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Abstract. We generalize the theorems of Stein–Tomas and Strichartz about surface restrictions of Fourier transforms to systems of orthonormal functions with an optimal dependence on the number of functions. We deduce the corresponding Strichartz bounds for solutions to Schrödinger equations up to the endpoint, thereby solving an open problem of Frank, Lewin, Lieb and Seiringer. We also prove uniform Sobolev estimates in Schatten spaces, extending the results of Kenig, Ruiz, and Sogge. We finally provide applications of these results to a Limiting Absorption Principle in Schatten spaces, to the well-posedness of the Hartree equation in Schatten spaces, to Lieb–Thirring bounds for eigenvalues of Schrödinger operators with complex potentials, and to Schatten properties of the scattering matrix.

Introduction

A classical topic in harmonic analysis is the so-called *restriction problem*. Given a surface S embedded in $\mathbb{R}^N$, $N \geqslant 2$, one asks for which exponents $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant \infty$ the Fourier transform of a function $f \in L^p(\mathbb{R}^N)$ belongs to $L^q(S)$, where S is endowed with its $(N-1)$-dimensional Lebesgue measure $d\sigma$. More precisely, defining the restriction operator $\mathcal{R}_S$ as $\mathcal{R}_S f = \widehat{f}|_S$ for all f in the Schwartz class, the problem is to know when $\mathcal{R}_S$ can be extended as a bounded operator from $L^p(\mathbb{R}^N)$ to $L^q(S)$. The operator dual to $\mathcal{R}_S$ is called the *extension operator*, which we denote by $\mathcal{E}_S$, and satisfies the identity

$$\mathcal{E}_S f(x) = \frac{1}{(2\pi)^{N/2}} \int_S f(\xi) e^{i\xi\cdot x}\, d\sigma(\xi), \quad \forall x \in \mathbb{R}^N, \tag{1}$$

for all $f \in L^1(S)$. The restriction problem is thus equivalent to knowing when $\mathcal{E}_S$ is bounded from $L^{q'}(S)$ to $L^{p'}(\mathbb{R}^N)$. We refer to [33] for a wide review of results concerning this problem and its motivations.

A model case of the restriction problem which is often considered in the literature is the case $q = 2$. There are two types of surfaces for which this problem has been completely settled. For smooth compact surfaces with non-zero Gauss curvature, the celebrated Stein–Tomas theorem [30, 34] states that the restriction problem has a positive answer if and only if $1 \leqslant p \leqslant 2(N+1)/(N+3)$. For quadratic surfaces, Strichartz [32] gave a complete answer depending on the type of the surface (paraboloid-like, cone-like, or sphere-like, see below for a more precise definition). Hence, in these cases we know exactly for which exponents p the inequality

$$\|\mathcal{E}_S f\|_{L^{p'}(\mathbb{R}^N)} \leqslant C\, \|f\|_{L^2(S)} \tag{2}$$

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holds for all $f \in L^2(S)$ with $C>0$ independent of f. The question we want to address in this work is a generalization of (2) to systems of orthonormal functions. More precisely, let $(f_j)_{j\in J}$ a (possibly infinite) orthonormal system in $L^2(S)$, and let $(\nu_j)_{j\in J}\subset\mathbb{C}$ be a family of coefficients. We prove inequalities of the form

$$\left\|\sum_{j\in J}\nu_j|\mathcal{E}_S f_j|^2\right\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant C\left(\sum_{j\in J}|\nu_j|^\alpha\right)^{1/\alpha} \tag{3}$$

for some $1\leqslant p\leqslant 2, \alpha>1$, with $C>0$ independent of (f_j), (ν_j). To appreciate the difference between (2) and (3), notice that combining (2) with the triangle inequality in $L^{p'/2}(\mathbb{R}^N)$ leads to the estimate

$$\left\|\sum_{j\in J}\nu_j|\mathcal{E}_S f_j|^2\right\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant \sum_{j\in J}|\nu_j|\,\|\mathcal{E}_S f_j\|^2_{L^{p'}(\mathbb{R}^N)} \leqslant C\sum_{j\in J}|\nu_j|,$$

which is weaker than (3), especially when the number of non-zero ν_j is infinite.

Generalizing functional inequalities involving a single function to systems of orthonormal functions is not a new topic. It is strongly motivated by the study of many-body systems in quantum mechanics, where a simple description of M independent fermionic particles is given by M orthonormal functions in some L^2-space. It is then important to obtain functional inequalities on these systems which behaviour is optimal in the number M of such functions. Typically, this behaviour is better than the one given by the triangle inequality. The first example of such a generalization is the Lieb–Thirring inequality [24], which states that for any $f_1,\dots,f_M$ orthonormal in $L^2(\mathbb{R}^d)$ and for any non-negative coefficients $\nu_1,\dots,\nu_M$, we have

$$\left\|\sum_{j=1}^M\nu_j|f_j|^2\right\|_{L^{1+\frac{2}{d}}(\mathbb{R}^d)} \leqslant C\left(\sup_j\nu_j\right)^{\frac{2}{d+2}}\left(\sum_{j=1}^M\nu_j\,\|\nabla f_j\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{d}{d+2}}. \tag{4}$$

Its counterpart for a single function is the Gagliardo–Nirenberg–Sobolev inequality,

$$\|f\|_{L^{2+\frac{4}{d}}(\mathbb{R}^d)} \leqslant C\,\|f\|^{\frac{2}{d+2}}_{L^2(\mathbb{R}^d)}\,\|\nabla f\|^{\frac{d}{d+2}}_{L^2(\mathbb{R}^d)}, \tag{5}$$

which together with the triangle inequality implies

$$\left\|\sum_{j=1}^M\nu_j|f_j|^2\right\|_{L^{1+\frac{2}{d}}(\mathbb{R}^d)} \leqslant C\left(\sum_{j=1}^M\nu_j\right)^{\frac{2}{d+2}}\left(\sum_{j=1}^M\nu_j\,\|\nabla f_j\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{d}{d+2}}, \tag{6}$$

which is weaker than (4). Lieb–Thirring inequalities are a decisive tool for proving stability of matter [24, 23]. In particular, we emphasize that (6) is *not* enough to prove stability of matter. The homogeneous Sobolev inequality for $0<s<d/2$,

$$\left\|(-\Delta)^{-s/2}f\right\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \leqslant C\,\|f\|_{L^2(\mathbb{R}^d)}, \tag{7}$$

also has a generalization to several functions which was proved by Lieb [22]: for any $f_1, \dots, f_M$ orthonormal in $L^2(\mathbb{R}^d)$ and for any non-negative coefficients $\nu_1, \dots, \nu_M$, we have

$$\left\| \sum_{j=1}^M \nu_j \left|(-\Delta)^{-s/2} f_j\right|^2 \right\|_{L^{\frac{d}{d-2s}}(\mathbb{R}^d)} \leqslant C \left(\sup_j \nu_j\right)^{\frac{2s}{d}} \left(\sum_{j=1}^M \nu_j\right)^{\frac{d-2s}{d}}. \tag{8}$$

Again, combining (7) with the triangle inequality leads to the estimate

$$\left\| \sum_{j=1}^M \nu_j \left|(-\Delta)^{-s/2} f_j\right|^2 \right\|_{L^{\frac{d}{d-2s}}(\mathbb{R}^d)} \leqslant \sum_{j=1}^M \nu_j \left\|(-\Delta)^{-s/2} f_j\right\|^2_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \leqslant C \sum_{j=1}^M \nu_j,$$

which is weaker than (8). Finally, a recent work by Frank, Lewin, Lieb, and Seiringer [9] generalizes the Strichartz inequality

$$\left\|e^{it\Delta} f\right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left\|f\right\|_{L^2_x(\mathbb{R}^d)}, \ d \geqslant 1,\ p, q \geqslant 2,\ \frac{2}{p} + \frac{d}{q} = \frac{d}{2},\ (d,p,q) \neq (2,\infty,2), \tag{9}$$

to orthonormal functions $f_1, \dots, f_M$ in $L^2(\mathbb{R}^d)$ with complex coefficients $\nu_1, \dots, \nu_M$:

$$\left\| \sum_{j=1}^M \nu_j \left|e^{it\Delta} f_j\right|^2 \right\|_{L^{\frac{p}{2}}_t L^{\frac{q}{2}}_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left(\sum_{j=1}^M |\nu_j|^\alpha\right)^{1/\alpha}, \ 1 \leqslant \alpha \leqslant 2q/(q+2),\ q \leqslant 2 + \frac{4}{d}. \tag{10}$$

These generalized Strichartz estimates were used in [20, 21] to study the nonlinear evolution of quantum systems with an infinite number of particles.

Our motivation to prove inequalities of the form (3) is threefold. Harmonic analysis tools are widely used in nonlinear problems. In a perspective to study nonlinear many-body problems, it is thus natural to understand the model restriction problem in this many-body context. In a more concrete approach, it was noticed by Strichartz [32] that the restriction problem for some quadratic surfaces is linked to space-time decay estimates for some evolution equations. As a consequence, we will see that (3) when S is one of these quadratic surfaces also has an interpretation in terms of solutions to evolution equations. In particular, we provide a new proof of the Strichartz inequality (10) which furthermore includes the *full range of exponents* where it is valid. As in the original article of Strichartz, this also provides new Strichartz inequalities for orthonormal functions for the fractional Laplacian $(-\Delta)^{1/2}$ and for the pseudo-relativistic operator $(1-\Delta)^{1/2}$. Finally, we present a general principle which allows to obtain bounds of the kind (3), not necessarily in the context of the extension operator. The advantage of this principle is that it allows to deduce *automatically* bounds for orthonormal systems from bounds for a single function, if these latter were proved by a certain method, namely by complex interpolation. Let us now provide some insight about this principle.

One of the reasons why the case $q = 2$ of the restriction problem is better understood is that we can compose the maps $\mathcal{E}_S$ and $\mathcal{R}_S = (\mathcal{E}_S)^*$. In particular, $\mathcal{E}_S$ is bounded from $L^2(S)$ to $L^{p'}(\mathbb{R}^N)$ if and only if $T_S := \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p(\mathbb{R}^N)$ to $L^{p'}(\mathbb{R}^N)$. Stein [30] and Strichartz [32] prove the boundedness of T_S by introducing an analytic family of operators (T_z) defined on a strip $a \leqslant \operatorname{Re} z \leqslant b$ in the complex plane, such that $T_S = T_c$ for some

$c \in (a,b)$. They prove that T_z is a bounded operator from L^2 to L^2 on the line $\operatorname{Re} z = b$ and from L^1 to L^∞ on the line $\operatorname{Re} z = a$. Using Stein's interpolation theorem [29], they deduce that $T_S = T_c$ is bounded from L^p to $L^{p'}$ for some exponent p, which turns out to be the optimal one. Now notice that Hölder's inequality implies that T_S is bounded from $L^p(\mathbb{R}^N)$ to $L^{p'}(\mathbb{R}^N)$ if and only if for any $W_1, W_2 \in L^{2p/(2-p)}(\mathbb{R}^N)$, the operator $W_1 T_S W_2$ is bounded from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ with the estimate

$$\|W_1 T_S W_2\|_{L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)} \leqslant C \|W_1\|_{L^{2p/(2-p)}(\mathbb{R}^N)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{R}^N)}, \tag{11}$$

with $C > 0$ independent of W_1, W_2.

Our key contribution is to show that the operator $W_1 T_S W_2$ is more than a mere bounded operator on L^2, namely that it belongs to a Schatten class. Recall that the Schatten class $\mathfrak{S}^\alpha(L^2(\mathbb{R}^N))$, $\alpha > 0$, is defined as the space of all compact operators on $L^2(\mathbb{R}^N)$ such that the sequence of their singular values belongs to ℓ^α. The $\mathfrak{S}^\alpha$-norm of such an operator is then the ℓ^α-norm of its singular values. The estimate that we prove is

$$\|W_1 T_S W_2\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^N))} \leqslant C \|W_1\|_{L^{2p/(2-p)}(\mathbb{R}^N)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{R}^N)}, \tag{12}$$

with $C > 0$ independent of W_1, W_2. Of course, (12) implies (11) and is hence stronger. By a well-known duality argument (see Lemma 3 below), the bound (12) is equivalent to a bound of the kind (3) for systems of orthonormal functions. The estimate (12) follows from a general principle which states that, as soon as a linear operator T belongs to an analytic family of operators of the same type as the one we described above, then the operator $W_1 T W_2$ satisfies a bound of the form (12). As we explained, under such assumptions on T, Stein's interpolation theorem would typically imply that T is bounded from L^p to $L^{p'}$. Our main input is to notice that T actually satisfies a stronger Schatten bound (12).

This general principle does not depend on the fact that T_S can be decomposed as $T_S = \mathcal{E}_S(\mathcal{E}_S)^*$. This decomposition is only necessary to deduce (3) from (12). However, the Schatten bound (12) has an interest by itself, even when it holds for an operator T which is not of the form $T = AA^*$. An example of such an operator is given by the resolvent of the Laplacian on $\mathbb{R}^N$,

$$T = (-\Delta - z)^{-1}, \qquad z \in \mathbb{C} \setminus [0,\infty).$$

In [16], Kenig, Ruiz, and Sogge prove the bound

$$\left\|(-\Delta - z)^{-1}\right\|_{L^p(\mathbb{R}^N)\to L^{p'}(\mathbb{R}^N)} \leqslant C|z|^{-1+N\left(\frac{1}{p}-\frac{1}{2}\right)}, \tag{13}$$

for a range of exponents p which depends on N. When z varies away from a neighborhood of the origin, these bounds are uniform in z and hence were labeled *uniform Sobolev estimates*. In [16], the authors used them to prove unique continuation results for solutions to Schrödinger equations, but numerous other applications of these estimates were later found. For instance, they were part of Goldberg and Schlag's proof of the Limiting Absorption Principle in L^p spaces [13]. In [7], they were used to control the size of eigenvalues of Schrödinger operators with complex-valued potentials. In this paper, we prove a Schatten bound of the type (12) for $T = (-\Delta - z)^{-1}$, using our general principle. As an application, we then deduce from it a version of the Limiting Absorption Principle in Schatten spaces.

The organization of the article is the following, with the corresponding main results of each section:

- In Section 1, we first explain our general principle (Proposition 1) from which the Schatten bounds (12) follow. We then apply this principle to various situations.
- In Section 2, we prove restriction theorems for systems of orthonormal functions (Theorem 3). We also prove a result about the optimality of these estimates (Theorem 4).
- In Section 3, we extend the Strichartz inequalities of [9] to the full range of exponents (Theorem 6). We furthermore prove new Strichartz inequalities for systems of orthonormal functions, for the operators $(-\Delta)^{1/2}$ (Theorem 8) and $(1-\Delta)^{1/2}$ (Theorem 9).
- In Section 4, we prove uniform Sobolev estimates in Schatten spaces (Theorem 10), and apply them to prove a Limiting Absorption Principle in Schatten spaces (Theorem 11).
- In Section 5, we give an application of Strichartz estimates to well-posedness of the non-linear Hartree equation in Schatten spaces (Theorem 12).
- In Section 6, we give another application of uniform Sobolev estimates to Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials (Theorem 14).
- In Section 7, we finally give an application of the Limiting Absorption Principle to Schatten estimates of the Scattering Matrix (Theorem 15).

1. A complex interpolation estimate in Schatten spaces

In this section, we explain how to obtain Schatten bounds of the form (12) by a complex interpolation method. The advantage of this result is that it requires assumptions that are naturally proved when one wants to show that a given operator is bounded from $L^p(\mathbb{R}^N)$ to $L^{p'}(\mathbb{R}^N)$ by a complex interpolation method. Hence, it provides an automatic way to upgrade this $L^p \to L^{p'}$ bound into a stronger Schatten bound.

Let us first recall that a family of operators (T_z) on $\mathbb{R}^N$ defined on a strip $a \leqslant \operatorname{Re} z \leqslant b$ in the complex plane ($a < b$) is analytic in the sense of Stein [29] if for all simple functions f, g on $\mathbb{R}^N$ (that is, functions that take a finite number of nonzero values on sets of finite measure in $\mathbb{R}^N$), the map $z \mapsto \langle g, T_z f\rangle$ is analytic in $a < \operatorname{Re} z < b$, continuous in $a \leqslant \operatorname{Re} z \leqslant b$, and if

$$\sup_{a\leqslant x\leqslant b} |\langle g, T_{x+is} f\rangle| \leqslant C(s),$$

for some $C(s)$ with at most a (double) exponential growth in s.

We also recall the definition of Schatten spaces, see, e.g., [27]. Let $\mathfrak{H}$ be a complex Hilbert space. For any compact operator T on $\mathfrak{H}$, the non-zero eigenvalues of $\sqrt{T^*T}$ are called the singular values of T. They form an (at most) countable set that we denote by $(\mu_n(T))_{n\in\mathbb{N}}$. For $\alpha > 0$, the Schatten space $\mathfrak{S}^\alpha(\mathfrak{H})$ is defined as the space of all compact operators T on

$\mathfrak{H}$ such that $\sum_{n\in\mathbb{N}} \mu_n(T)^\alpha < \infty$. When $\alpha \geqslant 1$, it is a Banach space endowed with the norm

$$\|T\|_{\mathfrak{S}^\alpha(\mathfrak{H})} := \left(\sum_{n\in\mathbb{N}} \mu_n(T)^\alpha\right)^{1/\alpha}.$$

Our result is the following.

Proposition 1. *Let (T_z) be an analytic family of operators on $\mathbb{R}^N$ in the sense of Stein defined on the strip $-\lambda_0 \leqslant \operatorname{Re} z \leqslant 0$ for some $\lambda_0 > 1$. Assume that we have the bounds*

$$\|T_{is}\|_{L^2\to L^2} \leqslant M_0 e^{a|s|}, \quad \|T_{-\lambda_0+is}\|_{L^1\to L^\infty} \leqslant M_1 e^{b|s|}, \quad \forall s \in \mathbb{R}, \tag{14}$$

for some $a, b \geqslant 0$ and for some $M_0, M_1 \geqslant 0$. Then, for all $W_1, W_2 \in L^{2\lambda_0}(\mathbb{R}^N, \mathbb{C})$, the operator $W_1 T_{-1} W_2$ belongs to $\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^N))$ and we have the estimate

$$\|W_1 T_{-1} W_2\|_{\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^N))} \leqslant M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L^{2\lambda_0}(\mathbb{R}^N)} \|W_2\|_{L^{2\lambda_0}(\mathbb{R}^N)}. \tag{15}$$

Proof of Proposition 1. Let W_1, W_2 be non-negative, simple functions, and define the family of operators

$$S_z := W_1^{-z} T_z W_2^{-z}.$$

The family (S_z) is still analytic in the sense of Stein in the strip $-\lambda_0 \leqslant \operatorname{Re} z \leqslant 0$, and satisfies $S_{-1} = W_1 T_{-1} W_2$. For all $s \in \mathbb{R}$ we have the first bound

$$\|S_{is}\|_{L^2\to L^2} \leqslant \left\|W_1^{-is}\right\|_{L^\infty} \|T_{is}\|_{L^2\to L^2} \left\|W_2^{-is}\right\|_{L^\infty} \leqslant M_0 e^{a|s|}.$$

By the Dunford–Pettis theorem [6, Thm. 2.2.5], the operator $T_{-\lambda_0+is}$ has an integral kernel $T_{-\lambda_0+is}(x,y)$ satisfying

$$\|T_{-\lambda_0+is}(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} = \|T_{-\lambda_0+is}\|_{L^1(\mathbb{R}^N)\to L^\infty(\mathbb{R}^N)} \leqslant M_1 e^{b|s|}, \quad \forall s \in \mathbb{R}.$$

Hence, we deduce the Hilbert–Schmidt bound

$$\begin{aligned}\|S_{-\lambda_0+is}\|_{\mathfrak{S}^2}^2 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W_1(x)^{2\lambda_0} |T_{-\lambda_0+is}(x,y)|^2 W_2(y)^{2\lambda_0}\, dxdy \\ &\leqslant M_1^2 e^{2b|s|} \|W_1\|_{L^{2\lambda_0}(\mathbb{R}^N)}^{2\lambda_0} \|W_2\|_{L^{2\lambda_0}(\mathbb{R}^N)}^{\lambda_0}.\end{aligned}$$

By [27, Thm. 2.9], we deduce that S_{-1} belongs to $\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^N))$ with

$$\|S_{-1}\|_{\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^N))} \leqslant M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L^{2\lambda_0}(\mathbb{R}^N)} \|W_2\|_{L^{2\lambda_0}(\mathbb{R}^N)}.$$

Hence, we have proved (15) for W_1, W_2 non-negative and simple. The non-negativity assumption can be removed by writing $W_j = e^{i\varphi_j}|W_j|$ and estimating

$$\|W_1 T_{-1} W_2\|_{\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^N))} \leqslant \left\|e^{i\varphi_1}\right\|_{L^2\to L^2} \||W_1| T_{-1} |W_2|\|_{\mathfrak{S}^{2\lambda_0}} \left\|e^{i\varphi_2}\right\|_{L^2\to L^2} \leqslant \||W_1| T_{-1} |W_2|\|_{\mathfrak{S}^{2\lambda_0}},$$

and the simplicity assumption is removed by density. □

Remark 2. The previous proof shows that the conclusion of Proposition 1 also holds when $\lambda_0 = 1$: in this case, there is even no interpolation to perform.

When furthermore we can decompose $T_{-1} = AA^*$, we deduce from (15) a corresponding result for systems of orthonormal functions.

Lemma 3 (Duality principle). *Let $\mathfrak{H}$ be a separable Hilbert space. Assume that A is a bounded operator from $\mathfrak{H}$ to $L^{p'}(\mathbb{R}^N)$ for some $1 \leqslant p \leqslant 2$, and that we have the estimate*

$$\left\| WAA^*\overline{W} \right\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^N))} \leqslant C \left\| W \right\|^2_{L^{2p/(2-p)}(\mathbb{R}^N)}, \quad \forall W \in L^{2p/(2-p)}(\mathbb{R}^N, \mathbb{C}), \tag{16}$$

for some $C > 0$ independent of W and for some $\alpha \geqslant 1$. Then, for any orthonormal system $(f_j)_{j\in J}$ in $\mathfrak{H}$, and any $(\nu_j)_{j\in J} \subset \mathbb{C}$, the inequality

$$\left\| \sum_{j\in J} \nu_j \left| Af_j \right|^2 \right\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant C \left(\sum_{j\in J} |\nu_j|^{\alpha'} \right)^{1/\alpha'} \tag{17}$$

holds with the same constant C as in (16).

Proof. First, notice that it is enough to show (17) for non-negative coefficients (ν_j). Indeed, assuming (17) for non-negative coefficients and taking $(\nu_j) \subset \mathbb{C}$, one has by the triangle inequality in $\mathbb{C}$

$$\left\| \sum_{j\in J} \nu_j \left| Af_j \right|^2 \right\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant \left\| \sum_{j\in J} |\nu_j| \left| Af_j \right|^2 \right\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant C \left(\sum_{j\in J} |\nu_j|^{\alpha'} \right)^{1/\alpha'}.$$

Hence, let $(f_j)_{j\in J}$ an orthonormal system in $\mathfrak{H}$ and $(\nu_j)_{j\in J} \subset \mathbb{R}_+$. We define an operator γ on $\mathfrak{H}$ with eigenfunctions (f_j) and corresponding eigenvalues (ν_j). In Dirac's notation, we have

$$\gamma = \sum_{j\in J} \nu_j |f_j\rangle\langle f_j|,$$

where $|f_j\rangle\langle f_j|$ denotes the orthogonal projection on $\mathbb{C}f_j \subset \mathfrak{H}$. The estimate (16) is equivalent to

$$\left\| A^*|W|^2 A \right\|_{\mathfrak{S}^\alpha(\mathfrak{H})} \leqslant C \left\| W \right\|^2_{L^{2p/(2-p)}(\mathbb{R}^N)}, \quad \forall W \in L^{2p/(2-p)}(\mathbb{R}^N, \mathbb{C}). \tag{18}$$

Using (18) and Hölder's inequality in Schatten spaces [27, Thm. 2.8], we deduce that

$$\begin{aligned} \mathrm{Tr}_{L^2(\mathbb{R}^N)}(WA\gamma(WA)^*) = \mathrm{Tr}_{\mathfrak{H}}(\gamma A^*|W|^2 A) &\leqslant C \left\| \gamma \right\|_{\mathfrak{S}^{\alpha'}(\mathfrak{H})} \left\| W \right\|^2_{L^{2p/(2-p)}(\mathbb{R}^N)} \\ &= C \left(\sum_{j\in J} \nu_j^{\alpha'} \right)^{1/\alpha'} \left\| W \right\|^2_{L^{2p/(2-p)}(\mathbb{R}^N)}. \end{aligned}$$

Since we have the identity

$$\mathrm{Tr}_{L^2(\mathbb{R}^N)}(WA\gamma(WA)^*) = \int_{\mathbb{R}^N} \left(\sum_{j\in J} \nu_j |(Af_j)(x)|^2 \right) |W(x)|^2 \, dx,$$

we infer that for all $V \in L^{p/(2-p)}(\mathbb{R}^N)$ with $V \geqslant 0$,

$$\int_{\mathbb{R}^N} \left(\sum_{j\in J} \nu_j |(Af_j)(x)|^2 \right) V(x) \, dx \leqslant C \left(\sum_{j\in J} \nu_j^{\alpha'} \right)^{1/\alpha'} \left\| V \right\|_{L^{p/(2-p)}(\mathbb{R}^N)}.$$

The duality principle for L^p-spaces (or choosing $V \equiv 1$ when $p = 2$) leads to the result, since $(p/(2-p))' = p'/2$. ◻

Remark 4. The previous proof shows that (16) and (17) are equivalent to the following bound: for any $\gamma \in \mathfrak{S}^{\alpha'}(\mathfrak{H})$, we have

$$\|\rho_{A\gamma A^*}\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant C\,\|\gamma\|_{\mathfrak{S}^{\alpha'}(\mathfrak{H})}\,, \tag{19}$$

with $C > 0$ independent of γ, and where $\rho_{A\gamma A^*}$ is the *density* of the operator $A\gamma A^*$. It is defined for any finite-rank γ by duality,

$$\int_{\mathbb{R}^N} \rho_{A\gamma A^*}(x)V(x)\,dx := \mathrm{Tr}_{\mathfrak{H}}(\gamma A^* V A),$$

and extended to all $\gamma \in \mathfrak{S}^{\alpha'}(\mathfrak{H})$ using the density of finite-rank operators in $\mathfrak{S}^{\alpha'}(\mathfrak{H})$ and the estimate (19) valid for all finite-rank γ.

To illustrate this duality principle, let us consider the case of Young's inequality. The underlying bounded operator A from $\mathfrak{H} = L^2(\mathbb{R}^N)$ to $L^{p'}(\mathbb{R}^N)$ is $Af = g * f$ for some fixed $g \in L^{2p'/(2+p')}(\mathbb{R}^N)$. Then, the corresponding Schatten bound (16) is the Kato–Seiler–Simon inequality [27, Thm. 4.1],

$$\left\| W|\widehat{g}(-i\nabla)|^2\overline{W} \right\|_{\mathfrak{S}^{p/(2-p)}(L^2(\mathbb{R}^N))} \leqslant (2\pi)^{N(1-2/p)}\,\|W\|_{L^{2p/(2-p)}(\mathbb{R}^N)}\,\|\widehat{g}\|^2_{L^{2p/(2-p)}}\,. \tag{20}$$

We emphasize that our proof of Proposition 1 is based on complex interpolation much like the proof of (20) in [27]. Together with Lemma 3, (20) implies the following Young inequality for systems of orthonormal functions, which we have not encountered in the literature.

Theorem 1 (Young inequality for orthonormal functions)**.** *Let $N \geqslant 1$, $1 \leqslant p \leqslant 2$, and $g \in L^{2p'/(2+p')}(\mathbb{R}^N)$. Then, for any (possibly infinite) orthonormal system (f_j) in $L^2(\mathbb{R}^N)$ and for any $(\nu_j) \subset \mathbb{C}$, we have*

$$\left\| \sum_j \nu_j\,|g * f_j|^2 \right\|_{L^{p'/2}(\mathbb{R}^N)} \leqslant (2\pi)^{-2/p}\,\|\widehat{g}\|^2_{L^{2p/(2-p)}} \left(\sum_j |\nu_j|^{\frac{p'}{2}} \right)^{\frac{2}{p'}}. \tag{21}$$

Remark 5. By the Hausdorff–Young inequality, $\|\widehat{g}\|_{L^{2p/(2-p)}}$ is controlled by $\|g\|_{L^{2p'/(2+p')}}$. For this reason, (21), for a single function f, is somewhat stronger than Young's inequality.

2. Restriction theorems

2.1. **Restriction theorems for orthonormal functions.** As explained in the introduction, we consider the same surfaces as Stein [30] and Strichartz [32]. The surfaces considered by Stein are smooth, compact surfaces embedded in $\mathbb{R}^N$ ($N \geqslant 2$) with non-zero Gauss curvature, endowed with their $(N-1)$-dimensional Lebesgue measure that we denote by $d\sigma$. The quadratic surfaces considered by Strichartz are split into three categories:

- **Case I:** $S = \{\xi \in \mathbb{R}^N,\ \xi_N = \xi_1^2 + \cdots + \xi_a^2 - \xi_{a+1}^2 - \cdots - \xi_{N-1}^2\}$ where $a = 0, \ldots, N-1$. The model case of a surface of this kind is the paraboloid $(a = 0, N-1)$. In this case, the measure is chosen to be $(1 + 4\xi_1^2 + \cdots + 4\xi_{N-1}^2)^{-1/2} d\sigma(\xi)$, where $d\sigma$ is the induced $(N-1)$-dimensional Lebesgue measure on S.
- **Case II:** $S = \{\xi \in \mathbb{R}^N,\ \xi_1^2 + \cdots + \xi_a^2 - \xi_{a+1}^2 - \cdots - \xi_N^2 = 0\}$, where $a = 1, \ldots, N-1$. The model case here is the cone $(a = N-1)$. The measure is chosen to be $(2|\xi|)^{-1} d\sigma(\xi)$.

- **Case III:** $S=\{\xi\in\mathbb{R}^N,\ \xi_1^2+\cdots+\xi_a^2-\xi_{a+1}^2-\cdots-\xi_N^2=-1\}$, where $a=0,\ldots,N-1$. There are two model cases here: the sphere ($a=0$) and the two-sheeted hyperboloid ($a=N-1$). The measure is chosen to be $(2|\xi|)^{-1}d\sigma(\xi)$.

Notice that in the case of quadratic surfaces, the measure is not the usual surface measure $d\sigma$. Writing S as $S=\{\xi:\ R(\xi)=0\}$ where R is the degree two polynomial appearing in the definition of S, we see that the chosen measure is simply $|\nabla R(\xi)|^{-1}d\xi$.

In any of these two cases, the extension operator $\mathcal{E}_S$ is defined by (1), and we denote $T_S=\mathcal{E}_S(\mathcal{E}_S)^*$. Our Schatten bounds on T_S are the following.

Theorem 2 (Schatten properties of extension maps)**.** *Let $N\geqslant 2$ and S a surface embedded in $\mathbb{R}^N$. Then, the inequality*

$$\|W_1T_SW_2\|_{\mathfrak{S}^{2p/(2-p)}(L^2(\mathbb{R}^N))}\leqslant C\,\|W_1\|_{L^{2p/(2-p)}(\mathbb{R}^N)}\,\|W_2\|_{L^{2p/(2-p)}(\mathbb{R}^N)} \tag{22}$$

holds for all W_1,W_2 with a constant $C>0$ independent of W_1,W_2, under the following assumptions on the exponent p:

— If S is a smooth, compact surface with non-zero Gauss curvature, then $1\leqslant p\leqslant 2(N+1)/(N+3)$;

— If S is a quadratic surface, then

- *Case I: $p=2(N+1)/(N+3)$;*
- *Case II: $p=2N/(N+2)$;*
- *Case III:*
  - *(i) $a=0$ and $1\leqslant p\leqslant 2(N+1)/(N+3)$;*
  - *(ii) $a\neq 0$, $N\geqslant 3$, and $2N/(N+2)\leqslant p\leqslant 2(N+1)/(N+3)$;*
  - *(iii) $a=1$, $N=2$, and $1<p\leqslant 6/5$.*

Combining Theorem 2 and Lemma 3 with $\mathfrak{H}:=L^2(S,d\sigma)$, we deduce the following result on restriction properties of systems of orthonormal functions.

Theorem 3 (Restriction estimates for orthonormal functions)**.** *Let $N\geqslant 2$ and S a surface embedded in $\mathbb{R}^N$. Then, for any (possibly infinite) orthonormal system (f_j) in $L^2(S,d\sigma)$ and for any $(\nu_j)\subset\mathbb{C}$, we have*

$$\left\|\sum_j \nu_j\,|\mathcal{E}_Sf_j|^2\right\|_{L^{p'/2}(\mathbb{R}^N)}\leqslant C\left(\sum_j|\nu_j|^{\frac{2p'}{p'+2}}\right)^{\frac{p'+2}{2p'}}, \tag{23}$$

with $C>0$ independent of (ν_j) and (f_j). The exponent p satisfies the same assumptions as in Theorem 2, according to the type of S.

Furthermore, according to Remark 4, for any $\gamma\in\mathfrak{S}^{2p'/(p'+2)}(L^2(S,d\sigma))$, the inequality

$$\left\|\rho_{\mathcal{E}_S\gamma(\mathcal{E}_S)^*}\right\|_{L^{p'/2}(\mathbb{R}^N)}\leqslant C\,\|\gamma\|_{\mathfrak{S}^{2p'/(p'+2)}(L^2(S,d\sigma))}, \tag{24}$$

holds with $C>0$ independent of γ. We shall see later (Theorem 4) that the Schatten exponent $2p'/(2+p')$ is the best possible for $p=2(N+1)/(N+3)$.

2.2. **Proof of Theorem 2.**

2.2.1. *Compact case.* Let us first consider the case S compact with non-zero Gauss curvature. Thus, let $N \geqslant 2$ and $S \subset \mathbb{R}^N$ a compact hypersurface with non-zero Gauss curvature, which is endowed with its $(N-1)$-dimensional Lebesgue measure $d\sigma$. The operator T_S acting on functions on $\mathbb{R}^N$ is a convolution operator: $T_S f = K_S * f$ for all f, where K_S is the function

$$K_S(x) = \frac{1}{(2\pi)^N} \int_S e^{ix\cdot\xi} d\sigma(\xi), \quad \forall x \in \mathbb{R}^N.$$

Using a smooth and finite partition of unity, $1 = \sum_\ell \psi_\ell$ on S, the operator T_S can be decomposed as $T_S = \sum_\ell T_\ell$, where T_ℓ is the convolution operator by the function

$$K_\ell(x) = \frac{1}{(2\pi)^N} \int_S e^{ix\cdot\xi} \psi_\ell(\xi) d\sigma(\xi), \quad \forall x \in \mathbb{R}^N.$$

The partition of unity is chosen in the following fashion. We assume that on the interior of the support of ψ_ℓ, the surface S is the graph of a smooth and compactly supported function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$, so that (possibly after a rotation),

$$K_\ell(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{N-1}} e^{ix\cdot(\xi',\varphi(\xi'))} \psi_\ell(\xi', \varphi(\xi'))(1 + |\nabla\varphi(\xi')|^2)^{1/2} d\xi', \quad \forall x \in \mathbb{R}^N.$$

To prove (22), it is then enough to show the estimate

$$\|W_1 T_\ell W_2\|_{\mathfrak{S}^{2p/(2-p)}(L^2(\mathbb{R}^N))} \leqslant C \|W_1\|_{L^{2p/(2-p)}(\mathbb{R}^N)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{R}^N)}$$

for each ℓ. Hence, from now on we drop the index ℓ and write (T, K) instead of (T_ℓ, K_ℓ). To prove this Schatten estimate, we use Proposition 1 by defining the same analytic family (T_z) of operators as in [30]. More precisely, let T_z be the convolution operator with the function K_z defined as

$$K_z(x) := \zeta_z(x_N) K(x), \quad \forall x \in \mathbb{R}^N,$$

where

$$\zeta_z(y) := \frac{1}{(2\pi)^N} \frac{e^{(z+1)^2}}{\Gamma(z+1)} \int_0^\infty e^{ity} t^z \eta(t)\, dt, \quad \forall y \in \mathbb{R},$$

where η is a smooth an compactly supported function on $\mathbb{R}$ such that $\eta \equiv 1$ on a neighborhood of the origin. As explained in [31, Ch. IX, Sec. 1.2.3], the family (T_z) is an analytic family of operators in the strip $-\lambda_0 \leqslant \operatorname{Re} z \leqslant 0$, with $1 \leqslant \lambda_0 \leqslant (N+1)/2$, which satisfies the estimate

$$\|T_{is}\|_{L^2 \to L^2} + \|T_{-\lambda_0+is}\|_{L^1 \to L^\infty} \leqslant C(s),$$

for all $s \in \mathbb{R}$ and for some $C(s)$ growing exponentially in s. By Proposition 1 and the identity $T = T_{-1}$, this proves (22) in the compact case.

2.2.2. *Quadratic case.* All three kinds of quadratic surfaces considered by Strichartz [32] are of the form $\{\xi : R(\xi) = r\}$ for some degree two polynomial R and some $r \in \mathbb{R}$. Strichartz introduces the family of tempered distributions $(G_z)_{z\in\mathbb{C}}$ on $\mathbb{R}^N$ as

$$\langle G_z, \varphi \rangle := g(z) \int_{\mathbb{R}^N} (R(\xi) - r)_+^z \varphi(\xi) d\xi := g(z) \int_{\mathbb{R}} (a - r)_+^z \left(\int_{S_a} \varphi(\xi) d\mu_a(\xi) \right) da,$$

where $S_a = \{x : R(x) = a\}$ and $d\mu_a(\xi) = |\nabla R(\xi)|^{-1/2} d\sigma_a(\xi)$, with $d\sigma_a(\xi)$ the $(N-1)$-dimensional Lebesgue measure on S_a. The function $g(z)$ has adequate properties according to the type of the surface considered, but in all cases it has a simple zero at $z=-1$ to ensure that $G_{-1}\equiv\delta_S$. The family of operators T_z is then defined as Fourier multipliers by G_z, that is

$$T_z f(x)=\langle G_z,\widehat{f}(\xi)e^{i\xi\cdot x}\rangle,\quad \forall x\in\mathbb{R}^N.$$

Strichartz then shows the bounds

$$\|T_{is}\|_{L^2\to L^2}+\|T_{-\lambda_0+is}\|_{L^1\to L^\infty}\leqslant C(s),$$

for all $s\in\mathbb{R}$ and for $C(s)$ growing exponentially, with $\lambda_0=(N+1)/2$ (Case I), $\lambda_0=N/2$ (Case II), $\lambda_0\geqslant(N+1)/2$ (Case III(i)), $N/2\leqslant\lambda_0\leqslant(N+1)/2$ (Case III(ii)), and $1<\lambda_0\leqslant 3/2$ (Case III(iii)). Together with Proposition 1, this shows (22) in the quadratic case and the proof of Theorem 2 is over.

Remark 6. Case I of a quadratic surface can also be reduced from the compact case via scaling, as in [31, Sec. VIII.5.16].

2.3. **Optimality of the Schatten exponent in the compact case.** We also discuss the optimality of the Schatten space $\mathfrak{S}^{2p'/(2+p')}(L^2(S))$ in (24), in the following sense: given $1\leqslant p\leqslant 2(N+1)/(N+3)$, we give an upper bound on the highest Schatten exponent such that (24) can possibly hold. For the sake of brevity, we only treat the compact case.

Theorem 4 (Optimality of the Schatten exponent)**.** *Let $N\geqslant 2$. Assume that $S\subset\mathbb{R}^N$ is a smooth surface with non-zero Gauss curvature, and let $1\leqslant p\leqslant 2(N+1)/(N+3)$. Then, for any $r>\frac{(N-1)p}{2N(p-1)}$, we have*

$$\sup_{\substack{\gamma\in\mathfrak{S}^r(L^2(S)),\\ \gamma\neq 0}}\frac{\left\|\rho_{\mathcal{E}_S\gamma(\mathcal{E}_S)^*}\right\|_{L^{p'/2}(\mathbb{R}^N)}}{\|\gamma\|_{\mathfrak{S}^r(L^2(S))}}=+\infty. \tag{25}$$

Remark 7. When $1\leqslant p\leqslant\frac{2(N+1)}{N+3}$, we have $\frac{(N-1)p}{2N(p-1)}\geqslant\frac{2p'}{2+p'}$ with equality if and only if $p=\frac{2(N+1)}{N+3}$. This shows, in particular, that the Schatten exponent on the right side of (24) is optimal when $p=\frac{2(N+1)}{N+3}$.

Proof of Theorem 4. Let $h>0$. We construct a trial operator γ_h on $L^2(S)$, by defining its integral kernel:

$$\gamma_h(\omega,\omega')=\int_{\mathbb{R}^N}\mathbb{1}(k^2\leqslant h^{-2})e^{ik\cdot(\omega-\omega')}\,dk,\ \forall(\omega,\omega')\in S\times S.$$

Let $f\in L^2(S)$. Using the Agmon–Hörmander inequality [2], [25, Thm. 4.2], we have

$$\langle f,\gamma_h f\rangle=\int_{\mathbb{R}^N}\mathbb{1}(k^2\leqslant h^{-2})\left|\int_S f(\omega)e^{-ik\cdot\omega}\,d\sigma(\omega)\right|^2 dk\leqslant Ch^{-1}\int_S|f(\omega)|^2\,d\sigma(\omega).$$

This shows that γ_h is a non-negative bounded operator on $L^2(S)$ with

$$\|\gamma_h\|_{L^2(S)\to L^2(S)}\leqslant Ch^{-1}.$$

We also compute its trace norm,

$$\|\gamma_h\|_{\mathfrak{S}^1(L^2(S))}=\operatorname{Tr}\gamma_h=\int_S\gamma_h(\omega,\omega)\,d\omega=Ch^{-N}.$$

By Hölder's inequality in Schatten spaces, we deduce that $\gamma_h \in \mathfrak{S}^r$ for all $1 \leqslant r \leqslant +\infty$ and that

$$\|\gamma_h\|_{\mathfrak{S}^r(L^2(S))} \leqslant C(h^{-1})^{\frac{N+r-1}{r}}. \tag{26}$$

Let us compute the left side of (24) for γ_h. We have

$$\rho_{\mathcal{E}_S \gamma_h (\mathcal{E}_S)^*}(\xi) = \int_{\mathbb{R}^N} \mathbb{1}(k^2 \leqslant h^{-2}) \left| \int_S e^{-i\omega\cdot\xi} e^{ik\cdot\omega}\, d\sigma(\omega) \right|^2 = \int_{\mathbb{R}^N} \mathbb{1}(k^2 \leqslant h^{-2}) \left| \widehat{d\sigma}(\xi - k) \right|^2 dk,$$

for all $\xi \in \mathbb{R}^N$. First, let us use the lower bound

$$\begin{aligned}\left\|\rho_{\mathcal{E}_S\gamma_h(\mathcal{E}_S)^*}\right\|_{L^{p'/2}(\mathbb{R}^N)} &\geqslant \left(\int_{|\xi|\leqslant h^{-1}R} \rho_{\mathcal{E}_S\gamma_h(\mathcal{E}_S)^*}(\xi)^{p'/2}\, d\xi\right)^{2/p'} \\ &= h^{-2N/p'} \left(\int_{|\xi|\leqslant R} \rho_{\mathcal{E}_S\gamma_h(\mathcal{E}_S)^*}(h^{-1}\xi)^{p'/2}\, d\xi\right)^{2/p'},\end{aligned}$$

for some $R > 0$ to be chosen later on. Next, we use a lower bound on $\widehat{d\sigma}$ which can be found, for instance, in [28, p. 51]: there exists a non-empty open cone $\Gamma \subset \mathbb{R}^N$ such that for all $k \in \Gamma$ with $|k| \geqslant R'$ for some $R' > 0$ large enough, we have

$$|\widehat{d\sigma}(k)| \geqslant \frac{C}{(1+|k|)^{\frac{N-1}{2}}}. \tag{27}$$

Notice that in the case of the sphere, (27) can be proved directly using that $\widehat{d\sigma}$ is explicitly expressed in terms of a Bessel function. In any case, this implies that, if $h \leqslant 1$,

$$\begin{aligned}\rho_{E\gamma_h E^*}(h^{-1}\xi) &\geqslant \int_{\substack{k\in\Gamma \\ |k|\geqslant h^{-1}R'}} \mathbb{1}(|k - h^{-1}\xi| \leqslant h^{-1}) \left|\widehat{d\sigma}(k)\right|^2 dk \\ &\geqslant h^{-N} \int_{\substack{k\in\Gamma \\ |k|\geqslant R'}} \mathbb{1}(|k-\xi| \leqslant 1) \left|\widehat{d\sigma}(h^{-1}k)\right|^2 dk \\ &\geqslant Ch^{-1} \int_{\substack{k\in\Gamma \\ |k|\geqslant R'}} \frac{\mathbb{1}(|k-\xi|\leqslant 1)}{(1+|k|)^{N-1}}\, dk.\end{aligned}$$

Choosing $R = R'$, we infer that

$$\left\|\rho_{\mathcal{E}_S\gamma_h(\mathcal{E}_S)^*}\right\|_{L^{p'/2}(\mathbb{R}^N)} \geqslant Ch^{-\frac{2N}{p'}-1} \left(\int_{|\xi|\leqslant R'} \left(\int_{\substack{k\in\Gamma \\ |k|\geqslant R'}} \frac{\mathbb{1}(|k-\xi|\leqslant 1)}{(1+|k|)^{N-1}}\, dk\right)^{p'/2} d\xi\right)^{2/p'}.$$

The double integral on the right side is easily seen to be finite. Combining this estimate with (26), this shows that

$$\frac{\left\|\rho_{\mathcal{E}_S\gamma_h(\mathcal{E}_S)^*}\right\|_{L^{p'/2}(\mathbb{R}^N)}}{\|\gamma_h\|_{\mathfrak{S}^r(L^2(S))}} \geqslant c(h^{-1})^{\frac{2N}{p'}+1-\frac{N+r-1}{r}},$$

which diverges as $h \to 0$ if and only if $r > \frac{(N-1)p}{2N(p-1)}$. □

3. Strichartz inequalities

3.1. **Laplacian case.** An important application of the restriction estimates for quadratic surfaces proved in [32] concerns space-time decay estimates for solutions to evolution equations, which are known as *Strichartz inequalities* and are a widely used tool to study nonlinear versions of these equations. For instance, when S is the paraboloid

$$S = \{(\omega, \xi) \in \mathbb{R} \times \mathbb{R}^d,\, \omega = -|\xi|^2\},$$

one has the identity for all $f \in L^1(S, d\mu)$ and for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathcal{E}_S f(t,x) = \frac{1}{(2\pi)^{d+1}} \int_S e^{i(t,x)\cdot(\omega,\xi)} f(\omega,\xi) d\mu(\omega,\xi) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d} e^{-it|\xi|^2} e^{ix\cdot\xi} f(-|\xi|^2, \xi)\, d\xi,$$

where $d\mu$ is the measure defined at the beginning of Section 2.1, which in the case of the paraboloid (Case I) is simply $d\mu(\omega,\xi) = (1+4|\xi|^2)^{-1/2} d\sigma(\omega,\xi)$. Hence, choosing $f(\omega,\xi) = \widehat{\varphi}(\xi)$ for some $\varphi : \mathbb{R}^d \to \mathbb{C}$, one deduces that

$$\mathcal{E}_S f(t,x) = \frac{1}{2\pi} \left(e^{it\Delta}\varphi\right)(x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

Using that $\mathcal{E}_S$ is bounded from $L^2(S, d\mu)$ to $L^{2+4/d}(\mathbb{R}^{d+1})$, Strichartz obtains his famous bound

$$\left\| e^{it\Delta} f \right\|_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left\| f \right\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in L^2(\mathbb{R}^d), \quad \forall d \geqslant 1, \tag{28}$$

where $C > 0$ is independent of f. In the same fashion, applying Theorem 3 with $N = d+1$ and S a paraboloid (Case I), we recover Strichartz's bound for orthonormal functions [9]:

Theorem 5 (Strichartz estimates for orthonormal functions—diagonal case)**.** *Assume that $d \geqslant 1$. Then, for any (possibly infinite) orthonormal system (f_j) in $L^2(\mathbb{R}^d)$ and for any $(\nu_j) \subset \mathbb{C}$, we have*

$$\left\| \sum_j \nu_j \left| e^{it\Delta} f_j \right|^2 \right\|_{L^{1+2/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left(\sum_j |\nu_j|^{\frac{d+2}{d+1}} \right)^{\frac{d+1}{d+2}}, \tag{29}$$

with $C > 0$ independent of (ν_j) and (f_j).

Equivalently, according to Remark 4, for any $\gamma \in \mathfrak{S}^{(d+2)/(d+1)}(L^2(\mathbb{R}^d))$, the inequality

$$\left\| \rho_{e^{it\Delta}\gamma e^{-it\Delta}} \right\|_{L^{1+2/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left\| \gamma \right\|_{\mathfrak{S}^{(d+2)/(d+1)}(L^2(\mathbb{R}^d))}, \tag{30}$$

holds with $C > 0$ independent of γ. As explained in the introduction, this result was proved for the first time in [9], using a different method. We recover it as a consequence of more general restriction estimates for orthonormal functions, hence providing a different proof. Our method actually allows to go further and to answer a question left open in [9]:

Theorem 6 (Strichartz estimates for orthonormal functions—general case)**.** *Assume that $d \geqslant 1$ and that $p, q \geqslant 1$ are such that*

$$\frac{2}{p} + \frac{d}{q} = d, \quad 1 \leqslant q < 1 + \frac{2}{d-1}.$$

Then, for any (possibly infinite) orthonormal system (f_j) in $L^2(\mathbb{R}^d)$ and for any $(\nu_j) \subset \mathbb{C}$, we have

$$\left\| \sum_j \nu_j \left| e^{it\Delta} f_j \right|^2 \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left(\sum_j |\nu_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}, \tag{31}$$

with $C > 0$ independent of (ν_j) and (f_j).

Equivalently, according to Remark 4, for any $\gamma \in \mathfrak{S}^{2q/(q+1)}(L^2(\mathbb{R}^d))$, the inequality

$$\left\| \rho_{e^{it\Delta}\gamma e^{-it\Delta}} \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant C \left\| \gamma \right\|_{\mathfrak{S}^{2q/(q+1)}(L^2(\mathbb{R}^d))}, \tag{32}$$

holds with $C > 0$ independent of γ. In [9], this result was proved only for the range $1 \leqslant q \leqslant 1 + 2/d$, and was shown to *fail* for $q \geqslant 1 + 2/(d-1)$. Hence, Theorem 6 provides the *full range* of exponents of Strichartz estimates for orthonormal functions. Notice that this range is significantly smaller than the range for a single function which is $1 \leqslant q \leqslant 1+2/(d-2)$ for $d \geqslant 3$ [15]. In Section 5, we give an application of these inequalities to the well-posedness of the non-linear Hartree equation in Schatten spaces, in the spirit of [20].

Theorem 6 follows again from a Schatten bound coupled to Lemma 3:

Theorem 7 (Schatten bound with space-time norms)**.** *Let $d \geqslant 1$ and S be the paraboloid*

$$S := \{(\omega, \xi) \in \mathbb{R} \times \mathbb{R}^d,\ \omega = -|\xi|^2\}.$$

Then, for all exponents $p, q \geqslant 1$ satisfying the relations

$$\frac{2}{p} + \frac{d}{q} = 1, \qquad q > d + 1,$$

we have the Schatten bound

$$\left\| W_1 T_S W_2 \right\|_{\mathfrak{S}^q(L^2(\mathbb{R}^{d+1}))} \leqslant C \left\| W_1 \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^d)} \left\| W_2 \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^d)},$$

with $C > 0$ independent of W_1, W_2.

Proof of Theorem 7. We investigate more precisely the bounds on the family G_z introduced in the proof of Theorem 2 when S is the paraboloid. Strichartz [32] uses the definition

$$G_z(\omega, \xi) = \frac{1}{\Gamma(z+1)} (\omega - |\xi|^2)_+^z,\ \forall (\omega, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

which ensures that the Fourier multiplication operator with G_{-1} coincides with the operator T_S. As before, we have a first bound

$$\| T_{is} \|_{L^2(\mathbb{R}^{d+1}) \to L^2(\mathbb{R}^{d+1})} = \| G_{is} \|_{L^\infty(\mathbb{R}^{d+1})} \leqslant \left| \frac{1}{\Gamma(1+is)} \right| \leqslant C e^{\pi|s|/2}.$$

To prove that $T_{-\lambda_0+is}$ is bounded from L^1 to L^∞, Strichartz computes explicitly the (inverse) Fourier transform of G_z and obtains

$$\check{G}_z(t,x) = \pi^{-\frac{d+1}{2}} i e^{iz\pi/2} e^{-i\pi d/4} e^{-i|x|^2/4t} |t|^{-d/2} (-t - i0)^{-z-1},\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

He deduces from this formula that $\check{G}_z$ belongs to $L^\infty_{t,x}$ when $\operatorname{Re} z = -1-d/2$. We now explain how to obtain better results from this expression than the one obtained in Theorem 2. To do so, recall that the distribution $(-t-i0)^\lambda$ on $\mathbb{R}$ satisfies the identity

$$(-t-i0)^\lambda = t_-^\lambda + e^{-i\pi\lambda} t_+^\lambda$$

for $\operatorname{Re}\lambda > -1$ [12, Ch. I, Sec. 3.6], where $t_\pm^\lambda$ are the distributions given by the $L^1_{\rm loc}$-functions

$$t_+^\lambda = \begin{cases} t^\lambda & \text{for } t>0 \\ 0 & \text{for } t \leqslant 0, \end{cases} \qquad t_-^\lambda = \begin{cases} 0 & \text{for } t \geqslant 0 \\ (-t)^\lambda & \text{for } t<0. \end{cases}$$

In particular, the distribution $(-t-i0)^\lambda$ is also given by a $L^1_{\rm loc}$-function, and we deduce the bound

$$\left|(-t-i0)^\lambda\right| \leqslant \max\left(1, e^{\pi \operatorname{Im}\lambda}\right) |t|^{\operatorname{Re}\lambda}, \quad \forall t \in \mathbb{R},$$

valid for all $\operatorname{Re}\lambda > -1$. In our context, we have $\lambda = -z-1$ with $z = -\lambda_0 + is$, so that $\operatorname{Re}\lambda = \lambda_0 - 1 > 0$. We thus deduce the bound

$$\left|\check{G}_{-\lambda_0+is}(t,x)\right| \leqslant C \max\left(1, e^{-3\pi s/2}\right) |t|^{\lambda_0-1-d/2}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^d,$$

valid for all $s \in \mathbb{R}$ and for all $\lambda_0 > 1$. We now go back to the proof of Proposition 1 and provide another estimate for $\left\|W_1^{-z} T_z W_2^{-z}\right\|_{\mathfrak{S}^2}$ when $z = -\lambda_0 + is$ using the Hardy–Littlewood–Sobolev inequality:

$$\begin{aligned}
\left\|W_1^{\lambda_0-is} T_{-\lambda_0+is} W_2^{\lambda_0-is}\right\|_{\mathfrak{S}^2}^2 &= \int_{\mathbb{R}^{2(d+1)}} W_1(t,x)^{2\lambda_0} \left|\check{G}_{-\lambda_0+is}(t-t', x-x')\right|^2 W_2(t',x')^{2\lambda_0}\, dx dx' dt dt' \\
&\leqslant C \max\left(1, e^{-3\pi s/2}\right) \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|W_1(t)\|^{2\lambda_0}_{L^{2\lambda_0}_x(\mathbb{R}^d)} \|W_2(t')\|^{2\lambda_0}_{L^{2\lambda_0}_x(\mathbb{R}^d)}}{|t-t'|^{d+2-2\lambda_0}}\, dt dt' \\
&\leqslant C \max\left(1, e^{-3\pi s/2}\right) \|W_1\|^{2\lambda_0}_{L_t^{\frac{4\lambda_0}{2\lambda_0-d}} L_x^{2\lambda_0}(\mathbb{R}\times\mathbb{R}^d)} \|W_2\|^{2\lambda_0}_{L_t^{\frac{4\lambda_0}{2\lambda_0-d}} L_x^{2\lambda_0}(\mathbb{R}\times\mathbb{R}^d)},
\end{aligned}$$

provided that $0 \leqslant d+2-2\lambda_0 < 1$, that is $(d+1)/2 < \lambda_0 \leqslant 1 + d/2$. For this range of λ_0, we conclude as in the proof of Proposition 1 that

$$\|W_1 T_{-1} W_2\|_{\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^{d+1}))} \leqslant C \|W_1\|_{L_t^{\frac{4\lambda_0}{2\lambda_0-d}} L_x^{2\lambda_0}(\mathbb{R}\times\mathbb{R}^d)} \|W_2\|_{L_t^{\frac{4\lambda_0}{2\lambda_0-d}} L_x^{2\lambda_0}(\mathbb{R}\times\mathbb{R}^d)},$$

which is the desired estimate. □

Remark 8. The same proof actually gives the full range of Strichartz estimates for a single function, except for the endpoints. Hence, all Strichartz estimates (and not only the diagonal ones) are implicitly contained in Strichartz's original article, except the endpoints. We are not aware that this has been observed before.

3.2. **Square root of the Laplacian case.** When S is the cone $S=\{(\omega,\xi)\in\mathbb{R}\times\mathbb{R}^d,\ \omega^2=|\xi|^2\}$ endowed with the measure $d\mu(\omega,\xi)=(2|(\omega,\xi)|)^{-1}d\sigma(\omega,\xi)$, one has the identity for all $f\in L^1(S,d\mu)$ and for all $(t,x)\in\mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned}\mathcal{E}_S f(t,x) &= \frac{1}{(2\pi)^{d+1}}\int_S e^{i(t,x)\cdot(\omega,\xi)}f(\omega,\xi)\,d\mu(\omega,\xi)\\ &= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d} e^{it|\xi|}e^{ix\cdot\xi}f(|\xi|,\xi)\frac{d\xi}{2\sqrt{2}|\xi|}+\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d} e^{-it|\xi|}e^{ix\cdot\xi}f(-|\xi|,\xi)\frac{d\xi}{2\sqrt{2}|\xi|}.\end{aligned}$$

In particular, when one chooses $f(\omega,\xi)=2\sqrt{2}|\xi|\widehat{\varphi}(\xi)$ if $\omega>0$ and $f(\omega,\xi)=0$ if $\omega<0$, we have the identity

$$\mathcal{E}_S f(t,x)=\frac{1}{2\pi}\left(e^{it(-\Delta)^{1/2}}\varphi\right)(x),\quad \forall(t,x)\in\mathbb{R}\times\mathbb{R}^d.$$

Since $\mathcal{E}_S$ is bounded from $L^2(S,d\mu)$ to $L^{2(d+1)/(d-1)}(\mathbb{R}^{d+1})$, we deduce the following Strichartz inequality

$$\left\|e^{it(-\Delta)^{1/2}}\varphi\right\|_{L^{2(d+1)/(d-1)}_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\,\|\varphi\|_{\dot{H}^{1/2}(\mathbb{R}^d)},$$

with $C>0$ independent of φ. We obtain the corresponding version of this result for orthonormal functions.

Theorem 8 (Strichartz estimates for orthonormal functions—fractional Laplacian case)**.** *Assume that $d\geqslant 1$. Then, for any (possibly infinite) orthonormal system (f_j) in $\dot{H}^{1/2}(\mathbb{R}^d)$ and for any $(\nu_j)\subset\mathbb{C}$, we have*

$$\left\|\sum_j \nu_j\left|e^{it(-\Delta)^{1/2}}f_j\right|^2\right\|_{L^{\frac{d+1}{d-1}}_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\left(\sum_j|\nu_j|^{1+\frac{1}{d}}\right)^{\frac{d}{d+1}}, \tag{33}$$

with $C>0$ independent of (ν_j) and (f_j).

We also have the operator version of this inequality

$$\left\|\rho_{e^{-it(-\Delta)^{1/2}}\gamma e^{it(-\Delta)^{1/2}}}\right\|_{L^{\frac{d+1}{d-1}}_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\left\|(-\Delta)^{1/4}\gamma(-\Delta)^{1/4}\right\|_{\mathfrak{S}^{1+\frac{1}{d}}(L^2(\mathbb{R}^d))}, \tag{34}$$

which holds with $C>0$ independent of γ.

Proof of Theorem 8. If (f_j) is an orthonormal system in $\dot{H}^{1/2}(\mathbb{R}^d)$, the functions

$$(\omega,\xi)\mapsto g_j(\omega,\xi)=\mathbb{1}(\omega>0)2\sqrt{2}|\xi|f_j(\xi)$$

are orthonormal in $L^2(S,d\mu)$ as explained in the beginning of this section. We then apply Theorem 3 to this system, with S being a cone (Case II). □

3.3. **Pseudo-relativistic case.** Finally, when the surface S is the two-sheeted hyperboloid $S=\{(\omega,\xi)\in\mathbb{R}\times\mathbb{R}^d,\ \omega^2=1+|\xi|^2\}$, with the measure $d\mu(\omega,\xi)=(2|(\omega,\xi)|)^{-1}d\sigma(\omega,\xi)$, one has the identity for all $f\in L^1(S,d\mu)$ and for all $(t,x)\in\mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned}\mathcal{E}_S f(t,x)&=\frac{1}{(2\pi)^{d+1}}\int_S e^{i(t,x)\cdot(\omega,\xi)}f(\omega,\xi)\,d\mu(\omega,\xi)\\&=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d}e^{it\sqrt{1+|\xi|^2}}e^{ix\cdot\xi}f(\sqrt{1+|\xi|^2},\xi)\frac{d\xi}{2\sqrt{1+|\xi|^2}}\\&\quad+\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d}e^{-it\sqrt{1+|\xi|^2}}e^{ix\cdot\xi}f(-\sqrt{1+|\xi|^2},\xi)\frac{d\xi}{2\sqrt{1+|\xi|^2}}.\end{aligned}$$

In particular, when one chooses $f(\omega,\xi)=2\mathbb{1}(\omega>0)\sqrt{1+|\xi|^2}\widehat{\varphi}(\xi)$, we have the identity

$$\mathcal{E}_S f(t,x)=\frac{1}{2\pi}\left(e^{it\sqrt{1-\Delta}}\varphi\right)(x),\quad\forall(t,x)\in\mathbb{R}\times\mathbb{R}^d.$$

Since $\mathcal{E}_S$ is bounded from $L^2(S,d\mu)$ to $L^q(\mathbb{R}^{d+1})$, with $2+4/d\leqslant q\leqslant 2+4/(d-1)$ $(d\geqslant 2)$ and $6\leqslant q<\infty$ $(d=1)$, we deduce the following Strichartz inequality

$$\left\|e^{it\sqrt{1-\Delta}}\varphi\right\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\left\|\varphi\right\|_{H^{1/2}(\mathbb{R}^d)},$$

with $C>0$ independent of φ. We obtain the corresponding version of this result for orthonormal functions.

Theorem 9 (Strichartz estimates for orthonormal functions—pseudo-relativistic case)**.** *Assume that $d\geqslant 1$. Let $1+2/d\leqslant q\leqslant 1+2/(d-1)$ if $d\geqslant 2$ and $3\leqslant q<\infty$ if $d=1$. Then, for any (possibly infinite) orthonormal system (f_j) in $H^{1/2}(\mathbb{R}^d)$, and for any $(\nu_j)\subset\mathbb{C}$, we have*

$$\left\|\sum_j\nu_j\left|e^{it\sqrt{1-\Delta}}f_j\right|^2\right\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\left(\sum_j|\nu_j|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}},\tag{35}$$

with $C>0$ independent of (ν_j) and (f_j).

We also have the operator version of this inequality

$$\left\|\rho_{e^{-it\sqrt{1-\Delta}}\gamma e^{it\sqrt{1-\Delta}}}\right\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\left\|(1-\Delta)^{1/4}\gamma(1-\Delta)^{1/4}\right\|_{\mathfrak{S}^{\frac{2q}{q+1}}(L^2(\mathbb{R}^d))},\tag{36}$$

which holds with $C>0$ independent of γ.

Proof of Theorem 9. If (f_j) is an orthonormal system in $H^{1/2}(\mathbb{R}^d)$, the functions

$$(\omega,\xi)\mapsto g_j(\omega,\xi)=\mathbb{1}(\omega>0)2(1+|\xi|^2)^{1/2}f_j(\xi)$$

are orthonormal in $L^2(S,d\mu)$ as explained in the beginning of this section. We then apply Theorem 3 to this system, with S being a two-sheeted hyperboloid (Case III(ii-iii)). $\square$

4. Uniform Sobolev estimates and the Limiting Absorption Principle

Our last result concerns Schatten class properties of the resolvent $(-\Delta - z)^{-1}$ of the Laplace operator on $\mathbb{R}^N$.

Theorem 10 (Uniform resolvent bounds in Schatten spaces)**.** *Let $N \geqslant 2$ and assume that*

$$\begin{cases} 1 < q \leqslant 3/2 & \text{if } N = 2, \\ \frac{N}{2} \leqslant q \leqslant \frac{N+1}{2} & \text{if } N \geqslant 3. \end{cases}$$

Then, for all $z \in \mathbb{C} \setminus [0, \infty)$, we have the estimate

$$\left\| W_1(-\Delta - z)^{-1} W_2 \right\|_{\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))} \leqslant C|z|^{-1+\frac{N}{2q}} \left\| W_1 \right\|_{L^{2q}(\mathbb{R}^N)} \left\| W_2 \right\|_{L^{2q}(\mathbb{R}^N)}, \tag{37}$$

where $C > 0$ is independent of W and z.

Remark 9. The estimate (37) with the Schatten norm replaced by the operator norm was proved by Kenig, Ruiz, and Sogge [16]. Their result is only stated for $|z| \geqslant 1$ in [16]. The estimate we presented follows easily for all $z \neq 0$ by homogeneity. The case $N = 2$ is not contained in [16], but can be treated along the same lines [7].

Proof of Theorem 10. The Fourier multiplier T by the function $\xi \mapsto (|\xi|^2 - \mu)^{-1}$ ($\mu \in \mathbb{C}$, $|\mu| \geqslant 1$, $\operatorname{Im} \mu \neq 0$) is interpolated in [16] by the family of Fourier multipliers (T_z) by the functions

$$m_z(\xi) := \frac{e^{z^2}}{\Gamma(N/2 + z)} (|\xi|^2 - \mu)^z, \quad \forall \xi \in \mathbb{R}^N,$$

where $(\cdot)^z$ denotes the principal branch. Then, Kenig, Ruiz, and Sogge prove the uniform estimate

$$\|T_{is}\|_{L^2 \to L^2} + \|T_{-\lambda_0 + is}\|_{L^1 \to L^\infty} \leqslant C,$$

for all $s \in \mathbb{R}$, with $C > 0$ independent of s and μ, and for $N/2 \leqslant \lambda_0 \leqslant (N+1)/2$. Theorem 10 thus follows from Proposition 1. The proof for $N = 2$ is similar, see [7]. ☐

Remark 10. When $q = \frac{N+1}{2}$, the Schatten space $\mathfrak{S}^{2q}$ is optimal in (37). Indeed, it is well-known that

$$W_1 \left(\frac{1}{-\Delta - 1 - it} - \frac{1}{-\Delta - 1 + it} \right) W_2 \xrightarrow[t \to 0_+]{} W_1 T_{\mathbb{S}^{N-1}} W_2$$

weakly in the sense of operators on $L^2(\mathbb{R}^N)$, where we recall that $T_{\mathbb{S}^{N-1}}$ was defined in Section 2. In particular, the bound (37) implies a Schatten bound on $W_1 T_{\mathbb{S}^{N-1}} W_2$ by the non-commutative Fatou lemma [27, Thm. 2.7d)], for which we know the optimal exponent by Theorem 4.

As an application of this result, we prove a Limiting Absorption Principle in Schatten spaces.

Theorem 11 (Limiting Absorption Principle in Schatten spaces)**.** *Let $N \geqslant 2$ and assume that $V \in L^q(\mathbb{R}^N, \mathbb{R})$ with*

$$\begin{cases} 1 < q \leqslant 3/2 & \text{if } N = 2 \\ \frac{N}{2} \leqslant q \leqslant \frac{N+1}{2} & \text{if } N \geqslant 3\,. \end{cases}$$

Then $\sqrt{V}(-\Delta+V-z)^{-1}\sqrt{|V|}\in\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))$ *for every* $z\in\mathbb{C}\setminus[0,\infty)$, *where we used the notation* $\sqrt{V}:=V/\sqrt{|V|}$ *(with* $\sqrt{V}:=0$ *if* $V=0$*). The mapping* $\mathbb{C}\setminus[0,\infty)\ni z\mapsto\sqrt{V}(-\Delta+V-z)^{-1}\sqrt{|V|}\in\mathfrak{S}^{2q}$ *is analytic and extends continuously to the open interval* $(0,\infty)$ *(with possibly different boundary values from above and below). Moreover, under the additional assumption* $q>N/2$, *there is a constant* $C_{N,q}$ *such that if* $|z|^{-1+N/2q}\|V\|_{L^q(\mathbb{R}^N)}\leqslant C_{N,q}$ *then*

$$\left\|\sqrt{V}(-\Delta+V-z)^{-1}\sqrt{|V|}\right\|_{\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))}\leqslant 2C_{N,q}|z|^{-1+N/2q}\|V\|_{L^q(\mathbb{R}^N)}\,. \tag{38}$$

If $q=N/2$ *and* $N\geqslant 3$, *the bound* (38) *holds provided* $|z|\geqslant C(V)$ *for some constant* $C(V)$ *depending on* V.

Before proving Theorem 11, we need more information about the Birman–Schwinger operator $\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}$.

Lemma 11. *Let* $N\geqslant 2$ *and assume that* $V\in L^q(\mathbb{R}^N,\mathbb{R})$, *where* q *satisfies the assumptions of Theorem 11. Let* $\delta\subset(0,\infty)$ *be a compact interval. Then, the family* $A(z):=\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}\in\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))$ *is analytic on the half-strips* $S_\pm:=\{z\in\mathbb{C},\ \mathrm{Re}\,z\in\mathring{\delta},\ \pm\mathrm{Im}\,z>0\}$. *On each* $S_\pm$, *it is continuous up to* $\overline{S_\pm}$ *and we denote by* $\sqrt{V}(-\Delta-\lambda\pm i0)^{-1}\sqrt{|V|}$ *its extensions at* $\lambda>0$. *For all* $z\in\overline{S_\pm}$, *we have the estimate*

$$\|A(z)\|_{\mathfrak{S}^{2q}}\leqslant C|z|^{-1+\frac{N}{2q}}\|V\|_{L^q}\,, \tag{39}$$

with C *as in* (37) *(and in particular independent of* δ*). Finally, for all* $z\in\overline{S_\pm}$, *the operator* $1+A(z)$ *is invertible and the map* $S_\pm\ni z\mapsto(1+A(z))^{-1}$ *is an analytic family of bounded operators on* $L^2(\mathbb{R}^N)$, *which is continuous on* $\overline{S_\pm}$.

The proof of Lemma 11 relies on a deep theorem of Koch and Tataru [17].

Proof of Lemma 11. The family $\mathbb{C}\setminus[0,\infty)\ni z\mapsto\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}=A(z)\in\mathfrak{S}^{2q}$ is analytic: indeed, by the resolvent formula we have for any $z,z_0\in\mathbb{C}\setminus[0,\infty)$,

$$\begin{aligned}\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}-\sum_{n=0}^{N}(z-z_0)^n\sqrt{V}(-\Delta-z_0)^{-n-1}\sqrt{|V|}&\\ =\sqrt{V}(-\Delta-z)^{-1}(z-z_0)^{N+1}(-\Delta-z_0)^{-N-1}\sqrt{|V|}&.\end{aligned}$$

The right side of this equality goes to zero in $\mathfrak{S}^{2q}$ as $N\to\infty$ if $|z-z_0|$ is small enough by the Kato–Seiler–Simon inequality [27, Thm. 4.1] and the constraint $4q>N$,

$$\begin{aligned}&\left\|\sqrt{V}(-\Delta-z)^{-1}(-\Delta-z_0)^{-N-1}\sqrt{|V|}\right\|_{\mathfrak{S}^{2q}}\\ &\qquad\leqslant\left\|\sqrt{|V|}(-\Delta-z_0)^{-1}\right\|_{\mathfrak{S}^{2q}}^2\left\|(-\Delta-z_0)^{-1}\right\|^{N-1}\left\|(-\Delta-z)^{-1}\right\|\leqslant C^N\|V\|_{L^q}^2\,.\end{aligned}$$

The same estimate shows that the entire series we found has a nonzero radius of convergence, showing the desired analyticity. Next, let us notice that we can use the results of [14], since the argument given in the proof of Lemma 3.5 in [10] shows that V is an admissible perturbation in the sense of [14] and that $L^{2q/(q+1)}(\mathbb{R}^N)\subset X$, where X is the Banach space defined in the introduction of [14]. Using [14, Lemma 4.1 b)], for each $\lambda>0$ there exists an operator $(-\Delta-\lambda\pm i0)^{-1}$ bounded from $L^{2q/(q+1)}$ to $L^{2q/(q-1)}$ such that $z\mapsto A(z)$ can be extended as a continuous family on the strips $\overline{S_\pm}$, for the weak topology of operators. Let us show that this family is actually continuous for the Schatten topology $\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))$. To do so, let $z\in\overline{S_\pm}$ and $(z_n)\subset S_\pm$ such that $z_n\to z$ as $n\to\infty$. We show that $(A(z_n))$ is a Cauchy sequence for the Schatten norm, which then implies the Schatten norm continuity of $A(z)$ up to the real axis. Thus, let $\varepsilon>0$. Let $W_1,\widetilde{W_1}$ be bounded, compactly supported measurable functions such that we have the decompositions

$$\sqrt{V}=W_1+W_2,\quad \sqrt{|V|}=\widetilde{W_1}+\widetilde{W_2},\quad \|W_2\|_{L^{q/2}}+\left\|\widetilde{W_2}\right\|_{L^{q/2}}\leqslant\varepsilon.$$

Using (37), we may estimate

$$\|A(z_n)-A(z_m)\|_{\mathfrak{S}^{2q}}\leqslant\left\|W_1((-\Delta-z_n)^{-1}-(-\Delta-z_m)^{-1})\widetilde{W_1}\right\|_{\mathfrak{S}^{2q}}+C\varepsilon.$$

By [36, Prop. VII.1.22], the family $z\mapsto W_1(-\Delta-z)^{-1}\widetilde{W_1}$ is continuous on $\overline{S_\pm}$ for the $\mathfrak{S}^{2q}$-topology, and hence for n,m large enough, we have

$$\left\|W_1((-\Delta-z_n)^{-1}-(-\Delta-z_m)^{-1})\widetilde{W_1}\right\|_{\mathfrak{S}^{2q}}\leqslant\varepsilon.$$

This shows that $(A(z_n))$ is a Cauchy sequence for the $\mathfrak{S}^{2q}$-topology, and hence $z\mapsto A(z)\in\mathfrak{S}^{2q}$ is continuous up to the boundary. This in particular shows that $\sqrt{V}(-\Delta-\lambda\pm i0)^{-1}\sqrt{|V|}$ belongs to $\mathfrak{S}^{2q}$ for all $\lambda>0$. This also shows that the estimate (39) carries over to the real axis. We then apply analytic Fredholm theory [35, Thm. I.4.2 & I.4.3] to the family $(A(z))$ in the strips $\overline{S_\pm}$ to infer $z\mapsto(1+A(z))^{-1}$ is a meromorphic family of bounded operators on $S_\pm$ with poles at the points z where $-1\in\sigma(A(z))$. Furthermore, this family is continuous up to the real axis, except at the points $\lambda\in\delta$ such that $-1\in\sigma(A(\lambda))$. It thus only remains to show that $-1\notin\sigma(A(z))$ for all $z\in\overline{S_\pm}$. When $\operatorname{Im} z\neq 0$, this follows from a simple argument similar to the beginning of the proof of Lemma 4.6 in [14] based on the fact that V is real-valued. Assume that there exists a $\lambda>0$, a sign $\pm$, and an $f\in L^2(\mathbb{R}^N)$ satisfying

$$\sqrt{V}R_0(\lambda)\sqrt{|V|}f=-f,$$

where we used the notation $R_0(\lambda)=(-\Delta-\lambda\pm i0)^{-1}$. Defining $g:=R_0(\lambda)\sqrt{|V|}f$, we deduce the relation $Vg=\sqrt{|V|}f$ and

$$R_0(\lambda)Vg=-g. \tag{40}$$

Since $f\in L^2$ and $V\in L^q$, Hölder's inequality implies that $\sqrt{|V|}f=Vg\in L^{2q/(q+1)}(\mathbb{R}^N)$. By [14, Lemma 4.1 a,b)], the resolvent $R_0(\lambda)$ is a bounded operator from X to X^*, showing that $g=-R_0(\lambda)Vg\in X^*$. Using now [14, Lemma 4.4], we have the decay estimate

$$\left\|(1+|x|^2)^N g\right\|_{X^*}<\infty$$

for all $N\geqslant 0$. Since $X^*\subset L^{2q/(q-1)}(\mathbb{R}^N)$ and writing for N large enough

$$g=(1+|x|^2)^{-N}\times(1+|x|^2)^N g,$$

we deduce from Hölder's inequality that $g\in L^2(\mathbb{R}^N)$. Furthermore, using the integrability of g and Vg, we can show that (40) implies that g satisfies the Schrödinger equation $(-\Delta+V)g=zg$ in the sense of distributions on $\mathbb{R}^N$. Since $g\in L^{2q/(q-1)}(\mathbb{R}^N)$ and

$Vg \in L^{2q/(q+1)}(\mathbb{R}^N)$, we deduce that $g \in W^{2,2q/(q+1)}_{\rm loc}(\mathbb{R}^N) \subset H^1_{\rm loc}(\mathbb{R}^N)$. Finally, by [17, Thm. 3], we learn that $g \equiv 0$ and therefore $f \equiv 0$. Thus, $-1 \notin \sigma(A(\lambda))$. □

We are now ready to prove Theorem 11.

Proof of Theorem 11. We make use of the identity

$$\sqrt{V}(-\Delta + V - z)^{-1}\sqrt{|V|} = \frac{1}{1+\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}}\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}. \tag{41}$$

By Lemma 11, we know that the maps

$$z \mapsto \frac{1}{1+\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}} \in \mathcal{B}(L^2), \quad z \mapsto \sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|} \in \mathfrak{S}^{2q}$$

are analytic on $\mathbb{C}\setminus[0,\infty)$ and extend continuously to $(0,\infty)$ with possibly different boundary values from above and from below. We are thus left to prove (38). First assume that $q > N/2$. Then it follows from Theorem 10 that

$$\left\|\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}\right\|_{L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)} \leqslant \frac{1}{2}$$

provided $C|z|^{-1+N/2q}\|V\|_{L^q(\mathbb{R}^N)} \leqslant 1/2$. Thus, for such z,

$$\left\|\left(1+\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}\right)^{-1}\right\|_{L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)} \leqslant 2\,.$$

The claimed bound then follows from the identity (41) and the inequality (39).

Now assume that $q = N/2$ and $N \geq 3$. In this case we write $\sqrt{V} = W_1 + W_2$ and $\sqrt{|V|} = \widetilde{W}_1 + \widetilde{W}_2$ as in the proof of Lemma 11. Then using again Theorem 10

$$\left\|\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}\right\|_{L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)} \leqslant \left\|W_1(-\Delta - z)^{-1}\widetilde{W}_1\right\|_{L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)} + C\varepsilon$$

for all $z \in \mathbb{C}\setminus[0,\infty)$. Since

$$\left\|W_1(-\Delta - z)^{-1}\widetilde{W}_1\right\|_{L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)} \to 0$$

as $|z| \to \infty$, as we have just shown (note that $W_1, \widetilde{W}_1 \in L^q(\mathbb{R}^N)$ with $q > N/2$), we can argue as before and obtain the result in this case as well. This finishes the proof of Theorem 11. □

5. Application of Strichartz estimates: global well-posedness for the Hartree equation in Schatten spaces

We illustrate the usefulness of the Strichartz estimates obtained in Theorem 6 by showing well-posedness results in Schatten spaces for the non-linear Hartree equation. The main point here is that we can consider a system of *infinitely many* equations and that we do not even need a trace class assumption. This is of crucial importance when studying the dynamics of quantum gases at positive density [20, 21]. Using our improved set of Strichartz exponents from Theorem 6 we can extend the previous work of one of us [26, Chap. 4, App. A]. Our exposition here is somehow sketchy. Details as well as similar consequences of

Strichartz inequalities for the fractional Laplacian and for the pseudo-relativistic operator will be addressed in a forthcoming work of the second author.

Theorem 12. *Let $d \geqslant 1$, $1 \leqslant q < 1 + 2/(d-1)$, $p \geqslant 1$ such that $2/p + d/q = d$ and $w \in L^{q'}_x(\mathbb{R}^d)$. Then, for any $\gamma_0 \in \mathfrak{S}^{2q/(q+1)}$, there exists a unique $\gamma \in C^0_t(\mathbb{R}, \mathfrak{S}^{2q/(q+1)})$ satisfying $\rho_\gamma \in L^p_{\mathrm{loc},t}(\mathbb{R}, L^q_x(\mathbb{R}^d))$ and*

$$\begin{cases} i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma], \\ \gamma_{|t=0} = \gamma_0. \end{cases}$$

This result is a consequence of homogeneous and inhomogeneous Strichartz estimates. The homogeneous part is the content of Theorem 6, while the inhomogeneous one can be deduced from Theorem 7 using the same method as the proof of Corollary 1 in [9]:

Theorem 13. *Let $d \geqslant 1$, $1 \leqslant q < 1 + 2/(d-1)$, $p \geqslant 1$ such that $2/p + d/q = d$, and $\gamma_0 \in \mathfrak{S}^{2q/(q+1)}$. Let $\gamma = \gamma(t)$ be the solution to the equation*

$$\begin{cases} i\partial_t \gamma = [-\Delta, \gamma] + R(t), \\ \gamma_{|t=0} = \gamma_0. \end{cases}$$

Then, the inequality

$$\left\| \rho_{\gamma(t)} \right\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \leqslant C_{\mathrm{Stri}} \left(\|\gamma_0\|_{\mathfrak{S}^{2q/(q+1)}} + \left\| \int_{\mathbb{R}} e^{-is\Delta} |R(s)| e^{is\Delta}\, ds \right\|_{\mathfrak{S}^{2q/(q+1)}} \right)$$

holds for some constant $C_{\mathrm{Stri}} > 0$ independent of γ_0 and R.

Proof of Theorem 12. We use a standard fixed point method on the Duhamel formulation of the Hartree equation, see [20] for details. Let $R > 0$ such that $\|\gamma_0\|_{\mathfrak{S}^{2q/(q+1)}} \leqslant R$, and let $T = T(R) > 0$ to be chosen later on. We define a map

$$\Phi(\gamma, \rho) = \left(\Phi_1(\gamma, \rho), \rho[\Phi_1(\gamma, \rho)] \right),$$

and we show that, for a suitable T, Φ is a contraction on the space

$$X := \Big\{ (\gamma, \rho) \in C^0_t([0,T], \mathfrak{S}^{2q/(q+1)}) \times L^p_t([0,T], L^q_x(\mathbb{R}^d)), \\ \|\gamma\|_{C^0 \mathfrak{S}^{2q/(q+1)}} + \|\rho\|_{L^p_t L^q_x} \leqslant 4 \max(1, C_{\mathrm{Stri}}) R \Big\}.$$

The map Φ_1 is defined as

$$\Phi_1(\gamma, \rho)(t) = e^{it\Delta} \gamma_0 e^{-it\Delta} - i \int_0^t e^{i(t-s)\Delta} [w * \rho(s), \gamma(s)] e^{i(s-t)\Delta}\, ds.$$

For all $(\gamma, \rho) \in X$, we have

$$\begin{aligned} \|\Phi_1(\gamma, \rho)\|_{C^0_t \mathfrak{S}^{2q/(q+1)}} &\leqslant R + 2 \int_0^T \|w * \rho(s)\|_{L^\infty_x} \|\gamma(s)\|_{\mathfrak{S}^{2q/(q+1)}}\, ds \\ &\leqslant R + 2T^{1/p'} \|w\|_{L^{q'}_x} \|\rho\|_{L^p_t L^q_x} \|\gamma\|_{C^0_t \mathfrak{S}^{2q/(q+1)}} \\ &\leqslant R + 8T^{1/p'} \|w\|_{L^{q'}_x} \max(1, C^2_{\mathrm{Stri}}) R^2. \end{aligned}$$

By Theorem 13, we also have

$$\|\rho[\Phi_1(\gamma,\rho)]\|_{L^p_t L^q_x} \leqslant C_{\rm Stri} R + 8 C_{\rm Stri} T^{1/p'} \|w\|_{L^{q'}_x} \max(1, C^2_{\rm Stri}) R^2.$$

Hence, for $T = T(R) > 0$ small enough, Φ maps X to itself. A similar argument shows that Φ is a contraction on X, and thus has a unique fixed point on X which is a solution to the Hartree equation on $[0,T]$. We may extend it to a maximal solution on some interval $[0, T_{\max})$, and we have the blow-up criterion given by the local theory

$$T_{\max} < \infty \Longrightarrow \|\gamma(t)\|_{\mathfrak{S}^{2q/(q+1)}} \xrightarrow[t\to T_{\max}]{} +\infty.$$

By the result of Yajima [37], knowing that the potential $w * \rho_\gamma$ belongs to $L^p_{{\rm loc},t} L^\infty_x$ implies that there exists a unitary operator $U(t)$ on $L^2_x(\mathbb{R}^d)$ such that $\gamma(t) = U(t)\gamma_0 U(t)^*$ for all t. In particular, the $\mathfrak{S}^{2q/(q+1)}$ norm of $\gamma(t)$ is a conserved quantity and cannot blow-up, thus leading to global solutions. □

6. Application of uniform Sobolev estimates: eigenvalues of Schrödinger operators with complex-valued potentials

As an application of the uniform Sobolev estimates obtained in Theorem 10, we prove Lieb–Thirring-type inequalities for the discrete spectrum of a Schrödinger operator $-\Delta + V$ where V is a complex potential belonging to $L^q(\mathbb{R}^N, \mathbb{C})$, where q satisfies the assumptions of Theorem 11. It was noticed in [7] that the uniform Sobolev estimates of Kenig, Ruiz, and Sogge could be used to control the size of eigenvalues of non self-adjoint Schrödinger operators. More precisely, let $\lambda \in \mathbb{C} \setminus [0,\infty)$ be an eigenvalue of $-\Delta + V$. Then, [7] contains the estimate

$$|\lambda|^\gamma \leqslant D_{\gamma,N} \int_{\mathbb{R}^N} |V(x)|^{\gamma + N/2}\, dx, \tag{42}$$

for some constant $D_{\gamma,N} > 0$, for all $0 < \gamma \leqslant 1/2$ and for all $N \geqslant 2$ (see [1] for the earlier result for $\gamma = 1/2$ and $N = 1$). Since the Kenig–Ruiz–Sogge bound implies a control on the size of a single eigenvalue, it is natural to assume that the Schatten bounds of Theorem 10 would provide bounds on sums of such eigenvalues. This is the content of the following result.

Theorem 14. *Let $N \geqslant 2$ and assume that $V \in L^q(\mathbb{R}^N, \mathbb{C})$ with*

$$\begin{cases} 1 < q \leqslant 3/2 & \text{if } N = 2, \\ \frac{N}{2} \leqslant q \leqslant \frac{N+1}{2} & \text{if } N \geqslant 3. \end{cases}$$

Denote by Z the discrete set of eigenvalues of $-\Delta + V$ in $\mathbb{C} \setminus [0,\infty)$, and for $\lambda \in Z$ let m_λ be the corresponding algebraic *multiplicity. Then, we have the following bounds:*

- *If $N/2 < q < (N+1)/2$, then*

$$\sum_{\lambda \in Z} m_\lambda d(\lambda, [0,\infty)) < \infty; \tag{43}$$

- *If $q = (N+1)/2$, then for all $\tau > 0$,*
$$\sum_{\lambda \in Z} m_\lambda |\lambda|^\tau d(\lambda, [0,\infty)) < \infty; \tag{44}$$
- *If $q = N/2$, then*
$$\sum_{\lambda \in Z} m_\lambda \frac{\operatorname{Im}\sqrt{\lambda}}{1+|\lambda|} < \infty, \tag{45}$$
where the branch of the square root is chosen to have positive imaginary part.

Here, $d(\lambda, [0,\infty))$ denotes the distance of λ to $[0,\infty)$.

The previous result has to be compared with the usual Lieb–Thirring inequality for eigenvalues of self-adjoint Schrödinger operators, which states that when $V \in L^q(\mathbb{R}^N, \mathbb{R})$, then
$$\sum_{\lambda \in Z} m_\lambda |\lambda|^\gamma = \sum_{\lambda \in Z} m_\lambda d(\lambda, [0,\infty))^\gamma \leqslant K_{\gamma,N} \int_{\mathbb{R}^N} |V(x)|^{\gamma+N/2}\, dx \tag{46}$$
for all $\gamma \geqslant 0$ when $N \geqslant 3$, $\gamma > 0$ when $N = 2$, $\gamma \geqslant 1/2$ when $N = 1$. Our assumptions on V corresponds to the range $0 \leqslant \gamma \leqslant 1/2$ in the previous inequality. In this range, (46) is much stronger than (43), (44), and (45): when V is real-valued, the eigenvalues of $-\Delta + V$ can only accumulate at 0, and hence the information that $\sum |\lambda|^\gamma < \infty$ implies both (43), (44), and (45) in the real-valued case. In the complex-valued case, eigenvalues may in principle accumulate at any point in $[0,\infty)$, and (43), (44), and (45) give qualitative information on the speed of convergence to $[0,\infty)$ of a sequence of eigenvalues that accumulate at a point in $[0,\infty)$. In the case $q = N/2$, the bound (42) does not imply the boundedness of Z; hence it does not exclude that a sequence of eigenvalues diverges. The bound (45) controls such divergence.

Eigenvalues bounds for Schrödinger operators with complex-valued potentials have been the topic of many works, for instance [1, 8, 18, 4] and references therein. To prove Theorem 14, we use a method developed by Demuth, Hansmann and Katriel relying on estimates on zeros of holomorphic functions and nicely exposed in the review [5]. The basic idea is that any eigenvalue $z_0 \in \mathbb{C} \setminus [0,\infty)$ of the operator $-\Delta + V$ corresponds to a "zero" of the analytic function $z \mapsto 1 + \sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|}$. Hence, estimates on sums of eigenvalues of $-\Delta + V$ amount to estimates on sums of zeros of holomorphic functions. The first result of this kind is Jensen's inequality [11, Sec. II.2], which states that the zeros (z_n) of a bounded analytic function on the unit disk satisfy the bound
$$\sum_n (1 - |z_n|) < \infty,$$
which may be compared to be bounds obtained in Theorem 14. As explained in [5], one may obtain similar results for analytic functions that blow up at the boundary of the unit disk, according to the rate of this blow-up. This is the content of a theorem of Borichev, Golinskii, and Kupin [3], which generalizes Jensen's inequality to functions that may blow up at the boundary. We use these versions of Jensen's inequality to prove Theorem 14, in the spirit of [5]. The results of Theorem 14 are better than the previously obtained bounds of [18, 4] with respect to eigenvalues accumulating at a point in $(0,\infty)$: (43) and (44) shows

that the sequence of their imaginary parts is in ℓ^1, while (45) even shows that it is in $\ell^{1/2}$. In [18, 4], the best result gives that such a sequence is only in ℓ^p for some larger exponent p. Hence, our result shows a higher speed of convergence towards the real axis for sequences of eigenvalues accumulating at $(0,\infty)$. However, our result for sequences accumulating at 0 is weaker than the ones obtained in [4].

It would be natural to require an explicit bound in terms of the L^q-norm of the potential V in (43), (44), (45), as in the usual Lieb–Thirring inequality. Equivalently, this amounts to have explicit bounds in Jensen's inequalities. Such bounds can be obtained under a normalization constraint of the holomorphic functions that we consider (e.g. of the type $f(0)=1$). In our case, the natural normalization of the function $z\mapsto 1+\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}$ is at ∞, where we know that the value of this function is 1. The issue now is that, to our knowledge, no Jensen's inequality with explicit bound is known when the function is normalized at the boundary of the unit disk (we may think of the point ∞ as being at the boundary of the domain of definition of $z\mapsto 1+\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}$). This is why our approach does not lead to explicit bounds in terms of the potential V.

Proof of Theorem 14. As we saw in Section 4, the map

$$\mathbb{C}\setminus[0,\infty)\ni z\mapsto 1+\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}\in 1+\mathfrak{S}^{2q}$$

is analytic. Hence, the map

$$\mathbb{C}\setminus[0,\infty)\ni z\mapsto h(z):=\mathrm{Det}_{\lceil 2q\rceil}\left(1+\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}\right)\in\mathbb{C}$$

is also analytic, where $\mathrm{Det}_{\lceil 2q\rceil}$ denotes the regularized determinant, see for instance [4, Sec. 2.2]. Furthermore, we have the inequality (see, e.g. [27, Thm 9.2])

$$\log\left|\mathrm{Det}_{\lceil 2q\rceil}\left(1+\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}\right)\right|\leqslant C\left\|\sqrt{V}(-\Delta-z)^{-1}\sqrt{|V|}\right\|_{\mathfrak{S}^{2q}}^{2q}. \tag{47}$$

As explained in [19, Thm. 21], the zeros of h (counted with multiplicity) are exactly the eigenvalues of $-\Delta+V$ (counted with *algebraic* multiplicity). Hence, we prove bounds on sums of zeros of h, and to do so we use Jensen-type inequalities. We begin with the case $q=N/2$, for which we can apply directly the usual Jensen inequality. In this case, by (47) and Theorem 10 we can bound

$$\log|h(z)|\leqslant C,$$

for all $z\in\mathbb{C}\setminus[0,\infty)$, with $C>0$ independent of z. As a consequence, we can apply Jensen's inequality to the map $\mathbb{H}\ni w\mapsto h(w^2)$, where $\mathbb{H}$ denotes the upper half space. The version of Jensen's inequality for analytic functions on the upper half space bounded up to the real axis can be found for instance in [11, Eq. (2.3)], which gives exactly (45). When $N/2<q\leqslant(N+1)/2$, the Schatten bounds of Theorem 10 are not uniform anymore and blow-up at 0: according to (47) they yield

$$\log|h(z)|\leqslant C|z|^{N-2q}. \tag{48}$$

Hence, we have to use the result of [3] about Jensen's inequality for functions on the unit disk that blow-up at some point of the boundary. Since this result is stated for analytic functions on the unit disk and h is defined on $\mathbb{C} \setminus [0,\infty)$, we have to use some conformal map from the disk to $\mathbb{C} \setminus [0,\infty)$, as in [4, Sec. 4.4]: let

$$\psi : \mathbb{D} \ni w \mapsto a \left(\frac{1+w}{1-w}\right)^2 \in \mathbb{C} \setminus [0,\infty),$$

where $\mathbb{D}$ denotes the unit disk and $a \in \mathbb{C}\setminus[0,\infty)$ is chosen such that $h(a) \neq 0$. For instance, by (42) one may take a to be sufficiently negative such that $|\lambda| < |a|$ for all eigenvalue λ of $-\Delta + V$, hence ensuring that $h(a) \neq 0$. Define now the map

$$g : \mathbb{D} \ni w \mapsto h(\psi(w))/h(a).$$

It is an analytic function on the unit disk, and by (48) it satisfies the bound

$$\log|g(w)| \leqslant C\frac{|a|^{N-2q}}{|1+w|^{4q-2N}} - \log|h(a)| \leqslant \frac{\widetilde{C}}{|1+w|^{4q-2N}},$$

for all $w \in \mathbb{D}$ and a constant $\widetilde{C}$ depending on a. Let us first consider the case $N/2 < q < (N+1)/2$: we apply [3] with $\tau = 2 - 4q - 2N > 0$ and obtain

$$\sum_{w\in Z(g)} m_w(1-|w|)|1+w| < \infty,$$

where $Z(g)$ denotes the set of zeros of g and m_w denotes the multiplicity of the zero w. The corresponding result for eigenvalues of $-\Delta + V$ is thus

$$\sum_{\lambda\in Z} m_\lambda(1-|\psi^{-1}(\lambda)|)|1+\psi^{-1}(\lambda)| < \infty.$$

By [4, Thm. 4.3.4] and the proof of Theorem 4.4.3 in [4], we have the estimate

$$1-|\psi^{-1}(\lambda)| \geqslant \frac{d(\lambda,[0,\infty))}{8|a|^{1/2}(|a|+|\lambda|)^{3/2}}\frac{1}{|1+\psi^{-1}(\lambda)|} \geqslant \frac{d(\lambda,[0,\infty))}{8^{3/2}|a|^2}\frac{1}{|1+\psi^{-1}(\lambda)|},$$

showing (43). Finally, in the case $q = (N+1)/2$, we apply [3] with $\tau > 0$ and obtain

$$\sum_{w\in Z(g)} m_w(1-|w|)|1+w|^{1+\tau} < \infty,$$

leading by the same method to

$$\sum_{\lambda\in Z} m_\lambda d(\lambda,[0,\infty))|1+\psi^{-1}(\lambda)|^\tau < \infty.$$

As the proof of Theorem 4.4.3 in [4], we use

$$|1+\psi^{-1}(\lambda)| \geqslant \frac{|\lambda|^{1/2}}{|\lambda|+|a|} \geqslant \frac{|\lambda|^{1/2}}{2|a|}$$

to infer that

$$\sum_{\lambda\in Z} m_\lambda|\lambda|^{\tau/2}d(\lambda,[0,\infty)) < \infty$$

for all $\tau > 0$, which is (44). □

7. Application of the Limiting Absorption Principle: Schatten properties of the Scattering Matrix

An important object in scattering theory is the so-called scattering matrix. For every $\lambda > 0$ (with the physical interpretation of an energy) this is a bounded operator $S(\lambda)$ in $L^2(\mathbb{S}^{N-1})$ (corresponding to the Fermi sphere in physics). Under rather weak assumptions on the potential V, the scattering matrix differs from the identity by a compact operator. In the next result we prove quantitative information in terms of trace ideals properties.

Theorem 15. *Let $N \geqslant 2$ and assume that $V \in L^q(\mathbb{R}^N, \mathbb{R})$ with*

$$\begin{cases} 1 < q \leqslant 3/2 & \text{if } N = 2 \\ \frac{N}{2} \leqslant q \leqslant \frac{N+1}{2} & \text{if } N \geqslant 3\,. \end{cases}$$

Then $S(\lambda) - 1 \in \mathfrak{S}^{2q}(L^2(\mathbb{S}^{N-1}))$ for every $\lambda > 0$ and the mapping $(0,\infty) \ni \lambda \mapsto S(\lambda) - 1 \in \mathfrak{S}^{2q}(L^2(\mathbb{S}^{N-1}))$ is continuous. Moreover, under the additional assumption $q > N/2$, there is a constant $C_{N,q}$ such that if $\lambda^{-1+N/q}\|V\|_{L^q(\mathbb{R}^N)} \leqslant C_{N,q}$ then

$$\|S(\lambda) - 1\|_{\mathfrak{S}^{2q}(L^2(\mathbb{S}^{N-1}))} \leqslant 3C_{N,q}\lambda^{-1+N/2q}\|V\|_{L^q(\mathbb{R}^N)}\,. \tag{49}$$

If $q = N/2$ and $N \geqslant 3$, the bound (49) holds provided $\lambda \geqslant C(V)$ for some constant $C(V)$ depending on V. For $q = (N+1)/2$, the trace ideal $\mathfrak{S}^{2q}(L^2(\mathbb{S}^{N-1})$ is best possible for (49) to hold.

Remark 12. Remarkably, under the pointwise bound $|V(x)| \leqslant C(1+|x|^2)^{-\rho/2}$ for some $\rho > 1$, the trace ideal $\mathfrak{S}^{2q}$ can be improved to $\mathfrak{S}^r$ for any $r > (N-1)/(\rho-1)$ [36, Prop. 8.1.5] and this is best possible [36, Thm. 8.2.1]. Nevertheless, under an L^p condition on V, our theorem is best possible.

For the definition of the scattering matrix we refer to [36]. The formula that will be important to us is that

$$S(\lambda) = 1 - 2\pi i\Gamma_0(\lambda)\sqrt{|V|}\left(1 - \sqrt{V}(-\Delta + V - \lambda - i0)^{-1}\sqrt{|V|}\right)\sqrt{V}\Gamma_0(\lambda)^* \tag{50}$$

(see (6.6.19) in [36]). Here $\Gamma_0(\lambda)$ is the operator that maps functions ψ on $\mathbb{R}^N$ to functions on $\mathbb{S}^{N-1}$ by restricting the Fourier transform to the sphere of radius $\sqrt{\lambda}$,

$$\left(\Gamma_0(\lambda)\psi\right)(\omega) = 2^{-1/2}\lambda^{(N-2)/4}\widehat{\psi}(\sqrt{\lambda}\omega), \quad \forall \omega \in \mathbb{S}^{N-1}\,.$$

Under the conditions of the theorem the products $\Gamma_0(\lambda)\sqrt{|V|}$ and $\sqrt{V}\Gamma_0(\lambda)^* = (\Gamma_0(\lambda)\sqrt{V})^*$ are bounded operators between the corresponding L^2 spaces by the Stein–Tomas restriction theorem (see also the proof below). Therefore the right side in the above formula for $S(\lambda)$ is well-defined.

Proof of Theorem 15. It follows from Theorem 2 that $\mathcal{E}^*_{\mathbb{S}^{N-1}}V\mathcal{E}_{\mathbb{S}^{N-1}} \in \mathfrak{S}^{2q}(L^2(\mathbb{S}^{N-1}))$ with

$$\|\mathcal{E}^*_{\mathbb{S}^{N-1}}V\mathcal{E}_{\mathbb{S}^{N-1}}\|_{\mathfrak{S}^{2q}(L^2(\mathbb{S}^{N-1}))} \leqslant C_{N,q}\|V\|_{L^q(\mathbb{R}^N)}$$

under the assumptions on q in the theorem. Thus,

$$\left\|\mathcal{E}^*_{\mathbb{S}^{N-1}}\sqrt{|V|}\right\|^2_{\mathfrak{S}^{4q}(L^2(\mathbb{R}^N),L^2(\mathbb{S}^{N-1}))} = \left\|\sqrt{|V|}\mathcal{E}_{\mathbb{S}^{N-1}}\right\|^2_{\mathfrak{S}^{4q}(L^2(\mathbb{S}^{N-1}),L^2(\mathbb{R}^N))} \leq C_{N,q}\|V\|_{L^q(\mathbb{R}^N)}\,.$$

By scaling,

$$\left\|\Gamma_0(\lambda)\sqrt{|V|}\right\|^2_{\mathfrak{S}^{4q}(L^2(\mathbb{R}^N),L^2(\mathbb{S}^{N-1}))} = \left\|\sqrt{|V|}\Gamma_0(\lambda)^*\right\|^2_{\mathfrak{S}^{4q}(L^2(\mathbb{S}^{N-1}),L^2(\mathbb{R}^N))}$$
$$\leq 2^{-1}C_{N,q}\ \lambda^{-1+N/2q}\|V\|_{L^q(\mathbb{R}^N)}\,.$$

Thus, it follows that

$$\|\Gamma_0(\lambda)V\Gamma_0(\lambda)^*\| \leq 2^{-1}C_{N,q}\ \lambda^{-1+N/2q}\|V\|_{L^q(\mathbb{R}^N)}\,.$$

Moreover, by Hölder's inequality for trace ideals and Theorem 11 we have $\Gamma_0(\lambda)V(-\Delta+V-\lambda-i0)^{-1}V\Gamma_0(\lambda) \in \mathfrak{S}^q(L^2(\mathbb{R}^N))$, and for $|z|$ large enough we can bound the norm as follows,

$$\left\|\Gamma_0(\lambda)V(-\Delta+V-\lambda-i0)^{-1}V\Gamma_0(\lambda)\right\|_{\mathfrak{S}^q(L^2(\mathbb{R}^N))}$$
$$\leqslant \left\|\Gamma_0(\lambda)\sqrt{|V|}\right\|^2_{\mathfrak{S}^{4q}(L^2(\mathbb{S}^{N-1}),L^2(\mathbb{R}^N))}\left\|\sqrt{V}(-\Delta+V-\lambda-i0)^{-1}\sqrt{|V|}\right\|_{\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))}$$
$$\leqslant C'_{N,q}\lambda^{-2+N/q}\|V\|^2_{L^q(\mathbb{R}^N)}\,.$$

This proves the desired bound. Continuity follows immediately from the formula (50) and the continuity statement in Theorem 11. Let us prove optimality of the trace ideal $\mathfrak{S}^{2q}$ in the case $q=(N+1)/2$. It follows from (50) and the fact that $\sqrt{V}(-\Delta+V-\lambda-i0)^{-1}\sqrt{|V|}$ is compact by Theorem 11, that $S(\lambda)-1\in\mathfrak{S}^r$ iff $\Gamma_0(\lambda)V\Gamma_0(\lambda)^*\in\mathfrak{S}^r$ for any fixed r. Now Theorem 4 implies that $\Gamma_0(\lambda)V\Gamma_0(\lambda)^*\notin\mathfrak{S}^r$ if $r<2q$ when $q=(N+1)/2$. This completes the proof. ☐

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