

# Optimal Values of Multidimensional Mean-Payoff Games\*

Romain Brenguier, Jean-François Raskin

Université Libre de Bruxelles (U.L.B.), Belgium

**Abstract** We study multidimensional mean-payoff games. More precisely, we want to compute all the values that can be ensured by the first player. Because weights are given in multiple dimensions, there can be incomparable such values, and even an infinite number of incomparable ones. We show that the set the values can be represented by a finite union of convex sets. From the computational point of view, we show that deciding if the intersection of the set of values with points satisfying a set of linear inequations can be done in  $\Sigma_2$ -P. The problem is both NP-hard and coNP-hard.

## 1 Introduction

Games in computer science are used to model interaction in computerized systems. They have been successfully used for the synthesis of reactive systems. In this context, the opponent represents a hostile environment, and winning strategies provide controllers, that ensures the correctness of the system. There has been recently an effort to lift the study from the qualitative setting to the quantitative setting. For example a lot of work have been carried out on the study of mean-payoff Games, for instance in [4].

In a mean-payoff game players try to maximize the average of the reward they get at each step of the game. These objectives can be extended to a multidimensional setting [5], where the reward comes as a vector, and each component represent a reward of a different kind.

For example when trying to model a wireless network, where each player control one device, one dimension of the reward can represent the bandwidth and another dimension the power consumption of the device (or rather its opposite as we are trying to maximize it).

The study of these multidimensional mean-payoff games have been focused until now on checking whether the first player has a strategy to ensure at least a given value on each dimension [5,11]. In this paper, we show that it is possible to compute all the values that can be ensured by the first player. This makes the task of the designer easier as it can directly choose the best solution among the optimal ones.

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*Contribution.* In Sect. 2, we define turn-based games played by two player **Eve** and **Adam** and multidimensional mean-payoff objectives and the different decision problems we consider. Basically our problem is to decide if there is a value that can be ensure by **Eve** and that satisfy some other constraints given as input. In Sect. 3, we study the problem in the case where all objectives are given by the limit inferior of the average of the reward. In Thm. 1, we give a full characterization of the values that can be ensured by **Eve**. This set can be computed by operation on polyhedra and we show a  $\Sigma_2$ -P algorithm to decide our decision problems. In Sect 4, we generalize our study to the case where both limit inferior and limit superior are present and show that a similar, although more involved, characterization holds. Again we have  $\Sigma_2$ -P algorithm to decide our decision problems, this time the analysis relies on the fact that witnesses can be found with relatively simple coordinates.

## 2 Definitions

### 2.1 Arena

We define weighted arenas, played by two players **Eve** and **Adam**.

**Definition 1 (Arena).** An arena  $\mathcal{A}$  is a tuple  $\langle \text{Stat}_{\exists}, \text{Stat}_{\forall}, s_0, \text{Edg} \rangle$ , where:

- $\text{Stat} = \text{Stat}_{\exists} \cup \text{Stat}_{\forall}$  is a finite set of states;
- $s_0 \in \text{Stat}$  is the initial state;
- $\text{Edg} \subseteq \text{Stat} \times \text{Stat}$  is a finite set of edges;

In an arena  $\mathcal{A}$ , whenever we arrive at a state  $s$ :

- if  $s \in \text{Stat}_{\exists}$ , then **Eve** selects a state  $s'$  such that  $(s, s') \in \text{Edg}$ .
- if  $s \in \text{Stat}_{\forall}$ , then **Adam** selects a state  $s'$  such that  $(s, s') \in \text{Edg}$ .

The game then continues from  $s'$ . This process starts from  $s_0$  and is repeated ad infinitum to form an infinite sequence of states.

**Definition 2 (History and plays).** A history of the game  $\mathcal{G}$  is a finite non-empty sequence of states i.e. an element of  $\text{Stat}^* \cdot \text{Stat}$ . We write  $h_i$  the  $i$ -th state of  $h$ , starting from 0, i.e.  $h = h_0 \cdot h_1 \cdots h_n$ . The length  $|h|$  of such an history is  $n + 1$ . We write  $\text{last}(h)$  the last state of  $h$ , i.e.  $h_{|h|-1}$ . A play  $\rho$  is an infinite sequence of states, i.e. an element of  $\text{Stat}^\omega$ . We write  $\rho_{\leq n}$  for the prefix of  $\rho$  of length  $n + 1$ , i.e.  $\rho_0 \cdots \rho_n$ . We also write  $\rho_{\geq n}$  for the suffix of  $\rho$  after  $\rho_{\leq n-1}$ , i.e.  $\rho_n \cdot \rho_{n+1} \cdots$  and  $\rho_{[m, n]}$  for  $(\rho_{\leq n})_{\geq m}$ .

**Definition 3 (Strategies).** Let  $\mathcal{A}$  be an arena, a strategy for **Eve** maps histories ending in a state of  $\text{Stat}_{\exists}$  to a successor of that state. Formally it is a function  $\sigma_{\exists}: \text{Stat}^* \cdot \text{Stat}_{\exists} \rightarrow \text{Stat}$ , such that for all for all history  $h$  and state  $s$ ,  $(s, \sigma_{\exists}(h \cdot s)) \in \text{Edg}$ . Similarly, a strategy for **Adam** is a function  $\sigma_{\forall}: \text{Stat}^* \cdot \text{Stat}_{\forall} \rightarrow \text{Act}$ , such that for all for all history  $h$  and state  $s$ ,  $(s, \sigma_{\forall}(h \cdot s)) \in \text{Edg}$ . A strategy profile is a pair of a strategy for **Eve** and a strategy for **Adam**.

**Definition 4 (Outcomes).** Let  $\mathcal{A}$  be an arena, and  $\sigma_{\exists}$  a strategy for Eve. A play  $\rho$  is compatible with the strategy  $\sigma_{\exists}$  if, for all  $k \in \mathbb{N}$ , if  $\rho_k \in \text{Stat}_{\exists}$  then  $\rho_{k+1} = \sigma_{\exists}(\rho_{\leq k})$ . We write  $\text{Out}_{\mathcal{G}}(s, \sigma_{\exists})$  for the set of paths in  $\mathcal{G}$  that are compatible with strategy  $\sigma_{\exists}$  and have initial state  $s$  (i.e.  $\rho$  such that  $\rho_0 = s$ ), these plays are called outcomes of  $\sigma_{\exists}$  from  $s$ . We simply write  $\text{Out}_{\mathcal{G}}(\sigma_{\exists})$  when  $s = s_0$ .

## 2.2 Multidimensional Mean Payoff

**Definition 5 (Multidimensional weighted games).** A weighted game is an arena equipped with a weight function. Formally it is a tuple  $(\mathcal{A}, w)$ , where  $\mathcal{A}$  is an arena and  $w: \text{Edg} \mapsto \mathbb{Z}^d$ .

Given a weight function  $w$ , we write  $w_i$  the function that associate to an edge  $e$  the projection to the  $i$ -th dimension of  $w(e)$ . We now present mean payoff preferences which are given by the long term average of the weights.

**Definition 6 (Mean payoff).** Given a weight function  $w$ , the mean payoff inferior over dimension  $i$  of a play  $\rho$  is:

$$\underline{\text{MP}}_i(\rho) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq k < n} w_i(\rho_k, \rho_{k+1}).$$

Similarly the mean payoff superior over dimension  $i$  of a play  $\rho$  is:

$$\overline{\text{MP}}_i(\rho) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq k < n} w_i(\rho_k, \rho_{k+1}).$$

**Definition 7.** Let  $v \in \mathbb{R}^{2-d}$ , we say that a strategy  $\sigma_{\exists}$  ensures value  $v$  from state  $s$  if all outcomes  $\rho$  of  $\sigma_{\exists}$  from state  $s$  are such that on each dimension  $i$ ,  $\underline{\text{MP}}_i(\rho) \geq v_i$  and  $\overline{\text{MP}}_i(\rho) \geq v_{d+i}$ . We write  $\text{value}(s)$  for the set of vectors  $v$  such that there is a strategy of Eve that ensures value  $v$  from state  $s$ .

## 2.3 The Value Problem

Our goal is to find the strategy of Eve that ensures a value as height as possible. However, since the weight are multidimensional it is not clear how to compare different values. A natural problem, is to try to optimize the value we can ensure with respect to a linear function.

**Definition 8.** A linear function is given by a vector  $a \in \mathbb{R}^d$ . The associated function  $\alpha_a: \mathbb{R}^d \mapsto \mathbb{R}$  is the function such that  $\alpha_a(x) = \sum_{i \in [1, d]} a_i \cdot x_i$ . A linear equation is a pair  $(a, b)$  where  $a \in \mathbb{R}^d \setminus \{0\}$  and  $b \in \mathbb{R}$ . The half-space satisfying  $(a, b)$  is the set  $\text{hsp}(a, b) = \{x \in \mathbb{R}^d \mid \alpha_a(x) \geq b\}$ . The hyperplane defined by  $(a, b)$  is the set  $\text{hp}(a, b) = \{x \in \mathbb{R}^d \mid \alpha_a(x) = b\}$ .

Given  $a$ , we are looking for a strategy  $\sigma_{\exists}$  that ensures a value  $v \in \mathbb{R}^d$ , and such that there is no  $\sigma'_{\exists}$  that ensures a value  $v'$ , with  $\alpha(v') > \alpha(v)$ . To make this into a decision problem, we fix a real  $b$ , and ask if it is possible to ensure a value  $v$  such that  $\alpha(v) \geq b$ .

**Definition 9 (Value problem).** *Given a mean payoff game  $\mathcal{G}$  and a linear equation  $(a, b)$  over elements of  $\mathbb{R}^{2^d}$ , is there a strategy  $\sigma_{\exists}$  and a value  $v \in \mathbf{hsp}(a, b)$  such that  $\sigma_{\exists}$  ensures  $v$ ?*

*Remark 1.* The answer to this problem is also useful when computing Nash equilibria in (single dimension) mean payoff games. Indeed, thanks to the suspect transformation [2], a multiplayer concurrent game with mean payoff objectives can be transformed into a multidimensional mean payoff turn-based game. The existence of a Nash equilibria in the original game is then equivalent to the existence of a run which has a value which can be ensured by the first player of the game in the suspect game.

We also define a generalization of this problem that considers a system of equation instead of a single one.

**Definition 10 (Threshold problem).** *Given a mean payoff game  $\mathcal{G}$ , a system of linear equation  $\lambda = ((a_1, b_1), \dots, (a_l, b_l))$  over elements of  $\mathbb{R}^{2^d}$ , is there a strategy  $\sigma_{\exists}$  and a value  $v \in \cap_{k=1}^l \mathbf{hsp}(a_k, b_k)$  such that  $\sigma_{\exists}$  ensures  $v$ ?*

*Remark 2.* Other works ([11] for instance) usually focus on the 0-threshold problem, which can be specified here by  $a_1 = (1, 0, \dots, 0)$ ,  $a_2 = (0, 1, 0, \dots, 0)$ , ... and all  $b_k$  are equal to 0. In such a case, it is only needed to check that the value  $\mathbf{0}$  can be ensured. This is indeed simpler than having to find the values that can be ensured: we will show that the threshold problem is NP-hard and coNP-hard while the 0-threshold problem is coNP-complete [11].

Many algorithms require computing the winning region for some objective, this leads to the definition of the following decision problem.

**Definition 11 (Region problem).** *Given a mean payoff game  $\mathcal{G}$ , a set of states  $S$ , a system of linear equation  $\lambda = ((a_1, b_1), \dots, (a_l, b_l))$  over elements of  $\mathbb{R}^{2^d}$ , is there from each state  $s$  of  $S$ , a strategy  $\sigma_{\exists}^s$  and a value  $v \in \cap_{k=1}^l \mathbf{hsp}(a_k, b_k)$  such that  $\sigma_{\exists}^s$  ensures  $v$  from  $s$ ?*

## 2.4 Decomposition in Simple Cycles

Our analysis of mean payoff game will rely on finding good cycles for **Eve**. Our proof will use as a tool the decomposition of a play into *simple cycles*.

**Definition 12 (Simple cycles).** *A simple cycle is sequence  $s_0 \cdot s_1 \cdots s_n$ , such that  $s_0 = s_n$  and for all  $i$  and  $j$ ,  $0 \leq i < j < n$ ,  $s_i \neq s_j$ . We write  $\mathbb{C}(\mathcal{G})$  ( $\mathbb{C}$  when  $\mathcal{G}$  is clear from the context) for the set of simple cycles in the concurrent game structure  $\mathcal{G}$ .*

In this paragraph, we recall the notion of decomposition of a play into *simple cycles* [7]. A history  $h = s_0 \cdot s_1 \cdots s_n$  is a *cycle* if  $s_0 = s_n$ ,  $n \geq 1$ . A *simple cycle* is a cycle such that for all  $i$  and  $j$ ,  $0 \leq i < j < n$ ,  $s_i \neq s_j$ . We write  $\mathbb{C}(\mathcal{A})$  ( $\mathbb{C}$  when  $\mathcal{A}$  is clear from the context) for the set of simple cycles in the concurrent game  $\mathcal{A}$ .

Every history  $h$  of a finite game can be uniquely decomposed into a sequence of simple cycles, except for a finite part. The decomposition process maintains a *stack*,  $\mathbf{st}(h)$ , of distinct states and moves. We write the stack content  $s_1 \cdot s_2 \cdots s_n$  where  $s_1$  is at the bottom of the stack and  $s_n$  the top. We use the notation  $s \in \mathbf{st}(h)$  for  $s \in \{s_1, s_2, \dots, s_n\}$ . The *decomposition*,  $\mathbf{dec}(h)$ , is a set of simple cycles. We define  $\mathbf{dec}(h)$  and  $\mathbf{st}(h)$  inductively as follows:

- for the single state history  $s$ ,  $\mathbf{dec}(s) = \emptyset$  and  $\mathbf{st}(s) = s$ .
- let  $h' = h \cdot s$  be a history.
  - If  $s \in \mathbf{st}(h)$ , and  $\mathbf{st}(h) = \alpha \cdot s \cdot \beta$ , then  $\mathbf{st}(h') = \alpha \cdot s$  and  $\mathbf{dec}(h') = \mathbf{dec}(h) \cup \{s \cdot \beta \cdot s\}$ .
  - else  $\mathbf{dec}(h') = \mathbf{dec}(h)$ ,  $\mathbf{st}(h') = \mathbf{st}(h) \cdot s$ .

Note that the stack always contains distinct elements, therefore only simple cycles are added to the decomposition. The elements in the stack from the bottom to the top, form a history  $s_0 \cdot s_1 \cdots s_n$ , where  $n+1$  is the height of the stack. The decomposition of a play is the union of the decompositions of the finite prefixes of the play.

### 3 Case of limit inferior

We first analyze the case where all the mean-payoff values would be taken as limit inferior of the average cost. We define value by:

$$\mathbf{value}(s) = \{v \in \mathbb{R}^d \mid \exists \sigma_{\exists}. \forall \rho \in \text{Out}(s, \sigma_{\exists}). \forall i \in \llbracket 1, d \rrbracket. \underline{\text{MP}}_i(\rho) \geq v_i\}$$

#### 3.1 Downward closure

**Definition 13.** Let  $\downarrow S$  be the downward closure of  $S$ , i.e.  $\{x \mid \exists x' \in S. \forall i \in \llbracket 1, n \rrbracket. x_i \leq x'_i\}$ .

**Lemma 1.** – Operator  $\downarrow$  is monotonic, i.e. for any set  $X$  and  $Y$  such that  $X \subseteq Y$ , we have that  $\downarrow X \subseteq \downarrow Y$ .  
– Operator  $\downarrow$  is compatible with union, i.e. for any set  $X$  and  $Y$ :  $(\downarrow X) \cup (\downarrow Y) = \downarrow (X \cup Y)$ .

*Proof.* Assume  $X \subseteq Y$ . Let  $x \in \downarrow X$ , there is  $x' \in X$  such that for all  $i$ ,  $x'_i \geq x_i$ . As  $X \subseteq Y$ ,  $x' \in Y$ , and therefore  $x \in \downarrow Y$ .

To show  $(\downarrow X) \cup (\downarrow Y) \subseteq \downarrow (X \cup Y)$  We use the monotonicity property (Lem. 1): since  $X \subseteq X \cup Y$ ,  $\downarrow X \subseteq \downarrow (X \cup Y)$ .

In the other direction, let  $x \in \downarrow (X \cup Y)$ , there exists  $x' \in X \cup Y$  such that  $x \leq x'$ . If  $x' \in X$  then  $x \in \downarrow X$  and otherwise  $x \in \downarrow Y$ .

### 3.2 Characterizing values

We define memoryless strategies. We will use them to characterize values.

**Definition 14 (Memoryless strategies).** A strategy  $\sigma_{\forall}$  is said memoryless if for every history  $h$  and  $h'$ , and all state  $s$ ,  $\sigma_{\forall}(h \cdot s) = \sigma_{\forall}(h' \cdot s)$ . We write  $\mathbb{M} = \{\sigma_{\forall}^1, \dots, \sigma_{\forall}^{|\mathbb{M}|}\}$  for the set of the memoryless strategies of **Adam** in  $\mathcal{G}$ .

Let  $\sigma_{\forall}$  be a memoryless strategy. Strategy  $\sigma_{\forall}$  determines a subgraph of  $\mathcal{G}$ , let  $\text{SSC}(\sigma_{\forall})$  be the set of *strongly connected component* in this subgraph. We write:

$$\mathbb{C}(s, \sigma_{\forall}) = \{C \mid \exists S \in \text{SSC}(\sigma_{\forall}). C = \{c \mid c \text{ simple cycle of } S\} \wedge S \text{ reachable from } s\}.$$

**Definition 15 (Convex hull).** The convex hull of a set of points  $V$  is the set

$$\text{conv}(V) = \left\{ \sum_{v \in V} t_v \cdot v \mid \forall v \in V. t_v \in [0, 1] \wedge \sum_{v \in V} t_v = 1 \right\}.$$

We now give a characterization of the set of values that **Eve** is able to ensure.

**Theorem 1.**

$$\underline{\text{value}}(s) = \bigcap_{\sigma_{\forall} \in \mathbb{M}} \bigcup_{C \in \mathbb{C}(s, \sigma_{\forall})} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right)$$

We start by recalling two lemmas of [11] which we will use in our proof.

**Lemma 2 ([11, Lem. 10(2)]).** Let  $S$  be a strongly connected graph. If  $S$  does not have a non-negative multi-cycle (i.e.  $C \in \mathbb{C}(S)$  and  $\lambda \in \mathbb{N}^C$  such that  $\sum_{c \in C} \lambda(c) \cdot w(c) \geq 0$ ), then there exists a constant  $m_G \in \mathbb{N}$  and a real  $c_G > 0$  such that for all history  $h$  in the graph  $S$  we have  $\min\{w_i(h) \mid i \in I\} \leq m_G - c_G \cdot |h|$ .

**Lemma 3 ([11, Lem. 11]).** If there is a strongly connected component  $S$  reachable from  $s_0$  that has a non-negative multi-cycle, then **Eve** has a strategy to satisfy the mean-payoff-inf (i.e.  $\forall i \in I \cup J. \forall \rho \in \text{Out}_{s_0}(\sigma_{\exists}). \underline{\text{MP}}_i(\rho) \geq 0$ ).

**Lemma 4.** Let  $\rho$  be a path. If  $\text{Inf}(\rho) \subseteq C$  for some  $C \in \mathbb{C}(\sigma_{\forall})$ , then  $\underline{\text{MP}}(\rho) \in \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right)$ .

*Proof.* We consider the lexicographic order  $\leq_{\text{lex}}$  defined by  $v \leq_{\text{lex}} v' \Leftrightarrow v = v' \vee \exists i \leq d. v_i < v'_i \wedge \forall j < i. v_j = v'_j$ . We also consider for each  $i \in \llbracket 1, d \rrbracket$ , the preorder  $\leq_i$  defined by  $v \leq_i v' \Leftrightarrow v_i \leq v'_i$ . We define  $L(n) = \min_{\leq_{\text{lex}}} \left\{ \frac{1}{m} \cdot w(\rho_{\leq m}) \mid m \geq n \right\}$  and  $K_i(n) = \min_{\leq_i} \left\{ \frac{1}{m} \cdot w(\rho_{\leq m}) \mid m \geq n \right\}$ . We write  $\text{MPL}(\rho) = \lim_{n \rightarrow \infty} L(n)$ , this limit exists because  $(\mathbb{R}^d, \leq_{\text{lex}})$  forms a complete lattice.

We show that for all coordinate  $i$ ,  $\text{MPL}(\rho)_i \geq \text{MP}_i(\rho)$ . We write  $1_i$  for the vector in which all coordinates are equal to 0 except the  $i$ -th coordinate which is equal to 1. Let  $\varepsilon > 0$  and consider the vector  $L(n) + \varepsilon \cdot 1_i$ . Since  $L(n) <_{\text{lex}} L(n) + \varepsilon \cdot 1_i$ , there exists  $m \geq n$  such that  $\frac{1}{m} \cdot w(\rho_{\leq m}) \leq_{\text{lex}} L(n) + \varepsilon \cdot 1_i$ . We have that  $\frac{w_i(\rho_{\leq m})}{m} \leq$

$L(n)_i + \varepsilon$  because for all coordinate  $j < i$ ,  $\frac{w_j(\rho_{\leq m})}{m}$  has to be equal to  $L(n)_j$ . We also have that  $\frac{1}{m} \cdot w(\rho_{\leq m}) \geq_i K_i(n)$ . Hence for all  $\varepsilon > 0$ ,  $K_i(n) < L(n)_i + \varepsilon$ , which shows that  $\text{MPL}(\rho)_i \geq \text{MP}_i(\rho)$ .

We now show that  $\text{MPL}(\rho) \in \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$ . After a finite prefix  $\rho_{\leq m}$ , the play stays in the component  $C$ . We consider the decomposition in simple cycle of  $\rho_{\geq m}$ , for  $n > m$ :

$$w(\rho_{\leq n}) = w(\rho_{\leq m}) + w(\text{st}(\rho_{[m,n]})) + \sum_{c \in \text{dec}(\rho_{[m,n]})} w(c)$$

and for all  $c \in \text{dec}(\rho_{[m,n]})$ ,  $c$  belongs to  $C$ . Since the length of the stack is bounded by the number of states, we have that:

$$w(\rho_{\leq m}) - W \cdot |V| + \sum_{c \in \text{dec}(\rho_{[m,n]})} w(c) \leq w(\rho_{\leq n}) \leq w(\rho_{\leq m}) + W \cdot |V| + \sum_{c \in \text{dec}(\rho_{[m,n]})} w(c)$$

where  $W$  contains on each coordinate the maximum of absolute value of weights in this coordinate.

$$\frac{w(\rho_{\leq m})}{n} - \frac{W \cdot |V|}{n} + \sum_{c \in \text{dec}(\rho_{[m,n]})} \frac{1}{n} \cdot w(c) \leq \frac{w(\rho_{\leq n})}{n} \leq \frac{w(\rho_{\leq m})}{n} + \frac{W \cdot |V|}{n} + \sum_{c \in \text{dec}(\rho_{[m,n]})} \frac{1}{n} \cdot w(c)$$

$$\begin{aligned} \sum_{c \in \text{dec}(\rho_{[m,n]})} \frac{1}{n} \cdot w(c) &= \sum_{c \in \text{dec}(\rho_{[m,n]})} \frac{|c|}{n} \cdot \frac{1}{|c|} \cdot w(c) \\ &= \frac{\sum_{c \in \text{dec}(\rho_{[m,n]})} |c|}{n} \cdot \sum_{c \in \text{dec}(\rho_{[m,n]})} \frac{|c|}{\sum_{c \in \text{dec}(\rho_{[m,n]})} |c|} \cdot \frac{1}{|c|} \cdot w(c) \end{aligned}$$

We write  $v_n$  for  $\sum_{c \in \text{dec}(\rho_{[m,n]})} \frac{|c|}{\sum_{c \in \text{dec}(\rho_{[m,n]})} |c|} \cdot \frac{1}{|c|} \cdot w(c)$ , we have that  $v_n \in \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$ .

We will show that  $\frac{1}{n} \cdot w(\rho_n)$  comes closer and closer to  $v_n$ .

$$\frac{w(\rho_{\leq m})}{n} - \frac{W \cdot |V|}{n} + \frac{\sum_{c \in \text{dec}(\rho_{[m,n]})} |c|}{n} \cdot v_n \leq \frac{w(\rho_{\leq n})}{n} \leq \frac{1}{n} \cdot w(\rho_{\leq m}) + \frac{W \cdot |V|}{n} + \frac{\sum_{c \in \text{dec}(\rho_{[m,n]})} |c|}{n} \cdot v_n$$

Let  $i$  be a coordinate:

$$\begin{aligned} \left| \frac{1}{n} \cdot w_i(\rho_n) - v_n \right| &\leq \frac{w_i(\rho_{\leq m})}{n} + \frac{W_i \cdot |V|}{n} + \frac{n - \sum_{c \in \text{dec}(\rho_{[m,n]})} |c|}{n} \cdot |(v_n)_i| \\ &\leq \frac{m \cdot W_i}{n} + \frac{W_i \cdot |V|}{n} + \frac{m + |V|}{n} \cdot |W_i| \\ &\leq \frac{2 \cdot (m + |V|) \cdot W_i}{n} \end{aligned}$$

We show that there is a strictly increasing sequence  $u_n$  such that  $\text{MP}(\rho_{u_n})$  converges to  $\text{MPL}(\rho)$ . We define  $u_0$  by 0 then give  $u_n$  we set  $u_{n+1}$  to be the

smallest  $m \geq u_n + 1$  such that  $L(u_n + 1) = \frac{1}{m} \cdot w(\rho_{\leq m})$ . The sequence  $u_n$  is strictly increasing and  $L(n) = \frac{w(\rho_{\leq u_n})}{n} = \text{MP}(\rho_{\leq u_n})$  hence  $\text{MP}(\rho_{u_n})$  converges to  $\text{MPL}(\rho)$ .

Since  $\text{MP}(\rho_{u_n})$  converges and is  $\frac{2 \cdot (m+V) \cdot W_i}{n}$  close to  $v_n$ , and  $v_n$  belongs to  $\text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$  which is a closed set, the limit of  $\text{MP}(\rho_{u_n})$  is within  $\text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$  and so is  $\text{MPL}(\rho)$ .  $\square$

*Proof (of Thm. 1).*  $\square$  Let  $v \in \text{value}(s)$ , then there is a strategy  $\sigma_{\exists}$  of **Eve**, such that for all memoryless strategy  $\sigma_{\forall} \in \mathbb{M}$  of **Adam**,  $\forall i \in I \cup J$ .  $\text{MP}_i(\text{Out}(\sigma_{\exists}, \sigma_{\forall})) \geq v_i$ . Let  $\rho = \text{Out}(\sigma_{\exists}, \sigma_{\forall})$ . We have that  $\text{Inf}(\rho) \subseteq C$  for some  $C \in \mathbb{C}(\sigma_{\forall})$ . Lem. 4 shows that  $\text{MP}(\rho) \in \downarrow \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$  which implies that  $v \in \downarrow \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$ .

As  $\text{MPL}(\rho) \geq \text{MP}(\rho)$ , we have that  $\text{MP}(\rho) \in \downarrow \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$ . Moreover, since  $\text{MP}(\text{Out}(\sigma_{\exists}, \sigma_{\forall})) \geq v$ , it follows that  $v \in \downarrow \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$ .

$\square$  Let  $v \in \bigcap_{\sigma_{\forall} \in \mathbb{M}} \bigcup_{C \in \mathbb{C}(\sigma_{\forall})} \downarrow \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$ . We fix a memoryless strategy  $\sigma_{\forall}$  of **Adam**. We have that  $v$  is in  $\downarrow \text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$  for some  $C \in \mathbb{C}(\sigma_{\forall})$ .

We replace the weights of the game by  $w'$  such that  $w'_i(s) = w_i(s) - v_i$ . We have for all path  $\rho$  that if the weight  $w(\rho)$  is greater than  $v$  then  $w'(\rho) \geq 0$ . We know there exists some  $v' \geq v$  of the form  $\sum_{k=1}^{|C|} \frac{\lambda_k}{|c_k|} \cdot w(c_k)$ . Hence there exists  $\lambda_1, \dots, \lambda_{|C|}$  such that  $\sum_{k=1}^{|C|} \frac{\lambda_k}{|c_k|} \cdot w'(c_k) \geq 0$ .

The idea is to construct for all  $n$  a finite path of length greater than  $n$  that has a total weight by  $w'$  close to 0. Let  $n$  be an integer, there exist  $p_k \in \mathbb{N}$  such that  $\left|\frac{p_k}{n} - \lambda_k\right| \leq \frac{1}{n}$ . Let  $l_k = \prod_{k' \neq k} |c_{k'}|$ , we construct the history  $h$  that follows  $p_1 \cdot l_1$  times the cycle  $c_1$ , then goes to the beginning of cycle  $c_2$  with a path of length at most  $|\text{Stat}|$  then follows  $p_2 \cdot l_2$  times the cycle  $c_2$  and so on until the  $|C|$ -th cycle.

First, lets look at the length of this path.

$$\begin{aligned} \sum_{k=1}^{|C|} \frac{p_k}{n} &\geq \sum_{k=1}^{|C|} \lambda_k - \frac{|C|}{n} \\ \sum_{k=1}^{|C|} p_k &\geq n \sum_{k=1}^{|C|} \lambda_k - |C| = n - |C| \end{aligned}$$

Hence  $|h| \geq n - |C|$ , hence it is not bounded when  $n$  grows toward infinity.



Let  $W'$  be the maximum weight that appears in  $w'$ . On dimension  $i$ , the total weight of  $h$  is such that:

$$\begin{aligned} \sum_{k=1}^{|C|} p_k \cdot l_k \cdot w'_i(c_k) - |\text{Stat}| \cdot |C| \cdot |W'| &\leq w'_i(h) \\ \prod_{k=1}^{|C|} |c_k| \cdot \sum_{k=1}^{|C|} p_k \cdot \frac{w'_i(c_k)}{|c_k|} - |\text{Stat}| \cdot |C| \cdot |W'| &\leq w'_i(h) \\ \prod_{k=1}^{|C|} |c_k| \cdot \sum_{k=1}^{|C|} \left( \lambda_k - \frac{1}{n} \right) \cdot \frac{w'_i(c_k)}{|c_k|} - |\text{Stat}| \cdot |C| \cdot |W'| &\leq w'_i(h) \\ -\frac{1}{n} \cdot \prod_{k=1}^{|C|} |c_k| \cdot \sum_{k=1}^{|C|} \frac{w'_i(c_k)}{|c_k|} - |\text{Stat}| \cdot |C| \cdot |W'| &\leq w'_i(h) \end{aligned}$$

Hence  $w'_i(h)$  is bounded on all dimension  $i$  when we make  $n$  grow.

By Lem. 10(2) of [11]<sup>1</sup> this implies there is a non negative cycle in  $C$  for the weight  $w'$ . By Lem. 11 of [11],  $\sigma_V$  is not a winning strategy. We know thanks to Theorem 8 of [11] that multi-mean payoff games are determined under memoryless strategies of **Adam**. Therefore **Adam** has no winning strategy. Since these games are also determined [8], this means that **Eve** has a winning strategy. Therefore  $v \in \underline{\text{value}}(s)$ .

### 3.3 Half-spaces representation

The goal of this section is to show that the equation:  $\underline{\text{value}}(s) \cap \text{hsp}(\lambda) \neq \emptyset$  can be solved using a relatively simple half-space representation (it might still be exponential compared with the size of the arena).

The set  $\text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$  is a convex hull and it is well known these can be written as finite intersection of half-spaces [9].

**Definition 16 (Polyhedron).** *The boundary of  $\text{hsp}(a, b)$  is the hyperplane:  $\text{bndr}(a, b) = \{x \in \mathbb{R}^d \mid \sum_{i \in [1, d]} a_i \cdot x_i = b\}$ . If  $P = \cap_{i=1}^k H_i$  where for all  $i$ ,  $H_i$  is a half-space,  $P$  is called a polyhedron. A bounded polyhedron is called a polytope. A face  $F$  of  $P$  is a subset of  $P$  of the form  $F = P \cap \text{bndr}(H_F)$ , where  $H_F$  is a half-space such that  $P \subseteq H_F$ . In that case, say that  $H_F$  defines face  $F$  of  $P$ . A face of dimension 1 is called a vertex. If  $P$  has dimension  $d'$ , then a face of dimension  $d' - 1$  is called a facet. A complete set of facet defining half-spaces  $\mathcal{F}$  contains for each facet  $F$  a half-space  $H_F$  such that  $P \cap \text{bndr}(H) = F$  and  $P \subseteq H$ .*

**Definition 17 (Affine Hull).** *A affine subspace is a set of the form  $x + L$  with  $x \in \mathbb{R}^d$  and  $L$  is a linear subspace of  $\mathbb{R}^d$  (i.e. a subset closed under addition of vectors and under multiplication by real numbers). The affine hull of a set*

<sup>1</sup> Note that in [11] the weights are integer, which may not be the case for us. However the proof of Lem. 10(2) still works in the case of arbitrary reals.

$X \subseteq \mathbb{R}^d$  is the intersection of all affine subspaces of  $\mathbb{R}^d$  containing  $X$ , it is written  $\mathbf{aff}(X)$ . It can be described as the set of linear combinations of points of  $X$  [9].

We prove a proposition, similar in spirit to Helly's theorem (see [9]), that we will use to restrict the search to certain points.

**Lemma 5.** *Let  $H_1, \dots, H_n$  be  $n$  half-spaces of  $\mathbb{R}^d$  with  $n \geq d \geq 1$ , and  $S$  be an affine subspace of  $\mathbb{R}^d$  of dimension  $k$ , with  $1 \leq k \leq d$ . If  $S \cap \bigcap_{i=1}^n H_i \neq \emptyset$  then there is  $i_1, \dots, i_k$  such that:*

$$S \cap \bigcap_{j=1}^k \mathbf{bndr}(H_{i_j}) \subseteq S \cap \bigcap_{i=1}^n H_i.$$

Notice in particular that this lemma applies to the case where  $S = \mathbb{R}^d$ .

*Proof.* We show the proposition by induction over  $k$ . If  $k = 0$ , then if  $S \cap \bigcap_{i=1}^n H_i \neq \emptyset$  then every  $H_i$  contain  $S$  since it is reduced to one point. Therefore  $S \cap \bigcap_{i=1}^{k+1} H_i = S = S \cap \bigcap_{i=1}^n H_i$ .

Assume now  $k > 0$  and that the property holds for all affine subspace of dimension  $k' < k$ . We proceed by induction over  $n$ . If  $n = k$ , then since  $\mathbf{bndr}(H_i) \subseteq H_i$  for all  $i$ , we have that  $\bigcap_{i=1}^k \mathbf{bndr}(H_i) \subseteq \bigcap_{i=1}^k H_i$  which shows the property.

Assume now that the property holds for  $n$ , we consider a half-space  $H_{n+1}$ .

– If  $S \subseteq \mathbf{bndr}(H_{n+1})$  then we use the induction hypothesis for  $n$ , we have

$$\begin{aligned} \bigcap_{j=1}^k (\mathbf{bndr}(H_{i_j}) \cap S) &\subseteq S \cap \bigcap_{i=1}^n H_i \\ &\subseteq S \cap \mathbf{bndr}(H_{n+1}) \cap \bigcap_{i=1}^n H_i \\ &\subseteq S \cap H_{n+1} \cap \bigcap_{i=1}^n H_i \\ &\subseteq S \cap \bigcap_{i=1}^{n+1} H_i \end{aligned}$$

– Otherwise  $\mathbf{bndr}(H_{n+1}) \cap S$  forms a affine subspace of dimension  $k - 1$ , to which we can apply the induction hypothesis.

• If  $S \cap \mathbf{bndr}(H_{n+1}) \cap \bigcap_{i=1}^n H_i \neq \emptyset$  then there is  $i_1, \dots, i_{k-1}$  such that:

$$\begin{aligned} (\mathbf{bndr}(H_{n+1}) \cap S) \cap \bigcap_{j=1}^{k-1} \mathbf{bndr}(H_{i_j}) &\subseteq S \cap \mathbf{bndr}(H_{n+1}) \cap \bigcap_{i=1}^n H_i \\ S \cap \bigcap_{j=1}^{k-1} \mathbf{bndr}(H_{i_j}) \cap \mathbf{bndr}(H_{n+1}) &\subseteq S \cap \bigcap_{i=1}^{n+1} H_i \end{aligned}$$

Which shows the property.

- Otherwise  $(\bigcap_{i=1}^n H_i \cap S) \cap \text{bndr}(H_{n+1}) = \emptyset$ , then since  $\bigcap_{i=1}^n H_i \cap S$  is convex it is either totally inside  $H_{n+1}$  or totally outside of it. If it is outside then the intersection is empty. Otherwise,  $\bigcap_{i=1}^n H_i \cap S \subseteq H_{n+1}$ . We apply the induction hypothesis over  $n$ : there is  $i_1, \dots, i_k$  such that:

$$\bigcap_{j=1}^k \text{bndr}(H_{i_j}) \cap S \subseteq \bigcap_{i=1}^n H_i \cap S \subseteq S \cap \bigcap_{i=1}^{n+1} H_i$$

Which proves the result.

**Theorem 2** ([6, Thm. 3.(b)]). *Let  $P$  be a polytope. Let  $\mathcal{F}$  be complete set of facet defining half-spaces, then  $P$  is the intersection of  $\text{aff}(P)$  with all half-spaces in  $\mathcal{F}$ .*

The set  $\text{conv}\left(\left\{\frac{1}{|c|} \cdot w(c) \mid c \in C\right\}\right)$  is a convex hull and can be written as finite intersection of half-spaces [9] but the number of half-spaces that is required can be exponential with respect to the dimension  $d$ : it is of the order of  $n^d$  where  $n$  is the number of points given as input. In the following, we will describe how these half-spaces are obtained.

**Definition 18.** *We write  $\pi_j(x)$  for the projection of  $x$  over the plane  $\{x \mid x_j = 0\}$ , that is  $\pi_j(x) = x'$  such that for all  $i \neq j$ ,  $x'_i = x_i$  and  $x'_j = 0$ . We also write  $\pi_J(x)$  for the projection of  $x$  over the plane  $\{x \mid \forall j \in J. x_j = 0\}$ . We write  $\downarrow_i X$  for  $\{v \mid \exists x \in X, \lambda \geq 0. v = x - \lambda \cdot \mathbf{1}_i\}$ . Let also  $\uparrow_K S$  be the set  $\{x \mid \exists x' \in S. \forall j \in [1, n] \setminus K. x'_j = x_j\}$ .*

**Lemma 6.** *Let  $P$  be a polyhedron, and  $\mathcal{F}(\pi_j(P))$  be a set of half-spaces of  $\text{aff}(\pi_j(P))$  such that  $\pi_j(P) = \bigcap_{H \in \mathcal{F}(\pi_j(P))} H$ .*

$$\uparrow_j P = \bigcap_{H \in \mathcal{F}(\pi_j(P))} H + \mathbb{R} \cdot \mathbf{1}_j$$

*Proof.*

$$\begin{aligned} x \in \uparrow_j P &\Leftrightarrow \exists x' \in P. \pi_j(x) = \pi_j(x') \\ &\Leftrightarrow \pi_j(x) \in \bigcap_{H \in \mathcal{F}(\pi_j(P))} H \\ &\Leftrightarrow \pi_j(x) + \mathbb{R} \cdot \mathbf{1}_j \subseteq \bigcap_{H \in \mathcal{F}(\pi_j(P))} H + \mathbb{R} \cdot \mathbf{1}_j \\ &\Leftrightarrow x \in \bigcap_{H \in \mathcal{F}(\pi_j(P))} H + \mathbb{R} \cdot \mathbf{1}_j \end{aligned}$$

□

**Lemma 7.** *Let  $P$  be a polyhedron and  $\mathcal{F}(P)$  be set of half-spaces such that  $P = \bigcap_{H \in \mathcal{F}(P)} H$ . Let  $\mathcal{F}_-^j(P) \subseteq \mathcal{F}(P)$  be the set of half-spaces of  $\mathcal{F}(P)$  that contain the direction  $-\mathbf{1}_j$ , i.e.  $H - \mathbb{R}^+ \cdot \mathbf{1}_j = H$ .*

$$\downarrow_j P = \uparrow_j P \cap \bigcap_{H \in \mathcal{F}_-^j(P)} H$$

*Proof.*  $\square$  Let  $x \in \downarrow_j P$ . We have that  $x \in \uparrow_j P$ . There is  $x' \in P$  such that  $x = x' - \lambda \cdot \mathbf{1}_j$  with  $\lambda \geq 0$ . Let  $f \in \mathcal{F}_-^j$ , since  $x' \in H_f$  we have that  $x \in H_f - \mathbb{R}^+ \cdot \mathbf{1}_j = H_f$ . Therefore  $x \in \downarrow_j P \cap \bigcap_{f \in \mathcal{F}_-^j} H_f$ .

$\square$  Let  $x \in \uparrow_j P \cap \bigcap_{f \in \mathcal{F}_-^j} H_f$ . There exists  $x' \in P$  such that  $\pi_j(x) = \pi_j(x')$ . Consider the line segment  $[x, x']$ , it does not intersect any  $\mathbf{bndr}(H_f)$  with  $f \in \mathcal{F}_-^j$  since both  $x$  and  $x'$  are inside these half-spaces.

If  $[x, x']$  does not intersect any  $\mathbf{bndr}(H_f)$  where  $f$  is a facet of  $P$  then  $x$  is in the intersection of these half-spaces, and using Thm. 2, this means that  $x \in P$ .

Otherwise  $[x, x']$  intersects some  $\mathbf{bndr}(H_f)$  with  $f$  facet of  $P$  which is not in  $\mathcal{F}_-^j$ . Since  $H_f$  contains  $x'$  but not  $x$ , and these two only differs by their  $j$  coordinates, this means that  $x_j \leq x'_j$  and therefore  $x \in \downarrow_j P$ .  $\square$

We write  $B_J$  for the subspace  $\sum_{j \in J} \mathbb{R} \cdot \mathbf{1}_j$  and  $C_J$  for the cone  $\sum_{j \in J} \mathbb{R}_+ \cdot \mathbf{1}_j$ . If  $\mathcal{F}$  is a set of half-spaces, we write  $\mathcal{F}_-^J \subseteq \mathcal{F}$  the half-spaces of  $\mathcal{F}$  that contain the direction  $-C_J$ , i.e.  $H - C_J = H$ .

**Lemma 8.**

$$\downarrow_K P = \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_J(P))} H + B_J$$

with the convention that  $\pi_\emptyset(P) = P$ ,  $\mathcal{F}_-^\emptyset(P) = \mathcal{F}(P)$  and  $B_\emptyset = \mathbf{0}$ .

*Proof.* We prove the result by induction over the size of  $K$ . The case where  $|K| = 1$  is a direct consequence of Lem. 6 and 7.

Assume the equality holds for some set  $K$ . Consider a dimension  $j \notin K$ .

$$\pi_j(\downarrow_K P) = \downarrow_K \pi_j(P) = \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))} H + B_J$$

So we can chose:

$$\mathcal{F}(\pi_j(\downarrow_K P)) = \bigcup_{J \subseteq K} \{H + B_J \mid H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))\} \quad (1)$$

We also have by induction hypothesis:

$$\mathcal{F}(\downarrow_K P) = \bigcup_{J \subseteq K} \{H + B_J \mid H \in \mathcal{F}_-^{K \setminus J}(\pi_J(P))\}$$

$$\mathcal{F}_-^j(\downarrow_K P) = \bigcup_{J \subseteq K} \{H + B_J \mid H \in \mathcal{F}_-^{K \cup \{j\} \setminus J}(\pi_J(P))\} \quad (2)$$

Using Lem. 6 and 7:

$$\begin{aligned} \downarrow_{K \cup \{j\}} P &= \bigcap_{H \in \mathcal{F}(\pi_j(\downarrow_K P))} H + B_j \cap \bigcap_{H \in \mathcal{F}_-^j(\downarrow_K P)} H \\ &= \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))} H + B_J + B_j \cap \bigcap_{H \in \mathcal{F}_-^j(\downarrow_K P)} H \quad \text{By (1)} \\ &= \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \setminus J}(\pi_{J \cup \{j\}}(P))} H + B_{J \cup \{j\}} \cap \bigcap_{J \subseteq K} \bigcap_{H \in \mathcal{F}_-^{K \cup \{j\} \setminus J}(\pi_J(P))} H + B_J \quad \text{By (2)} \\ &= \bigcap_{J \subseteq K \cup \{j\}} \bigcap_{H \in \mathcal{F}_-^{K \cup \{j\} \setminus J}(\pi_J(P))} H + B_J \end{aligned}$$

Which proves the property for  $K \cup \{j\}$  and concludes the induction.  $\square$

**Corollary 1.**

$$\downarrow P = \bigcap_{J \subseteq [1, d]} \bigcap_{H \in \mathcal{F}_-^{[1, d] \setminus J}(\pi_J(P))} H + B_J$$

### 3.4 Algorithm

In order to solve the value problem, our algorithm is going to guess a vector of a certain form which satisfies the linear equation and check that it belong to value. For this last step, we reuse an algorithm from [11], which is coNP as we recall here.

**Theorem 3.** [11, Thm. 7.2] *For multi-weighted games with objective  $\{\rho \mid \underline{\text{MP}}(\rho) \geq (0, \dots, 0)\}$ , the problem of deciding whether a given state is winning for Eve is coNP-complete.*

**Corollary 2.** *Given a vector  $x \in \mathbb{R}^d$  and a state  $s$ , we can decide with a coNP algorithm whether  $x \in \underline{\text{value}}(s)$ .*

*Proof.* This is done by subtracting  $x$  from all the weights and using the preceding algorithm to decide if  $s$  is winning for the objective  $\{\rho \mid \underline{\text{MP}}(\rho) \geq (0, \dots, 0)\}$ .

$\underline{\text{value}}(s) \cap \text{hsp}(\lambda) \neq \emptyset$  if, and only if, there is a function  $f: \mathbb{M} \mapsto \mathbb{C}$ , such that  $f(\sigma_v) \in \mathbb{C}(s, \sigma_v)$  for all strategy  $\sigma_v$  and

$$\text{hsp}(\lambda) \cap \bigcap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(\sigma_v) \right\} \right) \neq \emptyset \quad (3)$$

By Corollary 1,  $\downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(\sigma_v) \right\} \right)$  is the intersection of the half-spaces in  $\mathcal{H}^{f(\sigma_v)}$ , where for a set of simple cycles  $C$ :

$$\mathcal{H}^C = \left\{ H + B_J \mid J \subseteq [1, d] \wedge H \in \mathcal{F}_-^{[1, d] \setminus J}(\pi_J(P)) \wedge P = \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right) \right\}$$

**Lemma 9.** *Let  $\mathcal{H} = \{\text{hsp}(\lambda)\} \cup \bigcup_{\sigma_v \in \mathbb{M}} \mathcal{H}^{f(\sigma_v)}$ . The three following propositions are equivalent:*

1.  $\underline{\text{value}}(s) \cap \text{hsp}(\lambda) \neq \emptyset$ ;
2. there exists  $d$  half-spaces  $H_1, \dots, H_d$  in  $\mathcal{H}$  such that  $\bigcap_{1 \leq i \leq d} \text{bndr}(H_i) \neq \emptyset$  and  $\exists x \in \bigcap_{1 \leq i \leq d} \text{bndr}(H_i)$ ,  $x \in \underline{\text{value}}(s) \cap \text{hsp}(\lambda)$ .
3. there exists  $d$  half-spaces  $H_1, \dots, H_d$  in  $\mathcal{H}$  such that  $\bigcap_{1 \leq i \leq d} \text{bndr}(H_i) \neq \emptyset$  and  $\forall x \in \bigcap_{1 \leq i \leq d} \text{bndr}(H_i)$ ,  $x \in \underline{\text{value}}(s) \cap \text{hsp}(\lambda)$ .

*Proof.* The implications (3)  $\implies$  (2)  $\implies$  (1) are obvious.

We now prove the implication (1)  $\implies$  (3). By Thm. 1

$$\begin{aligned} \underline{\text{value}}(s) &= \bigcap_{\sigma_V \in \mathbb{M}} \bigcup_{C \in \mathbb{C}(s, \sigma_V)} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right) \\ &= \bigcup_{f: \mathbb{M} \rightarrow \mathbb{C}} \bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right) \end{aligned}$$

The intersection of  $\text{hsp}(\lambda)$  with  $\underline{\text{value}}(s)$  is non-empty if, and only if, there exists  $f: \mathbb{M} \rightarrow \mathbb{C}$ , such that  $\text{hsp}(\lambda) \cap \bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(\sigma_V) \right\} \right)$  is non-empty. By Corollary 1,  $\downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(\sigma_V) \right\} \right)$  is the intersection of the half-spaces in  $\mathcal{H}^{f(\sigma_V)}$ . So  $\text{hsp}(\lambda) \cap \bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(\sigma_V) \right\} \right)$  is non-empty if and only if  $\bigcap_{H \in \mathcal{H}^{f(\sigma_V)}} H$  is non empty. By Lem. 5 this is the case if and only if there is  $d$  half-spaces  $H_1, \dots, H_d$  in  $\mathcal{H}^{f(\sigma_V)}$  such that  $\bigcap_{1 \leq i \leq d} \text{bndr}(H_i) \subseteq \text{hsp}(\lambda) \cap \bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(\sigma_V) \right\} \right)$ . This implies  $\bigcap_{1 \leq i \leq d} \text{bndr}(H_i) \subseteq \text{hsp}(\lambda) \cap \underline{\text{value}}(s)$  which shows that (3) holds.  $\square$

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**Algorithm 1:** Algorithm for the value problem in multi-mean-payoff

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**Existentially guess  $d$  half-space  $H_1, \dots, H_d \in \mathcal{H}$**   
**if  $\bigcap_{j=1}^d \text{bndr}(H_j) = \emptyset$  then return false;**  
**else take  $x$  one arbitrary point in  $\bigcap_{j=1}^d \text{bndr}(H_j)$**   
**if  $x \notin \text{hsp}(\lambda)$  then return false;**  
**if  $x \in \text{value}(s)$  then return true**

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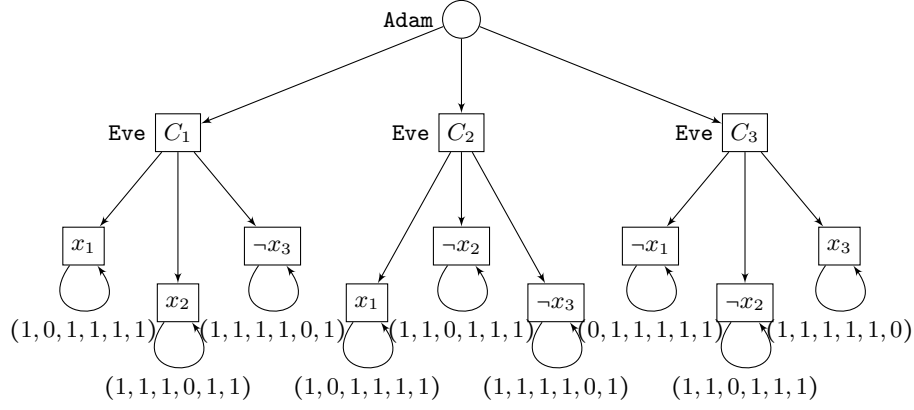
Based on this property we devise Algorithm 1 to decide the value problem. Its correctness holds from Lem. 9. Algorithm 1 is in  $\Sigma_2\text{P} = \text{NP}^{\text{NP}} = \text{NP}^{\text{coNP}}$ , since checking whether  $x \in \text{value}(s)$  can be done in  $\text{coNP}$  (by Corollary 2) and computing the intersection of hyperplanes can be done in polynomial time [10, Thm. 3.3]. We obtain from this, the following theorem.

**Theorem 4.** *The threshold problem is in  $\Sigma_2\text{P}$  for limit inferior objectives.*

### 3.5 Hardness

**Proposition 1.** *The threshold problem is NP-hard for limit inferior objectives.*

*Proof.* Let  $\phi = \exists x_1. \exists x_2. \dots \exists x_n. C_1 \wedge \dots \wedge C_n$  be a formula, where  $C_i = \ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3}$  with  $\ell_{i,j} \in \{x_k, \neg x_k \mid k \in \llbracket 1, n \rrbracket\}$ . We have two dimensions for each variable  $x_i$ , so  $d = 2 \cdot n$ . We write  $0_i$  for the vector  $v$  such that  $v_i = 0$  and  $v_j = 1$  for  $j \neq i$ . We define a family of vectors  $v_{i,j}$  where if  $\ell_{i,j} = x_k$  then  $v_{i,j} = 0_{2 \cdot k + 1}$  and if  $\ell_{i,j} = \neg x_k$  then  $v_{i,j} = 0_{2 \cdot k}$ . We have one initial state  $s_0$  controlled by Adam, one state for



**Figure 1.** Example of the encoding of 3SAT into the threshold problem, for formula  $\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$ .

each clause  $C_i$  controlled by **Eve**, and one state for each literal of the formula. The construction is illustrated in Figure 1.

We consider the constraints  $\lambda_i = (v_{2 \cdot i} + v_{2 \cdot i + 1} \geq 1)$ .

In the example, valuation  $x_1 \rightarrow \text{true}, x_2 \rightarrow \text{false}, x_3 \rightarrow \text{true}$  makes the formula valid, the corresponding value is  $v = (1, 0, 0, 1, 1, 0)$ . Consider the strategy of **Eve** that in  $C_1$  chooses  $x_1$ , in  $C_2$  chooses  $x_1$  and in  $C_3$  chooses  $\neg x_2$ . In all the cases this strategy ensures to **Eve** a payoff better than  $v$ . In contrast, valuation  $x_1 \rightarrow \text{false}, x_2 \rightarrow \text{false}, x_3 \rightarrow \text{true}$  does not make the formula valid, the corresponding value is  $v' = (0, 1, 0, 1, 1, 0)$ , and if **Adam** chooses  $C_1$  then **Eve** cannot ensure  $v'$ .

Let  $\xi$  be a valuation, we associate to it a vector  $v_\xi$  such that if  $\xi(x_i) = \text{true}$  then  $(v_\xi)_{2 \cdot i} = 1$  and  $(v_\xi)_{2 \cdot i + 1} = 0$ ; if  $\xi(x_i) = \text{false}$  then  $(v_\xi)_{2 \cdot i} = 0$  and  $(v_\xi)_{2 \cdot i + 1} = 1$ . We prove that if  $\xi$  makes  $\phi$  valid then **Eve** can ensure  $v_\xi$ .

In the other direction, assume  $v_\xi$  makes  $\phi$  valid. We consider a strategy for **Eve** that in each  $C_i$  chooses a state  $\ell_{i,j}$  such that  $\xi$  makes  $\ell_{i,j}$  true, this is possible because  $v_\xi$  makes  $\phi$  valid. If  $\ell_{i,j} = x_k$ , then  $\xi(x_k) = \text{true}$  and  $(v_\xi)_{2 \cdot k + 1} = 0$ . The payoff is then  $0_{2 \cdot k + 1}$  which is greater than  $v_\xi$ . Similarly, if  $\ell_{i,j} = \neg x_k$ , the payoff  $0_{2 \cdot k}$  is greater than  $v_\xi$ . This shows that **Eve** has a strategy to ensure  $v_\xi$ .

We can already conclude that if  $\phi$  is satisfiable then **Eve** can ensure a value that satisfies the equations in  $\lambda$ .

Now, reciprocally, assume there is some  $v$  that satisfies  $\lambda$  and that can be ensured by **Eve**. For each  $k \in \llbracket 1, n \rrbracket$ ,  $v_{2 \cdot k} > 0$  or  $v_{2 \cdot k + 1} > 0$ . We select a valuation  $\xi_v$  such that  $\xi_v(x_k) = \text{true} \Leftrightarrow v_{2 \cdot k} > 0$ . We prove that if **Eve** ensures  $v$  then  $\xi_v$  makes  $\phi$  valid.

Assume **Eve** ensures  $v$ . This means that from every  $C_i$ , she can find a  $\ell_{i,j}$  such that  $v_{i,j} \geq v$ . If  $\ell_{i,j} = x_k$ , then  $v_{i,j} = 0_{2 \cdot k + 1}$ , hence  $(v_\xi)_{2 \cdot k + 1} = 0$ . Therefore  $\xi_v(x_k) = \text{true}$ . Similarly if  $\ell_{i,j} = \neg x_k$ , then  $\xi_v(x_k) = \text{false}$ . This means that in each  $C_i$ , we can find a literal made true by  $\xi_v$  and therefore  $\xi_v$  makes  $\phi$  valid.

This proves that  $\phi$  is satisfiable if, and only if, Eve can ensure a value that satisfies the equations in  $\lambda$ .  $\square$

## 4 General case

### 4.1 Characterizing values

**Proposition 2.** *If  $J \neq \emptyset$ , then:*

$$\text{value}(s) = \left( \bigcap_{j \in J} \downarrow_{J \setminus \{j\}} \bigcap_{\sigma_V \in \mathbb{M}} \bigcup_{C \in \mathbb{C}(\sigma_V)} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right) \right)$$

In order to prove this result, we will reuse some properties of multi-mean-payoff games that were proved in [11].

**Lemma 10 ([11] Lem. 17).**

$$\begin{aligned} & \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \forall j \in J. \overline{\text{MP}}_j(\rho) \geq v_j \\ \Leftrightarrow & \forall j \in J. \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \overline{\text{MP}}_j(\rho) \geq v_j. \end{aligned}$$

**Lemma 11 ([11] Lem. 14).** *Let  $j \in J$ .*

$$\begin{aligned} & \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \overline{\text{MP}}_j(\rho) \geq v_j \\ \Leftrightarrow & \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \text{MP}_j(\rho) \geq v_j. \end{aligned}$$

We now proceed to the proof of the proposition.

*Proof (of Prop. 2).* Let  $v$  be a vector of  $\mathbb{R}^{2 \cdot d}$ .

$$\begin{aligned} v \in \text{value}(s) & \Leftrightarrow \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \forall j \in J. \overline{\text{MP}}_j(\rho) \geq v_j \\ & \Leftrightarrow \forall j \in J. \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \overline{\text{MP}}_j(\rho) \geq v_j \text{ (by Lem. 10)} \\ & \Leftrightarrow \forall j \in J. \exists \sigma_{\exists}. \forall \sigma_{\forall}. \forall i \in I. \text{MP}_i(\rho) \geq v_i \text{ and } \text{MP}_j(\rho) \geq v_j \text{ (by Lem. 11)} \\ & \Leftrightarrow \forall j \in J. v \in \downarrow_{J \setminus \{j\}} \underline{\text{value}}(s) \\ & \Leftrightarrow v \in \left( \bigcap_{j \in J} \downarrow_{J \setminus \{j\}} \underline{\text{value}}(s) \right) \\ & \Leftrightarrow v \in \downarrow_{J \setminus \{j\}} \bigcap_{\sigma_V \in \mathbb{M}} \bigcup_{C \in \mathbb{C}(\sigma_V)} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right) \text{ (by Thm. 1)} \end{aligned}$$

### 4.2 Algorithm

In the following proposition, the goal is to express the value problem in terms of intersection of convex sets in a space of dimension  $d = |I| + |J|$ . The number of such convex sets is exponential but if the intersection is not empty, there is a witness whose coordinates are expressible as a small rational. This leads the way toward a non-deterministic polynomial algorithm.



**Definition 19 (Sizes).** The size of a rational number  $r = \frac{p}{q}$  where  $p \in \mathbb{Z}, q \in \mathbb{Z}$ ,  $p$  and  $q$  are relatively prime, is:  $\|r\| = 1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(q + 1) \rceil$ . The size of a vector  $v = (r_1, \dots, r_d)$  is  $\|v\| = d + \sum_{i \in [1, d]} \|r_i\|$ . The size of a matrix  $A = (a_{ij})_{i=1, j=1}^d \times^n$  is  $\|A\| = m \cdot n + \sum_{i, j \in [1, d] \times [1, n]} \|a_{ij}\|$ .

**Theorem 5.**  $\text{value}(s) \cap [-W, W]^d$  is of the form  $\cup_{f \in F} \text{conv}(E_f)$  where  $F$  is finite and vertices of  $E_f$  have polynomially bounded size.

In order to prove this theorem, we will first state several intermediary propositions.

**Proposition 3 ([10, Corollary 3.2b]).** If the system  $Ax = b$  of rational linear equations has a solution, it has one of size polynomially bounded by the sizes of  $A$  and  $b$ .

**Proposition 4.** Given  $n$  hyperplanes  $H_1 = \text{hp}(a_1, b_1), \dots, H_n = \text{hp}(a_n, b_n)$ , if their intersection is non empty we can find a point in the intersection expressed by  $\frac{\alpha}{\beta}$  with  $\alpha, \beta$  polynomially bounded with respect to the size of the equations  $\sum_{j \in [1, n]} |b_j| + \sum_{i \in [1, d]} |a_{i, j}|$ .

*Proof.* Points at the intersection are the solutions of the system

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} X = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Then Prop. 3 guarantees that we can find a solution with polynomially bounded size.

**Proposition 5.** Let  $P$  be a polyhedron and  $\mathcal{F}$  be a complete set of facet defining half-spaces. If  $v$  is a vertex of  $P$ , then  $v$  is the intersection of at most  $d + 1$  boundary of half-spaces in  $\mathcal{F}$ , i.e. there is  $H_1, \dots, H_m \in \mathcal{F}$  with  $m \leq d + 1$  such that  $\{v\} = \cap_{j \in [1, m]} \text{bndr}(H_j)$ .

*Proof.* A vertex is the intersection of the facets containing it [9, Chapter 5]. Let  $v$  be a vertex of  $P$  and let  $F_v \subset \mathcal{F}$  be the set of facet defining half-spaces, corresponding to facets containing  $v$ . We have that  $\cap_{H \in F_v} \text{bndr}(H) = \{v\}$ . We then use Prop. 5, to conclude that  $d$  half-spaces are enough.

**Proposition 6.** Let  $V_1, \dots, V_n$  be set of points, and let  $P = \cap_{i=1}^n \text{conv}(V_i)$ . Then  $P = \text{conv}(X)$  where  $X$  is a finite set of points whose size of coordinates are polynomial with respect to  $d$  and size coordinates of points of  $\cup_i V_i$ .

*Proof.* By Prop. 5, vertices of  $P$  are at the intersection of at most  $d+1$  boundaries of facet defining half-spaces of  $P$ . These half-spaces can be taken as a subset of facet defining half-spaces of  $\{V_i \mid i \in [1, n]\}$ .

We first show that equation describing these half-spaces are not too complicated. Facets of  $\text{conv}(V_i)$  are given by taking at most  $d + 1$ , vertices of  $\text{conv}(V_i)$ .

If  $H$  is given by vertices  $v_1, \dots, v_d$  of  $V_i$ , then  $H = \{v_1 + \lambda_2 \cdot (v_2 - v_1) + \dots + \lambda_{d-J} \cdot (v_{d-J} - v_1) \mid \lambda_2, \dots, \lambda_d \in \mathbb{R}\}$ . This is equivalently described by an equation  $(l_1, l_2, \dots, l_d)$  where  $l_j$  is a solution of the equation  $\begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} l_j^T = \mathbf{1}_j^T$ . Note that this is only valid when  $v_1, \dots, v_d$  are linearly independent. Using Prop. 3, coordinates in this equation are polynomial with respect to the coordinates of vertices  $v_1, \dots, v_d$ .

Now, vertices of  $P$  are at the intersection of at most  $d$  of these hyperplanes. Therefore, using Prop.3, there coordinates are polynomially bounded.  $\square$

**Proposition 7.**  $[-W, W]^d \cap \bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_V) \right\} \right)$  is the convex hull of a finite set  $V(j, \sigma_V)$  where all points of  $V(j, \sigma_V)$  have coordinate polynomial with respect to  $W$ ,  $d$  and  $|\text{Stat}|$ .

*Proof.* By Corollary 1:

$$\bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_V) \right\} \right) = \bigcap_{\sigma_V \in \mathbb{M}, H \in \mathcal{H}^f(\sigma_V)} H$$

We also have  $[-W, W]^d$  as the intersection of half-spaces in  $\{F_i \mid i \in \llbracket 1, d \rrbracket\} \cup \{G_i \mid i \in \llbracket 1, d \rrbracket\}$  where  $F_i = \{x \in \mathbb{R}^d \mid x_i \geq -W\}$  and  $G_i = \{x \in \mathbb{R}^d \mid x_i \leq W\}$ . As a consequence, the set  $[-W, W]^d \cap \bigcap_{\sigma_V \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_V) \right\} \right)$  the intersection of half-spaces and it is bounded, thus it is a convex polytope. Therefore, it is the convex hull of its set of vertices  $V$  [9, Prop.5.3.2]. Moreover  $\mathcal{F} = \mathcal{H} \cup \bigcup_{i \in \llbracket 1, d \rrbracket} F_i \cup \bigcup_{i \in \llbracket 1, d \rrbracket} G_i$  is a complete set of facet defining half-spaces where  $\mathcal{H} = \bigcup_{\sigma_V \in \mathbb{M}} \mathcal{H}^f(\sigma_V)$ . Vertices in  $V$  are at the intersection of at most  $d + 1$  hyperplanes in  $\mathcal{F}$  by Prop. 5. Therefore by Prop. 3 their coordinates have size polynomially bounded by the size of  $W$  and coefficients appearing in the equations of half-spaces in  $\mathcal{H}$ .

The half-spaces of  $\mathcal{H}$  are of the form  $H + B_J$  where  $J \subseteq \llbracket 1, d \rrbracket$  and  $H$  belong to  $\mathcal{F}(\pi_J(\text{conv}(\frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_V))))$ . Vertices of the projection of the convex hull, so equation for these hyperplanes are given by  $d - |J|$  vertices of the projection. If  $H$  is given by vertices  $v_1, \dots, v_{d-J}$  of the projection, then  $H = \{v_1 + \lambda_2 \cdot (v_2 - v_1) + \dots + \lambda_{d-J} \cdot (v_{d-J} - v_1) \mid \lambda_2, \dots, \lambda_{d-J} \in \mathbb{R}\}$ . This is equivalently described by an equation  $(l_1, l_2, \dots, l_d)$  where  $l_i$  is a solution

of the equation  $\begin{pmatrix} v_1 \\ \vdots \\ v_{d-J} \end{pmatrix} l_i^T = \mathbf{1}_i^T$ . Note that this is only valid when  $v_1, \dots, v_{d-J}$  are

linearly independent. Using Prop. 3, coordinates in this equation are polynomial with respect to the coordinates of vertices  $v_1, \dots, v_{d-J}$  which are themselves of the form  $\frac{1}{|c|} \cdot w(c)$  for some  $c \in \mathbb{C}$  or equal to 0.

We can therefore conclude that coordinates of vectors in  $V$  are polynomial with respect to  $W$  and  $|\text{Stat}|$ .  $\square$

**Proposition 8.**  $[-W, W]^d \cap \downarrow_{J \setminus \{j\}} \cap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right) = [-W, W]^d \cap \downarrow_{J \setminus \{j\}} [-W, W]^d \cap \cap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$

*Proof.* Let  $v \in [-W, W]^d \cap \downarrow_{J \setminus \{j\}} \cap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$  then for all  $j \in J$  there is some  $v^j$  such that  $v^j \in \cap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$  and for all coordinates  $k \in J \setminus \{j\}$ ,  $v_k^j = v_k$ . We take a vector  $v'_k = -W$  if  $k \in J \setminus \{j\}$  and  $v_k$  otherwise. Obviously  $v'$  belongs to  $[-W, W]^d$ .

We show that  $v'$  also belongs to  $\downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$ . There is  $x \geq v^j$  such that  $x \in \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$ . We have that for all coordinate  $k$ ,  $x_k \geq -W$ . Hence, we have that  $v' \leq x$  and therefore it belongs to  $\downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$ .

Hence  $[-W, W]^d \cap \downarrow_{J \setminus \{j\}} [-W, W]^d \cap \cap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$  contains  $v$ . The inclusion in the other direction being trivial, this shows the equality.  $\square$

**Proposition 9.** *If  $Y \subseteq [-W, W]^d$  then  $[-W, W]^d \cap \downarrow_J \text{conv}(Y) = \text{conv}(Z)$  where  $Z = \{z \mid \exists y \in Y. \forall i \notin J. z_i = y_i \wedge \forall i \in J. z_i \in [-W, W]\}$ .*

*Proof.*  $\square$  Let  $x \in [-W, W]^d \cap \downarrow_J \text{conv}(Y)$ , there is  $y \in \text{conv}(Y)$  such that for all  $i \notin J$ ,  $x_i = y_i$ .

For  $j \in J$ , we define  $\mu_j = \frac{x_j + W}{2 \cdot W}$ , notice that  $\mu_j \in [0, 1]$  since  $x_j \in [-W, W]$ . We then define for  $v \in \{0, 1\}^{|J|}$  the coefficient  $\lambda_v = \prod_{v_j=1} \mu_j \prod_{v_j=0} (1 - \mu_j)$ . We have that for all  $v$ ,  $\lambda_v \in [0, 1]$  since this is the cause of all  $\mu_j$ . Moreover  $\sum_{v \in \{0, 1\}^{|J|}} \lambda_v = 1$ , this is proved by a simple induction:

- if  $|J| = 1$ , then let  $\{j\} = J$ ,  $\sum_{v \in \{0, 1\}^{|J|}} \lambda_v = \mu_j + (1 - \mu_j) = 1$ ;
- if  $|J| > 1$ , then let  $k \in J$ :

$$\begin{aligned} \sum_{v \in \{0, 1\}^{|J|}} \lambda_v &= \mu_k \cdot \sum_{v|v_k=1} \frac{\lambda_v}{\mu_k} + (1 - \mu_k) \cdot \sum_{v|v_k=0} \frac{\lambda_v}{1 - \mu_k} \\ &= \mu_k \cdot \sum_{v|v_k=1} \prod_{v_j=1|j \neq k} \mu_j \prod_{v_j=0|j \neq k} (1 - \mu_j) \\ &\quad + (1 - \mu_k) \cdot \sum_{v|v_k=0} \prod_{v_j=1|j \neq k} \mu_j \prod_{v_j=0|j \neq k} (1 - \mu_j) \\ &= \mu_k + (1 - \mu_k) \quad (\text{By induction hypothesis}) \\ &= 1 \end{aligned}$$

We can also prove that  $\sum_{v \in \{0, 1\}^{|J|}} \lambda_v \cdot v = (\mu_1, \dots, \mu_j)$ : let  $k \in J$ :

$$\begin{aligned} \left( \sum_{v \in \{0, 1\}^{|J|}} \lambda_v \cdot v \right)_k &= \sum_{v \in \{0, 1\}^{|J|} | v_k=1} \lambda_v \\ &= \mu_k \cdot \sum_{v \in \{0, 1\}^{|J|} | v_k=1} \frac{\lambda_v}{\mu_k} \\ &= \mu_k \end{aligned}$$

Lets write  $y = \sum_{j=1}^n \nu_j \cdot y^j$  with  $\nu_j \in [0, 1]$ ,  $\sum_{j=1}^n \nu_j = 1$ , and  $y^j \in Y$ . Then given  $v \in \{0, 1\}^{|J|}$ , we write  $z(j, v)$  for  $z$  such that  $\forall i \notin J. z_i = y_i^j$  and  $\forall i \notin J. z_i \in -W$  if  $v_i = 0$  and  $z_i = W$  if  $v_i = 1$ . We have that  $\pi_J(z(j, v)) = \pi_J(y^j)$ . Now, let  $x' = \sum_{j \in J, v \in \{0, 1\}^{|J|}} \nu_j \cdot \lambda_v \cdot z(j, v)$ .

– If  $i \notin J$ , then:

$$\begin{aligned} x'_i &= \sum_{j \in J, v \in \{0, 1\}^{|J|}} \nu_j \cdot \lambda_v \cdot y_i^j \\ &= \sum_{j \in J} \nu_j \cdot y_i^j = y_i = x_i. \end{aligned}$$

– If  $i \in J$ , then:

$$\begin{aligned} x'_i &= \sum_{j \in J, v \in \{0, 1\}^{|J|}} \nu_j \cdot \lambda_v \cdot z(j, v) \\ &= \sum_{j \in J} \nu_j \cdot \left( \sum_{v \in \{0, 1\}^{|J|} | v_i = 1} \lambda_v \cdot W - \sum_{v \in \{0, 1\}^{|J|} | v_i = 0} \lambda_v \cdot W \right) \\ &= \sum_{j \in J} \nu_j \cdot \left( -W + 2 \cdot W \cdot \left( \sum_{v \in \{0, 1\}^{|J|} | v_i = 1} \lambda_v \right) \right) \\ &= \sum_{j \in J} \nu_j \cdot (-W + 2 \cdot W \cdot \mu_i) \\ &= \sum_{j \in J} \nu_j \cdot (-W + x_i + W) \\ &= \sum_{j \in J} \nu_j \cdot x_i \\ &= x_i \end{aligned}$$

Hence  $x = x'$  which shows that  $x$  is in  $\text{conv}(Z)$ .

$\square$  All points  $z \in Z$  are in  $[-W, W] \cap \downarrow_{J \setminus \{j\}} \cap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$  therefore their convex hull also is.  $\square$

*Proof (of Thm. 5).* By Prop. 2:

$$\text{value}(s) = \bigcap_{j \in J} \downarrow_{J \setminus \{j\}} \bigcap_{\sigma_v \in \mathbb{M}} \bigcup_{C \in \mathbb{C}(\sigma_v)} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in C \right\} \right).$$

We can then reorder intersection and union operators by considering functions that selects for pairs  $(j, \sigma_v)$  a simple cycle among  $\mathbb{C}(\sigma_v)$ , i.e. functions of  $F$ , where  $F = \{f: J \times \mathbb{M} \mapsto \mathbb{C} \mid \forall j \in J. \forall \sigma_v \in \mathbb{M}. f(j, \sigma_v) \in \mathbb{C}(\sigma_v)\}$ .

We have:

$$\text{value}(s) = \bigcup_{f \in F} \bigcap_{j \in J} \downarrow_{J \setminus \{j\}} \bigcap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$$

$$\begin{aligned}
\text{value}(s) \cap [-W, W]^d &= [-W, W]^d \cap \bigcup_{f \in F} \bigcap_{j \in J} \downarrow_{J \setminus \{j\}} \bigcap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right) \\
&= \bigcup_{f \in F} \bigcap_{j \in J} [-W, W]^d \cap \downarrow_{J \setminus \{j\}} \bigcap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)
\end{aligned}$$

Using Prop. 8, this equals to:

$$\bigcup_{f \in F} \bigcap_{j \in J} [-W, W]^d \cap \downarrow_{J \setminus \{j\}} [-W, W]^d \cap \bigcap_{\sigma_v \in \mathbb{M}} \downarrow \text{conv} \left( \left\{ \frac{1}{|c|} \cdot w(c) \mid c \in f(j, \sigma_v) \right\} \right)$$

Using Prop. 7, this equals to

$$\bigcup_{f \in F} \bigcap_{j \in J} [-W, W]^d \cap \downarrow_{J \setminus \{j\}} \text{conv}(V(f, j))$$

Where all points of  $\cup_{f \in F, j \in J} V(f, j)$  have their coordinates polynomially bounded.

Using Prop. 9,  $[-W, W]^d \cap \downarrow_{J \setminus \{j\}} \text{conv}(Y(f, j)) = \text{conv}(Z(j, f))$  and points of  $Z(j, f)$  also have their coordinates polynomially bounded.

Then using Prop. 6, the vertices of  $\bigcap_{j \in J} [-W, W]^d \cap \downarrow_{J \setminus \{j\}} \text{conv}(V(f, j))$  have polynomially bounded coordinates. This concludes the proof of Thm. 5.  $\square$

**Theorem 6.** *Let  $S$  be a set of states and  $\lambda$  a system of linear equations.  $\bigcap_{s \in S} \text{value}(s) \cap [-W, W]^d \cap \text{hsp}(\lambda)$  is of the form  $\bigcup_{f \in F} \text{conv}(E_f)$  where  $F$  is finite and vertices of  $E_f$  have polynomially bounded size.*

*Proof.* This is a consequence of Thm. 5 and Prop. 4.

We write  $X$  for the set of points that have coordinates polynomially bounded by the same polynomial as for Thm. 6 We give a  $\Sigma_2\text{P}$  algorithm (Algorithm 2) that given a set  $S$  of states, a constraint  $\lambda$  on the payoffs as a system of linear equation, solves the region problem.

**Theorem 7.** *The region problem is in  $\Sigma_2\text{P}$ .*

*Proof.* The correctness of the Algorithm 2 is a consequence of Thm. 6.

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**Algorithm 2:** Algorithm for the region problem in multidimensional mean-payoff games

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**Existentially guess  $v$  in  $X$**

**if**  $x \in \bigcap_{s \in S} \text{value}(s) \cap \text{hsp}(\lambda)$  **then return true;**  
    **else return false;**

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