as high as possible on all dimensions. However, since the weights are multidimensional, there is not a unique maximal threshold in general. We use the concept of *Pareto optimality* to identify the most interesting thresholds. To define the set of Pareto optimal thresholds, we first define the set of thresholds that `Eve` can force:

$$\texttt{value}(\mathcal{G}, s) = \left\{ v \in \mathbb{R}^d \mid \exists \sigma_\exists \cdot \forall \rho \in \mathsf{Outcome}(s, \sigma_\exists) \cdot \begin{array}{l} \forall i \in I : \underline{\mathsf{MP}}_i(\rho) \geq v_i \\ \wedge \forall j \in J : \overline{\mathsf{MP}}_j(\rho) \geq v_j \end{array} \right\}.$$

A threshold $v \in \mathbb{R}^d$ is *Pareto optimal* from s if is maximal in the set $\texttt{value}(\mathcal{G}, s)$. So the set of Pareto optimal thresholds is defined as:

$$\mathsf{PO}(\mathcal{G}, s) = \{v \in \texttt{value}(\mathcal{G}, s) \mid \neg \exists v' \in \texttt{value}(\mathcal{G}, s) : v' > v\}.$$

We refer to this set as the *Pareto curve* of the game. Our goal is to compute a representation of this curve. Note that the set of thresholds that `Eve` can force is exactly equal to the downward closure of the Pareto optimal thresholds, i.e. $\texttt{value}(\mathcal{G}, s) = \downarrow \mathsf{PO}(\mathcal{G}, s)$.

Linear inequations Let $a \in \mathbb{Q}^d$ be a vector in d dimensions. The associated *linear function* $\alpha_a \colon \mathbb{R}^d \mapsto \mathbb{R}$ is the function $\alpha_a(x) = \sum_{i \in [\![1,d]\!]} a_i \cdot x_i$ that computes the weighted sum relative to a. A *linear inequation* is a pair (a, b) where $a \in \mathbb{Q}^d \setminus \{\mathbf{0}\}$ and $b \in \mathbb{Q}$. The *half-space* satisfying (a, b) is the set $\frac{1}{2}\texttt{space}(a, b) = \{x \in \mathbb{R}^d \mid \alpha_a(x) \geq b\}$. A *linear equation* is also given by a pair (a, b) where $a \in \mathbb{Q}^d \setminus \{\mathbf{0}\}$ and $b \in \mathbb{Q}$ but we associate to it the *hyperplane* $\texttt{hplane}(a, b) = \{x \in \mathbb{R}^d \mid \alpha_a(x) = b\}$. If $H = \frac{1}{2}\texttt{space}(a, b)$ is a half-space, we sometimes write $\texttt{hplane}(H)$ for the *associated hyperplane* $\texttt{hplane}(a, b)$. A *system of linear inequations* is a set $\lambda = \{(a_1, b_1), \ldots, (a_l, b_l)\}$ of linear inequations. The *polyhedron* generated by λ is the set $\texttt{polyhedron}(\lambda) = \bigcap_{(a,b) \in \lambda} \frac{1}{2}\texttt{space}(a, b)$.

A natural problem, is to try to optimize the threshold we can ensure with respect to a linear function $\alpha \colon \mathbb{R}^d \mapsto \mathbb{R}$. We are looking for a strategy $\sigma_\exists$ which ensures a threshold $v \in \mathbb{R}^d$, and such that there is no $\sigma'_\exists$ which ensures a threshold $v' \in \mathbb{R}^d$, with $\alpha(v') > \alpha(v)$. To make this into a decision problem, we fix a real b, and ask if it is possible to ensure threshold v such that $\alpha(v) \geq b$. We consider a generalization of this problem which considers a set of linear inequations instead of a single one.

Polyhedron value problem Given a mean-payoff game $\mathcal{G}$, a set of linear inequations λ over elements of $\mathbb{R}^d$, the *polyhedron value problem* asks whether there is a strategy $\sigma_\exists$ and a value $v \in \texttt{polyhedron}(\lambda)$ such that $\sigma_\exists$ ensures v. Note that this is equivalent to ask whether $\texttt{polyhedron}(\lambda)$ intersects $\texttt{value}(\mathcal{G}, s)$.

Remark 1. Other works ([17,8] for instance) focus on the 0-value problem, which is a special case of the polyhedron value problem (take as polyhedron the set $\mathbb{R}^d_+$). This special case is simpler: we will show that the polyhedron value problem is $\Sigma_2\mathsf{P}$-complete while the 0-value problem is coNP-complete [17].

Consider a system with n resources $R_1, \ldots, R_n$ that are shared among d agents $A_1, \ldots, A_d$. Two agents cannot access the same resource at the same time and can request one resource at any time. We want to control the access to the resources in a way that minimizes the time that is spent during the waiting period by the different agents. This situation can be seen as a d dimensional game, in which if A_i is waiting then the reward is -1 on the i-th dimension and 0 otherwise. A situation with two agents and one resource is represented in Figure 1.

On each dimension, the average corresponds to the opposite of the average waiting time of each agent. For limit inferior objectives the controller cannot ensure a payoff of 0 on all dimensions. However, it can ensure thresholds like $(-1, 0)$, $(0, -1)$, or $(-\frac{1}{2}, -\frac{1}{2})$, and in fact all the thresholds on the line segment from point $(-1, 0)$ to $(0, -1)$, or below it (this set is the set of *feasible thresholds*).

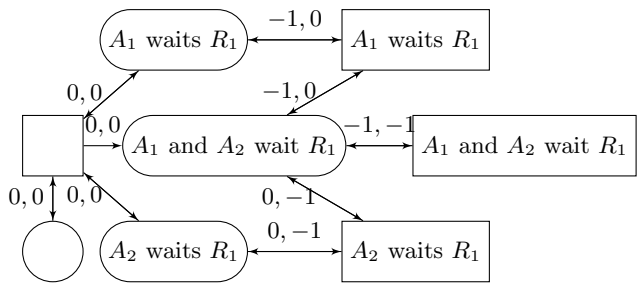

Figure 1. A two-dimensional mean-payoff game. Rounded states belong to `Eve` and rectangles to `Adam`.

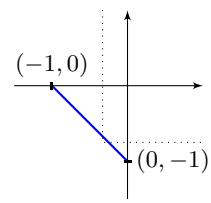

Figure 2. Pareto curve of the game of Figure 1.

Figure 2 shows the Pareto curve of the game. To illustrate the polyhedron value problem, assume we want a strategy which gives at least $-\frac{1}{3}$ on the first dimension, at least $-\frac{3}{4}$ on the second one: this corresponds to solution of the problems with $\lambda = ((1,0), -\frac{1}{3}), ((0,1), -\frac{3}{4})$. The frontier of this polyhedron is represented by dotted lines on the figure. This polyhedron has a non-empty intersection with the set of feasible thresholds, which means the problem has a solution.

3 Geometrical representations

Since our typical reader may not be familiar with all the notions of discrete geometry that we need, we summarize in this section useful notions and properties related to convex sets which are useful for our characterization of the sets $\mathsf{PO}(\mathcal{G}, s)$ and $\mathtt{value}(\mathcal{G}, s)$. For an introduction to discrete geometry, we refer the interested reader to [10]. We also prove new results in Theorem 1, 2 and 3 on manipulating and querying polytopes and systems of linear inequations. Those results are necessary to prove the main theorems of our paper, and we believe that they are of interest on their own.

To allow computational complexity measure, the size of the representations of geometrical objects is relevant. We give here the number of bits required to represent the objects that we manipulate. The *size* of a rational number $r = \frac{p}{q} \in \mathbb{Q}$ where $p \in \mathbb{Z}, q \in \mathbb{N}$, p and q are relatively prime, is: $||r|| = 1 + \lceil \log_2(|p|+1) \rceil + \lceil \log_2(q+1) \rceil$. The size of a vector $v = (r_1, \ldots, r_d)$ is $||v|| = d + \sum_{i \in [\![1,d]\!]} ||r_i||$. The size of an equation (a, b) is $||(a, b)|| = ||a|| + ||b||$ and the size of a system of equations λ is $||\lambda|| = \sum_{(a,b)\in\lambda} ||(a, b)||$.

A bounded polyhedron is called a *polytope*. A *face* F of P is a subset of P of the form $F = P \cap H_F$, where H_F is a half-space such that $P \subseteq H_F$. In that case, say that H_F *defines face* F of P. A face of dimension 1 is called a *vertex*. If P has dimension d', then a face of dimension $d' - 1$ is called a *facet*. Given a polytope P, a *complete set of facet-defining half-spaces* $\mathcal{F}$ contains for each facet F a half-space $H_F = \frac{1}{2}\mathtt{space}(a_F, b_F)$ such that $P \cap \mathtt{hplane}(a_F, b_F) = F$ and $P \subseteq H_F$. We will write $\mathcal{F}(P)$ for such a set. The *convex hull* of a set of points $X \subseteq \mathbb{R}^d$

is the set $\texttt{conv}(X) = \left\{\sum_{x\in X} t_x \cdot x \mid \forall x \in X.\ t_x \in [0,1] \wedge \sum_{x\in X} t_x = 1\right\}$. The *downward closure* of a set of points $X \subseteq \mathbb{R}^d$, is the set $\downarrow X = \{x \mid \exists x' \in X.\ \forall i \in [\![1,d]\!].\ x_i \leq x'_i\}$.

If X is a finite set of points, the convex hull $P = \texttt{conv}(X)$ can be written as a finite intersection of half-spaces [10], it is therefore a polytope. It can be represented either by its extremal points or as the intersection of its facet-defining half-spaces.

For our algorithms, it is important to be able to go from the half-space representation to the extremal point representation and vice-versa. We need also to bound the complexity of the objects that we obtain, i.e. we want to ensure that the half-spaces are defined with inequations of polynomial size and the extremal points to be representable with polynomial encodings. We will show in Theorem 3 that this is possible.

Small solutions of large systems of equations The following theorem establishes that if a system of linear equations has a solution, then it also has a solution with a small encoding.

Theorem 1. *There is a polynomial function P_1 such that for all system of equations λ of $\mathbb{R}^d$, if $\bigcap_{(a,b)\in\lambda} \texttt{hplane}(a,b) \neq \varnothing$, then there exists $x \in \bigcap_{(a,b)\in\lambda} \texttt{hplane}(a,b)$ whose representation has size smaller than $P_1(d)\cdot(1+\max\{||(a,b)|| \mid (a,b) \in \lambda\})$.*

The proof given in appendix relies on a result of [12] that we extend to non-singular, and non-square matrices of rational numbers rather than integers. Note also that our bound depends on the number of dimension d but not on the number of equations as in [12].

Small witnesses of large systems of inequations To decide the polyhedron value problem, our algorithm nondeterministically constructs solutions of large systems of inequations. We show in Theorem 2 that we can restrict the guesses to points whose representation is of polynomial size. The proof relies on Lemma 1 that says that if a system of inequations has a solution, then there is one at the intersections of at most d of the hyperplanes defined by the associated equations. This is illustrated in Figure 3: in two dimensions, if a collection of half-spaces (i.e. half-planes here) intersect (green shaded area in the picture), then either there is a point at the intersection of two boundary lines which is in the intersection (this is the case in our example for the blue points), or one of these lines is included in the intersection (this would be the case for instance if we only took parallel lines).

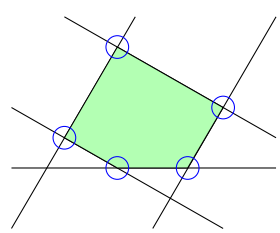

Figure 3. Illustration of Lemma 1 in the case of two dimensions: In this example, the possible witnesses of the property are circled.

Lemma 1. *Let $H_1, \ldots, H_n$ be n inequations of $\mathbb{R}^d$. If $\bigcap_{i=1}^{n} \frac{1}{2}\mathtt{space}(H_i) \neq \varnothing$ then there are $k \leq d$ indexes $i_1, \ldots, i_d$ such that:*

1. $\bigcap_{j=1}^{k} \mathtt{hplane}(H_{i_j}) \neq \varnothing$, *and*
2. $\bigcap_{j=1}^{k} \mathtt{hplane}(H_{i_j}) \subseteq \bigcap_{i=1}^{n} \frac{1}{2}\mathtt{space}(H_i)$.

From Lemma 1, we conclude that small solutions always exists for systems of linear inequations independently of the number of inequations. Note that the main difference between the next theorem and Theorem 1 is that we consider here systems of inequations rather than equations.

Theorem 2. *There is a polynomial function P_1 such that for all systems of inequations λ of $\mathbb{R}^d$, if $\mathtt{polyhedron}(\lambda) \neq \varnothing$ then there is a point $x \in \mathtt{polyhedron}(\lambda)$ whose representation has size smaller than $P_1(d) \cdot (1 + \max\{||(a,b)|| \mid (a,b) \in \lambda\})$.*

Size obtained when changing the representation of polyhedra As already recalled, it is well known that we can represent a polytope either as the intersection of half-spaces (solutions of a system of inequations) or by the finite set of its vertices (extremal points). Theorem 3 characterizes the complexity of one representation w.r.t. the other. Point 1 tells us how to bound the size of the inequations in the half-spaces representation as a function of the size of the representation of the points in the vertices representation. Point 2 does the same for the downward closure of the convex hull of the set of points. Point 3 tells us how to bound the size of the representation of the vertices as a function of the size of the inequations in the half-space representation. Proofs can be found in the appendix.

Theorem 3. *There are polynomial functions P_2 and P_3 such that:*

1. *given a finite set of points $V = \{v_1, \ldots, v_n\}$, there are $k \leq n^d$ inequations $(a_1, b_1), \ldots, (a_k, b_k)$ whose representations have size smaller than $P_2(d) \cdot (2 + \log_2(\max\{||v|| \mid v \in V\}))$ and such that $\bigcap_{i \in [\![1,k]\!]} \frac{1}{2}\mathtt{space}(a_i, b_i) = \mathtt{conv}(V)$.*
2. *given a finite set of points $V = \{v_1, \ldots, v_n\}$, there are $k \in \mathbb{N}$ inequations $(a_1, b_1), \ldots, (a_k, b_k)$ whose representations have size smaller than $P_2(d) \cdot (2 + \log_2(\max\{||v|| \mid v \in V\}))$ and such that $\bigcap_{1 \leq i \leq k} \frac{1}{2}\mathtt{space}(a_i, b_i) = \downarrow \mathtt{conv}(V)$.*
3. *given a polytope P (i.e a bounded polyhedron) represented by a system of inequations λ, such that $P = \bigcap_{(a,b) \in \lambda} \frac{1}{2}\mathtt{space}(a, b)$, there is a finite set V of points whose representations have size smaller than $P_3(d) \cdot (2 + \log_2(\max\{||(a,b)|| \mid (a,b) \in \lambda\}))$ and such that $\mathtt{conv}(V) = P$.*

4 The limit inferior case

Let us fix for this section a weighted game $\mathcal{G} = \langle \mathsf{States}_\exists, \mathsf{States}_\forall, \mathsf{Edges}, w, I, J \rangle$ with $J = \emptyset$, i.e. the averages for all dimensions are defined using $\liminf$. In this case, the set of thresholds that can be ensured by $\mathtt{Eve}$ from state $s \in \mathsf{States}$ is:

$$\mathtt{value}(\mathcal{G}, s) = \left\{ v \in \mathbb{R}^d \mid \exists \sigma_\exists.\ \forall \rho \in \mathsf{Outcome}(s, \sigma_\exists).\ \forall i \in [\![1, d]\!].\ \underline{\mathsf{MP}}_i(\rho) \geq v_i \right\}.$$

To obtain a geometrical characterization of this set, we first study the set of thresholds that `Eve` can ensure when `Adam` plays according to a fixed memoryless strategy. Then, we show that the set $\texttt{value}(\mathcal{G}, s)$ is the intersection of those sets for all the memoryless strategies of `Adam`. With the results of previous section, we deduce that if there is a solution to the *polyhedron value problem*, then there is one of bounded size which allows us to justify the correctness of a $\Sigma_2\mathsf{P}$-algorithm for this problem. Finally, we show that this algorithm has optimal worst-case complexity by providing a matching $\Sigma_2\mathsf{P}$-lower bound.

Playing against memoryless strategies of `Adam` Memoryless strategies for `Adam` are important as they are optimal for the threshold problem [17], i.e. if `Adam` has a strategy to prevent `Eve` from ensuring some threshold v then he has a memoryless one to do so. Our analysis relies on simple cycles. Let $S \subseteq \mathsf{States}$ be a subset of states of the arena of $\mathcal{G}$. A *simple cycle* within S is a finite sequence of states $s_0 \cdot s_1 \cdots s_n \in S^*$, such that $s_0 = s_n$, and for all i and j, $0 \leq i < j < n$, $s_i \neq s_j$. We write $\mathbb{C}(S)$ for the set of simple cycles of $\mathcal{A}$ within S. Let $\sigma_\forall \in \mathbb{M}$ be a memoryless strategy for `Adam`, this strategy induces the graph $\mathcal{G}(\sigma_\forall) = \langle \mathsf{States}, \mathsf{Edges}_{\sigma_\forall} \rangle$ where $\mathsf{Edges}_{\sigma_\forall} = \{(s, s') \in \mathsf{Edges} \mid s \in \mathsf{States}_\exists \vee (s \in \mathsf{States}_\forall \wedge \sigma_\forall(s) = s')\}$ which is a subgraph of the game arena in which `Adam` plays according to the memoryless strategy $\sigma_\forall$. We denote by $\texttt{SCC}(s, \sigma_\forall)$ the set of *strongly connected components* accessible from s in $\mathcal{G}(\sigma_\forall)$.

Lemma 2. *For all $\sigma_\forall \in \mathbb{M}$, for all infinite paths $\rho = \rho_0\rho_1 \ldots \rho_n \ldots$ in $\mathcal{G}(\sigma_\forall)$, let $S \in \mathit{SCC}(\rho_0, \sigma_\forall)$ be such that $\mathsf{Inf}(\rho) \subseteq S$, then*

$$\underline{\mathsf{MP}}(\rho) \in \downarrow \mathit{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right).$$

Proof. An *accumulation point* of a sequence $x_0, x_1, \ldots, x_n, \ldots$ of vectors in $\mathbb{R}^d$ is a vector $x \in \mathbb{R}^d$ such that for every open set containing x, there are infinitely many elements in the sequence which belong to the open set. It is proved in [1] that if a run ρ gets trapped for ever in the SCC S (i.e $\mathsf{Inf}(\rho) \subseteq S$) the set of accumulation points of the sequence $(\frac{1}{n} \cdot w(\rho_{\leq n}))_{\leq n}$ is included in $\texttt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right)$. Now, let us show that $\underline{\mathsf{MP}}(\rho)$ is smaller than any accumulation point of the infinite sequence of vectors $(\frac{1}{n} \cdot w(\rho_{\leq n}))_{\leq n}$. Indeed, if x be an accumulation point of that sequence, then for all dimension i the sequence $\frac{1}{n} w_i(\rho_{\leq n})$ comes infinitely often arbitrarily close to x_i. This implies that $\liminf \frac{1}{n} w_i(\rho_{\leq n})$ is smaller than x_i for all dimensions i. Therefore $\underline{\mathsf{MP}}(\rho) \leq x$ and $\underline{\mathsf{MP}}(\rho) \in \downarrow \{x\} \subseteq \downarrow \texttt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right)$. □

Note that it is not always the case that $\underline{\mathsf{MP}}(\rho)$ is in the convex hull of $\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}$. The example Figure 4 shows that the downward closure operator is necessary. In this example, the sequence

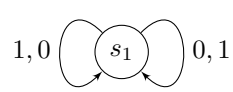

Figure 4. Example of a game where $\underline{\mathsf{MP}}(\rho)$ does not belong to $\texttt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right)$ for all path ρ.

of vectors $(1,0)^{2^1} \cdot (0,1)^{2^2} \cdot (1,0)^{2^3} \cdot (0,1)^{2^4} \cdots$ which can be obtained with a path ρ that cycles on state s_1 is such that $\underline{\mathsf{MP}}(\rho) = (0,0)$ which is not in the convex hull of $(1,0)$ and $(0,1)$ (this convex hull is the set of points $(t, 1-t)$ with $t \in [\![0,1]\!]$).

Lemma 3. *For all $\sigma_\forall \in \mathbb{M}$, for all $s \in \mathsf{States}$, for all $S \in \mathit{SCC}(s, \sigma_\forall)$, for all $v \in \mathit{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right)$, there exists an infinite path ρ of $\mathcal{G}(\sigma_\forall)$ starting from s such that $\underline{\mathsf{MP}}(\rho) = v$.*

Proof. Let $\{c_1, c_2, \ldots, c_n\}$ be a set of simple cycles in S such that $v = \sum_{i=1}^{n} \lambda_i \frac{1}{|c_i|} \cdot w(c_i)$, with $\sum_{i=1}^{n} \lambda_i = 1$, and $\lambda_i \in [0,1]$, i.e. v is a linear combination of the average weights of the simple cycles. It is proved in [17, Lemma 11], that we can build a path ρ that starts in s, reaches the SCC S and then cycles within S between the simple cycles $c_1, c_2, \ldots, c_n$ in such a way that the $\underline{\mathsf{MP}}$ on each dimension j, $1 \leq j \leq d$ is equal to $v_j = \sum_{i=1}^{n} \lambda_i \frac{1}{|c_i|} \cdot w(c_i)_j$. □

Characterizing the feasible thresholds As `Adam` can play optimally with memoryless strategies, the set of feasible thresholds that `Eve` can force is obtained by considering the intersection of all the sets of thresholds that she can enforce against those memoryless strategies of `Adam`.

Theorem 4. *Let $\mathcal{G}$ be a game and s a state of $\mathcal{G}$:*

$$\mathit{value}(\mathcal{G}, s) = \bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S \in \mathit{SCC}(s, \sigma_\forall)} \downarrow \mathit{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \;\mid\; c \in \mathbb{C}(S)\right\}\right)$$

Proof. For the left to right inclusion. Assume that $\sigma_\exists$ is a winning strategy of `Eve` for the threshold v. For all memoryless strategies $\sigma_\forall \in \mathbb{M}$ of `Adam`, we have that $\rho = \mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)$ is such that $\underline{\mathsf{MP}}(\rho) \geq v$. By Lemma 2, $\underline{\mathsf{MP}}(\rho)$ belongs to the set $\downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right)$, and as this set is downward closed it contains v.

For the right to left inclusion. Take any v in the set on the right. By Lemma 3, for all memoryless strategy $\sigma_\forall \in \mathbb{M}$ of `Adam`, we know that there exists an infinite path ρ starting from s in the graph $\mathcal{G}(\sigma_\forall)$ and such that $\underline{\mathsf{MP}}(\rho) = v$. This is equivalent to say that there exists a strategy $\sigma_\exists$ for `Eve` such that $\underline{\mathsf{MP}}(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) = v$. So, this means that memoryless strategies of `Adam` cannot force from s an outcome with $\underline{\mathsf{MP}}$ which is not at least equal to v. As memoryless strategies of `Adam` are optimal, it means that `Adam` cannot obtain from s an outcome with $\underline{\mathsf{MP}}$ which is not at least equal to v, no matter the strategy that he plays. As multidimensional mean-payoff games are determined [17], it means that `Eve` has a strategy to force outcomes from s with a $\underline{\mathsf{MP}}$ at least equal to v, which in turn implies that $v \in \mathtt{value}(\mathcal{G}, s)$. □

Remark 2. As a corollary of Thm. 4, notice that the set $\mathtt{value}(\mathcal{G}, s)$ is closed.

Small witnesses for polyhedron value problem We now show that if $\texttt{value}(\mathcal{G}, s) \cap \texttt{polyhedron}(\lambda) \neq \varnothing$, then there is a *witness* whose representation is polynomial.

Theorem 5. *There is a polynomial function P_4 such that, for all weighted game $\mathcal{G}$, for all states s, and system of linear inequations λ, if $\texttt{value}(\mathcal{G}, s) \cap \texttt{polyhedron}(\lambda) \neq \varnothing$ then there exists $x \in \mathbb{Q}^d$ such that:*

1. $x \in \texttt{value}(\mathcal{G}, s) \cap \texttt{polyhedron}(\lambda)$
2. $||x|| \leq P_4(d) \cdot (2 + \max\{||(a_j, b_j)|| \mid (a_j, b_j) \in \lambda\} + \log_2((W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1)))$.

Proof. It follows from Theorem 4 that $\texttt{value}(\mathcal{G}, s) \cap \texttt{polyhedron}(\lambda) \neq \varnothing$ if, and only if, there is a function $f \colon \mathbb{M} \mapsto 2^{\mathsf{States}}$, such that $f(\sigma_\forall) \in \texttt{SCC}(s, \sigma_\forall)$ for all strategy $\sigma_\forall$ and $\texttt{polyhedron}(\lambda)$ intersects $\bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \texttt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \;\mid\; c \in \mathbb{C}(f(\sigma_\forall))\right\}\right)$. The values $\frac{1}{|c|} \cdot w(c)$ such that $c \in \mathbb{C}(S)$ for some SCC S, are such that their numerator is smaller in absolute value than $W_{\mathcal{G}} \cdot (|\mathsf{States}|+1)$ and their denominator is smaller in absolute value than $|\mathsf{States}| + 1$. Let $a = (W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1)$. We know by Theorem 3.3 that the set $\downarrow \texttt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S)\right\}\right)$ can be written as the intersection of half-spaces $H_1, \ldots, H_k$ whose representation have size smaller than $P_2(d) \cdot (2 + \log_2(a))$. We conclude using Theorem 2 that there is a value $x \in \texttt{value}(\mathcal{G}, s) \cap \texttt{polyhedron}(\lambda)$ whose representation have size smaller than $d \cdot P_1(d) \cdot (1 + \max\{||H_i||, ||(a_j, b_j)|| \mid i \in [\![1, k]\!], (a_j, b_j) \in \lambda\})$ which is smaller than $d \cdot P_1(d) \cdot (1 + P_2(d) \cdot (2 + \log_2(a) + \max\{||(a_j, b_j)|| \mid (a_j, b_j) \in \lambda\}))$. We obtain the result for $P_4(d) = d \cdot P_1(d) \cdot (1 + P_2(d))$. □

Based on this property, we design a non-deterministic algorithm and characterize the complexity of our decision problem.

Theorem 6. *The polyhedron value problem is $\Sigma_2\mathsf{P}$-complete for mean-payoff inferior.*

Proof. **Easiness** Based on Theorem 5, our algorithm is: 1. guess in polynomial time a value v; 2. check in deterministic polynomial time that it satisfies the set of linear equations λ (see e.g. [16, Theorem 3.3]); and 3. check in *non-deterministic* polynomial time that v belong to $\texttt{value}(\mathcal{G}, s)$. This last check is based on the following result for the threshold problem: it is proved in [17, Theorem 7.2] that given a weighted game $\mathcal{G}$, a state s, and a threshold $v \in \mathbb{Q}^d$, the problem of deciding whether $\texttt{Eve}$ has a winning strategy for the objective $\{\rho \mid \underline{\mathsf{MP}}(\rho) \geq v\}$ is coNP-complete. Our algorithm is thus in $\Sigma_2\mathsf{P} = \mathsf{NP}^{\mathsf{NP}} = \mathsf{NP}^{\mathsf{coNP}}$.

Hardness We illustrate the reduction on an example. The full proof that the polyhedron value problem is $\Sigma_2\mathsf{P}$-hard can be found in the appendix.

Consider a QSAT_2 formula: $\phi = \exists x_1. \cdots \exists x_n. \forall y_1. \cdots \forall y_m. C_1 \vee \cdots \vee C_p$ where each C_i is the conjunction of at most three literals $C_i = \ell_{i,1} \wedge \ell_{i,2} \wedge \ell_{i,3}$ and literals are of the form: x_j, $\neg x_j$, y_j, or $\neg y_j$. The construction of the game is illustrated in Figure 5. There is one dimension for each literal of the formula.

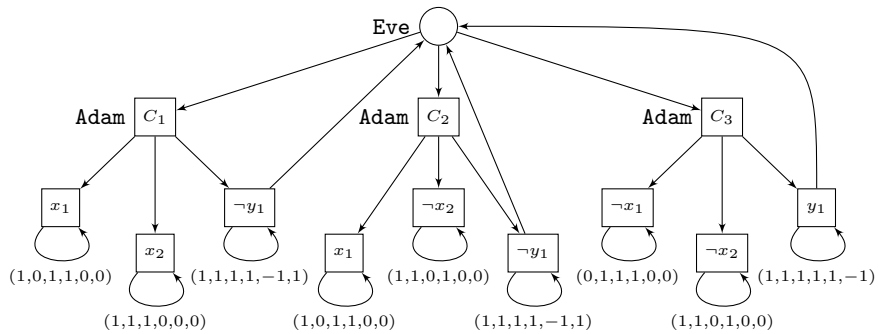

Figure 5. Example of the encoding of QSAT_2 into the polyhedron value problem, for formula $\phi = \exists x_1.\exists x_2.\forall y_1.\ (x_1 \wedge x_2 \wedge \neg y_1) \vee (x_1 \wedge \neg x_2 \wedge \neg y_1) \vee (\neg x_1 \wedge \neg x_2 \wedge y_1)$. In a vector $(v_1, v_2, v_3, v_4, v_5, v_6)$, v_1 is associated to x_1, v_2 to $\neg x_1$, v_3 to x_2, v_4 to $\neg x_2$, v_5 to y_1, and v_6 to $\neg y_1$.

We consider the constraint λ that enforces than on each dimension associated to a literal y_j or $\neg y_j$ the mean-payoff should be greater than 0, and the sum on the dimension associated to x_j and $\neg x_j$ (for j fixed) should be 1. It is possible to show that the polyhedron value problem is true if, and only if, the formula ϕ is valid.

Intuitively, if ϕ is valid, there is a partial valuation of the x variables that makes the remainder of the formula hold. From this partial valuation, we define a vector v that is 1 on dimensions associated to x literals that are true and 0 on the other dimensions. Such a vector satisfies the constraints λ. For each memoryless strategy of `Adam`, we can construct a counter strategy of `Eve` that is winning, and this is enough to show that `Eve` has a winning strategy. To construct this strategy, first notice that if the strategy of `Adam` chooses one literal in some clause and its negation in another clause, `Eve` can win by alternating between the two (this ensures 0 for this literal and 1 for the others). We can now assume that the strategy of `Adam` defines a valuation for the literals y, by setting those that are reachable to true. Then because ϕ is valid, `Eve` can choose a clause that holds under the valuation that we defined. Then `Adam` will always chose a literal that is true under this valuation and this ensures a payoff above v. □

Polynomial time algorithm for fixed number of dimensions We have seen in the previous paragraphs that the polyhedron value problem is $\Sigma_2\mathsf{P}$-complete. Now, we show that the problem has a much better worst-case complexity for fixed number of dimensions and polynomially bounded weights, two hypotheses which are reasonable in practice.

Theorem 7. *The polyhedron value problem is solvable in polynomial time for mean-payoff inferior with fixed number of dimensions, polynomially bounded*

weights and a system a linear constraints with polynomially bounded numerators and denominators.

Proof. Thanks to Theorem 5, the number of candidate witnesses in the polyhedron value problem is smaller than $2^{P_4(d)\cdot(2+\max\{||(a,b)||\mid(a,b)\in\lambda\}+\log_2((W_{\mathcal{G}}+1)\cdot(|\mathsf{States}|+1)))}$. Which is equal to $(4\cdot 2^{\max\{||(a,b)||\mid(a,b)\in\lambda\}}\cdot((W_{\mathcal{G}}+1)\cdot(|\mathsf{States}|+1)))^{P_4(d)}$. Note that the size of equations in λ are logarithmic in the values of numerators and denominators that appear in it, so $2^{\max\{||(a,b)||\mid(a,b)\in\lambda\}}$ is polynomial with respect to these values (but exponential with respect to d). The bound on the number of candidates is polynomial when the number of dimensions d is fixed and W is polynomially bounded. As the threshold problem for multidimensional weight mean-payoff games is solvable in polynomial time when the number of dimensions is fixed and the weights are polynomially bounded [8, Theorem 1], we obtain a polynomial time algorithm by simply testing all the polynomially many witnesses. □

5 Constructing the Pareto curve

Let $\mathcal{G}$ be a game of dimension d, we define the set of half-spaces $\mathcal{H}_{\mathcal{G}}$ and points $V_{\mathcal{G}}$ that are *relevant* for the representation of $\texttt{value}(\mathcal{G},s)$. $\mathcal{H}_{\mathcal{G}}$ and $V_{\mathcal{G}}$ are defined as the set of half-spaces and points with representation size bounded by $P_2(d)\cdot(2+\log_2((W_{\mathcal{G}}+1)\cdot(|\mathsf{States}|+1)))$. The following lemma explains why those sets are relevant.

Lemma 4. *Let $\mathcal{G}$ be a game of dimension d and s a state of $\mathcal{G}$. The set $\texttt{value}(\mathcal{G},s)$ can be written as a finite union of polyhedra, each of them definable as the intersection of half-spaces $H_1,\ldots,H_k\in\mathcal{H}_{\mathcal{G}}$. Moreover if $H_1,\ldots,H_k$ are half-spaces of $\mathcal{H}_{\mathcal{G}}$ and $\cap_{1\le j\le k}H_j\neq\varnothing$ then the intersection $\cap_{1\le j\le k}H_j$ contains a point of $V_{\mathcal{G}}$.*

Equivalence classes We say that two points x and y are equivalent with respect to the set of half-spaces $\mathcal{H}$, written $x\sim_{\mathcal{H}}y$, if they satisfy the same set of equations and inequations defined by $\mathcal{H}$. Formally $x\sim_{\mathcal{H}}y$ if for all $H\in\mathcal{H}$, $x\in H\Leftrightarrow y\in H$ and $x\in\texttt{hplane}(H)\Leftrightarrow y\in\texttt{hplane}(H)$. Given a point x, we write $[x]_{\mathcal{H}}=\{y\mid x\sim_{\mathcal{H}}y\}$ the equivalence class containing x. These equivalence classes are known in geometry as *cells* [15]. We write $C(\mathcal{H})$ the set of cells defined by $\mathcal{H}$. The following lemma, says that cells which intersect $\texttt{value}(\mathcal{G},s)$ are included in it.

Lemma 5. *Let $c\in C(\mathcal{H}_{\mathcal{G}})$ be a cell, $c\cap\texttt{value}(\mathcal{G},s)\neq\varnothing$ if, and only if, $c\subseteq\texttt{value}(\mathcal{G},s)$.*

From this we deduce a method to compute a representation of the set $\texttt{value}(\mathcal{G},s)$ as a finite union of cells in $C(\mathcal{H}_{\mathcal{G}})$. Given a tuple of at most $d+1$ points $X\subseteq V_{\mathcal{G}}$ with $|X|\le d+1$, we consider the geometrical center $b(X)=\sum_{x\in X}\frac{1}{d+1}\cdot x$. We write $B(V_{\mathcal{G}})$ the set of all these point, it contains at most $|V_{\mathcal{G}}|^{d+1}$ points.

The following lemma states that $\texttt{value}(\mathcal{G}, s)$ can be represented as the union of all cells that contain a point in $B(V_{\mathcal{G}})$ which is in $\texttt{value}(\mathcal{G}, s)$. Note that the fact that we do not need many points to cover each cell is coherent with Buck's theorem [4,15].

Lemma 6. *Let $\mathcal{G}$ be a game of dimension d, s a state of $\mathcal{G}$, We have that:*

$$[-W_{\mathcal{G}}, W_{\mathcal{G}}]^d \cap \texttt{value}(\mathcal{G}, s) = [-W_{\mathcal{G}}, W_{\mathcal{G}}]^d \cap \bigcup_{x \in B(V_{\mathcal{G}}) \cap \texttt{value}(\mathcal{G}, s)} [x]_{\mathcal{H}_{\mathcal{G}}}$$

As a corollary, we obtain an effective procedure to compute a representation of the set $\texttt{value}(\mathcal{G}, s)$. The complexity of this procedure is given in the following theorem both for the general case, and for fixed number of dimensions and polynomially bounded weights.

Theorem 8. *There is a deterministic exponential time algorithm that given a game $\mathcal{G}$ and a state s, constructs a effective representation of $[-W_{\mathcal{G}}, W_{\mathcal{G}}]^d \cap \texttt{value}(\mathcal{G}, s)$ as a union of cells. Moreover, when the dimension d is fixed and weights are polynomially bounded in the size of $\mathcal{G}$, then the algorithm works in deterministic polynomial time.*

Proof. The algorithm is based on the result of Lemma 6. We enumerate all subsets of $d+1$ points in $V_{\mathcal{G}}$. Because of their size, the number of points in $V_{\mathcal{G}}$ is bounded by $2^{P_4(d) \cdot (2 + \log_2((W_{\mathcal{G}}+1) \cdot (|\mathsf{States}|+1)))}$ which equals $(4 \cdot (W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1))^{P_4(d)}$. The number of subsets of $d+1$ points is thus bounded by $(4 \cdot (W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1))^{P_4(d) \cdot (d+1)}$. Note that it is exponential in general, but with polynomially bounded weights, $W_{\mathcal{G}}$ is polynomial in the sizes of the input, so that with d fixed this number of subsets is polynomially bounded.

Now for each of these subsets, we compute the geometrical center x, and test whether it is in $\texttt{value}(\mathcal{G}, s)$. Thanks to [8, Theorem 1], there is an algorithm that works in time $O(|\mathsf{States}|^2 \cdot |\mathsf{Edges}| \cdot d \cdot W_{\mathcal{G}} \cdot (d \cdot |\mathsf{States}| \cdot W_{\mathcal{G}})^{d^2 + 2 \cdot d + 1})$ to determine whether a point is in $\texttt{value}(\mathcal{G}, s)$. This is exponential in general, but polynomial when the number of dimension is fixed and weights are polynomially bounded.

Then, to determine the cell corresponding to the geometrical center x, we test for each $H \in \mathcal{H}_{\mathcal{G}}$ whether $x \in H$: the intersection of the half-spaces that contain x and the complement of those that do not contain is equal to the cell containing x. Since the sizes of the half-spaces in $\mathcal{H}$ are bounded by $P_2(d) \cdot (2 + \log_2((W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1)))$, we can test that x belongs to one of them in polynomial time and there are no more than $(4 \cdot (W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1))^{P_2(d)}$ such half-spaces. Therefore testing all half-spaces can be done in exponential time in general, and in polynomial time with fixed dimension and polynomially bounded weights. □

Pareto curve The Pareto curve is composed of the maximal points in $[-W_{\mathcal{G}}, W_{\mathcal{G}}]^d \cap \texttt{value}(\mathcal{G}, s)$. To describe this curve, we need to refine the cells in $C(\mathcal{H}_{\mathcal{G}})$: we add to $\mathcal{H}_{\mathcal{G}}$ the half-spaces that are necessary to represent the downward closure of cells in $C(\mathcal{H}_{\mathcal{G}})$ (details can be found in the appendix).

Theorem 9. *There is a deterministic exponential algorithm, that given a game $\mathcal{G}$ and a state s, computes an effective representation of $\mathsf{PO}(\mathcal{G}, s)$ as a union of cells. Moreover, when the dimension d is fixed and the weights are polynomially bounded then the algorithm works in deterministic polynomial time.*

Proof (Sketch). The algorithm works by computing a representation of $\mathtt{value}(\mathcal{G}, s)$ as a union of cells. Then, for each of these cells we check that there is no cell above by using the downward closure operator: this is where refining the cells is required. The number of those cells is exponential so these checks can be done in exponential time. Moreover in the case where the dimension is fixed and weights are polynomially bounded, this number is polynomial so the algorithm works in polynomial time. □

6 General case

We now consider the general case in which the average of dimensions in $I \subset [\![1, d]\!]$ are defined using lim inf and the average of dimensions in $J \subseteq [\![1, d]\!]$ are defined using lim sup. We give a characterization of the feasible thresholds as we did in Theorem 4. While the main ideas are similar, the characterization here is substantially more complicated and relies on a notion of subgame defined as follows. A *subarena for* $\mathtt{Eve}$ is a tuple $\langle \mathsf{States}'_\exists, \mathsf{States}'_\forall, \mathsf{Edges}' \rangle$ with $\mathsf{States}' \subseteq \mathsf{States}$, $\mathsf{Edges}' \subseteq \mathsf{Edges}$ and such that $\forall s \in \mathsf{States}'_\forall.\ (s, s') \in \mathsf{Edges} \Rightarrow (s, s') \in \mathsf{Edges}'$ (i.e. it does not restrict actions of $\mathtt{Adam}$). The game $\langle \mathcal{A}', w', I', J' \rangle$ is a subgame for $\mathtt{Eve}$ of $\langle \mathcal{A}, w, I, J \rangle$ if $\mathcal{A}'$ is a subarena for $\mathtt{Eve}$ of $\mathcal{A}$ and $w' = w$, $I' = I$, and $J' = J$. We write $\mathsf{Sub}(\mathcal{G}, s)$ the set of subgames for $\mathtt{Eve}$ which contain the state s.

Theorem 10. *Let $\mathcal{G}$ be a weighted game and s a state of $\mathcal{G}$, then $\mathtt{value}(\mathcal{G}, s)$ equals:*

$$\bigcup_{\mathcal{G}' \in \mathsf{Sub}(\mathcal{G}, s)} \bigcap_{s' \in \mathsf{States}'} \uparrow_J \left(\bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S' \in \mathsf{SCC}(s', \mathcal{G}'(\sigma_\forall))} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S') \right\} \right) \right)$$

where $\uparrow_J X = \{x \in \mathbb{R}^d \mid \forall j \in J.\ \exists x' \in X.\ \forall i \in I \cup \{j\}.\ x_i = x'_i\}$.

Example 1. Consider the example of Figure 6. We choose $J = \{1, 2\}$ and $I = \{3\}$, i.e. we consider the limit superior of the weights for the two first coordinate and the limit inferior for the last one. There is only one strategy of the adversary and one strongly connected component in this game. There are two simple cycles and their weight are $(2, -2, 0)$ and $(-2, 2, 1)$. We represented in Figure 7 and Figure 8 the feasible thresholds we can ensure with $z = 0$ and $z = \frac{1}{2}$.

For $z = 0$ the line segment between $(-2, 2, 0)$ and $(2, -2, 0)$ is below the convex hull of $(2, -2, 0)$ and $(-2, 2, 1)$. The downward closure this segment is the area that is below and left of this segment. The operator $\uparrow_{\{1,2\}}$ gives the whole area below of $(2, 2, 0)$ which is the Pareto optimum for $z = 0$. For $z = 1$

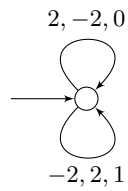

Figure 6. A one-state 3-dimensional mean-payoff game, controlled by `Eve`. We refer to the 3 dimensions by x, y and z respectively.

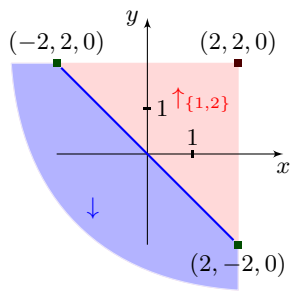

Figure 7. Pareto optimum for $z = 0$.

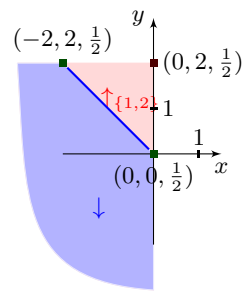

Figure 8. Pareto optimum for $z = \frac{1}{2}$.

only $(-2, 2, 1)$ is below the weight of a simple cycle therefore it will be the Pareto optimum for $z = 1$. The convex hull of $(-2, 2, 1)$ and $(2, -2, 0)$ is above the plane $z = \frac{1}{2}$ for coordinates of x and y between $(0, 0)$ and $(-2, 2)$. The operator $\uparrow_{\{1,2\}}$ will give the whole area below $(0, 2, \frac{1}{2})$ which is the Pareto optimum for $z = 0$.

Thanks to the characterization of Theorem 10, we can express the value problem in terms of intersection of convex sets with a small description and using techniques similar to the ones used in the case of limit inferior we can show the following:

Theorem 11. *The polyhedron value problem is $\Sigma_2\mathsf{P}$-complete.*

The algorithm uses Theorem 10 and relies on the same principle as for $\liminf$.

References

1. R. Alur, A. Degorre, O. Maler, and G. Weiss. On omega-languages defined by mean-payoff conditions. In *FoSSaCS'09*, pages 333–347, 2009.
2. P. Bouyer, R. Brenguier, N. Markey, and M. Ummels. Concurrent games with ordered objectives. In L. Birkedal, editor, *FoSSaCS'12*, volume 7213 of *Lecture Notes in Computer Science*, pages 301–315. Springer-Verlag, Mar. 2012.
3. L. Brim, J. Chaloupka, L. Doyen, R. Gentilini, and J.-F. Raskin. Faster algorithms for mean-payoff games. *Formal methods in system design*, 38(2):97–118, 2011.
4. R. Buck. Partition of space. *American Mathematical Monthly*, pages 541–544, 1943.
5. C. Carathéodory. Über den variabilitätsbereich der koeffizienten von potenzreihen, die gegebene werte nicht annehmen. *Mathematische Annalen*, 64(1):95–115, 1907.
6. A. Chakrabarti, L. de Alfaro, T. A. Henzinger, and M. Stoelinga. Resource interfaces. In *EMSOFT 2003*, volume 2855 of *Lecture Notes in Computer Science*. Springer, 2003.
7. K. Chatterjee, R. Majumdar, and T. A. Henzinger. Markov decision processes with multiple objectives. In *STACS 2006, 23rd Annual Symposium on Theoretical Aspects of Computer Science, Marseille, France, February 23-25, 2006, Proceedings*, volume 3884 of *Lecture Notes in Computer Science*, pages 325–336. Springer, 2006.

8. K. Chatterjee and Y. Velner. Hyperplane separation technique for multidimensional mean-payoff games. In *CONCUR 2013–Concurrency Theory*, pages 500–515. Springer, 2013.
9. M. Joswig. Beneath-and-beyond revisited. In *Algebra, Geometry and Software Systems*, pages 1–21. Springer, 2003.
10. J. Matoušek. *Lectures on discrete geometry*, volume 212. Springer, 2002.
11. L. Mirsky. *An introduction to linear algebra.* Dover Publications, 1955.
12. C. H. Papadimitriou. On the complexity of integer programming. *Journal of the ACM (JACM)*, 28(4):765–768, 1981.
13. C. H. Papadimitriou and M. Yannakakis. On the approximability of trade-offs and optimal access of web sources. In *Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on*, pages 86–92. IEEE, 2000.
14. A. Pnueli and R. Rosner. On the synthesis of a reactive module. In *POPL89*, pages 179–190. ACM Press, 1989.
15. J.-R. Sack and J. Urrutia. *Handbook of computational geometry.* Elsevier, 1999.
16. A. Schrijver. *Theory of linear and integer programming.* John Wiley & Sons, 1998.
17. Y. Velner, K. Chatterjee, L. Doyen, T. A. Henzinger, A. Rabinovich, and J.-F. Raskin. The complexity of multi-mean-payoff and multi-energy games. *CoRR*, abs/1209.3234, 2012.
18. U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *TCS*, 158(1):343–359, 1996.

A Additional definitions

We write $\rho_{\geq n}$ for the suffix of ρ after $\rho_{\leq n-1}$, i.e. $\rho_n \cdot \rho_{n+1} \cdots$ and $\rho_{[\![m,n]\!]}$ for $(\rho_{\leq n})_{\geq m}$.

B Geometrical representations

The size of a matrix $A = (a_{ij})_{1 \leq i \leq d, 1 \leq j \leq n}$ is $||A|| = d \cdot n + \sum_{i,j \in [\![1,d]\!] \times [\![1,n]\!]} ||a_{ij}||$.

A *affine subspace* is a set of the form $x + L$ with $x \in \mathbb{R}^d$ and L is a linear subspace of $\mathbb{R}^d$ (i.e. a subset closed under addition of vectors and under multiplication by real numbers). The *affine hull* of a set $X \subseteq \mathbb{R}^d$ is the intersection of all affine subspaces of $\mathbb{R}^d$ containing X, it is written $\mathtt{aff}(X)$. It can be described as the set of affine combinations of points of X [10]: $\mathtt{aff}(X) = \{\sum_{x \in X} t_x \cdot x \mid \forall x \in X.\ t_x \in \mathbb{R} \wedge \sum_{x \in X} t_x = 1\}$.

B.1 Bounding the sizes of solutions of system of equations

We recall that given a polytope P and $\mathcal{F}(P)$ be complete set of facet-defining half-spaces for P, then P is the intersection of $\mathtt{aff}(P)$ with all half-spaces in $\mathcal{F}(P)$ [9, Theorem 3.(b)].

Theorem 12 ([9, Theorem 3.(b)]). *Let P be a polytope. Let $\mathcal{F}(P)$ be complete set of facet-defining half-spaces for P, then P is the intersection of $\mathtt{aff}(P)$ with all half-spaces in $\mathcal{F}(P)$.*

B.2 Small solutions of large systems of equations

Our results is based on the following basic property from [12] which is said to be easily proved from Cramer's rule and states that systems of equations given by non-singular matrices have solutions of polynomial size.

Lemma 7 ([12, Lemma 1]). *Let A be a non-singular $m \times m$ integer matrix. The components of the solution of $Ax = b$ are all rationals with numerator and denominator bounded by $(m \cdot a)^m$ where $a = \max_{i,j}\{|A_{i,j}|, |b_j|\}$.*

We can prove a corollary that generalizes the result to non-singular matrices. Note in particular that we are not restricted to square matrices and that the bound depends on the number of columns d (that is the dimension of the space we consider) but not on the number of rows n (the number of equations in the system).

Corollary 1. *Let $A = \left(\frac{p_{i,j}}{q_{i,j}}\right)_{i,j \in [\![1,n]\!] \times [\![1,d]\!]}$ with $p_{i,j} \in \mathbb{Z}$ and $q_{i,j} \in \mathbb{N}$ be a $n \times d$ rational matrix and $b = \left(\frac{r_i}{s_i}\right)_{i \in [\![1,n]\!]}$ with $r_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ be a n rational vector. If $Ax = b$ has a solution, then it has one where all components are rationals with numerator and denominator bounded by $d^{2 \cdot d+2} d^{2 \cdot d^2 + 5 \cdot d + 2}$ where $a = \max_{i,j}\{|p_{i,j}|, |q_{i,j}|, |r_i|, |s_i|\}$.*

Proof. We consider the matrix A' and the vector b' obtained by multiplying each line i of A and b by $s_i \cdot \prod_{j \in \{1,\ldots,n\}} q_{i,j}$. We have that A' is an integer matrix and b' an integer vector, moreover the solutions of $Ax = b$ are the same than $A'x = b'$. The absolute value of the components of A' are smaller than $\max\{|p_{i,j}|\} \cdot \max\{|s_i|\} \cdot (\max\{q_{i,j}\})^d$, which we can bound by a^{d+2}. Similarly the absolute value of the components of b' are smaller than $\max\{|r_i|\} \cdot \max\{|s_i|\} \cdot (\max\{q_{i,j}\})^d$, which we can also bound by a^{d+2}.

Let $k = \mathsf{rank}(A') \leq d$, there are k linearly independent rows $A'_{i_1}, \ldots, A'_{i_k}$ of A', such that each row of A' is a linear combination of $A'_{i_1}, \ldots, A'_{i_k}$ (see for instance [11, Theorem 5.2.2]). We prove that if $A'x = b'$ has a solution then the equation:

$$\begin{pmatrix} A'_{i_1} \\ \vdots \\ A'_{i_k} \end{pmatrix} x = \begin{pmatrix} b'_{i_1} \\ \vdots \\ b'_{i,k} \end{pmatrix} \tag{1}$$

has the same set of solutions. Assume $A'x = b'$ has a solution and let X be a solution of $A'x = b'$. We obviously have that X is a solution of (1). Let A'_i be a row of A', there are coefficient $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $A'_i = \sum_{1 \leq k' \leq k} \lambda_{k'} A'_{i_{k'}}$. Since X is a solution, $A'_i X = b'_i$, hence $b'_i = \sum_{1 \leq k' \leq k} \lambda_{k'} A'_{i_{k'}} X = \sum_{1 \leq k' \leq k} \lambda_{k'} b'_{k'}$. Let

X' be a solution of (1), we have that $A'_i X' = \sum_{1\le k'\le k} \lambda_{k'} A'_{i_{k'}} X' = \sum_{1\le k'\le k} \lambda_{k'} b'_{k'} = b'_i$. This being true for all row i, X' is also a solution of $A'x = b'$.

The rows $A'_{i_1}, \dots, A'_{i_k}$ are linearly independent and can be completed in a base of $\mathbb{R}^n$ by some rows $r_{k+1}, \dots, r_m$, whose coefficients are all 0 or 1. The new matrix $A'' = \begin{pmatrix} A'_{i_1} \\ \vdots \\ A'_{i_k} \\ r_{k+1} \\ \vdots \\ r_m \end{pmatrix}$ is singular. Therefore the system $A''x = \begin{pmatrix} b'_{i_1} \\ \vdots \\ b'_{i,k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ has a solution and by Lemma 7, there is a solution where components are rationals with numerator and denominator bounded by $d \cdot (da^{d+2})^{2\cdot d+1}$. □

With the help of the preceding result, we can also bound the size of the solutions.

Theorem 13. *There is a polynomial function P_0 such that if $A = \left(\frac{p_{i,j}}{q_{i,j}}\right)_{i,j\in[\![1,m]\!]\times[\![1,n]\!]}$ with $p_{i,j} \in \mathbb{Z}$ and $q_{i,j} \in \mathbb{N}$ is a $m \times n$ rational matrix and $b = \left(\frac{r_i}{s_i}\right)_{i\in[\![1,m]\!]}$ with $r_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ is a m rational vector. and the system $Ax = b$ has a solution, then it has one where all components have size bounded by $P_0(n) \cdot (2 + \log_2(a))$ where $a = \max_{i,j}\{|p_{i,j}|, |q_{i,j}|, |r_i|, |s_i|\}$ and whose size is bounded by $n \cdot P_0(n) \cdot (1 + \max\{||A_{i,j}||, ||b_i|| \mid i \in [\![1,m]\!], j \in [\![1,n]\!]\})$.*

Proof. By the previous corollary 1, we can find a solution X such that for each $i \in [\![1,m]\!]$, $X_i = \frac{p_i}{q_i}$ with $|p_i| \le M$ and $0 < q_i \le M$ where $M = n^{2\cdot n+2} \cdot a^{n^2+5\cdot n+2}$.

Consider an index i. If $X_i = 0$ then $||X_i|| = 2$. Otherwise we have:

$$\begin{aligned}
||X_i|| &= 1 + \lceil \log_2(|p_i| + 1)\rceil + \lceil \log_2(q_i + 1)\rceil \\
&\le 1 + \lceil \log_2(|p_i|) + 1\rceil + \lceil \log_2(q_i) + 1\rceil \\
&\le 1 + \lceil \log_2(M) + 1\rceil + \lceil \log_2(M) + 1\rceil \\
&\le 1 + 2 \cdot \lceil (2\cdot n + 2)\cdot \log_2(n) + (n^2 + 5\cdot n + 2)\log_2(a) + 1\rceil \\
&\le 1 + 2 \cdot ((2\cdot n + 2)\cdot \log_2(n) + (n^2 + 5\cdot n + 2)\cdot \log_2(a) + 2)
\end{aligned}$$

One can check that for $n > 0$, $(2\cdot n + 2)\cdot \log_2(n) \le n^2 + 5\cdot n$, hence

$$||X_i|| \le 1 + 2\cdot (n^2 + 5\cdot n + 2)\cdot (1 + \log_2(a))$$

Therefore the size of X is bounded by $n \cdot (1 + 2\cdot (n^2 + 5\cdot n + 2)\cdot (1 + \log_2(a))$. As $\max\{||A_{i,j}||, ||b_i|| \mid i \in [\![1,m]\!], j \in [\![1,n]\!]\} \ge \log_2(\max_{i,j}\{|p_{i,j}|, |q_{i,j}|, |r_i|, |s_i|\}$, the property holds with $P_0(n) = 2\cdot n^2 + 10\cdot n + 5$. □

Then, Theorem 1 is an obvious consequence of the preceding theorem with $P_1(n) = n \cdot P_0(n)$.

Lemma 8. *(Lemma 1 in the paper) Let $H_1, \dots, H_n$ be n inequations of $\mathbb{R}^d$. If $\bigcap_{i=1}^n \frac{1}{2}\mathtt{space}(H_i) \ne \varnothing$ then there are $k \le d$ indexes $i_1, \dots, i_d$ such that:*

1. *$\bigcap_{j=1}^k \mathtt{hplane}(H_{i_j}) \ne \varnothing$, and*
2. *$\bigcap_{j=1}^k \mathtt{hplane}(H_{i_j}) \subseteq \bigcap_{i=1}^n \frac{1}{2}\mathtt{space}(H_i)$.*

Proof. We will prove by induction a slightly stronger version of the lemma as follows: let S be an affine subspace of $\mathbb{R}^d$ of dimension k, with $1 \le k \le d$. If $S \cap \bigcap_{i=1}^n \frac{1}{2}\mathtt{space}(H_i) \ne \varnothing$ then there are $k' \le k$ indexes $i_1, \dots, i_{k'}$ such that:

$$S \cap \bigcap_{j=1}^{k'} \mathtt{hplane}(H_{i_j}) \ne \varnothing \quad \text{and} \quad S \cap \bigcap_{j=1}^{k} \mathtt{hplane}(H_{i_j}) \subseteq S \cap \bigcap_{i=1}^{n} \frac{1}{2}\mathtt{space}(H_i).$$

The lemma correspond to the case where $S = \mathbb{R}^d$.

We show this property by induction other k. If $k = 0$, then if $S \cap \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \neq \varnothing$ then every $\frac{1}{2}\texttt{space}(H_i)$ contains S since it is reduced to one point. Therefore $S \cap \bigcap_{i=1}^{0} \frac{1}{2}\texttt{space}(H_i) = S = S \cap \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i)$.

Assume now $k > 0$ and that the property holds for all affine subspace of dimension $k' < k$. We show the property for k by induction over n. If $n = 0$ the property obviously holds.

We now assume that the property holds for $n \geq 0$ and prove it for $n + 1$. We consider a half-space H_{n+1}.

- If $S \subseteq \texttt{hplane}(H_{n+1})$ then we use the induction hypothesis for n: there are $k' \leq k$ indexes $i_1, \cdots, i_{k'}$ such that $\bigcap_{j=1}^{k'} \left(\texttt{hplane}(H_{i_j}) \cap S\right) \neq \varnothing$ and:

$$\begin{aligned}\bigcap_{j=1}^{k'} \left(\texttt{hplane}(H_{i_j}) \cap S\right) &\subseteq S \cap \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \\ &\subseteq S \cap \texttt{hplane}(H_{n+1}) \cap \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \\ &\subseteq S \cap \bigcap_{i=1}^{n+1} \frac{1}{2}\texttt{space}(H_i).\end{aligned}$$

  Thus the property holds for $n + 1$ in that case.
- Otherwise $\texttt{hplane}(H_{n+1}) \cap S$ forms a affine subspace of dimension $k-1$, to which we can apply the induction hypothesis.
  - If $S \cap \texttt{hplane}(H_{n+1}) \cap \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \neq \varnothing$ then there are $k' \leq k - 1$ indexes $i_1, \ldots, i_{k'}$ such that $(\texttt{hplane}(H_{n+1}) \cap S) \cap \bigcap_{j=1}^{k'} \texttt{hplane}(H_{i_j}) \neq \varnothing$ and :

$$(\texttt{hplane}(H_{n+1}) \cap S) \cap \bigcap_{j=1}^{k'} \texttt{hplane}(H_{i_j}) \subseteq S \cap \texttt{hplane}(H_{n+1}) \cap \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_{i)}$$

$$S \cap \bigcap_{j=1}^{k'} \texttt{hplane}(H_{i_j}) \cap \texttt{hplane}(H_{n+1}) \subseteq S \cap \bigcap_{i=1}^{n+1} \frac{1}{2}\texttt{space}(H_i)$$

    Which shows the property.
  - Otherwise $\left(\bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \cap S\right) \cap \texttt{hplane}(H_{n+1}) = \varnothing$. Let $S' = \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \cap S$, we show that it is either included in $\frac{1}{2}\texttt{space}(H_{n+1})$ or it does not intersect it: assume there is one point $x \in S'$ inside $\frac{1}{2}\texttt{space}(H_{n+1})$ and one point $x' \in S'$ outside of it, then $\alpha_a(x) \geq b$ and $\alpha_a(x') < b$. By continuity of linear functions, there would be $x'' \in [x, x']$ such that $\alpha_a(x'') = b$. Since S' is convex $x'' \in S'$, and therefore $x'' \in S' \cap \texttt{hplane}(H_{n+1})$ which contradicts the hypothesis.
    If S' is outside $\frac{1}{2}\texttt{space}(H_{n+1})$ then the intersection $S \cap \bigcap_{i=1}^{n+1} \frac{1}{2}\texttt{space}(H_i)$ is empty, so we have nothing to prove.
    Otherwise, $\bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \cap S \subseteq \frac{1}{2}\texttt{space}(H_{n+1})$. We apply the induction hypothesis over n: there are $k' \leq k$ indexes $i_1, \ldots, i_{k'}$ such that $\bigcap_{j=1}^{k'} \texttt{hplane}(H_{i_j}) \cap S \neq \varnothing$ and :

$$\bigcap_{j=1}^{k'} \texttt{hplane}(H_{i_j}) \cap S \subseteq \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i) \cap S \subseteq S \cap \bigcap_{i=1}^{n+1} \frac{1}{2}\texttt{space}(H_i)$$

    Which proves the result. □

Theorem 14. *(Theorem 2 in the paper) There is a polynomial function P_1 such that for all systems of inequations λ of $\mathbb{R}^d$, if $\texttt{polyhedron}(\lambda) \neq \varnothing$ then there is a point $x \in \texttt{polyhedron}(\lambda)$ whose representation has size smaller than $P_1(d) \cdot (1 + \max\{||(a, b)|| \mid (a, b) \in \lambda\})$.*

Proof. We prove in fact that a bound of $P_1(d) \cdot (1 + \max\{||a_{i,j}||, ||b_i|| \mid i \in [\![1, n]\!], j \in [\![1, d]\!]\}))$ is sufficient, and it implies the result.

The polynomial function P_1 comes from Theorem 1. Let $\lambda = \{(a_1, b_1), \dots, (a_n, b_n)\}$ be a system of inequations defining a non empty polyhedron. By Lemma 1, there is a set $H_1, \dots, H_k$ subset of $\{\frac{1}{2}\texttt{space}(a_i, b_i) \mid i \in [\![1, n]\!]\}$ with $k \leq d$ and such that: $\bigcap_{j=1}^{k} \texttt{hplane}(H_{i_j}) \neq \varnothing$ and $\bigcap_{j=1}^{k} \texttt{hplane}(H_{i_j}) \subseteq \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i)$. Since $\bigcap_{j=1}^{k} \texttt{hplane}(H_{i_j}) \neq \varnothing$ the system of *equation* $((a_{i_1}, b_{i_1}) \dots (a_{i_k}, b_{i_k}))$ has a solution. Using Theorem 1 there is a solution x where all components are bounded by $P_1(d) \cdot (1 + \max\{||A_{i,j}||, ||b_i|| \mid i \in [\![1, n]\!], j \in [\![1, m]\!]\}))$. Since $\bigcap_{j=1}^{k} \texttt{hplane}(H_{i_j}) \subseteq \bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i)$, the solution x is also inside the intersection of $\bigcap_{i=1}^{n} \frac{1}{2}\texttt{space}(H_i)$, and thus $x \in \texttt{polyhedron}(\lambda)$. □

B.3 Size obtained when changing the representation of polyhedra

We will decompose the proof of Theorem 3 in three parts.

From the point representation to the half-space representation

Theorem 15 (Theorem 3.1 in the paper).
There is a polynomial function P_2 (of degree 3) such that given a finite set of points $V = \{v_1, \dots, v_n\}$, with $v_i = \left(\frac{p_{i,j}}{q_{i,j}}\right)_{i,j \in [\![1,n]\!] \times [\![1,d]\!]}$, there are $k \leq n^d$ half-spaces $H_1, \dots, H_k$ whose inequations have size smaller than $P_2(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}| \mid i \in [\![1, n]\!], j \in [\![1, d]\!]\}))$ and such that $\cap_{i \in [\![1,k]\!]} H_i = \texttt{conv}(V)$.

Proof. Facets of $\texttt{conv}(V)$ are given by d' linearly independent vertices of V where d' is the dimension of $\texttt{conv}(V)$.

We first prove the case $d' = d$. Let f be a facet of $\texttt{conv}(V)$ given by $v_1, \dots, v_d$ which are d linearly independent points. We show how to obtain an equation of the half-space H^f defining this facet. The equation[2]

$$\begin{pmatrix} v_1^T \\ \vdots \\ v_d^T \end{pmatrix} A = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

has a solution and Theorem 1 guarantees that we can find a solution whose components have all size bounded by $P_1(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|\}))$. The equation $(A^T, 1)$ defines an hyperplane h. Moreover, for all v_j with $1 \leq j \leq k$, $A^T v_j = (v_j^T A)^T = 1$. This means that h contains all v_j and therefore either $\frac{1}{2}\texttt{space}(A^T, 1)$ or $\frac{1}{2}\texttt{space}(-A^T, -1)$ is defining the facet given by $v_1, \dots, v_d$. This gives us a description of H^f of size bounded by $d + P_1(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|\}))$. Since a polytope is the intersection of its facet-defining half-spaces, $\texttt{conv}(V)$ is the intersection of the H^f with f facet of $\texttt{conv}(V)$.

We now assume that $d' < d$ and that the property holds in a space of dimension $d-1$. We look for half-spaces defining the affine subspace $A = \texttt{aff}(\texttt{conv}(V))$ of dimension d' containing $\texttt{conv}(V)$. We take $d' + 1$ linearly independent points in V, which we write $v_1, \dots, v_{d'+1}$. Since they are linearly independent the equation:

$$\begin{pmatrix} v_1^T \\ \vdots \\ v_{d'+1}^T \end{pmatrix} B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

has a solution, and using Theorem 1, there is a solution B whose components all have size bounded by $P_1(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|\}))$. The equation $(B^T, 1)$ defines an affine subspace of dimension $d-1$ containing $\texttt{conv}(V_i)$, so we can apply the induction hypothesis inside it. We obtain $H'_1, \dots H'_{k'}$ half-spaces of A whose equations have size smaller than $d + P_1(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|\}))$ and such that $A \cap \bigcap_{i \in [\![1,k']\!]} H'_i = \texttt{conv}(V)$. The equation defining $H'_1, \dots, H'_{k'}$ also define half-spaces of $\mathbb{R}^d$. The convex hull of V_i is then the intersection of $H'_1, \cdots, H'_{k'}$ and the half-spaces given by $(B^T, 1)$ and $(-B^T, -1)$. The sizes of all the corresponding equations are bounded by $d + P_1(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|\}))$. The desired property is obtained for $P_2(d) = d + d \cdot P_1(d) = 2 \cdot d^3 + 10 \cdot d^2 + 11 \cdot d$. □

[2] When A is a matrix, A^T represent the *transpose* of A

From the half-space representation to the point representation

Theorem 16 (Theorem 3.3 in the paper). *There is a polynomial function P_3 (of degree 3) such that a polytope (i.e a bounded polyhedron) given by its set of facet-defining half-spaces $\mathcal{F} = \frac{1}{2}\mathit{space}(a_1, b_1), \ldots, \frac{1}{2}\mathit{space}(a_k, b_k)$, with $a_i = \left(\frac{p_{i,j}}{q_{i,j}}\right)_{j \in [\![1,d]\!]}$ with $p_{i,j} \in \mathbb{Z}$, $q_{i,j} \in \mathbb{N}$ and $b_i = \frac{r_i}{q_i}$ with $r_i \in \mathbb{Z}$, $s_i \in \mathbb{N}$, can be written as the convex hull $\mathit{conv}(V)$ of a finite set V of points whose representation have size smaller than $P_3(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|, |r_i|, |s_i| \mid 1 \leq i \leq k, 1 \leq j \leq d\}))$.*

We first prove that equations of $\mathtt{aff}(P)$ can be obtained from a subset of the equations $(a_1, b_1), \ldots, (a_k, b_k)$.

Lemma 9. *If $P = \frac{1}{2}\mathit{space}(a_1, b_1) \cap \cdots \cap \frac{1}{2}\mathit{space}(a_k, b_k)$, then there exists $i_1, \ldots, i_{k'} \in [\![1, k]\!]$ such that $\mathit{aff}(P) = \frac{1}{2}\mathit{space}(a_{i_1}, b_{i_1}) \cap \cdots \cap \frac{1}{2}\mathit{space}(a_{i_{k'}}, b_{i_{k'}})$.*

Proof. We first prove that given (a_i, b_i), if $\mathtt{aff}(P) \not\subseteq \mathtt{hplane}(a_i, b_i)$ then there exists $x \in P$ such that $a_i \cdot x < b_i$. Assume toward a contradiction that there is no such $x \in P$, then for all point x of P, $a_i \cdot x \geq b_i$. Since $P \subseteq \frac{1}{2}\mathtt{space}(a_i, b_i)$, we also have $a_i \cdot x \leq b_i$, and therefore $a_i \cdot x = b_i$. Therefore the affine space $\mathtt{hplane}(a_i, b_i)$ contains P. Since $\mathtt{aff}(P)$ is the smallest affine space that contains P, it is included in $\mathtt{hplane}(a_i, b_i)$. This contradicts the fact that $\mathtt{aff}(P) \not\subseteq \mathtt{hplane}(a_i, b_i)$.

We now prove that if $P = Q \cap \frac{1}{2}\mathtt{space}(a_i, b_i)$ and $\mathtt{aff}(P) \not\subseteq \mathtt{hplane}(a_i, b_i)$ then $\mathtt{aff}(Q) = \mathtt{aff}(P)$. What since $P \subseteq Q$, what we have to show is that $Q \subseteq \mathtt{aff}(P)$. Let $y \in Q$, we will prove that $y \in \mathtt{aff}(P)$. If $y \in P$ then $y \in \mathtt{aff}(P)$. Assume now $y \in Q \setminus P$, we have $y \notin \frac{1}{2}\mathtt{space}(a_i, b_i)$ and therefore $a_i \cdot y > b_i$. If $\mathtt{aff}(P) \not\subseteq \mathtt{hplane}(a_i, b_i)$, then we proved there exists $x \in P$, such that $a_i \cdot x < b_i$. Consider the line segment $[x, y]$. Since $a_i \cdot x < b_i$ and $a_i \cdot y > b_i$, there is $z \in [x, y]$ such that $z \in \mathtt{hplane}(a_i, b_i)$. Since both x and y are in Q, z also is, and as $z \in \frac{1}{2}\mathtt{space}(a_i, b_i)$, $z \in P$. We have $x \neq z$, $[x, z] \subseteq P$, therefore $\mathtt{aff}(P)$ must contain the line passing through x and z, and therefore contains y. This proves $y \in \mathtt{aff}(P)$. Hence $Q \subseteq \mathtt{aff}(P)$ and $\mathtt{aff}(Q) = \mathtt{aff}(P)$.

From this we can deduce that $\mathtt{aff}(P) = \bigcap_{(a_i,b_i) \mid \mathtt{aff}(P) \subseteq \mathtt{hplane}(a_i,b_i)} \frac{1}{2}\mathtt{space}(a_i, b_i)$. Consider the set Q given by $\bigcap_{(a_i,b_i) \mid \mathtt{aff}(P) \subseteq \mathtt{hplane}(a_i,b_i)} \frac{1}{2}\mathtt{space}(a_i, b_i)$, by the property we proved, for each (a_i, b_i) such that $\mathtt{aff}(P) \not\subseteq \mathtt{hplane}(a_i, b_i)$, $\mathtt{aff}(Q \cap \frac{1}{2}\mathtt{space}(a_i, b_i)) = \mathtt{aff}(Q)$. By a simple induction we can deduce

$$\mathtt{aff}\left(Q \cap \bigcap_{(a_i,b_i) \mid \mathtt{aff}(P) \not\subseteq \mathtt{hplane}(a_i,b_i)} \frac{1}{2}\mathtt{space}(a_i, b_i)\right) = \mathtt{aff}(Q).$$

Therefore $Q \subseteq \mathtt{aff}(P)$, moreover since for all (a_i, b_i) such that $\mathtt{aff}(P) \subseteq \mathtt{hplane}(a_i, b_i)$ we have $\mathtt{aff}(P) \subseteq \frac{1}{2}\mathtt{space}(a_i, b_i)$, this means that $\mathtt{aff}(P) \subseteq Q$. Hence $Q = \mathtt{aff}(P)$. □

Proof (of Theorem 3). Let P be the polytope $\bigcap_{1 \leq i \leq k} \frac{1}{2}\mathtt{space}(a_i, b_i)$ and d' be the dimension of $\mathtt{aff}(P)$. By [10, Proposition 5.3.2 (i)], a polytope is the convex hull of its vertices (its faces of dimension 0). Vertices are obtained by the intersection of $\mathtt{aff}(P)$ with d' hyperplane corresponding to facet-defining half-spaces. Lemma 9 proves that equations of $\mathtt{aff}(P)$ can be obtained from a subset of the equations $(a_1, b_1), \ldots, (a_k, b_k)$. Therefore, for each vertex v, there is system of equations given by a subset of the $(a_i, b_i)_{1 \leq i \leq k}$, such that v is the unique solution. Thanks to Theorem 1, the representation of this solution has size smaller than $P_1(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}|, |r_i|, |s_i| \mid i \in [\![1, k]\!], j \in [\![1, d]\!]\})$. This shows the result. □

Downward closure We now show that the representation in term of half-spaces of the downward closure of a convex hull is of polynomial size.

Lemma 10. *Operator $\downarrow$ is monotonic, i.e. for any set X and Y such that $X \subseteq Y$, we have that $\downarrow X \subseteq \downarrow Y$.*

Proof. Assume $X \subseteq Y$. Let $x \in \downarrow X$, there is $x' \in X$ such that for all i, $x'_i \geq x_i$. As $X \subseteq Y$, $x' \in Y$, and therefore $x \in \downarrow Y$.

Lemma 11. *Operator $\downarrow$ is compatible with union, i.e. for any set X and Y: $(\downarrow X) \cup (\downarrow Y) = \downarrow (X \cup Y)$.*

Proof. To show $(\downarrow X) \cup (\downarrow Y) \subseteq \downarrow (X \cup Y)$ We use the monotonicity property (Lemma 10): since $X \subseteq X \cup Y$, $\downarrow X \subseteq \downarrow (X \cup Y)$.

In the other direction, let $x \in \downarrow (X \cup Y)$, there exists $x' \in X \cup Y$ such that $x \leq x'$. If $x' \in X$ then $x \in \downarrow X$ and otherwise $x \in \downarrow Y$.

Equations of the downward closure of a polyhedron Our goal is from the representation in term of half-spaces of a polyhedron, to obtain a representation of its downward closure. We will show that the downward closure of convex hull can also be represented by relatively small (i.e. polynomial) equations. We will prove several lemmas to obtain the full characterization of Corollary 2. But we first introduce some notations.

Let X be a set of points and $J \subseteq [\![1, d]\!]$, we write:

- $\pi_J(X)$ for the set $\{x' \mid \exists x \in X.\ \forall i \in [\![1, d]\!] \setminus J.\ x'_i = x_i \text{ and } \forall j \in J.\ x'_j = 0\}$,
- $\downarrow_J (X)$ for the set $\{x' \mid \exists x \in X. \forall i \in [\![1, d]\!] \setminus J.\ x'_i = x_i \text{ and } \forall j \in J.\ x'_i \leq x_i\}$,
- $\updownarrow_J (X)$ for the set $\{x' \mid \exists x \in X. \forall i \in [\![1, d]\!] \setminus J.\ x'_i = x_i \text{ and } \forall j \in J.\ x'_i \in \mathbb{R}\}$;

We also write $\mathbf{1}_i$ for the point that has coordinate 1 on dimension i and 0 on the other dimensions. We write C_J for the cone $\sum_{j \in J} \mathbb{R}_+ \cdot \mathbf{1}_j$. If P is a polyhedron and $\mathcal{F}(P)$ a set of facet-defining half-spaces of P in the subspace $\texttt{aff}(P)$, we write $\mathcal{F}^J_-(P) \subseteq \mathcal{F}(P)$ the half-spaces H of $\mathcal{F}(P)$ that contain the direction $-C_J$, i.e. $H - C_J = H$.

Lemma 12. *Let P be a polyhedron, $J \subseteq [\![1, d]\!]$, and $\mathcal{F}(\pi_J(P))$ be a set of half-spaces of $\texttt{aff}(\pi_J(P))$. We have that:*

$$\updownarrow_J P = \bigcap_{H \in \mathcal{F}(\pi_J(P))} H + \mathbb{R}^J$$

where $\mathbb{R}^J$ denotes the subspace $\sum_{j \in J} \mathbb{R} \cdot \mathbf{1}_j$.

Proof.

$$\begin{aligned} x \in \updownarrow_J P &\Leftrightarrow \exists x' \in P.\ \pi_J(x) = \pi_J(x') \\ &\Leftrightarrow \pi_J(x) \in \bigcap_{H \in \mathcal{F}(\pi_J(P))} H \\ &\Leftrightarrow \pi_J(x) + \sum_{j \in J} \mathbb{R} \cdot \mathbf{1}_j \subseteq \bigcap_{H \in \mathcal{F}(\pi_J(P))} H + \sum_{j \in J} \mathbb{R} \cdot \mathbf{1}_j \\ &\Leftrightarrow x \in \bigcap_{H \in \mathcal{F}(\pi_J(P))} H + \mathbb{R}^J \end{aligned}$$

□

We show how to obtain facet-defining half-spaces of the downward closure of P with respect to one dimension.

Lemma 13. *Let P be a polyhedron, $\mathcal{F}(P)$ be a set of facet-defining half-spaces of P and $j \in [\![1, d]\!]$. We have:*

$$\downarrow_j P = \updownarrow_j P \cap \bigcap_{H \in \mathcal{F}^j_-(P)} H$$

Proof. $\boxed{\subseteq}$ Let $x \in \downarrow_j P$. We have that $x \in \updownarrow_j P$. There is $x' \in P$ such that $x = x' - \lambda \cdot \mathbf{1}_j$ with $\lambda \geq 0$. Let $f \in \mathcal{F}^j_-$, since $x' \in H_f$ we have that $x \in H_f - \mathbb{R}^+ \cdot \mathbf{1}_j = H_f$. Therefore $x \in \updownarrow_j P \cap \bigcap_{f \in \mathcal{F}^j_-} H_f$.

$\boxed{\supseteq}$ Let $x \in \updownarrow_j P \cap \bigcap_{f \in \mathcal{F}^j_-} H_f$. There exists $x' \in P$ such that $\pi_{\{j\}}(x) = \pi_{\{j\}}(x')$. Consider the line segment $[x, x']$, it does not intersect any $\texttt{hplane}(H_f)$ with $f \in \mathcal{F}^j_-$ since both x and x' are inside these half-spaces.

If $[x, x']$ does not intersect any $\texttt{hplane}(H_f)$ where f is a facet of P then x is in the intersection of these half-spaces, and using Theorem 12, this means that $x \in P$.

Otherwise $[x, x']$ intersects some $\texttt{hplane}(H_f)$ with f facet of P which is not in $\mathcal{F}^j_-$. Since H_f contains x' but not x, and these two only differs by their j coordinates, this means that $x_j \leq x'_j$ and therefore $x \in \downarrow_j P$. □

We now generalize the previous lemma to the downward closure with respect to several dimensions.

Lemma 14.

$$\downarrow_K P = \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}^{K \setminus J}_-(\pi_J(P))} H + \mathbb{R}^J$$

with the convention that $\pi_\varnothing(P) = P$, $\mathcal{F}^\varnothing_-(P) = \mathcal{F}(P)$ and $B_\varnothing = \mathbf{0}$.

Proof. We prove the result by induction over the size of K. The case where $|K| = 1$ is a direct consequence of Lemma 12 and 13.

Assume the equality holds for some set K. Consider a dimension $j \notin K$.

$$\pi_{\{j\}}(\downarrow_K P) = \downarrow_K \pi_{\{j\}}(P) = \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))} H + B_J$$

So we can chose:

$$\mathcal{F}(\pi_{\{j\}}(\downarrow_K P)) = \bigcup_{J \subseteq K} \{H + \mathbb{R}^J \mid H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))\} \tag{2}$$

We also have by induction hypothesis:

$$\mathcal{F}(\downarrow_K P) = \bigcup_{J \subseteq K} \{H + \mathbb{R}^J \mid H \in \mathcal{F}_-^{K \setminus J}(\pi_{\{j\}}(P))\}$$

$$\mathcal{F}_-^j(\downarrow_K P) = \bigcup_{J \subseteq K} \{H + \mathbb{R}^J \mid H \in \mathcal{F}_-^{K \cup \{j\} \setminus J}(\pi_{\{j\}}(P))\} \tag{3}$$

Using Lemma 12 and 13:

$$\begin{aligned}
\downarrow_{K \cup \{j\}} P &= \bigcap_{H \in \mathcal{F}(\pi_{\{j\}}(\downarrow_K P))} H + B_j \cap \bigcap_{H \in \mathcal{F}_-^j(\downarrow_K P)} H \\
&= \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))} H + \mathbb{R}^J + B_j \cap \bigcap_{H \in \mathcal{F}_-^j(\downarrow_K P)} H \qquad \text{By (2)} \\
&= \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))} H + B_{J \cup \{j\}} \cap \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \cup \{j\} \setminus J}(\pi_{\{j\}}(P))} H + \mathbb{R}^J \qquad \text{By (3)} \\
&= \bigcap_{J \subseteq K \cup \{j\}} \bigcap_{H \in \mathcal{F}_-^{K \cup \{j\} \setminus J}(\pi_J(P))} H + B_J
\end{aligned}$$

Which proves the property for $K \cup \{j\}$ and concludes the induction. □

From this lemma we deduce a representation of the downward closure of a polyhedron in term of half-spaces.

Corollary 2. *If P is a polyhedron, then:*

$$\downarrow P = \bigcap_{J \subseteq [\![1,d]\!]} \bigcap_{H \in \mathcal{F}_-^{[\![1,d]\!] \setminus J}(\pi_J(P))} H + \mathbb{R}^J$$

Theorem 17 (Theorem 3.2 in the paper). *There is a polynomial function P_2 such that given a finite set of points $V = \{v_1, \ldots, v_n\}$, with $v_i = \left(\frac{p_{i,j}}{q_{i,j}}\right)_{i,j \in [\![1,n]\!] \times [\![1,d]\!]}$, there is a finite collection of half-spaces $H_1, \ldots, H_k$ which are described by equations of size smaller than $P_2(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}| \mid i \in [\![1,n]\!], j \in [\![1,d]\!]\}))$ and such that $\cap_{i \in [\![1,k]\!]} H_i = \downarrow \mathtt{conv}(V)$.*

Proof. We will in fact use the same polynomial function P_2 than for Theorem 3.1.

We use Corollary 2, the collection $H_1, \ldots, H_k$ is given by the half-spaces of the form $H + \mathbb{R}^J$ with $J \subseteq [\![1,d]\!]$, and $H \in \mathcal{F}_-^{[\![1,d]\!] \setminus J}(\pi_J(\mathtt{conv}(V)))$. We show that these half-spaces are described by small equations.

We fix such J and $H \in \mathcal{F}_-^{[\![1,d]\!] \setminus J}(\pi_J(\mathtt{conv}(V)))$. The polytope $\pi_J(\mathtt{conv}(V))$ can be obtained as the convex hull of $\pi_J(V)$. The set $\pi_J(V)$ is a finite set of points whose components can be written with numerator and denominator smaller in absolute value than $a = \max\{|p_{i,j}|, |q_{i,j}| \mid i \in [\![1,n]\!], j \in [\![1,d]\!]\}$. Using Theorem 3.2, (Theorem 1 in the appendix), half-spaces of $\mathcal{F}(\pi_J(\mathtt{conv}(V)))$ can be described with size smaller than $P_2(d) \cdot (2 + \log_2(a))$. The set of facet $\mathcal{F}_-^{[\![1,d]\!] \setminus J}(\pi_J(\mathtt{conv}(V)))$ is include in $\mathcal{F}(\pi_J(\mathtt{conv}(V)))$ and therefore the size of the equation of H is also bounded by $P_2(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}| \mid i \in [\![1,n]\!], j \in [\![1,d]\!]\}))$.

Now let (A, b) be such an equation of H, it also defines a half-space H' of $\mathbb{R}^d$ which obviously contains H. We define A' to be such that for all $i \in [\![1, d]\!] \setminus J$, $A'_i = A_i$ and for all $j \in J$, $A'_j = 0$. Note that the size of A' is bounded by that of A and therefore the size of the equation is bounded by $P_2(d) \cdot (2 + \log_2(a))$. If $A' = (0, \dots, 0)$ this contradicts the fact that H is defining a facet of $\mathtt{aff}(\pi_J(\mathtt{conv}(V)))$. Otherwise the equation (A', b) defines a half-space which we write $H' = \frac{1}{2}\mathtt{space}(A', b)$. We have that $H' \cap \pi_J(\mathbb{R}^d) = H$ since all points of π_J have coordinate on J equal to 0 and only coefficient on these dimensions differ between A and A'. Moreover it contains the direction $\mathbb{R}^J$ since for all $j \in J$, $A' \cdot \mathbf{1}_j = 0$. Therefore H' is equal to $H + \mathbb{R}^J$. This shows that $\mathtt{conv}(V)$ can be described by equation that have size smaller than $P_2(d) \cdot (2 + \log_2(\max\{|p_{i,j}|, |q_{i,j}| \mid i \in [\![1, n]\!], j \in [\![1, d]\!]\}))$. □

C The limit inferior case

We recall the characterization given in Section 4.

Theorem 18. *(Theorem 4 in the paper) Let $\mathcal{G}$ be a game and s a state of $\mathcal{G}$:*

$$\mathit{value}(\mathcal{G}, s) = \bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S \in \mathit{SCC}(s, \sigma_\forall)} \downarrow \mathit{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \;\middle|\; c \in \mathbb{C}(S)\right\}\right)$$

Theorem 19. *The polyhedron value problem is $\Sigma_2\mathsf{P}$-hard for mean-payoff inferior.*

Proof. Consider a QSAT_2 formula, that is a formula of the form

$$\phi = \exists x_1. \cdots \exists x_n. \forall y_1. \cdots \forall y_m. C_1 \vee \cdots \vee C_p$$

where each C_i is the conjunction of at most three literals $C_i = \ell_{i,1} \wedge \ell_{i,2} \wedge \ell_{i,3}$ and literals are of the form: x_j, $\neg x_j$, y_j, or $\neg y_j$.

We consider the game where `Eve` starts by choosing a clause then `Adam` chooses a literal that appears in that clause. Then if the literal is a y_j or $\neg y_j$ then the game starts again from the initial state, and if the literals is a x_j or $\neg x_j$ then the game loops in the same state (`Adam` is forced to chose the same literal forever). The construction of the game is illustrated in Figure 5.

We consider weights in dimension $2(m + n)$, that is there are as many dimensions as possible literals, the literal x_j corresponds to dimension $2 \cdot j - 1$, $\neg x_j$ to dimension $2 \cdot j$, y_j to dimension $2 \cdot n + 2 \cdot j - 1$ and $\neg y_j$ to $2 \cdot n + 2 \cdot j$. The weights are given by the choice of `Adam`:

- if `Adam` chooses a literal ℓ of the form x_j or $\neg x_j$ then the weight is 0 in the dimension associated to the negation of ℓ, 0 on all dimensions associated to $y_{j'}$ literals, and 1 on all dimensions $x_{j'}$ or $\neg x_{j'}$ different from the negation of ℓ;
- if `Adam` chooses a literal ℓ of the form y_j or $\neg y_j$ then the weight is 1 on the dimension associated to ℓ, -1 on its negation, 0 on all $y_{j'}$ and $\neg y_{j'}$ with $j' \neq j$, and 1 on all dimensions associated to x_j or $\neg x_j$ for all $j \in [\![1, n]\!]$.

We consider the constraints λ that says that on all dimension k associated with a y_j or a $\neg y_j$, $v_k \geq 0$, and for all $j \in [\![1, n]\!]$, $v_{2 \cdot j - 1} + v_{2 \cdot j} \geq 1$ (that is, the sum on dimensions associated to x_j and $\neg x_j$ is greater to 1).

We now prove that the answer to the polyhedron value problem is true if, and only if, the formula ϕ is valid.

$\boxed{\Leftarrow}$ Assume that there is a (partial) valuation v of $x_1, \dots, x_n$ that makes the formula $\forall y_1. \cdots \forall y_m. C_1 \vee \cdots \vee C_p$ true. We consider the associated vector u that is 0 on coordinates associated to literals y_j and $\neg y_j$ and such that:

- if $v(x_i)$ is true, then the coordinate of u associated to x_i is 1, and that associated to $\neg x_i$ is 0;
- if $v(x_i)$ is false, then the coordinate of u associated to x_i is 0, and that associated to $\neg x_i$ is 1.

Obviously u satisfies the constraint λ. We now prove that `Eve` can ensure u. According to [17, Theorem 8], it is enough to show it for all memoryless strategy $\sigma_\forall$ of `Adam`.

Will we consider all memoryless strategies of `Adam`, but distinguish between those that can be associated to correct valuations of the variables and those that cannot, that is they chose y_j in some state and $\neg y_j$ in some other.

If the strategy of `Adam` chooses a literal y_j for some clause and $\neg y_j$ for another clause, then by alternating between these two clauses `Eve` ensures a payoff that is 0 on y_j and $\neg y_j$ and also all dimensions $y_{j'}$ and $y_{\neg j'}$, and that is 1 on all dimensions associated with x_i or $\neg x_i$. Hence, u is ensured in that case.

Now assume that `Adam` never chooses a literal y_j for some clause and $\neg y_j$ for another one. Then the memoryless strategy of `Adam` corresponds to a valid assignment of the truth value of the variables $y_1, \ldots, y_m$. This valuation is the following: we extend the valuation v to the variables $y_1, \ldots, y_m$ such that $v(y_j)$ is false if there is some clause where y_j is chosen by `Adam` and $v(y_j)$ is true otherwise. Since $\forall y_1. \cdots \forall y_m. C_1 \vee \cdots \vee C_p$ is true for the partial valuation v of $x_1, \ldots, x_n$, it means that the full valuation v makes $C_1 \vee \cdots \vee C_p$ true. Hence `Eve` can chose a C_i that is true for valuation v. Then `Adam` does not chose a literal y_j or $\neg y_j$ for that clause (if he chooses y_j then $v(y_j)$ is false by definition of v and this contradicts that C_i is true; and similarly if he chooses $\neg y_j$). Hence in that state, he chooses a literal x_i or $\neg x_i$ and by definition of the valuation v, we have that this ensure a payoff greater than u.

$\boxed{\Rightarrow}$ Assume now that there is u satisfying the constraints, that can be ensured by a strategy $\sigma_\exists$ of `Eve`. We define the (partial) valuation v of the variables $x_1, \ldots, x_n$:

a. if the coordinate of u associated to x_i is strictly greater than 0, then $v(x_i)$ is true;
b. if the coordinate of u associated to x_i is smaller or equal to 0, then $v(x_i)$ is false.

We will show that for all valuations of the y_i, the global valuation obtained makes some clause true.

Let v' be a valuation of $x_1, \ldots, x_n, y_1, \ldots, y_m$ compatible with v for variables $x_1, \ldots, x_n$. Assume toward a contradiction that the valuation v' does not make any of the clauses true. Then in each C_i `Adam` can chose a literal of the clause for which the valuation is false. Consider a strategy $\sigma_\forall$ that in each C_i chose a literal for which the valuation is false. Consider the outcome of the strategies $(\sigma_\exists, \sigma_\forall)$, either: (1) it will reach a literal x_i or $\neg x_i$ and stay there, or (2) it never reaches such a state and comes back infinitely often to the initial state. We distinguish between these two cases:

(1) If the run ends in a literal ℓ of the form x_i or $\neg x_i$ then by the way we defined $\sigma_\forall$, $v(\ell)$ is false which means that:
   - if $\ell = x_i$ then $v(x_i)$ is false and by (b) of the definition of v the coordinate of u associated to x_i is smaller or equal 0. By constraint λ the coordinate associated to $\neg x_i$ has to be at least 1. However, by the way we defined the weight in the game, the weight of the run on the dimension associated to $\neg x_i$ will be 0 . This is strictly smaller than 1 and contradicts that `Eve` ensures vector u.
   - if $\ell = \neg x_i$ then $v(x_i)$ is true and by (a) of the definition of v the coordinate of u associated to x_i is strictly greater than 0. However, by the way we defined the weight in the game, the weight of the run on the dimension associated to x_i will be equal to 0 which contradicts that `Eve` ensures vector u.

(2) In the other case the run never reaches such a state, it comes back infinitely often to the initial state, and only visits literals of the form y_j or $\neg y_j$.
   Note that since $\sigma_\forall$ only chooses literals that evaluate to false according to v, if y_j is chosen at some point then $\neg y_j$ is never chosen and reciprocally. We show that some the average weight on some dimension associated to one of these y_j, or $\neg y_j$ is strictly smaller than 0, which will show that `Eve` does not ensure u.
   Let ρ be the outcome of $(\sigma_\exists, \sigma_\forall)$, and $w_i(\rho_{\leq k})$ denote the total weight of the prefix of ρ of size $k+1$ on the dimension i. Assume towards a contradiction that $\liminf \frac{w_{+i}(\rho_{\leq k})}{k} \geq 0$ for each dimension $i \in [\![2 \cdot n+1, 2 \cdot (n+m)]\!]$ (these are dimensions associated to literals of the form y_j or $\neg y_j$). Then there is an index k, such that for all $k' \geq k$ and for all dimension $i \in [\![2 \cdot n+1, 2 \cdot (n+m)]\!]$, $\frac{w_i(\rho_{\leq k'})}{k'} > -\frac{1}{8 \cdot m}$. This means that for all $k' \geq k$, $\sum_{i \in [\![1, 2 \cdot m]\!]} \frac{w_{2 \cdot n+i}(\rho_{\leq k'})}{k'} > -\frac{1}{4}$. On the other hand, since one y_j or $\neg y_j$ is chosen at each turn of `Adam` and the negation of the one that are chosen are never chosen, we have that for all k', $\sum_{i \in [\![1, 2 \cdot m]\!]} w_{2 \cdot n+i}(\rho_{\leq k'}) \leq 1 - \frac{k'}{3}$ (the factor $\frac{1}{3}$ comes from the fact that we have three steps to come back to the initial state). This means that $\sum_{i \in [\![1, 2 \cdot m]\!]} \frac{w_{2 \cdot n+i}(\rho_{\geq k'})}{k'} \leq \frac{1}{k'} - \frac{1}{3}$. And this contradicts the fact that $\frac{w_i(\rho_{\geq k'})}{k'} > -\frac{1}{8 \cdot m}$ for all $k' > k$ and all dimension $i \in [\![2 \cdot n+1, 2 \cdot (n+m)]\!]$.
   Hence `Eve` cannot ensure 0 on all dimensions $i \in [\![2 \cdot n+1, 2 \cdot (n+m)]\!]$, thus she cannot ensure vector u.

This contradiction means that valuation v' makes at least one clause true. This being true for all valuation v' compatible with the partial valuation v, we can conclude that ϕ is valid. $\square$

D Constructing the Pareto curve

Lemma 15. *Let $\mathcal{G}$ be a game of dimension d and s a state of $\mathcal{G}$. The set $\mathtt{value}(\mathcal{G}, s)$ can be written as a finite union of polyhedra, each of them definable as the intersection of half-spaces $H_1, \ldots, H_k \in \mathcal{H}_\mathcal{G}$. Moreover if $H_1, \ldots, H_k$ are half-spaces of $\mathcal{H}_\mathcal{G}$ and $\cap_{1 \leq j \leq k} H_j \neq \varnothing$ then the intersection $\cap_{1 \leq j \leq k} H_j$ contains a point of $V_\mathcal{G}$.*

Proof. We can rewrite the expression of value to:

$$\bigcup_{f\,:\,\mathbb{M} \mapsto 2^{\mathsf{States}} \mid f(\sigma_\forall) \in \mathtt{SCC}(s, \sigma_\forall)} \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \;\mid\; c \in \mathbb{C}(f(\sigma_\forall))\right\}\right).$$

When f is fixed, the set $C_f = \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \;\mid\; c \in \mathbb{C}(f(\sigma_\forall))\right\}\right)$ is a polyhedron. As stated in the proof or Theorem 5, the set $\downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(f(\sigma_\forall))\right\}\right)$ can be written as the intersection of half-spaces $H_1, \ldots, H_k$ whose inequations have size smaller than $P_2(d) \cdot (2 + \log_2((W_\mathcal{G} + 1) \cdot (|\mathsf{States}| + 1)))$. Because of their size, these half-spaces are in $\mathcal{H}_\mathcal{G}$. Moreover the intersection for each $\sigma_\forall$ of these sets is also described by half-spaces with the same property. Therefore each polyhedra C_f can be written as the intersection of some half-spaces $H_1, \ldots, H_k \in \mathcal{H}_\mathcal{G}$. This prove the first point of the lemma.

Now, using Theorem 2, if there is a point at the intersection of a subset of these half-spaces, then there is such a point x which has size bounded by $P_1(d) \cdot (1 + P_2(d) \cdot (2 + \log_2((W_\mathcal{G} + 1) \cdot (|\mathsf{States}| + 1)))) \leq P_4(d) \cdot (2 + \log_2((W_\mathcal{G} + 1) \cdot (|\mathsf{States}| + 1)))$. Because of its size, x belongs to $V_\mathcal{G}$, which proves the last point of the lemma. □

Lemma 16. *Let $c \in C(\mathcal{H}_\mathcal{G})$ be a cell, $c \cap \mathtt{value}(\mathcal{G}, s) \neq \varnothing$ if, and only if, $c \subseteq \mathtt{value}(\mathcal{G}, s)$.*

Proof. Let $c \in C(\mathcal{H}_\mathcal{G})$ and assume $c \cap \mathtt{value}(\mathcal{G}, s) \neq \varnothing$. Then let $x \in c \cap \mathtt{value}(\mathcal{G}, s)$. As a consequence of Lemma 15, x belongs to a polyhedron P that can be written as the intersection of a subset of half-spaces in $\mathcal{H}_\mathcal{G}$ and P is included in $\mathtt{value}(\mathcal{G}, s)$.

Let now y be a point in c, we will show that y belongs to $\mathtt{value}(\mathcal{G}, s)$. Since $x \sim_{\mathcal{H}_\mathcal{G}} y$, they satisfy the same set of inequations corresponding to half-spaces of $\mathcal{H}_\mathcal{G}$, and in particular y belongs to P. As P is included in $\mathtt{value}(\mathcal{G}, s)$ we have that $y \in \mathtt{value}(\mathcal{G}, s)$. □

Lemma 17. *(Theorem 6 in the paper) Let $\mathcal{G}$ be a game of dimension d, s a state of $\mathcal{G}$, We have that:*

$$[-W_\mathcal{G}, W_\mathcal{G}]^d \cap \mathtt{value}(\mathcal{G}, s) = [-W_\mathcal{G}, W_\mathcal{G}]^d \cap \bigcup_{x \in B(V_\mathcal{G}) \cap \mathtt{value}(\mathcal{G}, s)} [x]_{\mathcal{H}_\mathcal{G}}$$

Proof. We show that the set $B(V_\mathcal{G})$ contains at least one point in each cell, the result then follows from Lemma 16.

Let c be a cell and k its dimension. Since $[-W_\mathcal{G}, W_\mathcal{G}]^d$ is bounded, the intersection of the cell with $[-W_\mathcal{G}, W_\mathcal{G}]^d$ correspond to the inside of a polytope P of dimension k defined by some half-spaces of $\mathcal{H}$. The vertices of this polytope are at the intersection of facet defining half-spaces which are half-spaces of $\mathcal{H}$. The intersection of these half-spaces and therefore vertices of P are points of $V_\mathcal{G}$.

Let x be a point in the cell c. We write $I = \{h \in \mathcal{H} \mid x \in h\}$ and $J = \{h \in \mathcal{H} \mid x \notin h\}$. We have that $c = [x]_\mathcal{H} = \bigcap_{h \in I} h \setminus \bigcup_{h \in J} h$. Given an half-space $h = \frac{1}{2}\mathtt{space}(a, b)$, we write h^- the half-space $\frac{1}{2}\mathtt{space}(-a, -b)$, we have that $h^- = (\mathbb{R}^d \setminus h) \cup \mathtt{hplane}(h)$ and $h^- \in \mathcal{H}$. Consider the polytope $P = \bigcap_{h \in I} h \cap \bigcap_{h \in J} h^-$, we have that $c \subseteq P$ and therefore $x \in P$. The vertices of P are points of $V_\mathcal{G}$ and x is in the convex hull of this vertices. By Carathéodory's theorem [5,10], x is in the convex hull of at most $k+1$ vertices $x_1, \ldots, x_{k+1}$ of P. The point $b(x_1, \ldots, x_{k+1})$ is in the convex hull of these vertices and therefore also in the same polytope as x.

We will show that $y = b(x_1, \ldots, x_{k+1})$ is not in any $h \in J$. Assume towards a contradiction that $y \in h \in J$. Let (a, b) be the inequation of h, as $y \in h$, $\alpha_a(y) \geq b$. Since the vertices of P are in h^-, for all $i \in [\![1, k+1]\!]$, $\alpha_a(x_i) \leq b$. If there is x_i such that $\alpha_a(x_i) < b$ then, as $y = \frac{1}{k} \cdot \sum_{1 \leq i \leq k+1} x_i$, we would also have $\alpha_a(y) < b$ which contradicts $y \in h$. We can therefore conclude that for all $i \in [\![1, k+1]\!]$, $x_i \in \mathtt{hplane}(h)$ and as $y \in \mathtt{conv}(\{x_1, \ldots, x_{k+1}\})$ we also have $y \in \mathtt{hplane}(h)$. Since x is in the convex hull of $x_1, \ldots, x_{k+1}$, we also have $x \in \mathtt{hplane}(h)$ and therefore $x \in h$ which contradicts $h \in J$.

Therefore $b(x_1, \ldots, x_{k+1})$ is in the same cell c as x. This proves that $B(V_\mathcal{G})$ contains at least one point in each cell. □

We consider the half-spaces $\mathcal{F}^i_-(c)$ (as defined in Section B.3) for $i \in [\![1,d]\!]$ and $c \in C(\mathcal{H}_\mathcal{G})$ such that $c \subseteq [-W,W]^d$. We take $\mathcal{H}^+_\mathcal{G} = \mathcal{H}_\mathcal{G} \cup \bigcup_{i\in[\![1,d]\!], c\in C(\mathcal{H}_\mathcal{G})} \mathcal{F}^i_-(c)$. We show that the cells defined by $\mathcal{H}^+_\mathcal{G}$ are compatible with downward closure.

Lemma 18. *Let x, y be two points of $\mathbb{R}^d$. If $x \in\downarrow y$, then $[x]_{\mathcal{H}^+_\mathcal{G}} \subseteq\downarrow [y]_{\mathcal{H}_\mathcal{G}}$ (notice that the second equivalence class considers $\mathcal{H}_\mathcal{G}$ and not $\mathcal{H}_\mathcal{G}$).*

Proof. Let $y \in \mathbb{R}^d$ and $c = [y]_{\mathcal{H}_\mathcal{G}}$. There is a subset of half-spaces $\{H_1, \dots, H_k\} \subseteq \mathcal{H}^+_\mathcal{G}$, that describes $\downarrow c$. If $z \in [x]_{\mathcal{H}_\mathcal{G}}$, then it satisfies the same equations and inequations from $\mathcal{H}^+_\mathcal{G}$ as x, and in particular $H_1, \dots, H_k$, therefore $z \in\downarrow [y]_{\mathcal{H}_\mathcal{G}}$. □

The following lemma identifies cells that contain only maximal values.

Lemma 19. *Let c be a cell in $C(\mathcal{H}^+_\mathcal{G})$. If $c \subseteq \mathtt{value}(\mathcal{G}, s)$ and for all $x \in \mathtt{value}(\mathcal{G}, s) \setminus c$ we have that $c \cap \downarrow [x]_{\mathcal{H}^+_\mathcal{G}} = \varnothing$, then $c \subseteq \mathsf{PO}(\mathcal{G}, s)$.*

Proof. Let c be a cell in $C(\mathcal{H}^+_\mathcal{G})$ that satisfies that $c \subseteq \mathtt{value}(\mathcal{G}, s)$ and for all $x \in \mathtt{value}(\mathcal{G}, s) \setminus c$, $c \cap \downarrow [x]_{\mathcal{H}^+_\mathcal{G}} = \varnothing$. Towards a contradiction assume that there is $y \in c$ that is not Pareto optimal. Then there is $x \in \mathtt{value}(\mathcal{G}, s)$ such that $y < x$. We have that $y \in c \cap \downarrow [x]_{\mathcal{H}^+_\mathcal{G}}$, by hypothesis, this means that $x \in c$.

We consider the half-line $L = \{y + t \cdot (x - y) \mid t \in \mathbb{R}^+\}$. All points on this half-line are greater than y. Let $t_1 = \inf\{t \mid y + t \cdot (x - y) \notin c\}$. There is $h \in \mathcal{H}^+_\mathcal{G}$ such that $\{y + t \cdot (x - y) \mid t < t_1\} \cap h = \varnothing$ and $\{y + t \cdot (x - y) \mid t > t_1\} \cap h = \varnothing$. Let $z = t_1 \cdot (x - y)$, since $\mathtt{value}(\mathcal{G}, s)$ is a closed set, z is also in $\mathtt{value}(\mathcal{G}, s)$. We have that $z \in \mathtt{hplane}(h)$ but $y \notin \mathtt{hplane}(h)$. We have $z \in \mathtt{value}(\mathcal{G}, s) \setminus c$, and $y \in\downarrow [z]$. This contradicts that for all $z \in \mathtt{value}(\mathcal{G}, s) \setminus c$, $c \cap \downarrow [z]_{\mathcal{H}^+_\mathcal{G}} = \varnothing$. □

We now establish the other direction.

Lemma 20. *Let c be a cell in $C(\mathcal{H}^+_\mathcal{G})$. If $c \subseteq \mathsf{PO}(\mathcal{G}, s)$ then $c \subseteq \mathtt{value}(\mathcal{G}, s)$ and for all $x \in \mathtt{value}(\mathcal{G}, s) \setminus c$, $c \cap \downarrow [x]_{\mathcal{H}^+_\mathcal{G}} = \varnothing$.*

Proof. If $c \subseteq \mathsf{PO}(\mathcal{G}, s)$ then by definition of Pareto optimality $c \subseteq \mathtt{value}(\mathcal{G}, s)$. Let $x \in \mathtt{value}(\mathcal{G}, s) \setminus c$. Since $c \subseteq \mathsf{PO}(\mathcal{G}, s)$, for all $y \in c$, if $x \in \mathtt{value}(\mathcal{G}, s)$ then $y \not< x$. Therefore $c \cap \downarrow x = \varnothing$ and by Lemma 18, $c \cap \downarrow [x]_{\mathcal{H}_\mathcal{G}} = \varnothing$. □

We deduce from this, a way to represent the Pareto curve as a union of cells in $C(\mathcal{H}^+_\mathcal{G})$.

Corollary 3. *The Pareto curve is the union of cells c in $C(\mathcal{H}^+_\mathcal{G})$ such that $c \subseteq \mathtt{value}(\mathcal{G}, s)$ and for all $c' \in C(\mathcal{H}^+_\mathcal{G}) \setminus \{c'\}$ if $c' \subseteq \mathtt{value}(\mathcal{G}, s)$ then $c \cap \downarrow c' = \varnothing$.*

Then, the correctness of Theorem 9 follows from the characterization of Corollary 3.

E General case

Theorem 20. *(Theorem 10 in the paper) Let $\mathcal{G}$ be a weighted game and s a state of $\mathcal{G}$, then $\mathtt{value}(\mathcal{G}, s)$ equals:*

$$\bigcup_{\mathcal{G}' \in \mathsf{Sub}(\mathcal{G},s)} \bigcap_{s' \in \mathsf{States}'} \uparrow_J \left(\bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S' \in \mathsf{SCC}(s', \mathcal{G}'(\sigma_\forall))} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S') \right\} \right) \right)$$

where $\uparrow_J X = \{x \in \mathbb{R}^d \mid \forall j \in J.\ \exists x' \in X.\ \forall i \in I \cup \{j\}.\ x_i = x'_i\}$.

Before proceeding to the poof, we recall here some properties of multi-mean-payoff games which were proved in [17].

Lemma 21 ([17, Lemma 17]).

$$\forall s \in \mathsf{States}.\ \exists \sigma_\exists.\ \forall \sigma_\forall.\ \forall i \in I.\ \underline{\mathsf{MP}}_i(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_i \text{ and } \forall j \in J.\ \overline{\mathsf{MP}}_j(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_j$$
$$\Leftrightarrow \forall s \in \mathsf{States}.\ \forall j \in J.\ \exists \sigma_\exists.\ \forall \sigma_\forall.\ \forall i \in I.\ \underline{\mathsf{MP}}_i(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_i \text{ and } \overline{\mathsf{MP}}_j(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_j.$$

Lemma 22 ([17, Lemma 14]). *Let $j \in J$ and $s \in \mathsf{States}$.*

$$\begin{aligned} &\exists \sigma_\exists.\ \forall \sigma_\forall.\ \forall i \in I.\ \underline{\mathsf{MP}}_i(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_i \text{ and } \overline{\mathsf{MP}}_j(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_j \\ \Leftrightarrow\ &\exists \sigma_\exists.\ \forall \sigma_\forall.\ \forall i \in I.\ \underline{\mathsf{MP}}_i(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_i \text{ and } \underline{\mathsf{MP}}_j(\mathsf{Outcome}(s, \sigma_\exists, \sigma_\forall)) \geq v_j. \end{aligned}$$

The proof of the theorem relies on the following lemma which is illustrated by Example 1. From this lemma, we prove inclusion in Theorem 10 by taking for $\mathcal{G}'$ the restriction of $\mathcal{G}$ to states accessible when `Eve` plays a strategy that ensures the given value. We prove the inclusion in the other direction by noticing that it is easier to win in $\mathcal{G}$ than in a subgame $\mathcal{G}'$ and conclude using Lemma 23.

Lemma 23. *Let $\mathcal{G}$ be a weighted game with $J \neq \varnothing$, and s a state of $\mathcal{G}$, then $\bigcap_{s \in \mathsf{States}} \mathtt{value}(\mathcal{G}, s)$ is equal to:*

$$\bigcap_{s \in \mathsf{States}} \uparrow_J \left(\bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S \in \mathsf{SCC}(s, \sigma_\forall)} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S) \right\} \right) \right)$$

Proof. Let v be a point in $\mathtt{value}(\mathcal{G}, s)$ for all state s, then by definition: $\forall i \in I$, $\underline{\mathsf{MP}}_i(\rho) \geq v_i$ and $\forall j \in J$, $\overline{\mathsf{MP}}_j(\rho) \geq v_j$. Thanks to [17, Lemma 17], this is equivalent to $\forall s \in \mathsf{States}.\ \forall j \in J.\ \exists \sigma_\exists.\ \forall \sigma_\forall.\ \forall i \in I.\ \underline{\mathsf{MP}}_i(\rho) \geq v_i$ and $\overline{\mathsf{MP}}_j(\rho) \geq v_j$. Then by [17, Lemma 14], this is also equivalent to $\forall s \in \mathsf{States}.\ \forall j \in J.\ \exists \sigma_\exists.\ \forall \sigma_\forall.\ \forall i \in I.\ \underline{\mathsf{MP}}_i(\rho) \geq v_i$ and $\underline{\mathsf{MP}}_j(\rho) \geq v_j$. This means that $\forall s \in \mathsf{States}$, $v \in \uparrow_J \mathtt{value}(\mathcal{G}', s)$, where game $\mathcal{G}'$ is obtained from $\mathcal{G}$ by setting $I' = I \cup J$ and $J' = \varnothing$. We then obtain the result using the characterization of Theorem 4. ◻

Proof (of Theorem 10). We first prove inclusion. Let $v \in \mathtt{value}(\mathcal{G}, s)$, and $\sigma_\exists$ a strategy which ensures it. Let S be the set of states that is reachable from s by following $\sigma_\exists$. We have $\{s\} \subseteq S \subseteq \mathsf{States}$. We also have $v \in \cap_{s' \in S} \mathtt{value}(s')$, since otherwise v would not be ensured by $\sigma_\exists$. Restricting the game to vertices of S defines a subgame $\mathcal{G}'$, since only edges of `Eve` have been removed, it is a subgame for `Eve`. In a subgame for `Eve` the values can only be lower than in the original game. Using Lemma 23 on this game, we obtain that v belongs to $\bigcap_{s \in S} \uparrow_J \bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S \in \mathtt{SCC}(s, \sigma_\forall)} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S) \right\} \right)$, which shows the inclusion.

Now, in the other direction, let v be an element of:

$$\bigcup_{\{s\} \subseteq S \subseteq \mathsf{States}} \bigcap_{s' \in S} \uparrow_J \left(\bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S' \in \mathtt{SCC}(s, \mathcal{G}'(\sigma_\forall))} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S') \right\} \right) \right).$$

Let $\mathcal{G}'$ be a subgame for `Eve` such that $v \in \bigcap_{s \in \mathsf{States}'} \uparrow_J \bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S' \in \mathtt{SCC}(s, \mathcal{G}'(\sigma_\forall))} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S') \right\} \right)$. We use Lemma 23 on this subgame $\mathcal{G}'$, to obtain:

$$\bigcap_{s \in S} \mathtt{value}(s, \mathcal{G}') = \bigcap_{s \in S} \uparrow_J \left(\bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S \in \mathtt{SCC}(s, \mathcal{G}'(\sigma_\forall))} \downarrow \mathtt{conv}\left(\left\{ \frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S) \right\} \right) \right)$$

Since the actions of `Adam` have not been restricted, it is harder to ensure a given threshold in $\mathcal{G}'$ than in $\mathcal{G}$. Therefore as v is ensured in $\mathcal{G}'$ from all states of S, it can also be ensured in $\mathcal{G}$ from all states of States', which shows the reverse inclusion. ◻

E.1 Expressing thresholds as a union of convex sets

We first prove the following lemma that we will use to reorder union and $\uparrow$ *operators* in the equation of Theorem 10.

Lemma 24. *Operator $\updownarrow$ (defined in appendix B.3) is compatible with union, i.e. for all $J \subseteq [\![1, d]\!]$ and all set X and Y: $(\updownarrow_J X) \cup (\updownarrow_J Y) = \updownarrow_J (X \cup Y)$.*

Proof. Let $x \in (\updownarrow_J X) \cup (\updownarrow_J Y)$. If $x \in \updownarrow_J X$, there is $x' \in X$ such that $\pi_J(x') = \pi_J(x)$. We also have $x' \in X \cup Y$, which proves $x \in \updownarrow_J (X \cup Y)$. The case where $x \in \updownarrow_J Y$ is similar.

In the other direction, let $x \in \updownarrow_J (X \cup Y)$, there exists $x' \in X \cup Y$ such that $\pi_J(x) = \pi_J(x')$. If $x' \in X$ then $x \in \updownarrow_J X$ and otherwise $x \in \updownarrow_J Y$. Therefore $x \in (\updownarrow_J X) \cup (\updownarrow_J Y)$.

We use the following to restrict our study to bounded polygon, so that we can easily represent them by their extremal points.

Lemma 25.

$$\mathit{value}(\mathcal{G}, s) = \downarrow ([-W_{\mathcal{G}}, W_{\mathcal{G}}]^d \cap \mathit{value}(\mathcal{G}, s))$$

Proof. Let $v \in \mathtt{value}(\mathcal{G}, s)$, there is a strategy $\sigma_\exists$ whose outcome all have weight greater than v. Since all outcome ρ of the game on all coordinate i, a weight $w_i(\rho)$ greater than $-W_{\mathcal{G}}$, we can replace all v_i that are smaller than $-W_{\mathcal{G}}$ by $-W_{\mathcal{G}}$ and we obtain a $v' \geq v$ such that $v' \in \mathtt{value}(s)$. Similarly, all weights are smaller than $W_{\mathcal{G}}$ so for all coordinate i, v_i cannot be greater than W. Therefore $v' \in [-W, W]^d \cap \mathtt{value}(\mathcal{G}, s)$ and $v \downarrow ([-W, W]^d \cap \mathtt{value}(\mathcal{G}, s))$.

For the other inclusion, this is simply because $\mathtt{value}(\mathcal{G}, s)$ is downward closed and $[-W, W]^d \cap \mathtt{value}(\mathcal{G}, s) \subseteq \mathtt{value}(\mathcal{G}, s)$.

We will also use the following lemma, to apply the $\updownarrow$ operator only on bounded polytopes.

Lemma 26. *Given a set $X \subseteq \mathbb{R}^d$, a bound W and $J \subseteq [\![1, d]\!]$, we have that:*

$$[-W, W]^d \cap (\updownarrow_J [-W, W]^d \cap (\downarrow [-W, W]^d \cap X)) = [-W, W]^d \cap (\updownarrow_J \downarrow ([-W, W]^d \cap X))$$

Proof. The inclusion holds because $[-W, W]^d \cap (\downarrow [-W, W]^d \cap X) \subseteq \downarrow [-W, W]^d \cap X$ and $\updownarrow_J$ is monotonic with respect to inclusion.

In the other direction, let $x \in [-W, W]^d \cap (\updownarrow_J \downarrow ([-W, W]^d \cap X))$. There is $x' \in \downarrow ([-W, W]^d \cap X)$ such that $\pi_J(x') = \pi_J(x)$. There is $x'' \in [-W, W]^d \cap X$ such that $x'' \geq x'$. Let y be such that $y_j = x''_j$ if $j \in J$ and $y_j = x_j = x'_j \leq x''_j$ otherwise. We have $\pi_J(y) = x$ and $y \in [-W, W]^d$ because both x'' and x do. We also have $y \in \downarrow [-W, W]^d \cap X$ because it is smaller than x''. This shows that $x \in \updownarrow_J \{y\} \subseteq [-W, W]^d \cap (\updownarrow_J [-W, W]^d \cap (\downarrow [-W, W]^d \cap X))$. ∎

Theorem 21. *There are polynomial functions P_5 and P_6 such that for all weighted games $\mathcal{G}$ and states s of $\mathcal{G}$, $\mathit{value}(\mathcal{G}, s)$ is a finite union of convex set which can be given as the intersection of a finite set of half-spaces $H_1^g, \ldots, H_{k_g}^g$ whose associated inequations have size bounded by $P_5(d) + P_6(d) \cdot (2 + \log_2((W_{\mathcal{G}} + 1) \cdot (|\mathsf{States}| + 1)))$.*

Proof. By Theorem 10 $\mathtt{value}(\mathcal{G}, s)$ is equal to:

$$\bigcup_{\{s\} \subseteq S \subseteq \mathsf{States}} \bigcap_{s' \in S} \bigcap_{j \in J} \updownarrow_{J \setminus \{j\}} \bigcap_{\sigma_\forall \in \mathbb{M}} \bigcup_{S' \in \mathtt{SCC}(s', \sigma_\forall)} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in \mathbb{C}(S')\right\}\right)$$

We can then reorder intersection and union operators using the fact that $\updownarrow$ operators are compatible with union (Lemma 24). We do this by considering functions that selects for a tuple $(s', j, \sigma_\forall)$ a strongly connected component of $\mathtt{SCC}(s', \sigma_\forall)$, i.e. functions of F, where $F = \{f \colon \mathsf{States} \times J \times \mathbb{M} \mapsto 2^{\mathsf{States}} \mid \forall s' \in \mathsf{States}, j \in J, \sigma_\forall \in \mathbb{M}.\ f(s', j, \sigma_\forall) \in \mathtt{SCC}(s', \sigma_\forall)\}$. Note that the set F can be huge, but is always finite. We then have that $\mathtt{value}(\mathcal{G}, s)$ is equal to:

$$\bigcup_{\{s\} \subseteq S \subseteq \mathsf{States}} \bigcup_{f \in F} \bigcap_{s' \in S} \bigcap_{j \in J} \updownarrow_{J \setminus \{j\}} \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right).$$

We use Lemma 25 to restrict the difficult operations ($\downarrow$ and $\updownarrow$) to bounded polytopes. This is because their results are easier to express using the vertex representation, which requires bounded polytopes. We express $\mathtt{value}(\mathcal{G}, s)$ as:

$$\downarrow [-W, W]^d \cap \bigcup_{\{s\} \subseteq S \subseteq \mathsf{States}} \bigcup_{f \in F} \bigcap_{s' \in S} \bigcap_{j \in J} \updownarrow_{J \setminus \{j\}} \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right).$$

Note that we can also use Lemma 25 to show:

$$\downarrow [-W, W]^d \cap \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right) = \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \mathtt{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right)$$

since this set is the $\texttt{value}$ set for the game where J would be empty (see 4). We therefore write $\texttt{value}(\mathcal{G}, s)$ as:

$$\downarrow [-W,W]^d \cap \bigcup_{\{s\}\subseteq S\subseteq \mathsf{States}} \bigcup_{f\in F} \bigcap_{s'\in S} \bigcap_{j\in J} \updownarrow_{J\setminus\{j\}} \downarrow [-W,W]^d \cap \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \texttt{conv}\left(\left\{\frac{1}{|c|}\cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right).$$

We now use Lemma 26 to introduce a cube $[-W,W]^d$ between the $\updownarrow$ and $\downarrow$ operators. Then $\texttt{value}(\mathcal{G}, s)$ can be written as:

$$\bigcup_{\{s\}\subseteq S\subseteq \mathsf{States}} \bigcup_{f\in F} \downarrow [-W,W]^d \cap \bigcap_{s'\in S} \bigcap_{j\in J} \updownarrow_{J\setminus\{j\}} [-W,W]^d$$
$$\cap \downarrow [-W,W]^d \cap \bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \texttt{conv}\left(\left\{\frac{1}{|c|}\cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right).$$

We obtain therefore a finite union of convex intersections. We now use geometrical arguments to show that the inequations defining these polyhedra are of polynomial size which will conclude the proof.

Thanks to Theorem 3.2, the set $\bigcap_{\sigma_\forall \in \mathbb{M}} \downarrow \texttt{conv}\left(\left\{\frac{1}{|c|}\cdot w(c) \mid c \in f(s', j, \sigma_\forall)\right\}\right)$ can be given by the intersection of half-spaces whose inequations have size bounded by $P_2(d)\cdot(2+\log_2((W+1)(|\mathsf{States}|+1)))$. The intersection with the cube $[-W,W]^d$ is obtained by adding inequations of size $d+||W||$ which is smaller than the previous bound. To apply $\downarrow$ we go back to the point representation, using Theorem 3.3, sizes of the vertices are bounded by $P_3(d)\cdot(2+P_2(d)\cdot(2+\log_2((W_\mathcal{G}+1)(|\mathsf{States}|+1))))$. Then using Theorem 3.2 once again, the downward closure gives us half-spaces which have size bounded by $P_2(d)\cdot(2+P_3(d)\cdot(2+(P_2(d)\cdot(2+\log_2((W_\mathcal{G}+1)(|\mathsf{States}|+1))))))$. Taking the intersection with the cube $[-W,W]^d$ again adds inequation that are within these bounds. We go back to a point representation in order to apply $\uparrow_{J\setminus\{j\}}$, which gives, thanks to Theorem 3.3, points of size bounded by $P_3(d)\cdot(2+P_2(d)\cdot(2+P_3(d)\cdot(2+(P_2(d)\cdot(2+\log_2((W_\mathcal{G}+1)(|\mathsf{States}|+1)))))))$.

We show that the operator $\uparrow_{J\setminus\{j\}}$ gives half-spaces whose size is polynomially bounded. We have seen in Lemma 12 that given a polyhedron P:

$$\updownarrow_J P = \bigcap_{H\in\mathcal{F}(\pi_J(P))} H + B_J$$

When P is given by its vertices, the projection π_J is obtained by projecting points one by one, which means replacing coordinates of J by 0 and therefore does not increase the size of the points. The equations to describe half-spaces of $\mathcal{F}(\pi_J(P))$, can be of size bounded by $P_2(d)$ times 2 plus the size of the vertices, thanks to Theorem 3.1. The same system of inequations, gives the sets $H+\mathbb{R}^J$ when lifted to $\mathbb{R}^d$. Therefore the size the inequations we obtain after applying $\uparrow_{J\setminus\{j\}}$ is bounded by $P_5(d)+P_6(d)\cdot(2+\log_2((W_\mathcal{G}+1)\cdot(|\mathsf{States}|+1))) = P_2(d)\cdot(2+P_3(d)\cdot(2+P_2(d)\cdot(2+P_3(d)\cdot(2+(P_2(d)\cdot(2+\log_2((W_\mathcal{G}+1)(|\mathsf{States}|+1))))))))$. Then, the intersection with the cube $[-W,W]^d$ does not add any complexity and thus the previous bound also bounds the size of the half-spaces that describes the convex set of the expression of $\texttt{value}(\mathcal{G}, s)$. □

Theorem 22. *There is a polynomial function P such that if S is a set of states and λ a system of linear equations, then $\bigcap_{s\in S} \texttt{value}(\mathcal{G}, s) \cap [-W,W]^d \cap \texttt{polyhedron}(\lambda)$ is of the form $\bigcup_{f\in F} \texttt{conv}(E_f)$ where F is finite and for each $f\in F$, E_f is a set of vertices which all have size bounded by $P(|\mathsf{States}|, d, ||\lambda||, ||W||)$.*

Proof. By Theorem 21 we have a finite union of polyhedra whose equations have size bounded by a polynomial function. Let $\lambda = (a_1, b_1)\dots(a_k, b_k)$. We use Theorem 2 to say that if $\texttt{polyhedron}(\lambda)$ intersects this union of polyhedra, then there is a solution of size bounded by $d\cdot P_1(\max\{||a_i, b_i|| \mid 1\le i\le k\}, P_6(||W||, ||\mathsf{States}||, d), d)$. □