



**HAL**  
open science

# Input-state feedback linearization of single-input nonlinear time-delay systems

Kamel Baibeche, Claude Moog

► **To cite this version:**

Kamel Baibeche, Claude Moog. Input-state feedback linearization of single-input nonlinear time-delay systems. ECC 2014, Jun 2014, Strasbourg, France. pp.460 - 465, 10.1109/ECC.2014.6862315 . hal-00976759

**HAL Id: hal-00976759**

**<https://hal.science/hal-00976759>**

Submitted on 10 Apr 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Input-State Feedback Linearization of Single-Input Nonlinear Time-Delay Systems

Kamel BAIBECHE and Claude H. MOOG

LUNAM, IRCCyN, UMR C.N.R.S 6597  
 1 rue de la Noë, BP 92101, 44321 Nantes Cedex 3, France.  
 e-mail : *Kamel.Baibeche/Claude.Moog@irccyn.ec-nantes.fr*

**Abstract** : In this paper, the input-state linearization problem for a class of single-input nonlinear time-delay systems is studied for the first time. The problem is solved by searching a linearizing output with full relative degree. Necessary and sufficient conditions are given for the existence of such an output which yields a solution to the linearization problem.

**Keywords** : Input-state feedback linearization, Nonlinear time-delay systems, Linearizing output.

## I. INTRODUCTION

Nonlinear time-delay models have to be considered for real systems for example in telecommunications and teleoperation [1].

Despite the remarkable development on the theory of linear time-delay systems which has been widely studied and some general results are now available ([2], [3]), there is virtually no general theory for nonlinear time-delay systems. This has motivated the research for this class of systems ([4], [5], [6]).

In this paper, we solve the input-state linearization problem for a class of single-input nonlinear time-delay systems. It is a control technique which allows to use methods available for linear time-delay systems. Our trick consists on the search of a dummy output with full relative degree and to compute an input-output linearizing feedback. To solve completely this problem, some necessary and sufficient conditions are derived for the first time ever in this work. They are used in the search of a state feedback control and a change of variables which linearize the full dynamics ([7], [8], [9], [10]).

In the case of linear systems without delays, this

problem reduces to the search of a controllable canonical form or the form of Brunovsky. However, for nonlinear time-delay systems, the transformation is difficult to obtain, additional conditions are required.

This paper is organized as follows. In section II, we present the problem statement of the input-state linearization problem for a class of single-input nonlinear time-delay systems. Section III will show the solution of the problem, start by solving some subproblems which considered like special cases, and finishing by some Lemmas and the main theorem which solve the problem. Finally, the conclusion is presented in section IV.

## II. PROBLEM STATEMENT

Consider the following class of single-input nonlinear time-delay systems, with the assumption that the delays are commensurable :

$$\dot{x}(t) = f(x(\cdot)) + g(x(\cdot))u(t) \quad (1)$$

where  $x(\cdot) = (x(t), x(t-1), \dots, x(t-k))$

we want to find a state feedback control  $u(t)$  and a change of variables  $z(t)$  :

$$\begin{cases} u(t) = \alpha((x(\cdot), v(\cdot))) \\ z(t) = \varphi(x(\cdot)) \end{cases} \quad (2)$$

with

$$x(t) = \varphi^{-1}(z(\cdot)) \quad (3)$$

such that, the closed-loop system reads

$$\dot{z}(t) = A(\delta)z + B(\delta)v \quad (4)$$

where, the state  $x \in R^n$ , the control input  $u \in R$ , the entries of  $f, g, \alpha$  and  $\varphi$  are meromorphic functions of their arguments,  $A(\delta)$  and  $B(\delta)$  are polynomial matrices with the assumption that the pair  $(A(\delta), B(\delta))$  is weakly controllable,  $v(\cdot)$  is the new control input,  $\delta$  represents the backward time-shift operator and

$$\begin{cases} v(\cdot) = (v(t), v(t-1), \dots, v(t-k)) \\ z(\cdot) = (z(t), z(t-1), \dots, z(t-k)) \end{cases}$$

To solve our problem we are interested in the question of the existence of a state feedback control and a change of variables which has to be invertible.

The problem consists in deriving some necessary and sufficient conditions, which proof the existence of this linearizing output.

We start by the following motivating example.

**Example 1 :** Consider the system (5)

$$\Sigma : \begin{cases} \dot{x}_1 = x_2(t-1) + x_1(t-1)u(t) \\ \dot{x}_2 = x_1(t-2)u(t-1) + x_3(t) \\ \dot{x}_3 = x_1(t) - x_1(t-1)u(t) \end{cases} \quad (5)$$

The problem is solved as follows :

We search a state feedback control

$$u(t) = \frac{v(t)}{x_1(t-1)} \quad (6)$$

the closed-loop of the system is

$$\Sigma_e : \begin{cases} \dot{x}_1 = x_2(t-1) + v(t) \\ \dot{x}_2 = x_3(t) + v(t-1) \\ \dot{x}_3 = x_1(t) - v(t) \end{cases} \quad (7)$$

We search the matrices  $A(\delta)$  and  $B(\delta)$

$$A(\delta) = \begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B(\delta) = \begin{bmatrix} 1 \\ \delta \\ -1 \end{bmatrix}$$

where

$$\text{rank}[B(\delta), A(\delta)B(\delta)] = \text{rank} \begin{bmatrix} 1 & \delta^2 \\ \delta & -1 \\ -1 & 1 \end{bmatrix} = 2.$$

Which implies that the pair  $(A(\delta), B(\delta))$  is weakly controllable.

The static state feedback input-state linearization problem for the system (5) is solvable, if there exists a linearizing output with full relative degree.

We can find a linearizing output  $y$  with full relative degree  $n = 3$  for the system (5) as follows

$$y = [B(\delta), A(\delta)B(\delta)]^\perp = \begin{bmatrix} 1 - \delta & 1 + \delta^2 & 1 + \delta^3 \end{bmatrix}$$

$$y = x_1(t) - x_1(t-1) + x_2(t) + x_2(t-2) + x_3(t) + x_3(t-3)$$

To find the change of variables, we take

$$\begin{cases} z_1(t) = x_1(t) \\ z_2(t) = x_2(t) \\ z_3(t) = x_3(t) \end{cases}$$

Finally, the description of the system in the new coordinates is

$$\begin{cases} \dot{z}_1 = z_2(t-1) + v(t) \\ \dot{z}_2 = z_3(t) + v(t-1) \\ \dot{z}_3 = z_1(t) - v(t) \end{cases} \quad (8)$$

### III. SOLUTION OF THE PROBLEM

To solve our problem we start by solving these different subproblems. First, we start by searching all  $h(x(\cdot))$  with relative degree  $\geq 2$ . After, we search all  $h(x(\cdot))$  with relative degree  $\geq 3$  until arrive to the general case where the relative degree of  $h(x(\cdot)) \geq n$ .

By solving these subproblems, we can find some necessary and sufficient conditions for the existence of a linearizing output  $h(x(\cdot))$ , which yields a solution to the linearizing problem.

To find the relative degree, we differentiate the output  $y = h(x(\cdot))$  until  $u$  appears in one of the equations.

**Subproblem 1 : (Relative degree  $\geq 2$ )**

Consider the system described by

$$\dot{x}(t) = f(x(t), x(t-1)) + g(x(t), x(t-1))u(t)$$

Find all  $h(x(\cdot))$  with relative degree  $\geq 2$ .

The condition to find  $h(x(\cdot))$  is :

$$\frac{\partial \dot{h}(x(\cdot))}{\partial u(t)} = 0 \quad (9)$$

After differentiating the equation (9) , we obtain

$$\begin{cases} \frac{\partial h}{\partial x(t)}g(\cdot) = 0 \\ \frac{\partial h}{\partial x(t-1)}g^-(\cdot) = 0 \\ \vdots \end{cases}$$

where  $g^-(\cdot) = \delta g(\cdot)$  is the function  $g$  shifted.

From this result, we can conclude :

1.1/ Find  $h(x(t))$  with relative degree  $\geq 2$

The condition to find  $h(x(t))$  is

$$dh \in \overline{g_e^2}^\perp$$

where  $\overline{g_e^2}^\perp$  is defined as follows

$$\overline{g_e^2}^\perp = \overline{\begin{bmatrix} g(\cdot) & \emptyset \\ \emptyset & I_d \end{bmatrix}}^\perp$$

The number of elements in the left kernel is

$$2n - \dim \overline{g_e^2}$$

1.2/ Find  $h(x(t),x(t-1))$  with relative degree  $\geq 2$

The condition to find  $h(x(t),x(t-1))$  is

$$dh \in \overline{g_e^3}^\perp$$

where  $\overline{g_e^3}^\perp$  is defined by

$$\overline{g_e^3}^\perp = \overline{\begin{bmatrix} g(\cdot) & \emptyset & \emptyset \\ \emptyset & g^-(\cdot) & \emptyset \\ \emptyset & \emptyset & I_d \end{bmatrix}}^\perp$$

The number of elements in the left kernel is

$$3n - \dim \overline{g_e^3}$$

1.3/ Find  $h(x(\cdot))$  with relative degree  $\geq 2$

The condition to find  $h(x(\cdot))$  is

$$\forall k \in N : dh \in \overline{g_e^k}^\perp$$

where  $\overline{g_e^k}^\perp$  is defined as

$$\overline{g_e^k}^\perp = \overline{\begin{pmatrix} g(\cdot) & \emptyset & \dots & \emptyset \\ \emptyset & g^-(\cdot) & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & \emptyset & \dots & I_d \end{pmatrix}}^\perp$$

The number of elements in the left kernel (the maximum) is

$$kn - \dim \overline{g_e^k}$$

**Example 2 :** Consider the system (10)

$$\Sigma : \dot{x}(t) = f(\cdot) + \begin{bmatrix} x_2(t-1) \\ x_3(t) \\ 1 \end{bmatrix} u(t) \quad (10)$$

Find all  $h(x(t))$  with relative degree  $\geq 2$ .

The problem can be solved as follows :

The new extended system associated with the original system is :

$$\Sigma_e : \begin{cases} \dot{x}(t) = f(\cdot) + \begin{bmatrix} x_2(t-1) \\ x_3(t) \\ 1 \end{bmatrix} u(t) \\ \dot{x}(t-1) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} u(t) \end{cases}$$

we form the following matrix :

$$g_e^2 = \begin{pmatrix} x_2(t-1) & 0 & 0 & 0 \\ x_3(t) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The involutive closure of  $g_e^2$  gives the following distribution :

$$\overline{g_e^2} = \begin{pmatrix} x_2(t-1) & 0 & 0 & 0 & 1 \\ x_3(t) & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

the number of elements in the left kernel is :  
( $2n - \dim \overline{g_e^2} = \dim \overline{g_e^2}^\perp = 1$ )

$$\text{with } \overline{g_e^2}^\perp = \left( 0 \ 1 \ -x_3(t) \ 0 \ 0 \ 0 \right)$$

Finally, we find  $h(x(t))$  with relative degree  $\geq 2$  :

$$h(x(t)) = x_2(t) - \frac{x_3^2(t)}{2}$$

**Subproblem 2 : (Relative degree  $\geq 3$ )**

Consider the system described by

$$\dot{x}(t) = f(x(t), x(t-1)) + g(x(t), x(t-1))u(t)$$

Find all  $h(x(t))$  with relative degree  $\geq 3$ .

The condition to find  $h(x(t))$  is

$$dh \in \overline{\begin{bmatrix} g(\cdot) & \emptyset & [f, g] & [f, I_d] \\ \emptyset & I_d & \emptyset & \emptyset \end{bmatrix}}^\perp$$

where  $[f, g]$  is the Lie bracket of  $f$  and  $g$ .

$$\begin{aligned} \begin{pmatrix} [f, g] \\ \emptyset \end{pmatrix} &= \begin{pmatrix} f \\ \emptyset \end{pmatrix}, \begin{pmatrix} g \\ \emptyset \end{pmatrix} \\ \begin{pmatrix} [f, I_d] \\ \emptyset \end{pmatrix} &= \begin{pmatrix} f \\ \emptyset \end{pmatrix}, \begin{pmatrix} \emptyset \\ I_d \end{pmatrix} \end{aligned}$$

More generally, we search all  $h(x(\cdot))$  with relative degree  $\geq n$ .

From these results, necessary and sufficient conditions for the existence of a solution by input-state linearization can be obtained.

**Lemma 1 :**

$\exists h(x(\cdot))$  with relative degree  $n$  if and only if,  $\exists k$  such that :

(i)  $\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-2} g_e^k\}$  involutive

(ii)  $dim\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-1} g_e^k\} = nk$

where

$$f_e^k = \begin{pmatrix} f \\ \emptyset \\ \vdots \\ \emptyset \end{pmatrix}_{nk}$$

condition (ii) ensures the controllability.

**Proof :**

• **Sufficiency**

We have  $\exists k \mid \{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-2} g_e^k\}$  involutive which means that is completely integrable.

So, there exists a linearizing output  $h(x(\cdot))$  with

$$dh \in \{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-2} g_e^k\}^\perp$$

To find the relative degree, we differentiate the output  $y = h(x(\cdot))$  until  $u$  appears in one of the equations.

$$y = h(x(\cdot))$$

$$\dot{y} = \dot{h}(x(\cdot)) = \frac{\partial h(x(\cdot))}{\partial x(\cdot)} \dot{x}(\cdot) = \frac{\partial h(x(\cdot))}{\partial x(\cdot)} (f_e^k + g_e^k u) = L_{f_e^k} h + L_{g_e^k} h u = L_{f_e^k} h$$

with  $(L_{g_e^k} h u = 0$  because  $dh \perp g_e^k$ )

$$y^{(2)} = \frac{\partial L_{f_e^k} h}{\partial x(\cdot)} \dot{x}(\cdot) = \frac{\partial L_{f_e^k} h}{\partial x(\cdot)} (f_e^k + g_e^k u) = L_{f_e^k}^2 h + L_{g_e^k} L_{f_e^k} h u = L_{f_e^k}^2 h$$

with  $(L_{g_e^k} L_{f_e^k} h u = 0$  because  $dh \perp [f_e^k, g_e^k]$ )

$\vdots$

$$y^{(n-1)} = L_{f_e^k}^{n-1} h$$

with  $(L_{g_e^k} L_{f_e^k}^{n-2} h u = 0$  because  $dh \perp ad_{f_e^k}^{n-2} g_e^k$ )

$$y^{(n)} = \frac{\partial L_{f_e^k}^{n-1} h}{\partial x(\cdot)} \dot{x}(\cdot) = \frac{\partial L_{f_e^k}^{n-1} h}{\partial x(\cdot)} (f_e^k + g_e^k u) = L_{f_e^k}^n h + L_{g_e^k} L_{f_e^k}^{n-1} h u$$

with  $(L_{g_e^k} L_{f_e^k}^{n-1} h u \neq 0)$ . which means that we have a relative degree  $\geq n$  but with condition (ii) :

$\exists k \mid dim\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-1} g_e^k\} = nk$   
the relative degree equals  $n$ .

• **Necessity**

We will proof the necessity by contradiction.

We have  $\exists h(x(\cdot))$  with relative degree  $n$ .

Assume :

(i)  $\exists k \mid \{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-2} g_e^k\}$  not involutive

(ii)  $\exists k \mid dim\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-1} g_e^k\} = nk$  which proof that all the vectors are independent and  $dim\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-2} g_e^k\} = nk - 1$

As a consequence of the condition (i)

$$dh \notin \{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{n-2} g_e^k\}^\perp$$

Which means that the relative degree of  $h(x(\cdot)) < n$  and it is not equal to  $n$  (contradiction).

Which implies the necessity of the conditions of Lemma 1.

To solve completely the input-state linearization problem, we have to solve also the input-output linearization problem which will be used in the final Theorem.

**Lemma 2 :**

Given  $y = h(x(\cdot))$  with relative degree  $n$ .

$$y^{(n)} = h^{(n)}(x(\cdot)) = a(x(\cdot)) + \sum_{i=0}^k b_i(x(\cdot)) u(t-i)$$

the input-output linearization problem is solvable if :

(i)  $\exists i \mid b_i \neq 0$  and  $b_j = 0 \quad \forall j \neq i$

(ii)  $\forall j < i \mid \frac{\partial h^{(n)}(\cdot)}{\partial x(t-j)} = 0$

$$(iii) \frac{\partial}{\partial x(\cdot)} \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \text{ unimodular}$$

**Proof :**

The solution of the problem can be found as follows : if the conditions of Lemma 2 are satisfied, so

$y^{(n)} = v(t)$  and we find :

$$u(t) = \frac{-a(x(\cdot)) + v(t)}{b_i(x(\cdot))}$$

After replacing  $u(t)$  in the system, we can find

the linearizing state  $z(t)$  by solving the closed-loop of the system. This change of variables has to be invertible.

**Remark 1 :**

The condition (ii) of Lemma 2 ensures the causality of  $u(t)$  which means that the system don't depend on future input .

Condition (iii) ensures that the change of variables is invertible.

The conditions of Lemma 2 are not necessary as it is shown in the following example.

**Example 3 :** Consider the system (11)

$$\begin{cases} \dot{x}_1 = x_2(t-1) + x_1^2(t-1) + u(t) \\ \dot{x}_2 = -u(t) \end{cases} \quad (11)$$

$$y = x_1(t) + x_2(t).$$

We compute  $y^{(2)}$  :

$$\begin{aligned} y &= x_1(t) + x_2(t) \\ \dot{y} &= x_2(t-1) + x_1^2(t-1) \\ y^{(2)} &= 2x_1(t-1)[x_2(t-2) + x_1^2(t-2)] + [2x_1(t-1) - 1]u(t-1) \end{aligned}$$

We verify the conditions of Lemma 2 :

(i)  $b_1 = 2x_1(t-1) - 1 \neq 0$  (condition (i) is satisfied)

(ii)  $\forall j < 1 \mid \frac{\partial y^{(2)}}{\partial x(t-j)} = 0$  (condition (ii) is satisfied)

$$(iii) \frac{\partial}{\partial x(\cdot)} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2x_1(t-1)\delta & \delta \end{pmatrix}$$

not unimodular (condition (iii) is not satisfied).

So, the change of variables is not invertible, thus the conditions of Lemma 2 are not fulfilled. Nevertheless the state feedback linearizes the input-output relation is :

$$u(t) = \frac{v(t) - 2x_1(t)[x_2(t-1) + x_1^2(t-1)]}{2x_1(t) - 1}$$

with  $(2x_1(t) - 1 \neq 0)$

From these results, sufficient conditions to solve the input-state linearization problem for a class of single-input nonlinear time-delay systems can be obtained.

**Theorem :** Given system (1)

The input-state linearization problem is solvable if,  $\exists k$  such that :

(i)  $\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{m-2} g_e^k\}$  involutive

(ii)  $\dim\{\overline{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{m-1} g_e^k}\} = nk$

(iii)  $y = h(x(\cdot))$  satisfies Lemma 2, where  $dh \in \{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{m-2} g_e^k\}^\perp$

**Proof :**

If we have  $h(x(\cdot))$  with relative degree  $n$  satisfies lemma 1, where  $dh \in \{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{m-2} g_e^k\}^\perp$

By using lemma 2, we can find a state feedback control  $u(t) = \frac{-a(x(\cdot))+v(t)}{b_i(x(\cdot))}$  that can solve the input-state linearization problem.

Now, we only have to prove that condition (iii) does not depend on the choice of  $h(x(\cdot))$ .

**Remark 2 :**

Consider two functions  $h(x(\cdot))$  and  $\tilde{h}(x(\cdot))$ , both have relative degree  $n$ .

from condition (i) of the Theorem, we have  $\dim\{g_e^k, [f_e^k, g_e^k], \dots, ad_{f_e^k}^{m-2} g_e^k\} = nk - 1$ , so there exists one solution in the left kernel and we can write

$$d h(x(\cdot)) = \lambda d \tilde{h}(x(\cdot))$$

$$\text{so, } \tilde{h}(x(\cdot)) = \Phi(h(x(\cdot)))$$

by differentiation the equation we find

$$\tilde{h}^{(n)}(x(\cdot)) = \tilde{a}(x(\cdot))h^{(n)}(x(\cdot)) + \tilde{b}(x(\cdot))$$

where  $\tilde{a}(x(\cdot)) = \frac{d\Phi}{dh}$  and there is no  $u(\cdot)$  in  $\tilde{b}(x(\cdot))$ .

So, if  $h(x(\cdot))$  satisfies Lemma 2, then  $\tilde{h}(x(\cdot))$  satisfies Lemma 2 as well.

Which implies that condition (iii) of the Theorem does not depend on the choice of  $h(x(\cdot))$ .

The conditions of the Theorem are illustrated by the following example.

**Example 4 :** Consider the system (12)

$$\begin{cases} \dot{x}_1 = x_2(t) \\ \dot{x}_2 = \sin(x_1^3(t-1)) + x_2(t-1)u(t) \end{cases} \quad (12)$$

where  $n = 2$  and  $k = 2$ .

We will prove that the system (12), can be linearized by static state feedback, by showing that the conditions of the Theorem are satisfied.

Let,

$$f_e^2 = \begin{bmatrix} x_2(t) \\ \sin(x_1^3(t-1)) \\ 0 \\ 0 \end{bmatrix}, \quad g_e^2 = \begin{bmatrix} 0 & 0 & 0 \\ x_2(t-1) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dim\{g_e^2\} = nk - 1 = 3$ ,  $\{g_e^2\}$  is involutive (condition (i) of the Theorem is satisfied)

$$\dim\{\overline{g_e^2}, [\overline{f_e^2}, \overline{g_e^2}]\} = \dim \begin{bmatrix} 0 & 0 & 0 & x_2(t-1) \\ x_2(t-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\dim\{\overline{g_e^2}, [\overline{f_e^2}, \overline{g_e^2}]\} = nk = 4$  (condition (ii) of the Theorem is satisfied).

We search the linearizing output

$$y = (g_e^2)^\perp = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

we find  $y = x_1(t)$ .

We compute  $y^{(2)}$

$$y = x_1(t)$$

$$\dot{y} = x_2(t)$$

$$y^{(2)} = \sin(x_1^3(t-1)) + x_2(t-1)u(t) = v(t)$$

We verify the condition (iii) of the Theorem :

$$(i) b_1 = x_2(t-1) \neq 0 \text{ (satisfied)}$$

$$(ii) \forall j < 0 \mid \frac{\partial y^{(2)}}{\partial x(t-j)} = 0 \text{ (satisfied)}$$

$$(iii) \frac{\partial}{\partial x(\cdot)} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ unimodular (satisfied).}$$

So, condition (iii) of the Theorem is satisfied.

The state feedback control is

$$u(t) = \frac{v(t) - \sin(x_1^3(t-1))}{x_2(t-1)}$$

with  $x_2(t-1) \neq 0$

The change of variables is

$$\begin{cases} z_1 = x_1(t) \\ z_2 = x_2(t) \end{cases} \quad (13)$$

Finally, the closed-loop of the system reads

$$\begin{cases} \dot{z}_1 = z_2(t) \\ \dot{z}_2 = v(t) \end{cases} \quad (14)$$

Which means that the input-state linearization problem of the system (12) is solved by the Theorem.

#### IV. CONCLUSION

Necessary and sufficient conditions for the existence of a linearizing output with full relative degree have been obtained in this work. They have been used as sufficient conditions to solve the input-state linearization problem for a class of single-input nonlinear time-delay systems.

Future work has to be done to search some necessary conditions to solve the input-state

linearization problem for this class of systems.

#### REFERENCES

- [1] J. P. Richard, Time-delay systems : an overview of some recent advances and open problems, *Automatica*, vol. 39, pp. 1667-1694, October 2003.
- [2] P. Picard, Sur l'observabilité et la commande des systèmes linéaires à retards modélisés sur un anneau, *Ph.D. dissertation*, Univ. de Nantes, Nantes, France, 1996.
- [3] M. Malabre and R. Rabah, Structure at infinity, model matching and disturbance rejection for linear systems with delays, *Kybernetika*, vol. 29, pp. 485-498, 1993.
- [4] L. A. Marquez Martinez, Analyse et commande des systèmes non linéaires à retards, *Ph.D. dissertation*, Ecole Centrale de Nantes, Nantes, France, 2000.
- [5] C. Califano, L. A. Marquez Martinez and C. H. Moog, Extended Lie Brackets for Nonlinear Time-delay Systems, *IFAC Word Congress*, 2011, Milan, Italy.
- [6] A. Germani, C. Manes and P. Pepe, Linearization and decoupling of nonlinear delay systems, *American Control Conference*, vol. 3, pp. 1948 - 1952, 1998.
- [7] L. A. Marquez Martinez and C. H. Moog, Input-output feedback linearization of time-delay systems, *IEEE Trans. Automatic Control*, vol. 49, pp. 781-785, May 2004.
- [8] T. Oguchi, A. Watanabe and T. Nakamizo, Input-output linearization of retarded non-linear systems by using an extension of Lie derivative, *International J. of Control*, vol. 75, pp. 582-590, 2002.
- [9] L. A. Marquez Martinez and C. H. Moog, On the input-output linearization of nonlinear time-delay systems, *Proc. of the IFAC Conf. Systems Structure Control*, Nantes, France, vol. 3, pp. 575-580, 1998.
- [10] J. C. Huijberts, C. H. Moog and R. Pothin, Input-output decoupling of nonlinear systems by static measurement feedback, *Systems Control Letters*, vol. 39, pp. 109-114, 2000.