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Pascal Auscher, Mihalis Mouroglou

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REPRESENTATION AND UNIQUENESS FOR BOUNDARY VALUE ELLIPTIC PROBLEMS VIA FIRST ORDER SYSTEMS

PASCAL AUSCHER AND MIHALIS MOURGOLOU

ABSTRACT. Given any elliptic system with t -independent coefficients in the upper-half space, we obtain representation and trace for the conormal gradient of solutions in the natural classes for the boundary value problems of Dirichlet and Neumann types with area integral control or non-tangential maximal control. The trace spaces are obtained in a natural range of boundary spaces which is parametrized by properties of some Hardy spaces. This implies a complete picture of uniqueness vs solvability and well-posedness.

In memory of B. Dahlberg

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1. INTRODUCTION

The goal of this article is to classify **all** weak solutions in the natural classes for the boundary value problems of t -independent elliptic systems (1) in the upper half space. The classification is obtained via use of first order systems and consists in a semigroup representation for the conormal gradients of such solutions. As a consequence, this will settle a number of issues concerning relationships between solvability, uniqueness and well-posedness.

This classification will be done independently of any solvability issue, which seems a surprising assertion. To understand this, let us recall the situation for harmonic functions, found for example in Chapter III, Appendix 4 of Stein's book [Ste], but presented in a form that suits our goals. Let u be a harmonic function on the upper half-space, t will be the vertical variable and x the horizontal one. We want to know about the trace of the gradient of u , $\nabla u = (\partial_t u, \partial_{x_1} u, \dots, \partial_{x_n} u)$, at the boundary and whether $\nabla u(t, x)$ can be recovered from its trace. For $\frac{n}{n+1} < p < \infty$, the non-tangential maximal control $\|(\nabla u)^*\|_p < \infty$ is equivalent to ∇u at the boundary being in L^p if $p > 1$ and in the Hardy space H^p if $p \leq 1$, and $\nabla u(t, x)$ can be written as the (vector-valued) Poisson extension of its trace. If $1 < p < \infty$ (we restrict the range for convenience), the area integral control $\|S(t\nabla u)\|_p < \infty$, together with a mild control at ∞ , is equivalent to ∇u belongs to the Sobolev space $\dot{W}^{-1,p}$ on the boundary and $\nabla u(t, x)$ is given by the Poisson extension of its trace in this topology. If one replaces area integral by a Carleson measure, again assuming a mild control of u at ∞ , this is equivalent to $\nabla u \in BMO^{-1}$ at the boundary and $\nabla u(t, x)$ is the Poisson extension of its trace. There is even a Hölder space version of this.

In the familiar situation of the Laplace equation, knowledge of u or of $\partial_t u$ at the boundary suffices to find the harmonic function u inside by a Poisson extension: this is a solvability property of this equation. With the above formulation, it seems one does not need to know solvability of the Dirichlet or Neumann problem in any sense when one works with the complete gradient; however, this is not clear from the available proofs because solvability could be implicit (use of Fatou type results, commutation of operators...). This observation is of particular interest when dealing with the more general systems (1) below, as one may not *a priori* know solvability (but would like to have it), which is usually difficult to establish; still there is room for proving existence of a trace and representation of the gradient as a first step. Solving the boundary value problem, that is, constructing a solution from the knowledge of one component of the gradient at the boundary amounts to an inverse problem on the boundary in a second step, and this is a different (albeit related) question.

Let us introduce our notation. We denote points in \mathbb{R}^{1+n} by boldface letters $\mathbf{x}, \mathbf{y}, \dots$ and in coordinates in $\mathbb{R} \times \mathbb{R}^n$ by (t, x) etc. We set $\mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n$. Consider the system of m equations given by

$$(1) \quad \sum_{i,j=0}^n \sum_{\beta=1}^m \partial_i (A_{i,j}^{\alpha,\beta}(x) \partial_j u^\beta(\mathbf{x})) = 0, \quad \alpha = 1, \dots, m$$

in \mathbb{R}_+^{1+n} , where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$ if $i = 1, \dots, n$. For short, we write $Lu = -\operatorname{div} A \nabla u = 0$ to mean (1), where we always assume that the matrix

$$(2) \quad A(x) = (A_{i,j}^{\alpha,\beta}(x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{m(1+n)})),$$

is bounded and measurable, independent of t , and satisfies the strict accretivity condition on the subspace \mathcal{H} of $L^2(\mathbb{R}^n; \mathbb{C}^{m(1+n)})$ defined by $(f_j^\alpha)_{j=1, \dots, n}$ is curl free in \mathbb{R}^n for all α , that is, for some $\lambda > 0$

$$(3) \quad \int_{\mathbb{R}^n} \operatorname{Re}(A(x)f(x) \cdot \overline{f(x)}) dx \geq \lambda \sum_{i=0}^n \sum_{\alpha=1}^m \int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 dx, \quad \forall f \in \mathcal{H}.$$

The system (1) is always considered in the sense of distributions with weak solutions, that is $H_{loc}^1(\mathbb{R}_+^{1+n}; \mathbb{C}^m) = W_{loc}^{1,2}(\mathbb{R}_+^{1+n}; \mathbb{C}^m)$ solutions. We remark that the equation (1) is intrinsic in the sense that it does not depend on the many choices that can be taken for A to represent L . Any such A with the required properties will be convenient. Our method will be in some sense A dependent. We will come back to this when stating the well-posedness results.

It was proved in [AA] that weak solutions of $Lu = 0$ in the classes

$$\mathcal{E}_0 = \{u; \|\tilde{N}_*(\nabla u)\|_2 < \infty\}$$

or

$$\mathcal{E}_{-1} = \{u; \|S(t\nabla u)\|_2 < \infty\}$$

(where $\tilde{N}_*(f)$ and $S(f)$ stand for a non-tangential maximal function and square function: definitions will be given later) have a semigroup representation in their conormal gradient

$$(4) \quad \nabla_A u(t, x) := \begin{bmatrix} \partial_{\nu_A} u(t, x) \\ \nabla_x u(t, x) \end{bmatrix}.$$

More precisely, one has

$$(5) \quad \nabla_A u(t, \cdot) = S(t)(\nabla_A u|_{t=0})$$

for a certain semigroup $S(t)$ acting on the subspace \mathcal{H} of L^2 in the first case and in the corresponding subspace in \dot{H}^{-1} , where \dot{H}^s is the homogeneous Sobolev space of order s , in the second case. Actually, another equivalent representation was obtained for $u \in \mathcal{E}_{-1}$ and this one was only explicitly derived in subsequent works ([AMcM, R2]) provided one defines the conormal gradient at the boundary in this subspace of \dot{H}^{-1} . In [R2], the semigroup representation was extended to intermediate classes of solutions defined by $\mathcal{E}_s = \{u; \|S(t^{-s}\nabla u)\|_2 < \infty\}$ for $-1 < s < 0$ and the semigroup representation holds in \dot{H}^s . In particular, for $s = -1/2$, this is the class of energy solutions used in [AMcM, AM] (other ‘‘energy’’ classes were defined in [KR] and used in [HKMP2]). So this allows one to deal with any problem involving energy solutions using this first order method, but this is restrictive.

In [AS], a number of *a priori* estimates was proved concerning the solutions enjoying the representation (5) for general systems (1). In certain ranges of p , $\nabla_A u|_{t=0}$ belongs to an identified boundary space and its norm is equivalent to one of these interior controls. Thus, it remained to eliminate this *a priori* information. This is what we do here by showing existence of the trace and semigroup representation for conormal gradients of solutions in the classes $\|\tilde{N}_*(\nabla u)\|_p < \infty$ or $\|S(t\nabla u)\|_p < \infty$ for $p \neq 2$ (and more) and for the ranges of p described in [AS]. No other assumption than t -independence and ellipticity is required. Hence, what we are after here are uniqueness results for the **initial value problem** of the first order equation (6) below. The case $p = 2$ in both cases was done in [AA].

To formulate the results, we need to recall the main discovery of [AAMc] that the system (1) is in correspondence with a first order system of Cauchy-Riemann type

$$(6) \quad \partial_t \nabla_A u + DB \nabla_A u = 0,$$

where

$$(7) \quad D := \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix},$$

and

$$(8) \quad B = \hat{A} := \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}$$

whenever we write

$$(9) \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and L in the form

$$(10) \quad L = - \begin{bmatrix} \partial_t & \nabla_x \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \partial_t \\ \nabla_x \end{bmatrix}.$$

The operators D and B satisfy the necessary requirements so that DB , a perturbed Dirac type operator, is a bisectorial operator and, by a result in [AKMc], has bounded holomorphic functional calculus on L^2 : the semigroup $S(t)$ is built from an extension of $e^{-t|DB|}$, the semigroup generated by $-|DB|$ on L^2 , from the closure of the range of DB in L^2 . We note that for L^* , the associated system is not given by B^* but by $\tilde{B} = NB^*N$ where $N = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

In [AS], the Hardy space H_{DB}^p associated to DB was exploited: on this space the semigroup has a bounded extension by construction. For the boundary value problems the two natural spectral subspaces $H_{DB}^{p,+}$ and $H_{DB}^{p,-}$ obtained as the ranges of the (extensions of the) bounded projections $\chi^+(DB)$ and $\chi^-(DB)$ respectively come into play. Formally, a solution to $Lu = 0$ on the upper half-space can be constructed from $\nabla_A u(t, \cdot) = e^{-tDB} \chi^+(DB) F_0$ for some $F_0 \in H_{DB}^p$ and a solution to $Lu = 0$ on the lower half-space from $\nabla_A u(t, \cdot) = e^{tDB} \chi^-(DB) F_0$ for some $F_0 \in H_{DB}^p$. In [R1], these operators are called Cauchy extension operators because this is exactly what is obtained for the $L = -\Delta$ in two dimensions: the formula with $\chi^+(DB)$ gives the analytic extension of functions on the real line to the upper half-space and the one with $\chi^-(DB)$ the analytic extension to the lower half-space.

However, the Hardy spaces are by definition abstract completions. To relate this to classical boundary spaces, one needs to have information on these Hardy spaces. It was the main thesis of [AS] to obtain a range of p , an open interval called $I_L = (a, p_+(DB)) \subset (\frac{n}{n+1}, \infty)$, for which $H_{DB}^p = H_D^p$, a closed and complemented subspace of L^p if $p > 1$ and H^p , the real Hardy space, if $p \leq 1$. The number $p_+(DB)$ has a certain meaning there. In fact, the notation I_L could be misleading as it depends on the choice of B thus of A , but we use it for convenience. This allowed to obtain comparisons between trace estimates at the boundary and interior control in classical function spaces at the boundary and in the interior. Remark that for any p , the method in [AS] still furnishes weak solutions of the system (1) (this is not explicitly written there) for data in a subspace of H_{DB}^p . However, for $p \notin I_L$, H_{DB}^p

is not a space of distributions, so we are unable to compare with classical situations (because limits, taken in different ambient spaces, cannot be identified).

Our goal here is to go backward: begin with an arbitrary solution in some class and prove the desired representation in the restricted range of exponents imposed by the Hardy space theory. In all, this gives two classification results.

Theorem 1.1. *Let $n \geq 1$, $m \geq 1$. Let $\frac{n}{n+1} < p < p_+(DB)$ be such that $H_{DB}^p = H_D^p$ with equivalence of norms. Then, for any weak solution u to $Lu = 0$ on \mathbb{R}_+^{1+n} , the following are equivalent:*

- (i) $\|\tilde{N}_*(\nabla u)\|_p < \infty$.
- (ii) $\|S(t\partial_t \nabla u)\|_p < \infty$ and $\nabla_A u(t, \cdot)$ converges to 0 in the sense of distributions as $t \rightarrow \infty$.
- (iii) $\exists! F_0 \in H_{DB}^{p,+}$, called the conormal gradient of u at $t = 0$ and denoted by $\nabla_A u|_{t=0}$, such that $\nabla_A u(t, \cdot) = S_p(t)(\nabla_A u|_{t=0})$ for all $t \geq 0$.
- (iv) $\exists F_0 \in H_D^p$ such that $\nabla_A u(t, \cdot) = S_p^+(t)F_0$ for all $t \geq 0$.

Here, $S_p(t)$ is the bounded extension to H_D^p of the semigroup $e^{-t|DB|}$ originally defined on H_D^2 , $S_p^+(t)$ is the extension to H_D^p of $e^{-tDB}\chi^+(DB)$ and both agree on $H_{DB}^{p,+}$. If any of the conditions above hold, then

$$(11) \quad \|\tilde{N}_*(\nabla u)\|_p \sim \|S(t\partial_t \nabla u)\|_p \sim \|\nabla_A u|_{t=0}\|_{H^p} \sim \|\chi_p^+ F_0\|_{H^p},$$

where χ_p^+ is the continuous extension of $\chi^+(DB)$ on H_D^p .

This has the following corollary.

Corollary 1.2. *Let p under the conditions of Theorem 1.1 and u be a weak solution to $Lu = 0$ on \mathbb{R}_+^{1+n} with $\|\tilde{N}_*(\nabla u)\|_p < \infty$. Then, we have the following regularity properties: $t \mapsto \nabla_A u(t, \cdot) \in C_0([0, \infty); H_{DB}^{p,+}) \cap C^\infty(0, \infty; H_{DB}^{p,+})$ and*

$$(12) \quad \|\nabla_A u|_{t=0}\|_{H^p} \sim \sup_{t \geq 0} \|\nabla_A u(t, \cdot)\|_{H^p}.$$

Moreover, if $p < n$, $u = \tilde{u} + c$, where $t \mapsto \tilde{u}(t, \cdot) \in C_0([0, \infty); \dot{H}^{1,p} \cap L^{p^*}) \cap C^\infty(0, \infty; \dot{H}^{1,p} \cap L^{p^*})$ and $c \in \mathbb{C}^m$, and

$$(13) \quad \sup_{t \geq 0} \|\tilde{u}(t, \cdot)\|_{p^*} \lesssim \|\nabla_A u|_{t=0}\|_{H^p}.$$

If $p \geq n$, then $t \mapsto u(t, \cdot) \in C_0([0, \infty); \dot{H}^{1,p} \cap \dot{\Lambda}^s) \cap C^\infty(0, \infty; \dot{H}^{1,p} \cap \dot{\Lambda}^s)$ with $s = 1 - \frac{n}{p}$ and

$$(14) \quad \sup_{t \geq 0} \|u(t, \cdot)\|_{\dot{\Lambda}^s} \lesssim \|\nabla_A u|_{t=0}\|_{L^p}.$$

In addition, we have the almost everywhere limits when $p \geq 1$,

$$(15) \quad \lim_{t \rightarrow 0} \iint_{W(t,x)} \nabla_A u(s, y) ds dy = \lim_{t \rightarrow 0} \int_{B(x,t)} \nabla_A u(t, y) dy = \nabla_A u|_{t=0}(x)$$

and similarly for the time derivatives $\partial_t u$, and the almost everywhere limit for u whatever p ,

$$(16) \quad \lim_{t \rightarrow 0} \iint_{W(t,x)} u(s, y) ds dy = \lim_{t \rightarrow 0} \int_{B(x,t)} u(t, y) dy = u|_{t=0}(x).$$

Here, we adopt the convention that $H^p = L^p$ if $p > 1$. The notation C_0 stands for continuous functions that vanish at ∞ . The exponent $p^* = \frac{np}{n-p}$ is the Sobolev exponent. The Whitney regions will be defined in Section 2.

If (iii) holds, then (iv) holds clearly and conversely (iv) implies (iii) with $\nabla_A u|_{t=0} = \chi_p^+ F_0$. The second condition in (ii) is mild and only meant to control the growth of the gradient at infinity. It cannot be avoided. It follows from the results in [AS], that (iii) implies (i) and (iii) implies (ii) and in this case (11) holds. Here, we prove the converses: existence of the semigroup equation and trace in (iii) for the indicated topology.

We remark that the estimates (12), (13) and (14) come *a posteriori* in Corollary 1.2: they are regularity results for the class of solutions in (i) or (ii). We are not sure we could run the argument taking the condition (12) or even the weaker one $\sup_{t \geq 0} \|\int_{[t, 2t]} \nabla_A u(s, \cdot) ds\|_{H^p} < \infty$ as a starting point, except if $p = 2$ (this is observed in [AA]) or p near 2. We shall not attempt to prove this.

A second theorem requires the use of negative order Sobolev and Hölder spaces. We will recall the definitions later.

Theorem 1.3. *Let $n \geq 1$, $m \geq 1$. Let $\frac{n}{n+1} < q < p_+(D\tilde{B})$ be such that $H_{D\tilde{B}}^q = H_D^q$ with equivalence of norms. Let u be a weak solution to $Lu = 0$ on \mathbb{R}_+^{1+n} .*

First when $q > 1$ and $p = q'$, the following are equivalent:

- (α) $\|S(t\nabla u)\|_p < \infty$ and $u(t, \cdot)$ converges to 0 in \mathcal{D}' modulo constants as $t \rightarrow \infty$ if $p \geq 2^*$.
- (β) $\exists! F_0 \in \dot{W}_{DB}^{-1,p,+}$, called the conormal gradient of u at $t = 0$ and denoted by $\nabla_A u|_{t=0}$, such that $\nabla_A u(t, \cdot) = \tilde{S}_p(t)(\nabla_A u|_{t=0})$ for all $t \geq 0$.
- (γ) $\exists F_0 \in \dot{W}_D^{-1,p}$, such that $\nabla_A u(t, \cdot) = \tilde{S}_p^+(t)F_0$ for all $t \geq 0$.

Here, $\tilde{S}_p^+(t)$ is the extension of $e^{-tDB}\chi^+(DB)$ to $\dot{W}_D^{-1,p}$ which agrees with the extension $\tilde{S}_p(t)$ of $e^{-t|DB|}$ on $\dot{W}_{DB}^{-1,p,+}$. If any of the conditions above hold, then

$$(17) \quad \|S(t\nabla u)\|_p \sim \|\nabla_A u|_{t=0}\|_{\dot{W}^{-1,p}} \sim \|\tilde{\chi}_p^+ F_0\|_{\dot{W}^{-1,p}},$$

where $\tilde{\chi}_p^+$ is the bounded extension of $\chi^+(DB)$ on $\dot{W}_D^{-1,p}$.

Second when $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1) \in [0, 1)$, the following are equivalent:

- (a) $\|t\nabla u\|_{T_{2,\alpha}^\infty} < \infty$ and $u(t, \cdot)$ converges to 0 in \mathcal{D}' modulo constants as $t \rightarrow \infty$.
- (b) $\exists! F_0 \in \dot{\Lambda}_{DB}^{\alpha-1,+}$, called the conormal gradient of u at $t = 0$ and denoted by $\nabla_A u|_{t=0}$, such that $\nabla_A u(t, \cdot) = \tilde{S}_\alpha(t)(\nabla_A u|_{t=0})$ for all $t \geq 0$.
- (c) $\exists F_0 \in \dot{\Lambda}_D^{\alpha-1}$, such that $\nabla_A u(t, \cdot) = \tilde{S}_\alpha^+(t)F_0$ for all $t \geq 0$,

where $\tilde{S}_\alpha^+(t)$ is the extension (defined by weak-star duality) of $e^{-tDB}\chi^+(DB)$ to $\dot{\Lambda}_D^{\alpha-1}$ which agrees with the extension (also defined by weak-star duality) $\tilde{S}_\alpha(t)$ of $e^{-t|DB|}$ on $\dot{\Lambda}_{DB}^{\alpha-1,+}$. If any of the conditions above hold, then

$$(18) \quad \|t\nabla u\|_{T_{2,\alpha}^\infty} \sim \|\nabla_A u|_{t=0}\|_{\dot{\Lambda}^{\alpha-1}} \sim \|\tilde{\chi}_\alpha^+ F_0\|_{\dot{\Lambda}^{\alpha-1}},$$

where $\tilde{\chi}_\alpha^+$ is the bounded extension of $\chi^+(DB)$ on $\dot{\Lambda}_D^{\alpha-1}$.

Let us mention that in the case $\alpha = 0$, $\|t\nabla u\|_{T_{2,\alpha}^\infty} < \infty$ means that $|t\nabla u(t, x)|^2 \frac{dt dx}{t}$ is a Carleson measure and $\dot{\Lambda}^{-1}$ is the space BMO^{-1} .

The condition at ∞ in (α) is used to eliminate some constant solutions in t . When $p < 2^* = \frac{2n}{n-2}$, it follows from $\|S(t\nabla u)\|_p < \infty$ and thus is redundant. Our statement

is therefore in agreement with the $p = 2$ result of [AA]. Again, (β) is equivalent to (γ) and (β) implies (α) is proved in [AS]. Similarly, (b) is equivalent to (c) and (b) implies (a) is proved in [AS]. We show here the converses.

Corollary 1.4. *In the first case of Theorem 1.3,*

$$t \mapsto \nabla_A u(t, \cdot) \in C_0([0, \infty); \dot{W}_{DB}^{-1,p,+}) \cap C^\infty(0, \infty; \dot{W}_{DB}^{-1,p,+})$$

and

$$(19) \quad \|\nabla_A u|_{t=0}\|_{\dot{W}^{-1,p}} \sim \sup_{t \geq 0} \|\nabla_A u(t, \cdot)\|_{\dot{W}^{-1,p}},$$

and $u = \tilde{u} + c$, where $t \mapsto \tilde{u}(t, \cdot) \in C_0([0, \infty); L^p) \cap C^\infty(0, \infty; L^p)$ and $c \in \mathbb{C}^m$ and

$$(20) \quad \sup_{t \geq 0} \|\tilde{u}(t, \cdot)\|_p \lesssim \|\nabla_A u|_{t=0}\|_{\dot{W}^{-1,p}}.$$

In addition, we have the non-tangential maximal estimate

$$(21) \quad \|\tilde{N}_*(\tilde{u})\|_p \lesssim \|S(t\nabla u)\|_p$$

and the almost everywhere limit

$$(22) \quad \lim_{t \rightarrow 0} \iint_{W(t,x)} u(s, y) ds dy = \lim_{t \rightarrow 0} \int_{B(x,t)} u(t, y) dy = u|_{t=0}(x).$$

In the second case of Theorem 1.3,

$$t \mapsto \nabla_A u(t, \cdot) \in C_0([0, \infty); \dot{\Lambda}_{DB}^{\alpha-1,+}) \cap C^\infty(0, \infty; \dot{\Lambda}_{DB}^{\alpha-1,+}),$$

where $\dot{\Lambda}^{\alpha-1}$ is equipped with weak-star topology, and

$$(23) \quad \|\nabla_A u|_{t=0}\|_{\dot{\Lambda}^{\alpha-1}} \sim \sup_{t \geq 0} \|\nabla_A u(t, \cdot)\|_{\dot{\Lambda}^{\alpha-1}}.$$

Next $t \mapsto u(t, \cdot) \in C_0([0, \infty); \dot{\Lambda}^\alpha) \cap C^\infty(0, \infty; \dot{\Lambda}^\alpha)$, where $\dot{\Lambda}^\alpha$ is equipped with the weak-star topology, and

$$(24) \quad \sup_{t \geq 0} \|u(t, \cdot)\|_{\dot{\Lambda}^\alpha} \lesssim \|\nabla_A u|_{t=0}\|_{\dot{\Lambda}^{\alpha-1}}.$$

Moreover, $u \in \dot{\Lambda}^\alpha(\overline{\mathbb{R}_+^{1+n}})$ with

$$(25) \quad \|u\|_{\dot{\Lambda}^\alpha(\overline{\mathbb{R}_+^{1+n}})} \lesssim \|\nabla_A u|_{t=0}\|_{\dot{\Lambda}^{\alpha-1}}.$$

For $\alpha = 0$, this is a BMO estimate on \mathbb{R}_+^{1+n} .

Remark that the estimate (21) holds for any weak solution for which the right hand side is finite (for those p), and nothing else.

Note that we have defined a number of apparently different notions of conormal gradients at the boundary. They are all consistent. Namely, if a solution satisfies several of the conditions in the range of applicability of our results then the conormal gradients at the boundary associated to each of these conditions are the same. This is again because everything happens in the ambient space of Schwartz distributions.

Let us turn to boundary value problems for solutions of $Lu = 0$ or $L^*u = 0$ and formulate four such problems:

- (1) $(D)_Y^{L^*}$: $L^*u = 0$, $u|_{t=0} \in Y$, $t\nabla u \in \tilde{\mathcal{T}}$.
- (2) $(R)_X^L$: $Lu = 0$, $\nabla_x u|_{t=0} \in X$, $\tilde{N}_*(\nabla u) \in \mathcal{N}$.
- (3) $(N)_Y^{L^*}$: $L^*u = 0$, $\partial_{\nu_{A^*}} u|_{t=0} \in \dot{Y}^{-1}$, $t\nabla u \in \tilde{\mathcal{T}}$.

$$(4) (N)_X^L: Lu = 0, \partial_{\nu_A} u|_{t=0} \in X, \tilde{N}_*(\nabla u) \in \mathcal{N}.$$

Here we restrict ourselves to $q \in I_L$. Then $\mathcal{N} = L^q$, $X = L^q$ if $q > 1$ and $X = H^q$ if $q \leq 1$. Next, Y is the dual space X (we are ignoring whether functions are scalar or vector-valued; context is imposing it) and $\dot{Y}^{-1} = \text{div}_x(Y^n)$ with the quotient topology, equivalently \dot{Y}^{-1} is the dual of \dot{X}^1 defined by $\nabla f \in X$. Finally, $t\nabla u \in \tilde{\mathcal{T}}$ means that $t\nabla u$ belongs the tent space \mathcal{T} equal to $T_2^{q'}$ if $q > 1$, to the weighted Carleson measure space $T_{2,n(\frac{1}{q}-1)}^\infty$ if $q \leq 1$, and $u(t, \cdot)$ converges to 0 in \mathcal{D}' modulo constants if $t \rightarrow \infty$.

In each case, solving means finding a solution with control from the data. For example, for $(D)_Y^{L^*}$ we want $\|t\nabla u\|_{\mathcal{T}} \lesssim \|u|_{t=0}\|_Y$. If one can do this for all data then the open mapping theorem furnishes the implicit constant. The behavior at the boundary is the strong or weak-star convergence specified by our corollaries above (almost everywhere convergence of Whitney averages is available as a bonus). Uniqueness means that there is at most one solution (modulo constants for $(R)_X^L$ and $(N)_X^L$) in the specified class for a given boundary data. Well-posedness is the conjunction of both existence of a solution for all data and uniqueness. No additional restriction is imposed.

Recall that the trace h of a conormal gradient contains two terms: the first one is called the scalar part and denoted by h_\perp , and the second one called the tangential part and denoted by h_\parallel , which has the gradient structure, *i.e.*, $h = [h_\perp, h_\parallel]^T$. The two maps $N_\perp : h \mapsto h_\perp$ and $N_\parallel : h \mapsto h_\parallel$ are of importance in this context because they carry the well-posedness. Denote by X_\perp and X_\parallel the corresponding parts of the space X_D defined as the image of X under the (extension to X) of the orthogonal projection on the L^2 range of the Dirac operator D . Do similarly for \dot{Y}_D^{-1} .

Theorem 1.5. *We have the following assertions for $q \in I_L$.*

- (1) $(D)_Y^{L^*}$ is well-posed if and only if $N_\parallel : \dot{Y}_{D\bar{B}}^{-1,+} \rightarrow \dot{Y}_\parallel^{-1}$ is an isomorphism.
- (2) $(R)_X^L$ is well-posed if and only if $N_\parallel : X_{D\bar{B}}^+ \rightarrow X_\parallel$ is an isomorphism.
- (3) $(N)_{Y^{-1}}^{L^*}$ is well-posed if and only if $N_\perp : \dot{Y}_{D\bar{B}}^{-1,+} \rightarrow \dot{Y}_\perp^{-1}$ is an isomorphism.
- (4) $(N)_X^L$ is well-posed if and only if $N_\perp : X_{D\bar{B}}^+ \rightarrow X_\perp$ is an isomorphism.

In each case, ontoeness is equivalent to existence and injectivity to uniqueness.

As said earlier, our method is A dependent in the sense that it builds a representation on the conormal gradient for $\nabla_A u$, hence the trace space depends on A . However, the well-posedness of the Dirichlet and regularity problems are intrinsic: it does not depend on which A is chosen to represent the operator L , thus N_\parallel is invertible for any possible choice of A . It is tempting to think that there is a better choice of A than other which could lead to invertibility of N_\parallel . The Neumann problem is of course A dependent because we impose the conormal derivative. For example, for $Lu = 0$ being a real and symmetric equation, then one chooses A real and symmetric, which is the only possible choice with this property. Then (2) and (4) are well-posed when $X = L^2$ (existence follows from [KP] and uniqueness, together with an other proof of existence via layer potentials, follows from [AAHK]). It is even extendable to all systems with $A = A^*$ in complex sense combining [AAMc] and [AA].

We continue with a discussion on duality principles first studied in [KP, KP1], obtaining the sharpest possible version (in our context) compared to earlier results ([KP, KP1, S, KiS, Ma, HKMP2, AM]).

Theorem 1.6. *Let $n \geq 1$, $m \geq 1$. Let $q \in I_L$. If $q > 1$,*

- (1) $(D)_{\dot{Y}}^{L^*}$ is well-posed if and only if $(R)_X^L$ is well-posed.
- (2) $(N)_{\dot{Y}^{-1}}^{L^*}$ is well-posed if and only if $(N)_X^L$ is well-posed.

If $q \leq 1$, then the ‘if’ direction holds in both cases.

In fact, these duality principles are best seen from extensions of some abstract results on pairs of projections in [AAH]. In concrete terms, this corresponds to the Green’s formula; we will do a direct proof where in the way, we use a easy consequence of our method: any solution in \mathcal{N} or $\tilde{\mathcal{T}}$ can be approximated by an energy solution in the topology \mathcal{N} or $\tilde{\mathcal{T}}$ respectively. This is possible by approximating the conormal gradient at the boundary, not just the Neumann or Dirichlet data.

Theorem 1.7. *Let $n \geq 1$, $m \geq 1$. Let $q \in I_L$. Then, for any weak solution u to $Lu = 0$ on \mathbb{R}_+^{1+n} with $\tilde{N}_*(\nabla u) \in \mathcal{N}$ and any weak solution w to $L^*w = 0$ on \mathbb{R}_+^{1+n} with $t\nabla w \in \mathcal{T}$ and $w(t, \cdot)$ converging to 0 in \mathcal{D}' modulo constants as $t \rightarrow \infty$,*

$$(26) \quad \langle \partial_{\nu_A} u|_{t=0}, w|_{t=0} \rangle = \langle u|_{t=0}, \partial_{\nu_{A^*}} w|_{t=0} \rangle.$$

Here the first pairing is the $\langle X, Y \rangle$ (sesquilinear) duality while the second one is the $\langle \dot{X}^1, \dot{Y}^{-1} \rangle$ (sesquilinear) duality.

There is a notion of solvability attached to energy solutions which was the one originally introduced in [KP]. Indeed, the ‘‘classical’’ Dirichlet and Neumann problems for solutions with $\|\nabla u\|_2 < \infty$ are well-posed by Lax-Milgram theorem: for all ‘‘energy data’’, one obtains a unique (modulo constants) solution in this class. The *DB* method furnishes an energy solution for any ‘‘energy data’’ (see [R2] or [AMcM] or [AM]). It has to be the same solution as the Lax-Milgram solution by uniqueness. The boundary value problem is said to be solvable for the energy class if there is a constant C such that any energy solution satisfies the interior estimate $\|\tilde{N}_*(\nabla u)\|_{\mathcal{N}} \leq C\|data\|_X$ or $\|t\nabla u\|_{\mathcal{T}} \leq C\|data\|_{\dot{Y}^{-1}}$ with ‘‘data’’ being the Neumann data or the gradient of the Dirichlet data depending on the problem. In [AS] it was shown for each of the four boundary value problems that solvability for the energy class implies existence of a solution for all data. We can improve this result as follows. We can define a notion of **compatible well-posedness** (a terminology taken from [BM]) which is that there is well-posedness of the boundary value problem **and** that when the Dirichlet or Neumann data is also an ‘‘energy data’’, then the solution coincides with the energy solution obtained by Lax-Milgram. This restriction is unavoidable. Examples from [Ax, KR, KKPT] show that existence for all data does not imply solvability for the energy class. That is, there are examples with smooth Dirichlet data where a solution exists and belongs to the appropriate class \mathcal{N} (for the regularity problem) but the energy solution does not belong to this class. Hence this is strictly stronger. It is clear that compatible well-posedness implies solvability for the energy class. The converse holds.

Theorem 1.8. *For each of the four boundary value problems with $q \in I_L$, solvability for the energy class implies, hence is equivalent to, compatible well-posedness.*

In particular, combining this theorem with the extrapolation results in [AM] (which have extra hypotheses) gives extrapolation of compatible well-posedness under the assumptions there.

The Green's formula has a useful corollary as far as solvability vs uniqueness of the dual problem is concerned. This improves also the known duality principles of [HKMP2] and [AM].

Theorem 1.9. *Let $n \geq 1$, $m \geq 1$. Let $q \in I_L$. Then*

- (1) *If $(R)_X^L$ is solvable for the energy class, then $(D)_Y^{L^*}$ is compatibly well-posed.*
- (2) *If $(D)_Y^{L^*}$ is solvable for the energy class and $q \geq 1$, then $(R)_X^L$ is compatibly well-posed.*
- (3) *If $(N)_X^L$ is solvable for the energy class, then $(N)_Y^{L^*}$ is compatibly well-posed.*
- (4) *If $(N)_Y^{L^*}$ is solvable for the energy class and $q \geq 1$, then $(N)_X^L$ is compatibly well-posed.*

In the items (2) and (4), uniqueness holds in the conclusion when $q < 1$.

Note that we include the case $q = 1$ in (2) and (4), while the corresponding directions in Theorem 1.6 are unclear. It has to do with the fact that there always exists an energy solution with given “energy data”; we do not have to “build” it.

It follows from our results that any method of solvability will lead to optimal results in the range of exponents governed by I_L . Of course, what we do not address here is how to prove solvability. We mention that [AS] describes this range in a number of cases covering the case of systems with De Giorgi-Nash-Moser interior estimates (in fact, less is needed) for which $I_L = (1 - \varepsilon, 2 + \varepsilon')$. Another interesting situation is the case of boundary dimension $n = 1$: then $I_L = (\frac{1}{2}, \infty)$ so that we are able to describe all possible situations in this class of boundary value problems. Another situation is constant coefficients in arbitrary dimension: $I_L = (\frac{n}{n+1}, \infty)$. This will imply compatible well-posedness of each of the four boundary value problems in every possible class considered here using results in [AAMc]. For general systems, I_L is an open interval and contains $[\frac{2n}{n+2}, 2]$.

In [HKMP1], it is proved, and this is a beautiful result, that for real equations $L^*u = 0$, L^* -harmonic measure is A_∞ of the Lebesgue measure on the boundary and that $(D)_{L^p}^{L^*}$ is solvable for the energy class (in the sense of our definition). In [DKP], it is shown (this is in bounded domains, but there is no difference, and for t -dependent coefficients as well) that this A_∞ condition implies BMO solvability of the Dirichlet problem for the energy class, *i.e.*, $(D)_{BMO}^{L^*}$. In [AS], it is shown that $(R)_{H^1}^L$ solvability for the energy class implies N_{\parallel} is an isomorphism for neighboring spaces, which by our result above means well-posedness and we just assume solvability with our improvement. Thus well-posedness of the regularity problem can be shown for spaces near H^1 . We have a little more precise result, completing the ones in [HKMP2] for dimensions $1 + n \geq 3$ and in [KR, B] for dimension $1 + n = 2$. There is a result of this type in [BM].

Corollary 1.10. *Consider real equations of the form (1). The regularity problem is compatibly well-posed for L^q for some range of $q > 1$, H^1 , and H^q for some range of $q < 1$. This is the same for the Neumann problem when $n = 1$. The Dirichlet problem is compatibly well-posed on L^p for some $p < \infty$, BMO and $\dot{\Lambda}^\alpha$ for some $\alpha > 0$. In both situations, well-posedness remains for perturbations in L^∞ of the matrix coefficients.*

It is well-known that Neumann problems and regularity problems are the same when $1 + n = 2$ using conjugates, hence the result is valid for the Neumann problem in two dimensions. Solvability of the Neumann problem seems to be harder in dimensions $1 + n \geq 3$.

We mentioned the perturbation case for completeness of this statement but this is a mere consequence of the results in [AS], Section 14.2, at this stage. Such results imply that well-posedness is stable under L^∞ perturbation of the coefficients. But it is a question, even with our improvements here, whether compatible well-posedness is stable under L^∞ perturbation of the coefficients.

We note that this result, as any other one so far, only takes care of solvability issues on the upper-half space. Results on the lower half-space can be formulated using the negative spectral spaces X_{DB}^- , $\dot{Y}_{DB}^{-1,-}$. We leave them to the reader. In [HKMP2], the solvability on both half-spaces (this holds for real equations) is also used to show invertibility of the single layer potential in this range. The converse holds. This can be made a formal statement in our context for all four problems, extending the result in [HKMP2] for the single layer potential. We shall explain the notation when doing the proof.

Theorem 1.11. *Fix $q \in I_L$. Let $X = L^q$ or H^q , \dot{X}^1 be the homogeneous Sobolev space $\dot{W}^{1,q}$ or $\dot{H}^{1,q}$, Y be the dual space of X and \dot{Y}^{-1} be the dual space of \dot{X}^1 .*

- (1) $(R)_X^L$ is well-posed on both half-spaces if and only if \mathcal{S}_0^L is invertible from X to \dot{X}^1 .
- (2) $(D)_Y^{L^*}$ is well-posed on both half-spaces if and only if $\mathcal{S}_0^{L^*}$ is invertible from \dot{Y}^{-1} to Y .
- (3) $(N)_X^L$ is well-posed on both half-spaces if and only if $\partial_{\nu_A} \mathcal{D}_0^L$ is invertible from \dot{X}^1 to X .
- (4) $(N)_{Y^{-1}}^{L^*}$ is well-posed on both half-spaces if and only if $\partial_{\nu_{A^*}} \mathcal{D}_0^{L^*}$ is invertible from Y to \dot{Y}^{-1} .

We remark that the invertibility of the layer potentials $\mathcal{D}_{0\pm}^L$ on L^p is sufficient for the solvability of the Dirichlet problem $(D)_{L^p}^L$ (not L^*) on both half-spaces. This can also be seen from our method but it is a well-known fact. This invertibility is what is proved on L^2 for the Laplace equation on special Lipschitz domains in [V]. More recently, [AAHK] establishes the same invertibility property on L^2 for real and symmetric equations (1). It is however not clear whether it is necessary.

The main bulk of the paper is the proof of the classification described in Theorems 1.1 and 1.3. The other statements concerning solvability, although important of course in the theory, are functional analytic consequences of our classification. We shall follow the setup introduced in [AA] which proves the case $p = 2$ of our statements. This is, nevertheless, far more involved. First, there is an approximation procedure to pass from the weak formulation to a semigroup equation for conormal gradients which should look like

$$\nabla_A u(t + \tau, \cdot) = e^{-\tau|DB|} \nabla_A u(t, \cdot), \quad t > 0, \tau \geq 0.$$

In classical situations, one tests the equation against the adjoint fundamental solution and goes to the limit in appropriate sense. Here, we only have at our disposal an adjoint fundamental operator. Thus we argue more at operator level and do not use any integral kernel representation. One difficulty is that we do not know how

to interpret this identity at first. This is especially true when $p > 2$. Some weak version against appropriate test functions, which we will build, is first proved. Then one constructs an element f_t which satisfies this equation and has a trace in the appropriate topology. Then the goal is to prove that $\nabla_A u(t, \cdot)$ can be identified to such an f_t . This uses behavior at ∞ in the statement to eliminate residual terms in some asymptotic expansion.

The main difficulty here is that we have to reconcile two different calculi: the one on Schwartz distributions \mathcal{S}' and the one using the functional calculus of DB via the Hardy spaces. It is thus crucial that these spaces, and any other one in the process, are embedded in \mathcal{S}' and this is the purpose of this interval I_L that we carry all the way through. The article [AS] will be most important for that and one cannot read this article without having the other one at hand, as one finds a lot of estimates used along the way. Outside this interval of exponents, our arguments collapse.

A word on history to finish. The recent papers contain historical background as concerns solvability methods (*e.g.*, [AAAHK, HKMP2, AM, HMiMo]). We isolate a few points and refer to those articles for more complete quotes. We feel that the starting point of the study of such boundary value problems with non-smooth coefficients is the breakthrough work of Dahlberg on the Dirichlet problem for the Laplace equation on Lipschitz domains [Da], then followed by the ones of Dahlberg, Fabes, Jerison, Kenig, Pipher, etc., setting the theory in the classical context of real symmetric equations: see the book by Carlos Kenig [Ke] and references therein for a good overview of the techniques based on potential theory and Green's function. Layer potential techniques based on fundamental solutions (extending the ones known and used (see [Br, V]) for the Laplace equation on Lipschitz domains) were developed in the context of real equations and their perturbations under the impetus of Steve Hofmann (see [HK, AAAHK, HMiMo, BM]). The solution of the Kato problem for second order operators and systems [CMcM, HMc, AHLMcT, AHMcT] and its extension in [AKMc] to DB operators brought a wealth of new estimates and techniques. The estimates for square roots can be used directly in the second order setup (see [HKMP1, HKMP2, HMiMo]). The article [AAH] in a first step and, more importantly, [AAMc] made the explicit link between this class of second order equations and the first order Cauchy-Riemann type system (6) for boundary value problems with L^2 data. This and the Hardy space theory were exploited to build solutions with L^p data of (1) in [AS]: this also gave new results for the boundary layer potentials. The work [AA] is the first one where a converse on representation for conormal gradients is proved for $p = 2$. As for uniqueness results concerning the boundary value problems, one can find many statements in the literature in our context ([DaK, Br, AAAHK, KS, HMiMo, BM]...) but always assuming the De Giorgi-Nash-Moser regularity for solutions. Ours are without such an assumption. Moreover, had we even assumed such regularity hypotheses, our results improve the existing ones for t -independent equations by being less greedy on assumptions: no superfluous *a priori* assumption is taken and they apply to each boundary value problem individually, and by proposing possible representation and uniqueness for $p \leq 1$ for regularity and Neumann problems.

Our results are bi-Lipschitz invariant in standard fashion. We do not insist, but this does cover the case of the Laplace equation above Lipschitz graphs. It is likely that a similar theory can be developed on the unit ball as in [KP], adapting the setup of [AR]. This deserves some adaptation though.

We do not address here the perturbation of t -independent operators by some t -dependent ones and leave the corresponding issues as the ones in this article open for $p \neq 2$; for $p = 2$, they are treated in [AA], and recently [HMaMo] addresses the method of layer potentials in this context under De Giorgi-Nash-Moser assumptions.

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2. TECHNICAL LEMMAS IN TENT SPACES

For $0 < q \leq \infty$, T_2^q is the tent space of [CMS]. For $0 < q < \infty$, this is the space of $L_{\text{loc}}^2(\mathbb{R}_+^{1+n})$ functions F such that

$$\|F\|_{T_2^q} = \|SF\|_q < \infty$$

with for all $x \in \mathbb{R}^n$,

$$(27) \quad (SF)(x) := \left(\iint_{\Gamma_a(x)} |F(t, y)|^2 \frac{dt dy}{t^{n+1}} \right)^{1/2},$$

where $a > 0$ is a fixed number called the aperture of the cone $\Gamma_a(x) = \{(t, y); t > 0, |x - y| < at\}$. Two different apertures give equivalent T_2^q norms.

For $q = \infty$, T_2^∞ is defined via Carleson measures by $\|F\|_{T_2^\infty} < \infty$, where $\|F\|_{T_2^\infty}$ is the smallest positive constant C in

$$\iint_{T_{x,r}} |F(t, y)|^2 \frac{dt dy}{t} \leq C^2 |B(x, r)|$$

for all open balls $B(x, r)$ in \mathbb{R}^n and $T_{x,r} = (0, r) \times B(x, r)$. For $0 < \alpha < \infty$, $T_{2,\alpha}^\infty$ is defined by $\|F\|_{T_{2,\alpha}^\infty} < \infty$, where $\|F\|_{T_{2,\alpha}^\infty}$ is the smallest positive constant C in

$$\iint_{T_{x,r}} |F(t, y)|^2 \frac{dt dy}{t} \leq C^2 r^{2\alpha} |B(x, r)|$$

for all open balls $B(x, r)$ in \mathbb{R}^n . For convenience, we set $T_{2,0}^\infty = T_2^\infty$. Introduce also the Carleson function $C_\alpha F$ by

$$C_\alpha F(x) := \sup \left(\frac{1}{r^{2\alpha} |B(y, r)|} \iint_{T_{y,r}} |F(t, z)|^2 \frac{dt dz}{t} \right)^{1/2},$$

taken over all open balls $B(y, r)$ containing x , so that $\|F\|_{T_{2,\alpha}^\infty} = \|C_\alpha F\|_\infty$.

For $1 \leq q < \infty$ and p the conjugate exponent to q , T_2^p is the dual of T_2^q for the duality

$$(F, G) := \iint_{\mathbb{R}_+^{1+n}} F(t, y) \overline{G(t, y)} \frac{dt dy}{t}.$$

For $0 < q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$, $T_{2,\alpha}^\infty$ is the dual of T_2^q for the same duality form. Although not done explicitly there, it suffices to adapt the proof of [CMS, Theorem 1].

For $0 < p < \infty$, we also introduce the space N_2^p as the space of $L_{loc}^2(\mathbb{R}_+^{1+n})$ functions such that $\tilde{N}_*F \in L^p(\mathbb{R}^n)$, where

$$(28) \quad \tilde{N}_*F(x) := \sup_{t>0} \left(\iint_{W(t,x)} |F(s,y)|^2 dsdy \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

with

$$(29) \quad W(t,x) := (c_0^{-1}t, c_0t) \times B(x, c_1t),$$

for some fixed parameters $c_0 > 1$, $c_1 > 0$. Changing the parameters yields equivalent N_2^p norms.

Lemma 2.1. *Let $0 < p < \infty$. If $F \in T_2^p$ and $0 < a < b < Ka$ with fixed $K > 1$. Then $F1_{a<t<b} \in N_2^p$ uniformly in a, b . If $F \in L^\infty(0, \infty; L^2)$ and $0 < a < b < \infty$, then $F1_{a<t<b} \in T_{2,\alpha}^\infty$ for all $\alpha \geq 0$.*

These estimates are trivially verified.

Lemma 2.2. *If $0 < p < r \leq 2$ and $F \in N_2^p$, then*

$$\left(\iint_{\mathbb{R}_+^{1+n}} |F(t,x)|^r t^{n(\frac{r}{p}-1)} \frac{dt dx}{t} \right)^{1/r} \lesssim \|\tilde{N}_*F\|_p.$$

Proof. This statement for $1 < r = \frac{p(n+1)}{n} \leq 2$ is explicitly in [HMiMo]. For the other cases, as $r \leq 2$, we easily obtain

$$\iint_{\mathbb{R}_+^{1+n}} |F(t,x)|^r t^{n(\frac{r}{p}-1)} \frac{dt dx}{t} \lesssim \iint_{\mathbb{R}_+^{1+n}} \left(\iint_{\tilde{W}(s,y)} |F|^2 \right)^{r/2} s^{n(\frac{r}{p}-1)} \frac{ds dy}{s},$$

where $\tilde{W}(s,y)$ is some slightly smaller Whitney region contained in $W(s,y)$. We can apply the inequalities of [CT], Theorem 2.6, for the pointwise non-tangential maximal function, adjusting the aperture of the cone at vertex x containing (s,y) so that the pointwise non-tangential maximal function of the expression within parentheses is controlled by $\tilde{N}_*F(x)$. \square

Lemma 2.3. *If $1 < p \leq 2$ and $F \in N_2^p$ then $F \in L_{loc}^p(0, \infty; L^p)$ and*

$$\sup_{a>0} \int_{[a,2a]} \|F_t\|_p^p dt \lesssim \|\tilde{N}_*F\|_p^p.$$

Proof. Do as above with $r = p$ and use that the t -integral is between a and $2a$. \square

Proposition 2.4. *Let $0 < p < \infty$. Suppose $0 < a < b < \infty$ and $F \in T_2^p$ with support in $[a,b] \times \mathbb{R}^n$. Let ρ_k be a standard mollifier in \mathbb{R}^n : $\rho \in C^\infty(\mathbb{R}^n; [0,1])$, $\text{supp } \rho \in B(0,1)$, $\int \rho = 1$ and $\rho_k(x) = k^n \rho(kx)$ for $k \geq 1$. Then, $F_k(t,x) = F_t \star_{\mathbb{R}^n} \rho_k(x)$ belongs to T_2^p uniformly in k and converges to F in T_2^p .*

Proof. Let $\|S_a F\|_p$ be the T_2^p norm on cones of aperture a . Let $x \in \mathbb{R}^n$ and $(t,y) \in \mathbb{R}_+^{1+n}$ with $|x-y| < t$. Using $|F_t \star_{\mathbb{R}^n} \rho_k(y)|^2 \leq |F_t|^2 \star_{\mathbb{R}^n} \rho_k(y)$, the supports of F and ρ , and Fubini's theorem, we have

$$\begin{aligned} \iint_{\substack{a \leq t \leq b \\ |x-y| < t}} |F_t \star_{\mathbb{R}^n} \rho_k(y)|^2 \frac{dt dy}{t^{n+1}} &\leq \iint_{\substack{a \leq t \leq b \\ |x-z| < t+1/k}} |F(t,z)|^2 \int \rho_k(y-z) dy \frac{dt dz}{t^{n+1}} \\ &\leq \iint_{\substack{a \leq t \leq b \\ |x-z| < t(1+1/a)}} |F(t,z)|^2 \frac{dt dz}{t^{n+1}}. \end{aligned}$$

Thus $S_1 F_k \leq S_{1+1/a} F$ for all $k \geq 1$. Also clearly, $S_1 F_k(x)$ converges to $S_1 F(x)$, because $F_t \star_{\mathbb{R}^n} \rho_k(y)$ converges to $F(t, y)$ in L^2 of any compact set in \mathbb{R}_+^{1+n} . The conclusion follows by dominated convergence. \square

3. SLICE-SPACES

We now introduce spaces adapted to T_2^p and $T_{2,\alpha}^\infty$. We shall call them slice-spaces while, when they are Banach spaces, they are particular cases of Wiener amalgams (see for example the survey [Fei] and also [Hei]) first introduced by Wiener and further developed in relation with time-frequency analysis and sampling theory. Our terminology comes from the heuristic image of slicing the tent space norm at a fixed height. This relation to easily cover the quasi-Banach case that we need later on.

For $p \in (0, \infty]$ and $t > 0$, the slice-space E_t^p is the subspace of $L_{loc}^2(\mathbb{R}^n)$ functions f with

$$\|f\|_{E_t^p} = \left(\int_{\mathbb{R}^n} \left(\int_{B(x,t)} |f(y)|^2 dy \right)^{p/2} dx \right)^{1/p} < \infty,$$

with the usual modification taking the ess sup norm when $p = \infty$ (as averages on balls are continuous with respect to the center, one can take sup). For $p < 1$, this is only a quasi-normed space. Also, for $\alpha \in [0, 1)$ and $t > 0$, the slice-space $E_t^{\infty,\alpha}$ is the subspace of $L_{loc}^2(\mathbb{R}^n)$ functions g such that

$$\|g\|_{E_t^{\infty,\alpha}} = \sup_{x \in \mathbb{R}^n, r \geq t} \frac{1}{r^\alpha} \left(\int_{B(x,r)} |g(y)|^2 dy \right)^{1/2} < \infty.$$

It was pointed out to us by A. Amenta that standard coverings of balls with radii r by roughly $(r/t)^n$ balls with radii t when $r \geq t$ show that

$$\|g\|_{E_t^{\infty,\alpha}} \sim \sup_{x \in \mathbb{R}^n} \frac{1}{t^\alpha} \left(\int_{B(x,t)} |g(y)|^2 dy \right)^{1/2}.$$

Hence, we see that $E_t^{\infty,\alpha} = E_t^\infty$ with equivalent norms $\|g\|_{E_t^{\infty,\alpha}} \sim t^{-\alpha} \|g\|_{E_t^\infty}$. Averaging

$$(30) \quad \int_{\mathbb{R}^n} g(y) dy = \int_{\mathbb{R}^n} \int_{B(x,t)} g(y) dy dx$$

and using Hölder's inequality with exponent $2/p$, we obtain that $E_t^p \subset L^p(\mathbb{R}^n)$ for $p \in (0, 2]$, with

$$\|f\|_p \leq \|f\|_{E_t^p}.$$

Similarly, using Hölder's inequality with exponent $p/2$ and then (30), $L^p(\mathbb{R}^n) \subset E_t^p$ for $p \in [2, \infty)$, with

$$\|f\|_{E_t^p} \leq \|f\|_p.$$

Note that the constant is one, thus uniform in t , and $E_t^2 = L^2(\mathbb{R}^n)$ for any $t > 0$, isometrically.

We have trivial embedding and projection that allow us to carry the properties of tent spaces to the slice-spaces by retraction. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and set

$$\iota(f)(s, x) = f(x) 1_{[t, et]}(s).$$

For $G : \mathbb{R}_+^{1+n} \rightarrow \mathbb{C}$ we set

$$\pi(G)(x) = \int_t^{et} G(s, x) \frac{ds}{s}.$$

Clearly, for suitable f and G ,

$$(31) \quad \iint_{\mathbb{R}_+^{1+n}} \iota(f)(s, x) G(s, x) \frac{ds dx}{s} = \int_{\mathbb{R}^n} f(x) \pi(G)(x) dx,$$

that is, π is the dual of ι in some sense, and

$$(32) \quad \pi \circ \iota(f) = f.$$

Let $t > 0$. Then for $p \in (0, \infty)$ and $\alpha \in [0, 1)$, it is easy to see that $\iota : E_t^p / E_t^{\infty, \alpha} \rightarrow T_2^p / T_{2, \alpha}^\infty$ is bounded and the norm is uniform in t , where X/Y means X or Y respectively. Next, $\pi : T_2^p / T_{2, \alpha}^\infty \rightarrow E_t^p / E_t^{\infty, \alpha}$ is bounded also with uniform bound in t with the same norm on T_2^p . This yields that E_t^p and $E_t^{\infty, \alpha}$ are retracts of T_2^p and $T_{2, \alpha}^\infty$ respectively.

Remark 3.1. For E_t^p we may use instead rescaled ι and π (consider $s \in [1, e]$) and adapt the aperture of the cone in the norm on T_2^p (consider cones of aperture t). Both methods are equivalent.

In the following lemmas we summarize the consequences of the retraction property on slice-spaces.

Lemma 3.2 (Duality). *Fix $t > 0$. In the pairing $\int_{\mathbb{R}^n} f(x)g(x) dx$,*

- 1) for $p \in (1, \infty)$, $E_t^{p'}$ is the dual space of E_t^p , where $\frac{1}{p} + \frac{1}{p'} = 1$.
- 2) for $p \in (0, 1]$, $E_t^{\infty, \alpha}$ is the dual space of E_t^p , where $\alpha = n(\frac{1}{p} - 1)$.

Proof. 1) We first show that $E_t^{p'} \subseteq (E_t^p)'$. If $g \in E_t^{p'}$, then $f \mapsto \int_{\mathbb{R}^n} fg$ induces a bounded linear functional on E_t^p . Indeed, using (32), we have

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx = \iint_{\mathbb{R}_+^{n+1}} |\iota(f)(s, x)\iota(g)(s, x)| \frac{ds dx}{s},$$

for all $f \in E_t^p$. Therefore, by tent space duality, there holds that

$$\left| \int_{\mathbb{R}^n} f(x)g(x) dx \right| \lesssim \|\iota(f)\|_{T_2^p} \|\iota(g)\|_{T_2^{p'}} \lesssim \|f\|_{E_t^p} \|g\|_{E_t^{p'}},$$

with the implicit constants uniform in t . (Note that one could obtain this inequality with constant 1 by applying (30).)

We now prove the converse inclusion, *i.e.*, $(E_t^p)' \subseteq E_t^{p'}$. Suppose that ℓ is a bounded linear functional on E_t^p . Then $\tilde{\ell} = \ell \circ \pi$ is a bounded linear functional on T_2^p and by tent space duality, there exists $G \in T_2^{p'}$ so that

$$\tilde{\ell}(F) = \iint_{\mathbb{R}_+^{n+1}} F(x, s)G(x, s) \frac{dx ds}{s},$$

for all $F \in T_2^p$. Then, if we set $F = \iota(f)$, one can easily see that $\ell(f) = \int_{\mathbb{R}^n} f(x)g(x) dx$ with $g = \pi(G)$, which is an element of $E_t^{p'}$.

2) The proof follows from a simple modification of 1) using that the dual of T_2^p is $T_{2, \alpha}^{\infty, \alpha}$ and we omit it. \square

At this point we introduce the notion of E_t^p -atoms and E_t^p -molecules, and then we prove that any function in E_t^p has an atomic decomposition if $p \leq 1$.

Definition 3.3. We fix $t > 0$ and we let $p \in (0, 1]$. A function $a \in E_t^p$ is said to be an E_t^p -atom if it is supported in a ball B_r of radius $r \geq t$ and satisfies

$$(33) \quad \|a\|_{L^2(\mathbb{R}^n)} \leq r^{n(1/2-1/p)}.$$

A function $m \in E_t^p$ is said to be an E_t^p -molecule adapted to a ball B_r of radius $r \geq t$ if it satisfies

$$(34) \quad \|m\|_{L^2(8B_r)} \leq r^{n(1/2-1/p)},$$

and if there exists $\epsilon > 0$ such that

$$(35) \quad \|m\|_{L^2(2^{k+1}B_r \setminus 2^k B_r)} \leq 2^{-\epsilon k} (2^k r)^{n(1/2-1/p)}, \quad k \geq 3.$$

Lemma 3.4 (Atomic decomposition). *Let $p \in (0, 1]$ and $f \in E_t^p$. Then there exist a sequence of numbers $\{\lambda_j\}_{j \geq 1} \subset \ell^p$ and a sequence of E_t^p -atoms $\{a_j\}_{j \geq 1}$ so that $f = \sum_j \lambda_j a_j$, with convergence in E_t^p and $\|f\|_{E_t^p} \lesssim \|\{\lambda_j\}\|_{\ell^p}$. If $f \in E_t^p \cap E_t^2$ then it converges also in E_t^2 . Conversely, given $\{\lambda_j\}_{j \geq 1} \subset \ell^p$ and E_t^p -atoms $\{a_j\}_{j \geq 1}$, the series $\sum_j \lambda_j a_j$ converges in E_t^p and defines a function in E_t^p with norm controlled by $\|\{\lambda_j\}\|_{\ell^p}$. Any E_t^p -molecule belongs to E_t^p with uniform norm.*

Proof. Suppose that $f \in E_t^p$. Then $\iota(f)$ has an atomic decomposition in T_2^p , that is, $\iota(f) = \sum_j \lambda_j A_j$. Each A_j is supported in a tent region $\widehat{B}_j = \{(t, y); B(y, t) \subset B_j\}$ and satisfies the bound $\int_{\widehat{B}_j} |A_j|^2 dx dt / t \leq |B_j|^{1-2/p}$. But the support of $\iota(f)$ is contained in the strip $[t, et] \times \mathbb{R}^n$ and thus, if r_j is the radius of the ball B_j , we necessarily have that $r_j \geq t$. We now set $a_j = \pi(A_j)$, which is an E_t^p -atom as one easily shows. The convergence of $\iota(f) = \sum_j \lambda_j A_j$ is both in T_2^p by the atomic decomposition and also T_2^2 using the support of $\iota(f)$. The convergence of $f = \sum_j \lambda_j a_j$ in $E_t^p \cap E_t^2$ follows from the boundedness of $\pi : T_2^q \rightarrow E_t^q$ for all $q \in (0, \infty)$.

The converse is similar. As $\iota(a_j)$ is a T_2^p atom, $F = \sum_j \lambda_j \iota(a_j)$ converges in T_2^p and $\pi(F) = \sum_j \lambda_j a_j$ is an element on E_t^p .

An E_t^p -molecule has an atomic decomposition using the annuli of its definition. Thus, it belongs to E_t^p with uniform norm. \square

The next lemma shows that if a function is in E_t^p for some $t > 0$, then it belongs to E_s^p for all $s > 0$.

Lemma 3.5 (Change of norms). *If $0 < s, t < \infty$ and $p \in (0, \infty)$, then $E_t^p = E_s^p$ with*

$$(36) \quad \min(1, (t/s)^{n/2-n/p}) \|f\|_{E_t^p} \lesssim_{n,p} \|f\|_{E_s^p} \lesssim_{n,p} \max(1, (t/s)^{n/2-n/p}) \|f\|_{E_t^p}.$$

Proof. To see this we use that E_t^p norms are comparable to T_2^p norms with aperture t and use the precise comparison of T_2^p norms under change of aperture obtained in [A1] (some of these bounds were already in Torchinsky's book [Tor]). \square

We define M_b to be the operator of multiplication with a function $b \in L^\infty(\mathbb{R}^n)$ and \mathcal{C}_ϕ to be the convolution operator with an integrable function ϕ with bounded support. We now show some stability, density and embedding properties of slice-spaces.

Lemma 3.6. *Fix $t > 0$ and $p \in (0, \infty]$. Then the following hold.*

- 1) $M_b : E_t^p \rightarrow E_t^p$.
- 2) $\mathcal{C}_\phi : E_t^p \rightarrow E_t^p$.
- 3) $\mathcal{D}(\mathbb{R}^n)$ is dense in E_t^p when $p < \infty$.
- 4) E_t^p embeds in the space of Schwartz distributions \mathcal{S}' .

Proof. The first point is obvious. The second point follows easily using Lemma 3.5: we may assume that ϕ is supported in the ball $B(0, R)$ and set $t = R$. In this case, we have

$$\int_{B(x, R)} |\phi \star f|^2 \leq \int_{\mathbb{R}^n} |\phi \star 1_{B(x, 2R)} f|^2 \leq \|\phi\|_1^2 \int_{B(x, 2R)} |f|^2$$

and it suffices to integrate $p/2$ powers or to take the sup norm when $p = \infty$. To prove 3), we use the usual truncation and mollification arguments along with 1) and 2). To show 4), we write $\int_{\mathbb{R}^n} f(x)\varphi(x) dx = (\iota(f), \iota(\varphi))$ and observe that $\iota(\varphi)$ belongs to the dual of T_2^p when $\varphi \in \mathcal{S}(\mathbb{R}^n)$. \square

Remark 3.7. Since $\mathcal{D}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, 3) yields that $E_t^p \cap E_t^2$ is dense in E_t^p when $p < \infty$.

In the following lemma the derivatives are taken in the $W_{loc}^{1,2}$ sense. We shall need to use several times this unconventional integration by parts.

Lemma 3.8 (Integration by parts in slice-spaces). *Let $p \in (0, \infty)$ and suppose that ∂ is a first order differential operator with constant coefficients and ∂^* is its adjoint operator. If $f, \partial f \in E_t^p$ and $g, \partial^* g \in (E_t^p)'$, then $\int_{\mathbb{R}^n} \partial f(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} f(x) \overline{\partial^* g(x)} dx$.*

Proof. Take a smooth cut-off function χ_R which is 1 on $B(0, R)$, 0 outside $B(0, 2R)$ and $\|\nabla \chi_R\|_\infty \leq CR^{-1}$. Then using integration by parts for $W^{1,2}$ functions with bounded support

$$\int_{\mathbb{R}^n} \chi_R(x) \partial f(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} \chi_R(x) f(x) \overline{\partial^* g(x)} dx - \int_{\mathbb{R}^n} \partial \chi_R(x) f(x) \overline{g(x)} dx.$$

It remains to let $R \rightarrow \infty$ by using dominated convergence for each integral. \square

Finally, we state that E_t^p are real and complex interpolation spaces.

Lemma 3.9 (Interpolation). *For fixed $t > 0$, suppose that $0 < p_0 < p < p_1 \leq \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$. Then*

- (1) $[E_t^{p_0}, E_t^{p_1}]_\theta = E_t^p$ (complex method) with equivalent norms uniformly with respect to t .
- (2) $(E_t^{p_0}, E_t^{p_1})_{\theta, q} = E_t^p$, $q = p$ (real method) with equivalent norms uniformly with respect to t .

Proof. This follows from the fact that E_t^p is a retract of T_2^p and the results in [CMS], completed and corrected in [Be]. \square

All of this extends to \mathbb{C}^N -valued functions.

4. OPERATORS WITH OFF-DIAGONAL DECAY ON SLICE-SPACES

In this section, we investigate the boundedness on slice-spaces of operators with L^2 off-diagonal decay and prove some convergence results in those spaces.

Definition 4.1. A family of operators $(T_s)_{s>0}$ is said to have L^2 off-diagonal decay of order $N \in \mathbb{N}$ if there exists $C_N > 0$ such that

$$(37) \quad \|1_E T_s f\|_2 \leq C_N \langle \text{dist}(E, F)/s \rangle^{-N} \|f\|_2,$$

for all $s > 0$, whenever $E, F \subset \mathbb{R}^n$ are closed sets and $f \in L^2$ with $\text{supp } f \subset F$. We have set $\langle x \rangle := 1 + |x|$ and $\text{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$.

Proposition 4.2. Let $p \in (0, \infty]$. If $(T_s)_{s>0}$ is a family of linear operators with L^2 off-diagonal decay of order $N > \inf(n|1/p - 1/2|, n/2)$, then

$$T_s : E_t^p \rightarrow E_t^p, \quad \text{uniformly in } 0 < s \leq t.$$

Proof. **Case** $0 < p \leq 1$. To prove boundedness in E_t^p we claim that it suffices to prove that T_s maps E_t^p -atoms to E_t^p -molecules. Indeed, let $f \in E_t^p \cap E_t^2$. Then, by Lemma 3.4, f has an atomic decomposition with convergence in both E_t^p and E_t^2 . Since $T_s : E_t^2 \rightarrow E_t^2$ and $f = \sum_j \lambda_j a_j$ in E_t^2 , we have $T_s f = \sum_j \lambda_j T_s a_j$ in E_t^2 , hence $|(T_s f)(x)| \leq \sum_j |\lambda_j| |(T_s a_j)(x)|$ for almost every $x \in \mathbb{R}^n$. Taking E_t^p quasi-norms in both sides and using the claim we obtain that $T_s : E_t^p \cap E_t^2 \rightarrow E_t^p$. Since $E_t^p \cap E_t^2$ is dense in E_t^p , we extend T_s by continuity to a bounded operator $\tilde{T}_s : E_t^p \rightarrow E_t^p$, with bounds uniform in $0 < s \leq t$.

We shall now prove our claim. Suppose that a is an E_t^p -atom supported in a ball B_r of radius $r \geq t$. Let also $C_k(B_r) = 2^{k+1}B_r \setminus 2^k B_r$ and apply (37) with $E = C_k(B_r)$, $F = B_r$ and $f = a$. Then we have that

$$(38) \quad \begin{aligned} \|T_s a\|_{L^2(C_k)} &\lesssim \frac{s^N}{(2^k r)^N} \|a\|_{L^2} \lesssim \frac{r^N}{(2^k r)^N} r^{n(1/2-1/p)} \\ &= 2^{-k(N-n(1/p-1/2))} (2^k r)^{n(1/2-1/p)}, \end{aligned}$$

where in the second inequality we used that $s \leq t \leq r$ and (33). The fact that $\|T_s a\|_{L^2(8B_r)} \lesssim r^{n(1/2-1/p)}$ is immediate from (37) and (33), which in conjunction with (38) implies that $T_s a$ is a E_t^p -molecule.

Case $p = \infty$. As $E_t^\infty = E_t^{\infty,0}$ with equivalent norms uniformly in t , it is enough to prove $T_s : E_t^{\infty,0} \rightarrow E_t^{\infty,0}$. Suppose that $g \in E_t^{\infty,0}$ and let us fix an arbitrary ball B_r of radius $r \geq t$ that contains x . We decompose g so that $g := g_0 + \sum_k g_k = g_1 8B_r + \sum_k g_k 1_{C_k(B_r)}$, where $C_k(B_r) = 2^{k+1}B_r \setminus 2^k B_r$, $k \geq 3$. We utilize the L^2 boundedness of T_s (coming from (37)) to get

$$(39) \quad \left(\int_{B_r} |T_s g_0|^2 \right)^{1/2} \lesssim \left(\int_{8B_r} |g|^2 \right)^{1/2} \leq \|g\|_{E_t^{\infty,0}}$$

In view of (37) and $s \leq t$, we have that

$$(40) \quad \left(\int_{B_r} |T_s g_k|^2 \right)^{1/2} \lesssim 2^{-k(N-n/2)} \left(\int_{2^{k+1}B_r} |g|^2 \right)^{1/2} \leq 2^{-k(N-n/2)} \|g\|_{E_t^{\infty,0}}.$$

Combining (39) and (40) and $N > n/2$, we obtain that $T_s g \in E_t^\infty$, uniformly in $0 < s \leq t$.

Case $1 < p < \infty$. In light of Lemma 3.9, we conclude by interpolation that T_s extends to a bounded operator from E_t^p to E_t^p for all $1 < p < \infty$. Indeed, we assume off-diagonal decay of order $N > n/2$, which allows us to apply the $p = 1$ and $p = \infty$ cases. \square

Remark 4.3. One can probably obtain a sharper lower bound in N for a given fixed p , but this suffices for our needs. Also notice that for all $s \geq t$, one obtains that $T_s : E_t^p \rightarrow E_t^p$ but the norm may not be uniform any longer. This comes from the fact that $E_s^p = E_t^p$ with equivalent norms.

Proposition 4.4. *Fix $t > 0$. Let I be the identity operator and $(T_s)_{s>0}$ be a family of linear operators which has L^2 off-diagonal decay of order $N > \inf(n/p, n/2)$. If $T_s \rightarrow I$ strongly in L^2 for $s \rightarrow 0$, then for $0 < p < \infty$, $T_s \rightarrow I$ strongly in E_t^p for $s \rightarrow 0$.*

Proof. As we have a uniform estimate for $0 < s < t$ from Proposition 4.2, it suffices to check this for f being in a dense class. Thus, take $f \in L^2$ supported in a ball and we may assume without loss of generality that it has radius $R \geq 2t$. Let $C_j(B_R) = 2^{j+1}B_R \setminus 2^jB_R$ for $j \geq 1$. Using the support condition of f and (37), we have

$$\begin{aligned} \int_{(2B_R)^c} \left(\int_{B(x,t)} |T_s f - f|^2 \right)^{p/2} dx &= \int_{(2B_R)^c} \left(\int_{B(x,t)} |T_s f|^2 \right)^{p/2} dx \\ &= \sum_{j \geq 1} \int_{C_j(B_R)} \left(\int_{B(x,t)} |T_s f|^2 \right)^{p/2} dx \\ &\lesssim \sum_{j \geq 1} t^{-np/2} (2^j R)^n \left(\frac{s}{2^j R} \right)^{Np} \|f\|_2^p \lesssim_{t,R,f} s^{Np} \end{aligned}$$

using $N > n/p$. Next, we have

$$\int_{2B_R} \left(\int_{B(x,t)} |T_s f - f|^2 \right)^{p/2} dx \lesssim t^{-np/2} R^N \|T_s f - f\|_2^p.$$

Using the strong convergence of $T_s \rightarrow I$ in L^2 , this proves that $\|T_s f - f\|_{E_t^p} \rightarrow 0$ as $s \rightarrow 0$. \square

5. SOME PROPERTIES OF WEAK SOLUTIONS

Throughout, we assume without mention that the coefficients of (1) satisfy the ellipticity conditions (2) and (3).

Lemma 5.1 (Caccioppoli). *Any weak solution u of (1) in a ball $B = B(\mathbf{x}, r)$ with $B \subset \mathbb{R}_+^{1+n}$ enjoys the Caccioppoli inequality for any $0 < \alpha < \beta < 1$ and some C depending on the ellipticity constants, n, m, α and β ,*

$$(41) \quad \iint_{\alpha B} |\nabla u|^2 \leq Cr^{-2} \iint_{\beta B} |u|^2.$$

This is well-known. Note that the ball B can be taken with respect to any norm in \mathbb{R}^{1+n} . Thus αB can be a Whitney region $W(t, x)$ as in (29) and βB is a slightly enlarged region of the same type.

Lemma 5.2. *If u is a weak solution of (1), then so is $\partial_t u$.*

Proof. This is a consequence of the fact that L has t -independent coefficients. \square

Lemma 5.3. *Suppose u is a weak solution of (1), then $t \mapsto \nabla_A u(t, \cdot)$ is C^∞ from $t > 0$ into $L^2_{\text{loc}}(\mathbb{R}^n)$ and for all $t > 0$,*

$$(42) \quad \int_{B(x, c_1 t)} |\nabla_A u(t, x)|^2 dx \lesssim \iint_{W(t, x)} |\nabla_A u(s, y)|^2 ds dy.$$

Proof. Let K be a compact set in \mathbb{R}^n and $[a, b]$ a compact interval in $(0, \infty)$. Using the inequality $|f(t) - \int_{[a, b]} f(s) ds|^2 \leq (b-a) \int_{[a, b]} |f'(s)|^2 ds$ for all $t \in [a, b]$, we have

$$\int_K |\nabla_A u(t, x)|^2 dx \lesssim_{a, b} \iint_{[a, b] \times K} |\nabla_A u(s, x)|^2 ds dx + \iint_{[a, b] \times K} |\partial_s \nabla_A u(s, x)|^2 ds dx.$$

and we conclude that $\nabla_A u(t, \cdot) \in L^2_{\text{loc}}(\mathbb{R}^n)$ using $\partial_s \nabla_A u(s, x) = \nabla_A \partial_s u(s, x)$ and Lemma 5.2. Applying this to all t -derivatives of u implies the same conclusion for all t -derivatives of the conormal gradient.

The inequality (42) follows from applying the above inequality and Caccioppoli inequality with $[a, b] = [\tilde{c}_0^{-1}t, \tilde{c}_0 t]$ and $K = \overline{B}(x, \tilde{c}_1 t)$, the closed ball, where $1 < \tilde{c}_0 < c_0$, $\tilde{c}_1 < c_1$ and c_0, c_1 are the parameters in the Whitney region $W(t, x)$. \square

Corollary 5.4. *Suppose u is a weak solution of (1). Let $k \in \mathbb{N}$. For $0 < p < \infty$, we have that $\|t^k \nabla \partial_t^k u\|_{N_2^p} \lesssim \|\nabla u\|_{N_2^p}$, $\|t^{k+1} \nabla \partial_t^k u\|_{T_2^p} \lesssim \|t \nabla u\|_{T_2^p}$, and for $\alpha \geq 0$ $\|t^{k+1} \nabla \partial_t^k u\|_{T_{2, \alpha}^\infty} \lesssim \|t \nabla u\|_{T_{2, \alpha}^\infty}$.*

Proof. This follows from a repeated use of Caccioppoli inequality: starting from $\nabla \partial_t^k u$, eliminate ∇ and then control $\partial_t^k u$ by $\nabla \partial_t^{k-1} u$, and iterate. Details are easy and we omit them. \square

Let us observe that we can truncate in t in these inequalities provided we enlarge the truncation in the right hand side.

Corollary 5.5. *Suppose u is a weak solution of (1). Then*

$$(43) \quad \sup_{t, t' > 0, t/t' \sim 1} \|\nabla_A u(t, \cdot)\|_{E_{t'}^p} \lesssim \|\nabla u\|_{N_2^p}.$$

Similarly,

$$(44) \quad \sup_{t, t' > 0, t/t' \sim 1} t \|\nabla_A u(t, \cdot)\|_{E_{t'}^p} \lesssim \|t \nabla u\|_{T_2^p}.$$

In particular, if one of the right hand sides is finite, then for any $\delta > 0$, $t \mapsto \nabla_A u(t, \cdot)$ is C^∞ valued in E_δ^p .

Proof. The inequalities follow right away with $t' = c_1 t$ from (42) together with Lemma 2.1 for the second one. Then use the equivalence (36). For the regularity, Corollary 5.4 tells us we have that ∇u is infinitely differentiable with respect to t in N_2^p topology or T_2^p topology. Using (43) or (44), and the fact that $E_\delta^p = E_t^p$ for all t with equivalent norms, we can obtain that $t \mapsto \nabla_A u(t, \cdot)$ is C^∞ valued in E_δ^p , using Lebesgue differentiation theorem and induction. Details are standard and skipped. \square

6. REVIEW OF BASIC MATERIAL ON DB AND BD

All the material below can be found in [AS]. With D and B given in (7) and (8), the operators $T = DB$ and BD with natural domains $B^{-1}D_2(D)$ and $D_2(D)$ are bisectorial operators with bounded holomorphic functional calculus on $L^2 = L^2(\mathbb{R}^n; \mathbb{C}^N)$. Their restrictions to their closed ranges are injective.

An important operator here is the orthogonal projection $\mathbb{P} : L^2 \rightarrow \overline{\mathbb{R}_2(D)} = \overline{\mathbb{R}_2(DB)}$. It is a Calderón-Zygmund operator, thus extends to a bounded operator on L^p , $1 < p < \infty$, H^p , $0 < p \leq 1$, etc.

Recall that for $0 < p < \infty$, \mathbb{H}_{DB}^p is defined as the subspace of $\overline{\mathbb{R}_2(DB)}$ with $\|\psi(tDB)h\|_{T_2^p} < \infty$ for an allowable ψ and H_{DB}^p is its completion for this norm (or quasi-norm). Allowable means that the choice of ψ does not affect the set and the norm, up to equivalence. Similarly \mathbb{H}_{BD}^p is defined as the subspace of $\overline{\mathbb{R}_2(BD)}$ with $\|\psi(tBD)h\|_{T_2^p} < \infty$ with an allowable ψ and H_{BD}^p is its completion for this norm (or quasi-norm). Also $H_{DB}^2 = \mathbb{H}_{DB}^2 = \overline{\mathbb{R}_2(DB)}$ and similarly for BD . The projection \mathbb{P}_{BD} onto $\overline{\mathbb{R}_2(BD)}$ along $\mathbb{N}_2(BD) = \mathbb{N}_2(D)$ will play an important role in some proofs.

The semigroup $(e^{-t|DB|})_{t \geq 0}$ on H_{DB}^2 extends to a bounded, strongly continuous semigroup on H_{DB}^p for all $0 < p < \infty$, which is denoted by $(S_p(t))_{t \geq 0}$. As a matter of fact the H^∞ -calculus extends: for any $b \in H^\infty(S_\mu)$, $b(DB)$, defined on H_{DB}^2 , extends to a bounded operator on H_{DB}^p . In particular, we have two spectral subspaces $H_{DB}^{p,\pm}$ of H_{DB}^p obtained as completions of the images $\mathbb{H}_{DB}^{p,\pm}$ of \mathbb{H}_{DB}^p under $\chi^\pm(DB)$ where $\chi^\pm = 1_{\{\pm \operatorname{Re} z > 0\}}$. The restrictions $S_p^\pm(t)$ of $S_p(t)$ to $H_{DB}^{p,\pm}$ are the respective extensions of $e^{-tDB}\chi^+(DB)$ and $e^{tDB}\chi^-(DB)$.

A similar discussion can be made for BD . There is also a notion of Hölder space adapted to BD which is useful here. However, H_{BD}^p and its Hölder version may not be spaces of measurable functions.

The space H_D^p agrees with $\mathbb{P}(L^p) = \overline{\mathbb{R}_p(D)}$ if $p > 1$ and $\mathbb{P}(H^p)$ if $\frac{n}{n+1} < p \leq 1$. Thus it is a closed and complemented subspace of L^p if $p > 1$ and H^p if $p \leq 1$. When $1 < p < \infty$, H_D^p and $H_D^{p'}$ are dual spaces for the standard inner product $\langle f, g \rangle = \int_{\mathbb{R}^n} (f(x), g(x)) dx$. The pair (u, v) inside the integral is the standard complex inner product on \mathbb{C}^N where our functions take their values. For $p \leq 1$, the dual space of H_D^p is $\dot{\Lambda}_D^\alpha$, the image of the Hölder space $\dot{\Lambda}^\alpha$ if $\alpha = n(\frac{1}{p} - 1)$ or of $BMO = \dot{\Lambda}^0$ if $\alpha = 0$ under \mathbb{P} .

We use the notation $v = \begin{bmatrix} v_\perp \\ v_\parallel \end{bmatrix}$ for vectors in \mathbb{C}^N , $N = m(1+n)$, where $v_\perp \in \mathbb{C}^m$ is called the scalar part and $v_\parallel \in \mathbb{C}^{mn}$ the tangential part of v . The elements in a space X_D can be characterized as the elements f in X such that $\operatorname{curl}_x f_\parallel = 0$ in distribution sense.

We note that $\int_{\mathbb{R}^n} (f(x), g(x)) dx$ makes sense for pre-Hardy spaces $\mathbb{H}_{DB}^p, \mathbb{H}_{B^*D}^{p'}$ when $1 < p < \infty$ and these spaces are in duality. Similarly, under this pairing, $\mathbb{H}_{DB}^{p,\pm}, \mathbb{H}_{B^*D}^{p',\pm}$ are in duality $1 < p < \infty$. When $p \leq 1$, one replaces $\mathbb{H}_{B^*D}^{p'}$ by its Hölder version. When taking completions, the pairing becomes an abstract one and the dualities extend.

Theorem 6.1. *There is an open interval I_L in $(\frac{n}{n+1}, \infty)$ such that for $p \in I_L$, $H_{DB}^p = H_D^p$ with equivalence of norms. For $p \in I_L$, we also have $\mathbb{H}_{DB}^p = H_{DB}^p \cap L^2 = H_{DB}^p \cap H_{DB}^2$ and $\mathbb{H}_{DB}^p = \mathbb{H}_D^p$ with equivalence of norms.*

The semigroup $S_p(t)$ regularizes in the scale of Hardy spaces. More precisely, for $0 < p \leq q < \infty$ both in the interval I_L , $S_p(t)$ maps H_D^p to H_D^q with norm bounded by $Ct^{-\frac{n}{p}-\frac{n}{q}}$. If, moreover, $p \leq 2$, then $S_p(t)$ maps H_D^p to $\mathbb{H}_D^q = H_D^q \cap H_D^2$ equipped with the same norm as H_D^q .

This interval always contains $[\frac{2n}{n+2}, 2]$ but it could be larger. This is the case for an equation (instead of a system) in (1) with real coefficients: I_L contains $[1, 2]$.

Based on the observation that the semigroup $S_p(t)$ allows one to construct solutions of (1), the thesis of [AS] was to obtain estimates on such solutions in terms of their trace at time $t = 0$. We recall that our goal here is to show that all solutions with such estimates have a trace and are given from the semigroup acting on this trace.

More results related to DB and BD , and in particular the needed estimates, will be given along the way.

7. PREPARATION

For a function $(t, x) \mapsto f(t, x)$, we use the notation f_t or $f(t, \cdot)$ to designate the map $x \mapsto f(t, x)$.

Lemma 7.1 ([AAMc, AA]). *If u is a weak solution to (1) on \mathbb{R}_+^{1+n} , then $F = \nabla_A u$ is an $L_{loc}^2(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$ solution of*

$$(45) \quad \begin{cases} \operatorname{curl}_x F_{\parallel} &= 0, \\ \partial_t F + DBF &= 0. \end{cases}$$

The equations are interpreted in the sense of distributions in \mathbb{R}_+^{1+n} , and D and B are defined in (7) and (8). For the second one, it reads

$$(46) \quad \iint_{\mathbb{R}_+^{1+n}} (\partial_s \varphi(s, x) \cdot F(s, x)) ds dx = \iint_{\mathbb{R}_+^{1+n}} (B^*(x) D \varphi(s, x) \cdot F(s, x)) ds dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$ and the integrals exist in the Lebesgue sense with $|\partial_s \varphi| |F|$ and $|B^* D \varphi| |F|$ integrable. Conversely, if F is an $L_{loc}^2(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$ solution of (45) in \mathbb{R}_+^{1+n} , then there exists, unique up to a constant in \mathbb{C}^m , a weak solution u to $Lu = 0$ in \mathbb{R}_+^{1+n} such that $F = \nabla_A u$.

We will use the integral notation when it makes sense, that is having verified integrability. In general, we will be careful about justifying use of integrals or duality pairings.

Remark 7.2. By Lemma 5.3, $t \mapsto F_t$ belongs to $C^\infty(0, \infty; L_{loc}^2(\mathbb{R}^n; \mathbb{C}^N))$. Thus $\partial_t F_t \in L_{loc}^2(\mathbb{R}^n; \mathbb{C}^N)$ and the equality

$$(47) \quad \partial_t F_t = -DBF_t$$

holds in $L_{loc}^2(\mathbb{R}^n; \mathbb{C}^N)$ for all $t > 0$ by standard arguments. Similarly, we have $\operatorname{curl}_x(F_t)_{\parallel} = 0$ in $\mathcal{D}'(\mathbb{R}^n; \mathbb{C}^N)$ for all $t > 0$.

Remark 7.3. It suffices to test for φ with $\varphi, \partial_t \varphi, D \varphi \in L^2$ and compact support in \mathbb{R}_+^{1+n} . Indeed, one can regularize by convolution in (t, x) and obtain a test function in $C_0^\infty(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$ to which (46) applies and then pass to the limit using $F \in C^\infty(0, \infty; L_{loc}^2(\mathbb{R}^n; \mathbb{C}^N))$.

Lemma 7.4. *Assume $F \in L_{loc}^2(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$ is a solution of (45) in \mathbb{R}_+^{1+n} . Let $\phi_0 \in \mathbb{H}_{B^*D}^2$. Fix $t \in \mathbb{R}_+$ and set $\mathbb{R}_{+,t} = \mathbb{R}_+ \setminus \{t\}$. Let $\varphi(s, x) = \varphi_s(x)$ with*

$$(48) \quad \varphi_s := \begin{cases} e^{-(t-s)|B^*D|} \chi^+(B^*D) \phi_0 = e^{-(t-s)B^*D} \chi^+(B^*D) \phi_0, & \text{if } s < t, \\ -e^{-(s-t)|B^*D|} \chi^-(B^*D) \phi_0 = -e^{(s-t)B^*D} \chi^-(B^*D) \phi_0, & \text{if } s > t. \end{cases}$$

Let $\eta \in \text{Lip}(\mathbb{R}_+)$ with compact support in $\mathbb{R}_{+,t}$ and $\chi \in \text{Lip}(\mathbb{R}^n)$, with compact support, real-valued. Then, the functions

$$(s, x) \mapsto |\eta'(s)\chi(x)B^*(x)D\varphi(s, x)||F(s, x)|$$

and

$$(s, x) \mapsto |\eta(s)B^*(x)D_\chi(x)\partial_s\varphi(s, x)||F(s, x)|$$

are integrable in (s, x) and one has the identity

$$(49) \quad \begin{aligned} & \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)\chi(x)B^*(x)D\varphi(s, x) \cdot F(s, x)) dsdx \\ &= \iint_{\mathbb{R}_+^{1+n}} (\eta(s)B^*(x)D_\chi(x)\partial_s\varphi(s, x) \cdot F(s, x)) dsdx, \end{aligned}$$

where D_χ is a bounded, supported on $\text{supp}(\chi)$, matrix-valued function with $\|D_\chi\|_\infty \lesssim \|\nabla\chi\|_\infty$.

Proof. Observe that $\varphi \in C^\infty(\mathbb{R}_{+,t}; \mathbf{D}_2(B^*D))$ and that

$$(50) \quad \partial_s\varphi_s = B^*D\varphi_s, \quad s \neq t.$$

For $(s, x) \in \mathbb{R}_+^{1+n}$, set

$$\varphi^{n,\chi}(s, x) := \eta(s)\chi(x)\varphi_s(x).$$

Since the support of η does not contain t , it makes sense even if φ_t is not defined. The functions $\varphi^{n,\chi}$, $\partial_s\varphi^{n,\chi}$ and $D\varphi^{n,\chi}$ are well-defined compactly supported L^2 functions with

$$\begin{aligned} \partial_s\varphi^{n,\chi}(s, x) &= \eta'(s)\chi(x)\varphi_s(x) + \eta(s)\chi(x)\partial_s\varphi_s(x) \\ B^*(x)D\varphi^{n,\chi}(s, x) &= \eta(s)B^*(x)[D, \chi]\varphi_s(x) + \eta(s)\chi(x)B^*(x)D\varphi_s(x) \end{aligned}$$

where $[D, \chi]$ is the commutator between D and multiplication by χ : it is a multiplication with the function D_χ of the statement. It follows that $\varphi^{n,\chi}$ is a test function for (46) according to Remark 7.3. Plugging the expressions into (46) and using (50), we obtain

$$(51) \quad \begin{aligned} & \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)\chi(x)\varphi_s(x) \cdot F(s, x)) dsdx \\ &= \iint_{\mathbb{R}_+^{1+n}} (\eta(s)B^*(x)D_\chi(x)\varphi_s(x) \cdot F(s, x)) dsdx. \end{aligned}$$

Now, we remark that this applies to $\partial_s F$ in place of F . On the one hand, using $\partial_s F_s = -DBF_s$ for all $s > 0$, and integrating by parts in x (first use Fubini's theorem and $D = D^*$ in the integration by parts)

$$\begin{aligned} & \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)\chi(x)\varphi_s(x) \cdot \partial_s F(s, x)) dsdx \\ &= - \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)B^*(x)D_\chi(x)\varphi_s(x) \cdot F(s, x)) dsdx \\ &\quad - \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)\chi(x)B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx. \end{aligned}$$

On the other hand, integrating by parts in the s variable,

$$\begin{aligned} & \iint_{\mathbb{R}_+^{1+n}} (\eta(s)B^*(x)D_\chi(x)\varphi_s(x) \cdot \partial_s F(s, x)) dsdx \\ &= - \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)B^*(x)D_\chi(x)\varphi_s(x) \cdot F(s, x)) dsdx \\ & \quad - \iint_{\mathbb{R}_+^{1+n}} (\eta(s)B^*(x)D_\chi(x)\partial_s \varphi_s(x) \cdot F(s, x)) dsdx. \end{aligned}$$

Combining all this yields (49). \square

8. PROOF OF THEOREM 1.1: (I) IMPLIES (III)

We assume (i) : $\|\tilde{N}_*F\|_p < \infty$ where F is a solution to (45) and $p \in I_L$, that is, $H_{DB}^p = H_D^p$.

Step 1. Finding the semigroup equation.

This will be achieved by taking limits in (49) by selecting χ , η and ϕ_0 in the definition of φ_s in (48).

Step 1a. Limit in space. We show that if $\phi_0 \in \overline{\mathbb{R}_2(B^*D)}$, with in addition $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$ (or, equivalently, $\mathbb{P}\phi_0 \in \mathbb{H}_D^{p'}$) when $p > 1$, then

$$(52) \quad \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx = 0$$

where the integral is defined in the Lebesgue sense.

We replace χ in (49) by χ_R with $\chi_R(x) = \chi(x/R)$ where $\chi \equiv 1$ in the unit ball $B(0, 1)$, has compact support in the ball $B(0, 2)$ and let $R \rightarrow \infty$. As χ_R tends to 1 and D_{χ_R} to 0, it suffices by dominated convergence to show that $|\eta'(s)B^*D\varphi_s F|$ and $|\eta(s)\partial_s \varphi_s F|$ are integrable on \mathbb{R}_+^{1+n} . As $1_{\text{supp } \eta} F \in T_2^p$, it is enough to have that $\eta'(s)B^*D\varphi_s$ and $\eta(s)\partial_s \varphi_s$ belong to $(T_2^p)'$. As $\partial_s \varphi_s = B^*D\varphi_s$ on $\text{supp } \eta$, it suffices to invoke the following lemma.

Lemma 8.1. *Assume $\phi_0 \in \overline{\mathbb{R}_2(B^*D)} = \mathbb{H}_{B^*D}^2$ and, in addition if $p > 1$, $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$. Then $\eta(s)B^*D\varphi_s \in (T_2^p)'$ for all η bounded and compactly supported away from t .*

Proof. We begin with the case $p \leq 1$. Set $G_s := |\eta(s)B^*D\varphi_s|$ and $G(s, x) = G_s(x)$. The definition of φ_s and the properties of φ_s show that $G_s \in L^2(\mathbb{R}^n)$ uniformly in $s \in \mathbb{R}_+$. As $s \mapsto G_s$ also has compact support in \mathbb{R}_+ , it is easy to see that $C_\alpha G \in L^\infty$ (the bound depends on η) for any $\alpha \geq 0$. In particular, $G \in (T_2^p)'$.

We continue with the case $p > 1$ and we use the functional calculus for B^*D . We have

$$\eta(s)B^*D\varphi_s = \begin{cases} \frac{\eta(s)}{t-s}\psi_+((t-s)B^*D)\phi_0 & \text{if } s < t, \\ \frac{\eta(s)}{s-t}\psi_-((s-t)B^*D)\phi_0 & \text{if } t < s, \end{cases}$$

with ψ_\pm the bounded homomorphic functions defined by $\psi_\pm(z) = \pm z e^{\mp z} \chi^\pm(z)$. By geometric considerations as $t-s$ is bounded away from 0, we see that if (s, y) belongs to a cone Γ_x with $s \in \text{supp } (\eta) \cap (0, t)$ then $(t-s, y)$ belongs to a cone $\tilde{\Gamma}_x$ with (bad)

finite aperture depending on the support of η . Thus, setting $t - s = \sigma$ and using also that s is bounded below and $t - s$ bounded above on $\text{supp}(\eta)$, we obtain

$$\iint_{\Gamma_x, s < t} |\eta(s)B^*(y)D\varphi_s(y)|^2 \frac{dsdy}{s^{n+1}} \lesssim_\eta \iint_{\tilde{\Gamma}_x} |\psi_+(\sigma B^*D)\phi_0(y)|^2 \frac{d\sigma dy}{\sigma^{n+1}}.$$

The change of aperture allows us to use $\tilde{\Gamma}_x$ in estimating tent space norms. By Theorem 5.7 of [AS], we have that

$$\|\psi_+(\sigma B^*D)\phi_0\|_{T_2^{p'}} \lesssim \|\mathbb{P}\phi_0\|_{p'},$$

where \mathbb{P} is the orthogonal projection of L^2 onto $\overline{\mathbb{R}_2(D)}$. By the assumption on p , $\mathbb{P}\phi_0 \in \mathbb{H}_D^{p'}$, and in particular, $\mathbb{P}\phi_0 \in L^{p'}$ with $\|\mathbb{P}\phi_0\|_{p'} \sim \|\phi_0\|_{\mathbb{H}_{B^*D}^{p'}}$.

The argument is the same when $t < s$, replacing ψ_+ by ψ_- . \square

Step 1b. Limit in time. We assume the condition of Step 1a on ϕ_0 hold.

Now we select appropriate functions η in (52) depending on t , which is still a fixed positive real number. We follow [AA] for the choices but the limit process requires more care.

We let $0 < \varepsilon < \inf(t/4, 1/4, 1/t)$ and pick η piecewise linear and continuous with $\eta(s) = 0$ if $s < t + \varepsilon$, $\eta(s) = 1$ if $t + 2\varepsilon \leq s \leq t + \frac{1}{2\varepsilon}$ and $\eta(s) = 0$ if $s > t + \frac{1}{\varepsilon}$. Plugging this choice into (52) and reorganizing we obtain

$$(53) \quad \frac{1}{\varepsilon} \iint_{[t+\varepsilon, t+2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx \\ = 2\varepsilon \iint_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx.$$

We make a second choice of η , again piecewise linear and continuous, with $\eta(s) = 0$ if $s < \varepsilon$, $\eta(s) = 1$ if $2\varepsilon \leq s \leq t - 2\varepsilon$ and $\eta(s) = 0$ if $s > t - \varepsilon$. Plugging this choice into (52) with $4\varepsilon < t$, we obtain

$$(54) \quad \frac{1}{\varepsilon} \iint_{[t-2\varepsilon, t-\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx \\ = \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx.$$

As said our goal is to pass to the limit in both formulas.

The second integral in (53) converges to 0 as $\varepsilon \rightarrow 0$ for fixed t since $1_{[\varepsilon, 2\varepsilon]}F \in T_2^p$. Indeed, using the function ψ_- defined earlier, set $G(s, x) = G_s(x) = 1_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}]}(s)(s-t)^{-1}\psi_-((s-t)B^*D)\phi_0(x)$, under the conditions on ϕ_0 in Step 1a. As $t\varepsilon < 1$, then $s \in [t + \frac{1}{2\varepsilon}, t + \frac{1}{\varepsilon}]$ implies $s \in [\frac{1}{2\varepsilon}, \frac{2}{\varepsilon}]$. Remark that this integral is bounded by

$$2\varepsilon \|1_{[\frac{1}{2\varepsilon}, \frac{2}{\varepsilon}]}F\|_{T_2^p} \|sG\|_{(T_2^p)'} \lesssim 2\varepsilon \|\tilde{N}_*F\|_p \|sG\|_{(T_2^p)'}$$

It remains to show $\|sG\|_{(T_2^p)'} < \infty$. Assume first $p \leq 1$ and let $\alpha = n(\frac{1}{p} - 1)$. Since $\|G_s\|_2 \lesssim \varepsilon \|\phi_0\|_2$ for those s , we have $\|C_\alpha(sG)\|_\infty \lesssim \varepsilon^{\alpha+\frac{n}{2}} \|\phi_0\|_2$.

Next, consider $p > 1$. Then one sees that $\|sG\|_{(T_2^p)'} \lesssim \|\psi_-(\sigma B^*D)\phi_0\|_{T_2^{p'}} \lesssim \|\mathbb{P}\phi_0\|_{p'} \sim \|\phi_0\|_{\mathbb{H}_{B^*D}^{p'}}$.

Therefore the left hand side of (53) converges to 0 as well. However, the way to interpret this limit and also how we can pass to the limit in (54) depend on whether $p \leq 2$ or $p > 2$.

We break the continuation of this step in the two cases $p \leq 2$ and $p > 2$.

Case $p \leq 2$. Changing s to $t + s$ in the first integral, this shows that

$$(55) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} ((B^* D e^{sB^* D} \chi^-(B^* D) \phi_0)(x) \cdot F(t + s, x)) ds dx = 0.$$

Before we take this limit, we reinterpret this integral. We begin with

Lemma 8.2. *For $q = p$ if $1 < p \leq 2$ and $q = p \frac{n+1}{n} \in (1, 2]$ if $p \leq 1$, we have $F \in C^\infty(0, \infty; H_D^q)$.*

Proof. Assume $p \leq 1$. By Lemma 2.2 applied to F , we obtain $F \in L_{loc}^q(0, \infty; L^q)$. As one can apply the same argument replacing F by its t -derivatives, we have $F \in C^\infty(0, \infty; L^q)$. Now, $\text{curl}_x(F_t)_\parallel = 0$ in $\mathcal{D}'(\mathbb{R}^n)$ as remarked before. As $F_t \in L^q$, this means that $F_t \in \overline{\mathcal{R}_q(D)} = H_D^q$ for all $t > 0$ (see Section 6).

Next, assume $1 < p \leq 2$. We know that $F \in L_{loc}^p(0, \infty; L^p)$. The rest of the argument is the same. \square

Lemma 8.3. *Let q be as Lemma 8.2. For $\phi_0 \in \mathbb{H}_{B^*D}^q$, $\varepsilon > 0$, $t > 0$, we have*

$$\begin{aligned} & \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} ((B^* D e^{sB^* D} \chi^-(B^* D) \phi_0)(x) \cdot F(t + s, x)) ds dx \\ &= \int_{[\varepsilon, 2\varepsilon]} \langle B^* D e^{sB^* D} \chi^-(B^* D) \phi_0, F_{t+s} \rangle ds \end{aligned}$$

where the pairing is the (sesquilinear) duality between $H_{B^*D}^q$ and H_{DB}^q .

Proof. Note that $t, \varepsilon > 0$ are fixed so estimates are allowed to depend on them. Set $G(s, x) = (sB^* D e^{sB^* D} \chi^-(B^* D) \phi_0)(x)$ and $H(s, x) = 1_{[\varepsilon, 2\varepsilon]}(s) F(t + s, x)$. Because of the truncation in s , we know that $H \in T_2^p$ (with norm depending on t, ε) and $G \in (T_2^p)'$ so that the integral on the left hand side is a Lebesgue integral. We may apply Proposition 2.4, so that this integral is a limit of the integral where H has been mollified by convolution in the x -variable, that is, $H(s, x)$ is replaced by $H_k(s, x) = H_{k,s}(x) = H_s \star_{\mathbb{R}^n} \rho_k(x)$. But recall that $H_s \in H_D^q = \overline{\mathcal{R}_q(D)}$ so that $\mathbb{P}H_s = H_s$ for all $s > 0$. This condition is preserved by convolution since it commutes with \mathbb{P} because \mathbb{P} is also given by convolution. Also such mollifications map L^q into L^2 , thus $H_{k,s} \in \mathbb{H}_D^2$ as well so that $H_{k,s} \in \mathbb{H}_{DB}^q = \mathbb{H}_D^q$ (as $p \leq q \leq 2$, thus $q \in I_L$). As $G_s \in \mathbb{H}_{B^*D}^q$ (because $\phi_0 \in \mathbb{H}_{B^*D}^q$ and this space is preserved by $H^\infty(S_\mu)$ functions of B^*D) which is in duality with \mathbb{H}_{DB}^q for the standard L^2 duality (see Section 6),

$$\int_{\mathbb{R}^n} (G(s, x) \cdot H_k(s, x)) dx = \langle G_s, H_{k,s} \rangle$$

for any fixed s , where the pairing is the duality extended to $H_{B^*D}^q, H_{DB}^q$. But $H_{k,s}$ converges to H_s in L^q , hence in $H_D^q = H_{DB}^q$, as $k \rightarrow \infty$ and this is uniform in s . Thus $\langle G_s, H_{k,s} \rangle$ converges to $\langle G_s, H_s \rangle$ uniformly in s . It remains to integrate the equality above in s and pass to the limit. \square

We continue with Step 1b in this case ($p \leq 2$) and conclude for the limit of (53). Pick $\delta > 0$, replace ϕ_0 by $e^{-\delta|B^*D|}\phi_0$ in (55) and use Lemma 8.2 and 8.3, to obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{[\varepsilon, 2\varepsilon]} \langle B^* D e^{(s+\delta)B^*D} \chi^-(B^*D)\phi_0, F_{t+s} \rangle ds = 0.$$

Now, the map $s \mapsto F_{t+s}$ is continuous at $s = 0$ into $H_D^q = H_{DB}^q$ and the map $s \mapsto B^* D e^{(s+\delta)B^*D} \chi^-(B^*D)\phi_0$ is continuous at $s = 0$ into $\mathbb{H}_{B^*D}^{q'}$. For the last point, this is because we have the continuity of the semigroup in $\mathbb{H}_{B^*D}^{q'}$ ([AS], Proposition 4.5). It follows that the limit is the value at $s = 0$ of the integrand and we have obtained for all $\phi_0 \in \mathbb{H}_{B^*D}^{q'}$, all $\delta > 0$ and $t > 0$ that

$$\langle B^* D e^{\delta B^*D} \chi^-(B^*D)\phi_0, F_t \rangle = 0.$$

We deduce from this equation that $F_t \in H_{DB}^{q,+}$. Indeed, using semigroup theory, the vector space containing $\{B^* D e^{\delta B^*D} \chi^-(B^*D)\phi_0, \phi_0 \in \mathbb{H}_{B^*D}^{q'}, \delta > 0\}$ forms a dense subspace of $\mathbb{H}_{B^*D}^{q',-}$ [$\chi^-(B^*D)\phi_0 = \lim_{t \rightarrow 0} \int_0^t B^* D e^{\delta B^*D} \chi^-(B^*D)\phi_0 d\delta$ and approximate each integral by Riemann sums] which is dense in $H_{B^*D}^{q',-}$. As $F_t \in H_{DB}^q$, this shows that $F_t \in H_{DB}^{q,+}$ (See Section 6).

We turn to taking limits in (54) still in the case $p \leq 2$. Reinterpreting the dx integrals using the duality pairing between $H_{B^*D}^{q'}$ and H_{DB}^q as in Lemma 8.3 and reorganizing we obtain a second equation for fixed $t > 0$ and $\phi_0 \in \mathbb{H}_{B^*D}^{q'}$,

$$(56) \quad \int_{[\varepsilon, 2\varepsilon]} \langle B^* D e^{-sB^*D} \chi^+(B^*D)\phi_0, F_{t-s} \rangle ds \\ = \int_{[\varepsilon, 2\varepsilon]} \langle B^* D e^{-(t-s)B^*D} \chi^+(B^*D)\phi_0, F_s \rangle ds.$$

Next, if we replace ϕ_0 by $e^{-\delta|B^*D|}\phi_0$ with $\delta > 0$, we can pass to the limit as $\varepsilon \rightarrow 0$ as above and obtain for the integral in the left hand side of (56),

$$\langle B^* D e^{-\delta B^*D} \chi^+(B^*D)\phi_0, F_t \rangle.$$

Set

$$I_t^{\varepsilon, \delta} = \int_{[\varepsilon, 2\varepsilon]} \langle B^* D e^{-(t-s)B^*D} \chi^+(B^*D) e^{-\delta|B^*D|}\phi_0, F_s \rangle ds.$$

We thus have shown

$$\lim_{\varepsilon \rightarrow 0} I_t^{\varepsilon, \delta} = \langle B^* D e^{-\delta B^*D} \chi^+(B^*D)\phi_0, F_t \rangle$$

for all $t, \delta > 0$. We shall use this information together that $F_t \in H_{DB}^{q,+}$ to prove that for all $t > 0$, $F_t \in \mathbb{H}_{DB}^{q,+}$ and for all $t > 0$ and $\tau \geq 0$,

$$F_{t+\tau} = e^{-\tau DB} \chi^+(DB) F_t = e^{-\tau|DB|} F_t.$$

So we fix t, δ, τ . Observe that $I_{t+\tau}^{\varepsilon, \delta} = I_t^{\varepsilon, \delta+\tau}$. Thus, we obtain

$$\langle B^* D e^{-(\tau+\delta)B^*D} \chi^+(B^*D)\phi_0, F_t \rangle = \langle B^* D e^{-\delta B^*D} \chi^+(B^*D)\phi_0, F_{t+\tau} \rangle.$$

Because $F_t \in H_{DB}^{q,+}$, the left hand side can be rewritten as

$$\langle B^* D e^{-\delta B^*D} \chi^+(B^*D)\phi_0, S_q^+(\tau) F_t \rangle$$

where $S_q^+(\tau)$ is the bounded extension of $e^{-\tau DB}\chi^+(DB)$ to $H_{DB}^{q,+}$ described in Section 6. By density as before, this shows that for all $t > 0$ and $\tau \geq 0$

$$F_{t+\tau} = S_q^+(\tau)F_t.$$

Remark that as we already know that $F_t \in H_{DB}^{q,+}$, this is the same as

$$F_{t+\tau} = S_q(\tau)F_t.$$

By Corollary 8.3 in [AS], the semigroup maps H_{DB}^q into \mathbb{H}_{DB}^2 . This proves that $F_{t+\tau} \in \mathbb{H}_{DB}^2$ and, therefore, $F_t \in \mathbb{H}_{DB}^{q,+}$ for all $t > 0$ so that the original semigroup acts on F_t and we can write for all $t > 0$ and $\tau \geq 0$,

$$F_{t+\tau} = e^{-\tau DB}\chi^+(DB)F_t = e^{-\tau|DB|}F_t.$$

Step 2 when $p \leq 2$. Finding the trace at $t = 0$.

We distinguish two subcases.

Step 2a: $1 < p \leq 2$.

If $1 < p \leq 2$, recall that $q = p$ from Lemma 8.2. From Lemma 2.3, $\int_{[t,2t]} \|F_s\|_p ds \lesssim \|\tilde{N}_*F\|_p < \infty$. In particular $\tilde{F}_t = \int_{[t,2t]} F_s ds$ belongs to L^p with uniform bound with respect to t , thus it has a subsequence \tilde{F}_{t_k} converging weakly in L^p with $t_k \rightarrow 0$. Call the limit F_0 . Then $F_0 \in H_{DB}^{p,+}$ (which is the completion of $\mathbb{H}_{DB}^{p,+}$). Now $e^{-\tau|DB|}\tilde{F}_{t_k} = S_p^+(\tau)\tilde{F}_{t_k}$ converges weakly to $S_p^+(\tau)F_0$ by continuity of the semigroup $S_p^+(\tau)$ on $H_{DB}^{p,+}$, while $\int_{[t_k,2t_k]} F_{\tau+s} ds$ converges strongly to F_τ and we obtain

$$S_p^+(\tau)F_0 = F_\tau.$$

Step 2b: $p \leq 1$.

This case is more delicate. Let us explain the strategy. We introduce a new Banach space \tilde{H}_{DB}^p of Schwartz distributions which contains H_{DB}^p . This means that we will have the containments $H_{DB}^p \subset \tilde{H}_{DB}^p \subset \mathcal{S}'$ with **continuous inclusions**. We shall obtain the trace and representation of F_t in \tilde{H}_{DB}^p . Then we shall show that the trace actually belongs to the smaller space H_{DB}^p (a regularity result) and conclude from this for the representation $S_p^+(\tau)F_0 = F_\tau$ in the space H_{DB}^p .

We begin by building \tilde{H}_{DB}^p . Several choices are possible but in a very narrow window to match both the functional calculus of DB and the usual calculus of distributions. Recall that $p \in I_L$ and $\frac{n}{n+1} < p \leq 1$. Let $p_0 = \inf I_L$ and $s_0 = n(\frac{1}{p_0} - 1)$. We fix $n(\frac{1}{p} - 1) < s < s_0$ and we select $r > 1$ so that $n(\frac{1}{p} - \frac{1}{r}) < 1$ and

$$(57) \quad \beta = \frac{2}{r'} + s \left(1 - \frac{2}{r'}\right) > n \left(\frac{1}{p} - \frac{1}{r}\right) = \alpha.$$

The different possibilities are in the choice of this r .

Let $\tilde{\mathbb{H}}_{DB}^p = \{h \in \mathbb{H}_{DB}^2; \|h\|_{\tilde{\mathbb{H}}_{DB}^p} < \infty\}$ with

$$\|h\|_{\tilde{\mathbb{H}}_{DB}^p} = \left(\iint_{\mathbb{R}_+^{1+n}} |e^{-t|DB|h(x)}|^r t^{n(\frac{r}{p}-1)} \frac{dt dx}{t} \right)^{1/r} = \left(\int_0^\infty \|e^{-t|DB|h}\|_r^r t^{\alpha r} \frac{dt}{t} \right)^{1/r}.$$

Set $\tilde{\mathbb{H}}_{DB}^{p,\pm} = \tilde{\mathbb{H}}_{DB}^p \cap \mathbb{H}_{DB}^{2,\pm}$. Let \tilde{H}_{DB}^p and $\tilde{H}_{DB}^{p,\pm}$ be the respective completions with respect to this norm.

Lemma 8.4. *The space \tilde{H}_{DB}^p embeds in the space of Schwartz distributions.*

Proof. Let $Q_t = t^2 \Delta e^{t^2 \Delta}$ where Δ is the ordinary negative self-adjoint Laplacian on \mathbb{R}^n . We first claim that for $h \in \mathbb{H}_{DB}^2$,

$$(58) \quad \left(\int_0^\infty \|Q_t h\|_r^r t^{\alpha r} \frac{dt}{t} \right)^{1/r} \lesssim \left(\int_0^\infty \|e^{-t|DB|} h\|_r^r t^{\alpha r} \frac{dt}{t} \right)^{1/r}.$$

It is well-known that the norm on the left is equivalent to that of the homogeneous Besov space $\dot{B}_r^{-\alpha, r}$, which, as $-\alpha < 0$, can be identified to a subspace of \mathcal{S}' . We make this identification and consider now this Besov space as contained in \mathcal{S}' with continuous inclusion. To prove this inequality, we use the Calderón reproducing formula in the functional calculus of DB . Let ψ be the holomorphic function $\psi(z) = \pm 2ze^{\mp z}$ for $z \in S_{\mu^\pm}$ so that

$$\int_0^\infty \psi(sz) e^{\mp sz} \frac{ds}{s} = 1, \quad \forall z \in S_\mu,$$

hence, for all $h \in \mathbb{H}_{DB}^2$,

$$\int_0^\infty \psi(sDB) e^{-s|DB|} h \frac{ds}{s} = h$$

with convergence in L^2 . Applying Q_t yields

$$Q_t h = \int_0^\infty Q_t \psi(sDB) e^{-s|DB|} h \frac{ds}{s}$$

with convergence in L^2 . By the standard Schur argument, the conclusion will follow from the inequality

$$(59) \quad t^\alpha \|Q_t \psi(sDB) e^{-s|DB|} h\|_r \lesssim g(s/t) \|e^{-s|DB|} h\|_r s^\alpha,$$

with $g : (0, \infty) \rightarrow (0, \infty)$ independent of h, s, t such that $\int_0^\infty g(u) \frac{du}{u} < \infty$. Because $p < r < 2$, we have $r \in I_L$, hence by [AS], Theorem 4.19, we have $\|\psi(sDB) e^{-s|DB|} h\|_r \lesssim \|e^{-s|DB|} h\|_r$ uniformly in s . Thus, if $t \leq s$, (59) holds with $g(u) = u^{-\alpha}$ when $u > 1$. In the case $t > s$, observe that $\psi(z) = z\tilde{\psi}(z)$ on S_μ with $\tilde{\psi} \in H^\infty(S_\mu)$. Hence, $Q_t \psi(sDB) = sQ_t DB \tilde{\psi}(sDB)$ and observe that $Q_t DB$ is bounded on L^r with norm bounded by Ct^{-1} ($Q_t D$ is convolution with an L^1 function and B is bounded multiplication). Thus, using again [AS], Theorem 4.19, we obtain (59) with $g(u) = u^{1-\alpha}$ if $u < 1$. As $\alpha < 1$, the desired property holds for g . This proves that $\tilde{\mathbb{H}}_{DB}^p \subset \dot{B}_r^{-\alpha, r}$ continuously.

To prove the inclusion when taking completion, we have to show that if (h_ε) is a Cauchy sequence in $\tilde{\mathbb{H}}_{DB}^p$ that converges to 0 in $\dot{B}_r^{-\alpha, r}$ then it also converges to 0 in \tilde{H}_{DB}^p (in other words, this proves that the extension of the identity map is injective). Let $G_\varepsilon(t, x) = e^{-t|DB|} h_\varepsilon(x)$. As G_ε is a Cauchy sequence in $L^r(\mathbb{R}_+^{1+n}; \mathbb{C}^N, t^{\alpha r-1} dx dt)$, it converges to some G in this (complete) space. We have to show that $G = 0$ (For the moment, \tilde{H}_{DB}^p is defined as the closure in $L^r(\mathbb{R}_+^{1+n}; \mathbb{C}^N, t^{\alpha r-1} dx dt)$ of elements G with $G(t, x) = e^{-t|DB|} h(x)$, $h \in \tilde{\mathbb{H}}_{DB}^p$). It suffices to show this in the sense of

distributions on \mathbb{R}_+^{1+n} . Let $\chi \in C_0^\infty(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$. Then $(G - G_\varepsilon, \bar{\chi}) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Next, we have

$$(G_\varepsilon, \bar{\chi}) = \int_0^\infty \langle e^{-t|DB|} h_\varepsilon, \chi_t \rangle \frac{dt}{t} = \langle h_\varepsilon, \varphi \rangle$$

with

$$\varphi = \int_a^b \mathbb{P} e^{-t|B^*D|} \chi_t \frac{dt}{t}.$$

We used that $\text{supp } \chi \subset [a, b] \times \mathbb{R}^n$ and $\chi_t(x) = \chi(t, x)$. We also used that $h_\varepsilon \in \mathbb{H}_{DB}^2 = \mathbb{R}(\mathbb{P})$ to insert the orthogonal projection \mathbb{P} . For any t fixed in $[a, b]$, we have

$$D\mathbb{P} e^{-t|B^*D|} \chi_t = D e^{-t|B^*D|} \chi_t = e^{-t|DB^*|} D \chi_t \in L^2$$

and by [AS], Corollary 4.21, $\mathbb{P} e^{-t|B^*D|} \chi_t = \mathbb{P} e^{-t|B^*D|} \mathbb{P} \chi_t \in \dot{\Lambda}^s$ as $\mathbb{P} \chi_t \in \dot{\Lambda}^s \cap L^2$ and s can be taken as the one chosen before the statement. Thus $\varphi \in \mathbb{H}_D^2$, $D\varphi \in L^2$ and $\varphi \in \dot{\Lambda}^s$. The first two conditions imply that $\varphi \in W^{1,2}$, the usual Sobolev space, and the first and third that $\varphi \in L^\infty$, and by interpolation $\varphi \in L^{r'}$. Now, for some constant $c > 0$, one can use the usual Calderón reproducing formula to write

$$\langle h_\varepsilon, \varphi \rangle = c \int_0^\infty \langle Q_t h_\varepsilon, Q_t \varphi \rangle \frac{dt}{t}.$$

As $Q_t h_\varepsilon$ converges to 0 in $L^r(\mathbb{R}_+^{1+n}; \mathbb{C}^N, t^{\alpha r-1} dx dt)$, it is enough to show that $Q_t \varphi \in L^{r'}(\mathbb{R}_+^{1+n}; \mathbb{C}^N, t^{-\alpha r'-1} dx dt)$ to conclude that $\langle h_\varepsilon, \varphi \rangle \rightarrow 0$ as $\varepsilon \rightarrow 0$. The part for $t > 1$ follows from the boundedness of Q_t on $L^{r'}$ and $\varphi \in L^{r'}$ as $\alpha r' > 0$. For $t \leq 1$, we use

$$\|Q_t \varphi\|_2 \lesssim t \|\nabla \varphi\|_2, \quad \|Q_t \varphi\|_\infty \lesssim t^s \|\varphi\|_{\dot{\Lambda}^s}$$

hence,

$$\|Q_t \varphi\|_{r'} \lesssim \|Q_t \varphi\|_2^{2/r'} \|Q_t \varphi\|_\infty^{1-2/r'} \lesssim_\varphi t^\beta,$$

where β is the number defined in (57). The convergence when $t \leq 1$ follows from $\beta > \alpha$. \square

Having shown the embedding of \tilde{H}_{DB}^p in \mathcal{S}' , we decide to identify \tilde{H}_{DB}^p to a subspace of \mathcal{S}' . Note that as we have identified H_{DB}^p to H_D^p , which is also a subspace of \mathcal{S}' , we can now compare these two realizations of \tilde{H}_{DB}^p and H_{DB}^p safely.

- Lemma 8.5.** (1) *We have the spectral splitting $\tilde{H}_{DB}^p = \tilde{H}_{DB}^{p,+} \oplus \tilde{H}_{DB}^{p,-}$.*
 (2) *The identity map is an embedding of $H_{DB}^{p,\pm}$ into $\tilde{H}_{DB}^{p,\pm}$ respectively.*
 (3) *Let $S_p^+(\tau)$ and $\tilde{S}_p^+(\tau)$ be the respective bounded extensions by density of $e^{-\tau|DB|}$ on $H_{DB}^{p,+}$ and $\tilde{H}_{DB}^{p,+}$. If $h \in H_{DB}^p \cap \tilde{H}_{DB}^{p,+}$, then $h \in H_{DB}^{p,+}$ and $\tilde{S}_p^+(\tau)h = S_p^+(\tau)h$.*

Proof. For (1), let $h \in \tilde{\mathbb{H}}_{DB}^p$. Then we have $h = h^+ + h^-$ where $h^\pm = \chi^\pm(DB)h$ and

$$e^{-t|DB|} h = e^{-t|DB|} h^+ + e^{-t|DB|} h^- = \chi^+(DB) e^{-t|DB|} h + \chi^-(DB) e^{-t|DB|} h.$$

As $\chi^\pm(DB)$ are bounded operators on \mathbb{H}_{DB}^r equipped with L^r norm ([AS], Theorem 4.19, because $r \in I_L$), we obtain $\|\chi^\pm(DB) e^{-t|DB|} h\|_r \lesssim \|e^{-t|DB|} h\|_r$ for all $t > 0$. It follows that $\|h^\pm\|_{\tilde{\mathbb{H}}_{DB}^p} \lesssim \|h\|_{\tilde{\mathbb{H}}_{DB}^p}$. The splitting follows in the completion.

For (2), by [AS], Theorem 9.1, since $p \in I_L$, $H_{DB}^{p,+}$ is a closed subspace of $H_{DB}^p = H_D^p$ and

$$\|h\|_{\mathbb{H}_{DB}^p} \sim \|h\|_{H^p} \sim \|\tilde{N}_*(e^{-t|DB|} h)\|_p, \quad \forall h \in \mathbb{H}_{DB}^{p,+}.$$

Let $h \in \mathbb{H}_{DB}^{p,+}$. Combining this with Lemma 2.2 and the choice of r , $h \in \widetilde{\mathbb{H}}_{DB}^p$ and

$$\|h\|_{\widetilde{\mathbb{H}}_{DB}^p} \lesssim \|h\|_{\mathbb{H}_{DB}^p}.$$

By completion, this shows that the identity map extends to a continuous map from $H_{DB}^{p,+}$ to $\widetilde{H}_{DB}^{p,+}$. Now, both completions are embedded in \mathcal{S}' so this extended map must be injective: it is an embedding. The argument for the inclusion of $H_{DB}^{p,-}$ in $\widetilde{H}_{DB}^{p,-}$ is the same.

Next, the property (3) is now an easy exercise in functional analysis. We give it for the sake of completeness. Let $h \in \widetilde{H}_{DB}^{p,+} \cap H_{DB}^p$. There exists $h_\varepsilon \in \mathbb{H}_{DB}^p$ converging to h in H_{DB}^p . Write $h = h^+ + h^-$ according to the spectral splitting $H_{DB}^p = H_{DB}^{p,+} \oplus H_{DB}^{p,-}$. Thus $h_\varepsilon^\pm = \chi^\pm(DB)h_\varepsilon \in \mathbb{H}_{DB}^{p,\pm}$ and converge respectively to h^\pm for the H_{DB}^p topology. By the embedding in part (2), the convergence is also in $\widetilde{H}_{DB}^{p,\pm}$. Thus, we obtain that $h = h^+ + h^-$ also in \widetilde{H}_{DB}^p . Since $h \in \widetilde{H}_{DB}^{p,+}$, the splitting obtained in part (1) yields that $h^- = 0$ and $h = h^+$ and it follows that $h \in H_{DB}^{p,+}$. Now, one can assume that the h_ε , which converge to h , belong to $\mathbb{H}_{DB}^{p,+}$ to begin with. For fixed $\varepsilon > 0$ and $\tau > 0$, by definition of the extensions, $S_p^+(\tau)h_\varepsilon = e^{-\tau|DB|}h_\varepsilon = \widetilde{S}_p^+(\tau)h_\varepsilon$. If $\varepsilon \rightarrow 0$, the first term converges to $S_p^+(\tau)h$ in $H_{DB}^{p,+}$, while the last term converges to $\widetilde{S}_p^+(\tau)h$ in $\widetilde{H}_{DB}^{p,+}$. The equality follows again using the embedding in part (2). \square

We come back to the solution F_t . Recall that for $t > 0$, $F_t \in \mathbb{H}_{DB}^{2,+}$ and when $\tau \geq 0$,

$$e^{-\tau|DB|}F_t = F_{t+\tau},$$

hence for all $k \in \mathbb{N}$,

$$(-DB)^k F_t = \partial_t^k F_t.$$

Lemma 8.6. *F_t belongs to $\widetilde{\mathbb{H}}_{DB}^{p,+}$ uniformly in $t > 0$ and converges to some $h \in \widetilde{H}_{DB}^{p,+}$ as $t \rightarrow 0$, so that $F_t = \widetilde{S}_p^+(t)h$ for all $t > 0$.*

Proof. We are going to use another feature of the spaces $\widetilde{\mathbb{H}}_{DB}^p$, which is the possibility of changing the norm (exactly as with the Hardy spaces \mathbb{H}_{DB}^p given by square functions). Indeed, if $\psi, \tilde{\psi} \in \Psi_0^\tau(S_\mu)$ with $\tau > \alpha$, and ψ is non-degenerate on S_μ , then for all $h \in \mathbb{H}_{DB}^2$,

$$(60) \quad \left(\int_0^\infty \|\tilde{\psi}(tDB)h\|_r^r t^{\alpha r} \frac{dt}{t} \right)^{1/r} \lesssim_{\psi, \tilde{\psi}} \left(\int_0^\infty \|\psi(tDB)h\|_r^r t^{\alpha r} \frac{dt}{t} \right)^{1/r}.$$

The proof is roughly the same as the one of (58) but staying entirely within the functional calculus for DB . As ψ is non-degenerate, there exists $\theta \in \Psi_1^1(S_\mu)$ such that the Calderón reproducing formula

$$\int_0^\infty \theta(sDB)\psi(sDB)h \frac{ds}{s} = h$$

holds with convergence in \mathbb{H}_{DB}^2 , hence

$$\tilde{\psi}(tDB)h = \int_0^\infty \tilde{\psi}(tDB)\theta(sDB)\psi(sDB)h \frac{ds}{s}$$

with convergence in \mathbb{H}_{DB}^2 . Now, by [AS], Theorem 4.19 and functional calculus for DB we have

$$t^\alpha \|\tilde{\psi}(tDB)\theta(sDB)\psi(sDB)h\|_r \lesssim s^\alpha g(s/t) \|\psi(sDB)h\|_r$$

with $g(u) = \inf(u^{-\alpha}, u^{\tau-\alpha})$. The details are similar to the ones above and we skip them. We may apply this to $\psi(z) = z^k e^{-|z|}$ for any $k \in \mathbb{N}$. For $k = 0$, we recover the defining norm. But here we pick k with $(k + \alpha)r - 1 > 0$ and it gives an equivalent norm. Thus we have for fixed $t > 0$

$$\begin{aligned}
\|F_t\|_{\tilde{\mathbb{H}}_{DB}^{p,+}}^r &\sim \int_0^\infty \|\tau^k (DB)^k e^{-\tau|DB|} F_t\|_r^r \tau^{\alpha r} \frac{d\tau}{\tau} \\
&= \int_0^\infty \|(DB)^k F_{t+\tau}\|_r^r \tau^{(k+\alpha)r-1} d\tau \\
&\leq \int_t^\infty \|(DB)^k F_\tau\|_r^r (\tau - t)^{(k+\alpha)r-1} d\tau \\
&\leq \int_0^\infty \|(DB)^k F_\tau\|_r^r \tau^{(k+\alpha)r-1} d\tau \\
&= \int_0^\infty \|\tau^k \partial_\tau^k F_\tau\|_r^r \tau^{\alpha r-1} d\tau \\
&\lesssim \|\tilde{N}_*(\tau^k \partial_\tau^k F)\|_p^r \\
&\lesssim \|\tilde{N}_* F\|_p^r.
\end{aligned}$$

We used the change of variable $t + \tau \rightarrow \tau$ and $(k + \alpha)r - 1 > 0$ in the fourth line, Lemma 2.2 in the next to last inequality and Corollary 5.4 in the last inequality.

Next, for $0 < t' < t \leq \delta$, we wish to show that $\|F_t - F_{t'}\|_{\tilde{\mathbb{H}}_{DB}^{p,+}}$ tends to 0, which will imply the existence of the limit in the completion. First, by Minkowski inequality in L^r and a computation as above

$$\left(\int_0^\delta \|\tau^k (DB)^k e^{-\tau|DB|} (F_t - F_{t'})\|_r^r \tau^{\alpha r-1} d\tau \right)^{1/r} \leq 2 \left(\int_0^{2\delta} \|\tau^k (DB)^k F_\tau\|_r^r \tau^{\alpha r-1} d\tau \right)^{1/r}.$$

Secondly, for $\tau \geq \delta$, by the mean value inequality

$$\begin{aligned}
\|(DB)^k e^{-\tau|DB|} (F_t - F_{t'})\|_r &= \|\partial_\tau^k (F_{t+\tau} - F_{t'+\tau})\|_r \\
&\leq \int_{t'+\tau}^{t+\tau} \|\partial_s^{k+1} F_s\|_r ds \\
&\leq \left(\int_{t'+\tau}^{t+\tau} \|s^{k+1} \partial_s^{k+1} F_s\|_r^r ds \right)^{1/r} |t - t'|^{1/r'} \tau^{-(k+1)} \\
&\leq \left(\int_\tau^{2\tau} \|s^{k+1} \partial_s^{k+1} F_s\|_r^r ds \right)^{1/r} |t - t'|^{1/r'} \tau^{-(k+1)}.
\end{aligned}$$

Thus, as $\tau \sim s$, changing the order of integration, we get

$$\begin{aligned}
\int_\delta^\infty \|\tau^k (DB)^k e^{-\tau|DB|} (F_t - F_{t'})\|_r^r \tau^{\alpha r-1} d\tau &\leq |t - t'|^{r/r'} \int_\delta^\infty \|s^{k+1} \partial_s^{k+1} F_s\|_r^r \frac{s^{\alpha r-1}}{s^{r-1}} ds \\
&\lesssim \left(\frac{|t - t'|}{\delta} \right)^{r/r'} \|\tilde{N}_*(F)\|_p^r
\end{aligned}$$

arguing as above in the last inequality. This gives the desired limit 0 of $\|F_t - F_{t'}\|_{\tilde{\mathbb{H}}_{DB}^{p,+}}$ when $t, t' \rightarrow 0$.

Let h be the limit in $\tilde{H}_{DB}^{p,+}$ of F_t as $t \rightarrow 0$. As $F_{t+\tau} = e^{-t|DB|} F_\tau = \tilde{S}_p^+(t) F_\tau$, taking the strong limit in $\tilde{H}_{DB}^{p,+}$ as $\tau \rightarrow 0$ for fixed $t > 0$ yields $F_t = \tilde{S}_p^+(t) h$ for all $t > 0$. \square

It remains to show that $h \in H_{DB}^p$ to conclude that $F_t = S_p^+(t)h$ for all $t > 0$ by (3) in Lemma 8.5, which finishes the proof of (i) implies (iii) in this case. To do this we follow an idea in [HMiMo].

Lemma 8.7. *If h is the limit in $\tilde{H}_{DB}^{p,+}$ of F_t as $t \rightarrow 0$, we have $h \in H^p$ with $\|h\|_{H^p} \lesssim \|\tilde{N}_*F\|_p$, and also $h \in H_D^p$, which is the same as $h \in H_{DB}^p$.*

Proof. Let $\varphi_0 \in \mathcal{S}$. Let χ be a real, C^1 function with compact support in $[0, \infty)$ and $\chi(0) = 1$. Let $\varphi(s, y) = \varphi_s(y) = \chi(s)\varphi_0(y)$ for $s \geq 0$ and $y \in \mathbb{R}^n$. Observe that $\varphi \in C^1([0, \infty); \mathcal{S})$. We have $\langle h, \varphi_0 \rangle = \lim_{\tau \rightarrow 0} \langle F_\tau, \varphi_\tau \rangle$. Indeed,

$$\langle h, \varphi_0 \rangle - \langle F_\tau, \varphi_\tau \rangle = \langle h - F_\tau, \varphi_0 \rangle + (1 - \chi(\tau))\langle F_\tau, \varphi_0 \rangle$$

and strong convergence of F_τ to h in $\tilde{H}_{DB}^{p,+}$ implies convergence in \mathcal{S}' . Next, the pairing $\langle F_\tau, \varphi_\tau \rangle$ is now taken as the L^2 pairing and since $F \in C^1((0, \infty); L^2)$ and $\varphi \in C^1((0, \infty); L^2)$ with bounded support in s , we have

$$\langle F_\tau, \varphi_\tau \rangle = \int_\tau^\infty (-\langle F_s, \partial_s \varphi_s \rangle - \langle \partial_s F_s, \varphi_s \rangle) ds.$$

Using $-\langle \partial_s F_s, \varphi_s \rangle = \langle DBF_s, \varphi_s \rangle = \langle F_s, B^*D\varphi_s \rangle$ and taking the limit as $\tau \rightarrow 0$, we obtain that

$$\langle h, \varphi_0 \rangle = - \int_0^\infty \langle F_s, \partial_s \varphi_s \rangle ds + \int_0^\infty \langle F_s, B^*D\varphi_s \rangle ds = I + II.$$

The pairings inside the integrals are Lebesgue integrals on \mathbb{R}^n . The convergence of the s integral at 0 is in the sense prescribed above. However, note that we have put the action of D on φ in the process and we shall see that this way, for some choices of φ , we obtain *bona fide* Lebesgue integrals on \mathbb{R}_+^{1+n} as the next argument shows.

Now choose $\varphi_0(y) = \frac{1}{r^n} \phi(\frac{x-y}{r})$ with $\phi \in C_0^\infty$ supported in the ball $B(0, c_1)$ with mean value 1. We have that $\langle h, \varphi_0 \rangle = h \star \frac{1}{r^n} \phi(\frac{\cdot}{r})(x)$. By the Fefferman-Stein characterisation of H^p , we need to control $\sup_{r>0} |h \star \frac{1}{r^n} \phi(\frac{\cdot}{r})|$ in L^p to conclude that the Schwartz distribution h belongs to H^p . We use for that the integral representation of $\langle h, \varphi_0 \rangle$ above in which we take $\chi(t)$ supported in $[0, c_0r)$ with value 1 on $[0, c_0^{-1}r]$ and $\|\chi\|_\infty + r\|\chi'\|_\infty \lesssim 1$. Note that the integrand of I is supported in the Whitney box $W(r, x)$, so that looking at powers of r and applying Cauchy-Schwarz inequality, this integral is dominated by $(\tilde{N}_*F)(x)$. For II , using the boundedness of B , we obtain

$$|II| \lesssim \iint_T |F| \|\nabla_y \varphi\|_\infty \lesssim r^{-n-1} \iint_T |F|,$$

where $T := (0, c_0r) \times B(x, c_1r)$. Then, using the inequality in Lemma 2.2

$$\iint_{\mathbb{R}_+^{1+n}} |u| \lesssim \|\tilde{N}_*u\|_{\frac{n}{n+1}}$$

with $u = |F|1_T$ and by support considerations, we obtain

$$r^{-n-1} \iint_T |F| \lesssim \left(r^{-n} \int_{(1+c_0)B(x, c_1r)} (\tilde{N}_*F)^{\frac{n}{n+1}} \right)^{\frac{n+1}{n}} \lesssim (\mathbf{M}((\tilde{N}_*F)^{\frac{n}{n+1}}))^{\frac{n+1}{n}}(x),$$

where \mathbf{M} is the Hardy-Littlewood maximal operator. As $\frac{n}{n+1} < p$, we obtain the conclusion from the maximal theorem and $\tilde{N}_*F \in L^p$.

It remains to prove $h \in H_D^p$. Assume $\varphi_0 \in \mathcal{S}$ is such that $D\varphi_0 = 0$. Consider the extension $\varphi(s, y) = \chi(s)\varphi_0(y)$. Then $D\partial_s\varphi_s = 0$ and $\langle F_s, \partial_s\varphi_s \rangle = 0$ because F_s is orthogonal to the null space of D . Also $D\varphi_s = 0$ and $\langle F_s, B^*D\varphi_s \rangle = 0$. It follows that $\langle h, \varphi_0 \rangle = 0$ by the representation above. As $h \in H^p$, this means that $h \in H_D^p$. \square

Case $p > 2$. Here $p > 2$ means that $2 < p < p_+(DB)$ since we impose $H_{DB}^p = H_D^p$. For p in this range, $H_{B^*D}^{p'} = \overline{R_{p'}(B^*D)} = B^*\overline{R_{p'}(D)}$ is thus a closed subspace of $L^{p'}$, so that it is equipped with $L^{p'}$ norm and for all $h \in H_{B^*D}^{p'}$,

$$\|\mathbb{P}h\|_{p'} \sim \|h\|_{p'} \sim \|h\|_{\mathbb{H}_{B^*D}^{p'}}.$$

Recall also that $\mathbb{H}_{B^*D}^{p'} = H_{B^*D}^{p'} \cap H_{B^*D}^2$ and in this space we are able to compute without thinking about completions.

Recall that our goal is to interpret the limits in (53) and (54). Here, we do not *a priori* know that F_t belongs to some L^p space but only that $F_t \in E_t^p$ uniformly. In fact, we could suppose that F_t belongs to L^p for some $p < p_0$ with $p_0 > 2$. This follows from Meyers $W^{1,p}$ inequality for weak solutions. Thus the argument of subcase $p \leq 2$ would carry almost without change for $p < p_0$. However, we do not know the relation between p_0 and $p_+(DB)$ so that we would have to work in the range $2 < p < \inf(p_0, p_+(DB))$. We decide not to do this, in order to obtain the full range up to $p_+(DB)$. We shall use the slice-spaces E_t^p more extensively.

We shall rely on three technical lemmas.

Lemma 8.8. *Let $h \in \mathbb{H}_{B^*D}^{p'}$. For all $\delta > 0$, $e^{-\delta|B^*D|}h \in E_\delta^{p'}$ with uniform bound with respect to δ . More precisely, $\sup_{\delta > 0} \|e^{-\delta|B^*D|}h\|_{E_\delta^{p'}} \lesssim \|h\|_{p'}$.*

Proof. Theorem 9.3 of [AS] gives us the non-tangential maximal estimates

$$\|\tilde{N}_*(e^{-t|B^*D|}h)\|_{p'} \sim \|h\|_{p'}$$

for p' in our range. By Theorem 5.7 in [AS] we have the $T_2^{p'}$ estimate,

$$\|t|B^*D|e^{-t|B^*D|}h\|_{T_2^{p'}} \sim \|h\|_{p'},$$

hence

$$\|\tilde{N}_*(t|B^*D|e^{-t|B^*D|}h)\|_{p'} \lesssim \|h\|_{p'}.$$

As $t|B^*D|e^{-t|B^*D|}h = -t\partial_t e^{-t|B^*D|}h$, using a similar argument via a mean value inequality as in the proof of Lemma 5.3, we obtain the desired uniform estimates $e^{-\delta|B^*D|}h \in E_\delta^{p'}$. \square

Lemma 8.9. *Fix $\delta > 0$. Let $h \in \mathbb{H}_{B^*D}^{p', \pm} \cap E_\delta^{p'}$. Then $e^{-s|B^*D|}h$ converges to h in $E_\delta^{p'}$ when $s \rightarrow 0$.*

Proof. See Section 15. \square

Lemma 8.10. *Let $\varphi_0 \in \mathcal{S}$ and $\phi_0 = \mathbb{P}_{B^*D}\varphi_0$ where the projection \mathbb{P}_{B^*D} was defined in Section 6. Then $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$, $B^*D\phi_0 = B^*D\varphi_0 \in \mathbb{H}_{B^*D}^{p'}$ and $\chi^\pm(B^*D)B^*D\phi_0 \in \mathbb{H}_{B^*D}^{p', \pm} \cap E_\delta^{p'}$ for all $\delta > 0$.*

Proof. See Section 15. \square

We begin the argument. Let ϕ_0 satisfy $\phi_0, B^*D\phi_0 \in \mathbb{H}_{B^*D}^{p'}$ and $\chi^\pm(B^*D)B^*D\phi_0 \in \mathbb{H}_{B^*D}^{p', \pm} \cap E_\delta^{p'}$ for all $\delta > 0$. From $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$, we have the equalities (53) and (54) and we restart from those. Recall that one term in (53) tends to 0 as $\varepsilon \rightarrow 0$. Thus we need to calculate the limit of the other term in (53) and take the limit in (54). Our first goal is to obtain some identities on $\partial_t F_t$.

We begin with computing the limit of the first term in (53). As $\phi_0, B^*D\phi_0 \in \mathbb{H}_{B^*D}^{p'}$, for $s > 0$ we have $B^*De^{sB^*D}\chi^-(B^*D)\phi_0 = e^{sB^*D}\chi^-(B^*D)B^*D\phi_0$. It follows from Lemmas 8.8 and 8.9 that $e^{sB^*D}\chi^-(B^*D)B^*D\phi_0 \in E_t^{p'}$ and converges to $\chi^-(B^*D)B^*D\phi_0 = B^*D\chi^-(B^*D)\phi_0$ in this space as $s \rightarrow 0$. Since $s \mapsto F_{t+s}$ is continuous near $s = 0$ into $E_t^{p'}$ by Corollary 5.5, we obtain

$$\begin{aligned} & \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} ((B^*De^{sB^*D}\chi^-(B^*D)\phi_0)(x) \cdot F(t+s, x)) ds dx \\ &= \int_{[\varepsilon, 2\varepsilon]} \langle B^*De^{sB^*D}\chi^-(B^*D)\phi_0, F_{t+s} \rangle ds \\ &\rightarrow \langle B^*D\chi^-(B^*D)\phi_0, F_t \rangle, \quad \varepsilon \rightarrow 0, \end{aligned}$$

the pairings denoting the $E_t^{p'}, E_t^p$ duality. This is in fact a Lebesgue integral. It follows from (55) that

$$(61) \quad \langle B^*D\chi^-(B^*D)\phi_0, F_t \rangle = 0$$

for all such ϕ_0 and $t > 0$.

Similarly the first term in (54) converges to $\langle B^*D\chi^+(B^*D)\phi_0, F_t \rangle$ for all such ϕ_0 and $t > 0$. We set

$$\begin{aligned} I_{t, \phi_0}^\varepsilon &:= \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) ds dx \\ &= \int_{[\varepsilon, 2\varepsilon]} \langle B^*De^{-(t-s)B^*D}\chi^+(B^*D)\phi_0, F_s \rangle ds, \end{aligned}$$

where again the pairing can be interpreted using the $E_t^{p'}, E_t^p$ duality. Thus we have shown,

$$\lim_{\varepsilon \rightarrow 0} I_{t, \phi_0}^\varepsilon = \langle B^*D\chi^+(B^*D)\phi_0, F_t \rangle$$

for all $t > 0$. For $\tau \geq 0$, replacing t by $t + \tau$, we obtain

$$\lim_{\varepsilon \rightarrow 0} I_{t+\tau, \phi_0}^\varepsilon = \langle B^*D\chi^+(B^*D)\phi_0, F_{t+\tau} \rangle.$$

But at the same time, $I_{t+\tau, \phi_0}^\varepsilon = I_{t, \phi_\tau}^\varepsilon$ with $\phi_\tau = e^{-\tau|B^*D|}\phi_0$. As ϕ_τ satisfies the same requirements as ϕ_0 and $B^*D\chi^+(B^*D)\phi_\tau = B^*De^{-\tau B^*D}\chi^+(B^*D)\phi_0$, we obtain

$$\lim_{\varepsilon \rightarrow 0} I_{t+\tau, \phi_0}^\varepsilon = \langle B^*D\chi^+(B^*D)\phi_\tau, F_t \rangle = \langle B^*De^{-\tau B^*D}\chi^+(B^*D)\phi_0, F_t \rangle.$$

We have obtained the relation

$$(62) \quad \langle B^*De^{-\tau B^*D}\chi^+(B^*D)\phi_0, F_t \rangle = \langle B^*D\chi^+(B^*D)\phi_0, F_{t+\tau} \rangle.$$

Summing (61) at $t + \tau$ and (62), we have for all such $\phi_0, t > 0$ and $\tau \geq 0$,

$$(63) \quad \langle B^*D\phi_0, F_{t+\tau} \rangle = \langle B^*De^{-\tau B^*D}\chi^+(B^*D)\phi_0, F_t \rangle.$$

With these identities, we next show that $\partial_t F_t \in L^p$. Let $\varphi_0 \in \mathcal{S}$. By Lemma 8.10, the function $\phi_0 = \mathbb{P}_{B^*D}\varphi_0$ has the required properties to apply (63). Moreover,

$B^*D\varphi_0 = B^*D\phi_0$. Using the integration by parts on both sides, justified by Lemma 3.8, and $DBF_t = -\partial_t F_t$, we conclude that

$$(64) \quad \langle \varphi_0, \partial_t F_{t+\tau} \rangle = \langle e^{-\tau B^*D} \chi^+(B^*D)\phi_0, \partial_t F_t \rangle.$$

Now, using this equality with $\tau = t$ and using the $E_t^{p'}, E_t^p$ duality, we have

$$\begin{aligned} |\langle \varphi_0, \partial_t F_{2t} \rangle| &\leq \|e^{-tB^*D} \chi^+(B^*D)\phi_0\|_{E_t^{p'}} \|\partial_t F_t\|_{E_t^p} \\ &\lesssim \|\chi^+(B^*D)\phi_0\|_{p'} \|\partial_t F_t\|_{E_t^p} \\ &\lesssim \|\varphi_0\|_{p'} t^{-1}. \end{aligned}$$

In the second inequality, we used Theorem 9.3 of [AS]. In the last inequality, we used that $\chi^+(B^*D)$ and \mathbb{P}_{B^*D} are bounded on $L^{p'}$ for $p_-(B^*D) < p' < 2$ which is our range here. For the first operator, this is the functional calculus on $\mathbb{H}_{B^*D}^{p'}$ and for the second operator, this is because we have the kernel/range decomposition for B^*D in $L^{p'}$ and p' as above. We also used $\|t\partial_t F_t\|_{E_t^p} \lesssim \|t\partial_t F\|_{N_2^p} \lesssim \|F\|_{N_2^p} < \infty$ by hypothesis. As the inequality above holds for any Schwartz function, this means that $t\partial_t F_{2t}$ belongs to L^p with uniform norm with respect to t .

The next step is to prove the semigroup representation for $\partial_t F_t$. It follows that $t\partial_t F_t \in \overline{\mathbb{R}_p(D)}$ as $\partial_t F_t$ is a conormal gradient (hence satisfies the curl condition). Moreover, we can now use the extension $S_p^+(\tau)$ of $e^{-\tau DB} \chi^+(DB)$ in $\overline{\mathbb{R}_p(D)} = H_{DB}^p$ and for $\varphi_0 \in \mathcal{S}$,

$$\langle e^{-\tau B^*D} \chi^+(B^*D)\phi_0, \partial_t F_t \rangle = \langle \phi_0, S_p^+(\tau)\partial_t F_t \rangle = \langle \varphi_0, S_p^+(\tau)\partial_t F_t \rangle.$$

The last equality is because $\varphi_0 - \phi_0$ belongs to $\mathbb{N}_{p'}(B^*D) = \mathbb{N}_{p'}(D)$ which is the polar set of $\overline{\mathbb{R}_p(D)}$. Thus, for all $t > 0$ and $\tau \geq 0$, we have

$$(65) \quad \partial_t F_{t+\tau} = S_p^+(\tau)\partial_t F_t = S_p(\tau)\partial_t F_t.$$

The last equality is because we know that $\partial_t F_t \in H_{DB}^p$ and we can deduce from (61) that $\partial_t F_t \in H_{DB}^{p,+}$, $S_p(\tau)$ being the extension of $e^{-\tau|DB|}$ to H_{DB}^p .

It remains to integrate and obtain the trace at $t = 0$. To do this, we introduce a distribution $G_{t,\tau}$ by defining for $\varphi_0 \in \mathcal{S}$,

$$(66) \quad \langle \varphi_0, G_{t,\tau} \rangle = \langle \varphi_0, F_{t+\tau} \rangle - \langle e^{-\tau B^*D} \chi^+(B^*D)\mathbb{P}_{B^*D}\varphi_0, F_t \rangle.$$

By the same argument as above,

$$\begin{aligned} |\langle e^{-\tau B^*D} \chi^+(B^*D)\mathbb{P}_{B^*D}\varphi_0, F_t \rangle| &\leq \|e^{-\tau B^*D} \chi^+(B^*D)\mathbb{P}_{B^*D}\varphi_0\|_{E_t^{p'}} \|F_t\|_{E_t^p} \\ &\lesssim_{t,\tau} \|\varphi_0\|_{p'} \|\tilde{N}_* F\|_p. \end{aligned}$$

Notice that the implicit constant is uniform in t when $\tau = t$. It follows that there exists an element $f_{t,\tau} \in L^p$ such that for all $\varphi_0 \in \mathcal{S}$,

$$(67) \quad \langle \varphi_0, f_{t,\tau} \rangle = \langle e^{-\tau B^*D} \chi^+(B^*D)\mathbb{P}_{B^*D}\varphi_0, F_t \rangle.$$

Both pairings are in fact integrals and the equality extends to all $\varphi_0 \in L^{p'}$ by density. Taking $\varphi_0 \in \mathbb{N}_{p'}(D)$ shows $f_{t,\tau} \in \overline{\mathbb{R}_p(D)}$. As $L^p \subset E_t^p$ for any t , we have that $G_{t,\tau}$ is a well-defined element in \mathcal{S}' and $G_{t,\tau} = F_{t+\tau} - f_{t,\tau} \in E_t^p$.

We now show that $G_{t,\tau}$ is constant as a function of $t > 0$ and $\tau > 0$. It is quite clear using smoothness of $t \mapsto F_t$ in any fixed E_δ^p and of $\tau \mapsto e^{-\tau B^*D} \chi^+(B^*D)\mathbb{P}_{B^*D}\varphi_0$

in $E_\delta^{p'}$ that one can differentiate $\langle \varphi_0, G_{t,\tau} \rangle$ in $t > 0$ and $\tau > 0$ when $\varphi_0 \in \mathcal{S}$ and by (66),

$$\langle \varphi_0, \partial_t G_{t,\tau} \rangle = \langle \varphi_0, \partial_t F_{t+\tau} \rangle - \langle e^{-\tau B^* D} \chi^+(B^* D) \mathbb{P}_{B^* D} \varphi_0, \partial_t F_t \rangle = 0$$

and

$$\langle \varphi_0, \partial_\tau G_{t,\tau} \rangle = \langle \varphi_0, \partial_\tau F_{t+\tau} \rangle + \langle B^* D e^{-\tau B^* D} \chi^+(B^* D) \mathbb{P}_{B^* D} \varphi_0, F_t \rangle = 0.$$

We used again the integration by parts argument and $\partial_t F_t = -DBF_t$. We obtain $\partial_t G_{t,\tau} = \partial_\tau G_{t,\tau} = 0$. Let $G = G_{1,1} = G_{t,\tau}$.

Let us show that $G \in L^p$ and then that $G \in \overline{\mathbb{R}_p(D)}$. As observed, $\|f_{t,t}\|_p$ is uniformly bounded in t . In particular, we have $f_{t,t} \in E_t^p$ with $\sup_{t>0} \|f_{t,t}\|_{E_t^p} \lesssim \sup_{t>0} \|f_{t,t}\|_{L^p} \lesssim \|\tilde{N}_* F\|_p$. By taking the difference with F_{2t} , this implies that

$$\sup_{t>0} \left(\int_{\mathbb{R}^n} \left(\int_{B(x,t)} |G(y)|^2 dy \right)^{p/2} dx \right)^{1/p} < \infty.$$

Applying Fatou's lemma when $t \rightarrow 0$ shows that $G \in L^p$. This implies that $F_{2t} = G + f_{t,t}$ belongs to L^p , hence $F_{2t} \in \overline{\mathbb{R}_p(D)}$ (because it satisfies the curl condition) and it follows that $G \in \overline{\mathbb{R}_p(D)}$.

Remark that as we know that $G \in L^p$ and $F_{t+\tau} \in L^p$, one can extend the definition of $\langle \varphi_0, G_{t,\tau} \rangle$ to any $\varphi_0 \in L^{p'}$ by density because all pairings make sense. Taking the derivative in τ in (66) implies that

$$(68) \quad 0 = \langle \varphi_0, -DBF_{t+\tau} \rangle + \langle B^* D e^{-\tau B^* D} \chi^+(B^* D) \mathbb{P}_{B^* D} \varphi_0, F_t \rangle \\ = -\langle B^* D \varphi_0, F_{t+\tau} \rangle + \langle e^{-\tau B^* D} \chi^+(B^* D) \mathbb{P}_{B^* D} B^* D \varphi_0, F_t \rangle = -\langle B^* D \varphi_0, G_{t,\tau} \rangle.$$

The last equality holds because $B^* D \varphi_0 \in L^{p'}$ and we use the extension mentioned above. Thus $DBG = 0$ in the sense of Schwartz distributions.

We have seen that $DBG = 0$ in \mathcal{S}' and $G \in L^p$. Thus $G \in \mathbb{N}_p(DB)$ and we conclude that $G = 0$ from the splitting $L^p = \mathbb{N}_p(DB) \oplus \overline{\mathbb{R}_p(D)}$ which holds in our range of p . We can now write for $\varphi_0 \in \mathcal{S}$ using (67)

$$\langle \varphi_0, f_{t,\tau} \rangle = \langle e^{-\tau B^* D} \chi^+(B^* D) \mathbb{P}_{B^* D} \varphi_0, F_t \rangle = \langle \mathbb{P}_{B^* D} \varphi_0, S_p^+(\tau) F_t \rangle = \langle \varphi_0, S_p^+(\tau) F_t \rangle,$$

and as we have just shown that $\langle \varphi_0, f_{t,\tau} \rangle = \langle \varphi_0, F_{t+\tau} \rangle$, we have obtained

$$F_{t+\tau} = S_p^+(\tau) F_t$$

in \mathcal{S}' for all $t > 0, \tau > 0$ and as both terms are in L^p , this also holds in L^p . Using the uniform bound on $F_t = f_{t/2,t/2}$ in L^p we can use a weak limit argument as in previous cases to deduce the existence of $F_0 \in H_{DB}^{p,+}$ such that $F_\tau = S_p^+(\tau) F_0$ for all $\tau > 0$. This concludes the proof of this case.

9. PROOF OF THEOREM 1.1: (II) IMPLIES (III)

We assume (ii) and set $F = \nabla_A u$. Thus, $t \partial_t F \in T_2^p$, and F is a solution of (45) in \mathbb{R}_+^{1+n} . We also assume that F_t converges to 0 in \mathcal{D}' as $t \rightarrow \infty$. Recall that $p \in I_L$, that is, $H_{DB}^p = H_{DB}^p$. We will use Theorem 6.1 repeatedly.

Step 1. Finding the semigroup equation.

This will be achieved by taking limits in (49) with $\partial_s F_s$ replacing F_s by selecting χ , η and ϕ_0 .

Step 1a. Limit in space. We show that if $\phi_0 \in \overline{\mathbb{R}_2(B^*D)}$, with $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$ (or, equivalently, $\mathbb{P}\phi_0 \in \mathbb{H}_D^{p'}$) if $p > 1$, then

$$(69) \quad \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)B^*(x)D\varphi_s(x) \cdot \partial_s F(s, x)) dsdx = 0$$

and the integral is defined in the Lebesgue sense.

We replace χ by χ_R with $\chi_R(x) = \chi(x/R)$ where $\chi \equiv 1$ in the unit ball $B(0, 1)$, has compact support in the ball $B(0, 2)$ and let $R \rightarrow \infty$. As χ_R tends to 1 and $D\chi_R$ to 0, it suffices by dominated convergence to show that $|\eta'(s)B^*D\varphi_s\partial_s F|$ and $|\eta(s)\partial_s\varphi_s\partial_s F|$ are integrable on \mathbb{R}_+^{1+n} . As $s\partial_s F \in T_2^p$, it is enough to have that $\eta'(s)B^*D\varphi_s$ and $\eta(s)\partial_s\varphi_s$ belong to $(T_2^p)'$. As $\partial_s\varphi_s = B^*D\varphi_s$ on $\text{supp } \eta$, it suffices to invoke Lemma 8.1 again.

Step 1b. Limit in time.

For fixed $t > 0$, $0 < \varepsilon < \inf(t/4, 1/4, 1/t)$, we obtain by making the same choices of η as in the proof of (i) implies (iii),

$$(70) \quad \frac{1}{\varepsilon} \iint_{[t+\varepsilon, t+2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot \partial_s F(s, x)) dsdx \\ = 2\varepsilon \iint_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot \partial_s F(s, x)) dsdx,$$

and

$$(71) \quad \frac{1}{\varepsilon} \iint_{[t-2\varepsilon, t-\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot \partial_s F(s, x)) dsdx \\ = \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot \partial_s F(s, x)) dsdx.$$

We show that the second integral in (70) converges to 0 as $\varepsilon \rightarrow 0$ for fixed t . Using the function ψ_- defined earlier, set $G(s, x) = G_s(x) = 1_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}]}(s)(s-t)^{-1}\psi_-((s-t)B^*D)\phi_0(x)$, under the conditions on ϕ_0 in Step 1. Remark that this integral is bounded by

$$2\varepsilon \|s\partial_s F\|_{T_2^p} \|G\|_{(T_2^p)'}$$

Assume first $p \leq 1$ and let $\alpha = n(\frac{1}{p} - 1)$. As $t\varepsilon < 1$ and $s \in [t + \frac{1}{2\varepsilon}, t + \frac{1}{\varepsilon}]$, we have $s \in [\frac{1}{2\varepsilon}, \frac{2}{\varepsilon}]$. Since $\|G_s\|_2 \lesssim \varepsilon \|\phi_0\|_2$ for those s , we have $\|C_\alpha(G)\|_\infty \lesssim \varepsilon^{\alpha+\frac{n}{2}+1} \|\phi_0\|_2$.

Next, consider $p > 1$. Then one sees that $\|G\|_{T_2^{p'}} \lesssim \varepsilon \|\psi_-(\sigma B^*D)\phi_0\|_{T_2^{p'}} \lesssim \varepsilon \|\mathbb{P}\phi_0\|_{p'}$.

From now on, we distinguish the case $p \leq 2$ from $p > 2$.

Case $p \leq 2$. Using Lemma 8.2 with $\partial_s F$ instead of F , we have $\partial_s F \in C^\infty(0, \infty; L^q)$ and we can reinterpret the dx -integrals in (70) and (71) with the $L^{q'} - L^q$ duality. Copying *mutatis mutandi* the argument in the proof of (i) implies (iii), we can take the limit as $\varepsilon \rightarrow 0$ and obtain the following information: for all $t > 0$, $\partial_t F_t \in \mathbb{H}_{DB}^{2,+}$, and if $\tau \geq 0$

$$(72) \quad e^{-\tau|DB|} \partial_t F_t = e^{-\tau DB} \chi^+(DB) \partial_t F_t = \partial_t F_{t+\tau}.$$

Note that this equation can be differentiated as many times as we want in both t and τ . More information can be obtained such as $tF_t \in \mathbb{H}_{DB}^{q,+}$ but we do not need this.

The argument used in section 8 was to integrate from (72), but it does not work the same here. Instead we look for a candidate f_t for F_t via Hardy space theory. We first prove that $\partial_t f_t = \partial_t F_t$ and then integrate and conclude.

Step 2. Defining an auxiliary function f_t .

We begin with an observation, valid whatever $p \in (0, \infty)$.

Lemma 9.1. *For each $t > 0$ and $N > \frac{n+1}{2}$, we have $(s, x) \mapsto s^N \partial_s^N F_{t+s}(x)$ belongs to T_2^p with uniform bound with respect to t . Moreover, it is C^∞ as a function of t valued in T_2^p . If $t \rightarrow 0$, then it converges to $(s, x) \mapsto s^N \partial_s^N F_s(x)$ in T_2^p and, if $t \rightarrow \infty$, it converges to 0 in T_2^p .*

Proof. Using Corollary 5.4, we have $\|s^N \partial_s^N F\|_{T_2^p} \lesssim \|s \partial_s F\|_{T_2^p} < \infty$. For fixed $t > 0$ and $x \in \mathbb{R}^n$ (recall that $\Gamma(x)$ denotes a cone with aperture 1 and vertex x),

$$\begin{aligned} \iint_{\Gamma(x)} s^{2N-n-1} |\partial_s^N F_{t+s}(y)|^2 ds dy &\leq \iint_{\Gamma(x)} (s+t)^{2N-n-1} |\partial_s^N F_{t+s}(y)|^2 ds dy \\ &= \iint_{\Gamma(x)+(t,0)} s^{2N-n-1} |\partial_s^N F_s(y)|^2 ds dy. \end{aligned}$$

We used that $2N - n - 1 > 0$ in the first inequality and the change of variable $s+t \rightarrow s$. Using the containment $\Gamma(x) + (t, 0) \subset \Gamma(x)$, we have

$$\|s^N \partial_s^N F_{t+s}\|_{T_2^p} \leq \|s^N \partial_s^N F_s\|_{T_2^p}.$$

The same argument plus dominated convergence shows that $\|s^N \partial_s^N F_{t+s}\|_{T_2^p} \rightarrow 0$ when $t \rightarrow \infty$.

Let us look at the limit at 0. Let $\Gamma_\delta(x)$ be the truncation of $\Gamma(x)$ for $s \leq \delta$ and $\Gamma^\delta(x) = \Gamma(x) \setminus \Gamma_\delta(x)$. Arguing as before with a crude estimate and assuming $t \leq \delta$, we have

$$\iint_{\Gamma_\delta(x)} s^{2N-n-1} |\partial_s^N (F_{t+s} - F_s)(y)|^2 ds dy \leq 2 \iint_{\Gamma_{2\delta}(x)} s^{2N-n-1} |\partial_s^N F_s(y)|^2 ds dy.$$

Next, by the mean value inequality and $s \geq \delta$,

$$\iint_{\Gamma^\delta(x)} s^{2N-n-1} |\partial_s^N (F_{t+s} - F_s)(y)|^2 ds dy \leq \frac{t^2}{\delta^2} \iint_{\Gamma^\delta(x)} s^{2N+2-n-1} |\partial_s^{N+1} F_s(y)|^2 ds dy.$$

By dominated convergence, the integral on $\Gamma_{2\delta}(x)$ goes to 0 in $L^{p/2}$ when $\delta \rightarrow 0$. Next, having chosen δ small, one makes the integral on $\Gamma^\delta(x)$ tend to 0 if $t \rightarrow 0$. This proves the limit at 0.

The C^∞ smoothness at any point $t > 0$ can be proved by iterating the second part of the argument. We skip details. \square

We now use the Hardy space theory associated to DB . Let N be as above and M another integer also chosen large. Let c be some constant chosen later. We claim that for all $t \geq 0$ there is an element f_t in $H_{DB}^{p,+}$ such that for all $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$ when $1 < p \leq 2$ or $\phi_0 \in \mathbb{L}_{B^*D}^\alpha$ when $p \leq 1$ (We are using the notation of [AS, Section 4.2]: see in particular Corollary 4.4.)

$$(73) \quad \begin{aligned} \langle \phi_0, f_t \rangle &= c \iint_{\mathbb{R}_+^{1+n}} (\tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0(x) \cdot (\tau^N \partial_\tau^N F_{t+\frac{\tau}{2}})(x)) \frac{d\tau dx}{\tau} \\ &= c \int_0^\infty \langle \phi_0, \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau} \end{aligned}$$

and

$$f_t = S_p^+(t) f_0 = S_p(t) f_0.$$

Here the expression $\partial_\tau^N F_{t+\frac{\tau}{2}}$ means that we differentiate N times with respect to τ the function $F_{t+\frac{\tau}{2}}$. We recall that $S_p^+(t)$ is the continuous extension of $e^{-tDB} \chi^+(DB)$ to H_{DB}^p and it agrees with the continuous extension $S_p(t)$ of $e^{-t|DB|}$ on $H_{DB}^{p,+}$. Both integrals converge in the Lebesgue sense (one is on \mathbb{R}_+ , the other one on \mathbb{R}_+^{1+n}).

Let us define $f_t \in H_{DB}^p$ for each $t \geq 0$. Truncating $\tau^N \partial_\tau^N F_{t+\frac{\tau}{2}}$ with χ_k being the indicator of $[\frac{1}{k}, k] \times B(0, k)$, we have an element of $T_2^2 \cap T_2^p$. If $M > |\frac{n}{p} - \frac{n}{2}|$ as $p \leq 2$, then [AS, Corollary 4.4] for DB with the map $\mathbb{S}_{\varphi, DB}$ and $\varphi(z) = cz^M e^{-\frac{1}{2}|z|}$ shows that

$$\begin{aligned} f_{t,k} &= c \int_0^\infty \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} (\chi_k \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}}) \frac{d\tau}{\tau} \\ &= \mathbb{S}_{\varphi, DB}(\chi_k \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}}) \end{aligned}$$

belongs to \mathbb{H}_{DB}^p . Clearly, $(f_{t,k})$ is a Cauchy sequence in \mathbb{H}_{DB}^p and, as $k \rightarrow \infty$, it converges to some element in H_{DB}^p . This defines f_t .

Let us prove the integral formulae (73). This is in fact the dual argument to the one above. By the L^2 theory, for all integers k , we have

$$\langle \phi_0, f_{t,k} \rangle = c \iint_{\mathbb{R}_+^{1+n}} (\tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0(x) \cdot (\chi_k \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}})(x)) \frac{d\tau dx}{\tau}.$$

As M is large enough, we may apply Theorem 5.7 in [AS] to conclude that $(\tau, x) \mapsto \tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0(x) \in (T_2^p)'$ with $(T_2^p)'$ norm bounded by $\|\mathbb{P}\phi_0\|_{p'}$ when $p > 1$ and $\|\mathbb{P}\phi_0\|_{\dot{\Lambda}^\alpha}$ and $\alpha = n(\frac{1}{p} - 1)$ when $p \leq 1$. Then we use Theorem 4.20 in [AS] to see that $\|\mathbb{P}\phi_0\|_{p'} \sim \|\phi_0\|_{\mathbb{H}_{B^*D}^{p'}}$ and $\|\mathbb{P}\phi_0\|_{\dot{\Lambda}^\alpha} \sim \|\phi_0\|_{\mathbb{L}_{B^*D}^\alpha}$ in the allowed range of p . Thus the above integral on \mathbb{R}_+^{1+n} converges (absolutely) as $k \rightarrow \infty$ by dominated convergence and we obtain the first line in (73). To see the second line, we use Fubini's theorem and rewrite the dx integral as a pairing for the L^2 duality for fixed τ because both entries in the written pairing are L^2 functions. Thus we also obtain

$$\langle \phi_0, f_t \rangle = c \int_0^\infty \langle \tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0, \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}$$

which is the same thing as the second line of (73) by taking adjoints (again for each τ , only the L^2 theory is used). This means that we have proved the formulae (73) for such ϕ_0 .

This will allow us to prove the semigroup formula for f_t . Let us see that now. Differentiating (72) (reverse the roles of t and τ) shows that as $N \geq 1$,

$$\tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} = e^{-tDB} \chi^+(DB) \tau^N \partial_\tau^N F_{\frac{\tau}{2}}.$$

Thus, we obtain

$$\langle \tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0, \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle = \langle \tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_t, \tau^N \partial_\tau^N F_{\frac{\tau}{2}} \rangle$$

with $\phi_t = e^{-tB^*D} \chi^+(B^*D) \phi_0$. As ϕ_t satisfies the same requirements as ϕ_0 above, we can use the definition of f_t and obtain

$$\langle \phi_0, f_t \rangle = \langle \phi_t, f_0 \rangle = \langle e^{-tB^*D} \chi^+(B^*D) \phi_0, f_0 \rangle = \langle \phi_0, S_p^+(t) f_0 \rangle.$$

Note that by Theorem 6.1 and $p \leq 2$, this implies that $f_t \in \mathbb{H}_{DB}^2 = \overline{\mathbb{R}_2(D)}$ for all $t > 0$ (qualitatively).

The next claim is that $\partial_t f_t = \partial_t F_t$ in \mathcal{S}' upon choosing the constant c appropriately. To do this, we have to make the connection between calculus in \mathcal{S}' and calculus in H_{DB}^p by showing that the second line of (73) holds for $\varphi_0 \in \mathcal{S}$ instead of ϕ_0 , still with absolutely convergent integral in τ .

Let $\varphi_0 \in \mathcal{S}$. As $\mathcal{S} \subset L^2$, define $\phi_0 = \mathbb{P}_{B^*D} \varphi_0 \in \mathbb{H}_{B^*D}^2$. We have that $\mathbb{P} \phi_0 = \mathbb{P} \varphi_0$ and since \mathbb{P} preserves $L^{p'}$ and $\dot{\Lambda}^\alpha$, it follows that ϕ_0 has the required property to compute $\langle \phi_0, f_t \rangle$ with the definition given above. Now, $\phi_0 - \varphi_0 \in \mathbf{N}_2(D)$, thus by L^2 theory,

$$\langle \varphi_0, f_t \rangle = \langle \phi_0, f_t \rangle$$

and also for each $\tau > 0$,

$$\langle \varphi_0, \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle = \langle \phi_0, \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle.$$

This proves our claim upon applying the second line of (73) for ϕ_0 .

We next show how to select the constant c appropriately to show our claim, that is $\langle \varphi_0, \partial_t f_t \rangle = \langle \varphi_0, \partial_t F_t \rangle$ for all $\varphi_0 \in \mathcal{S}$ and all $t > 0$. Indeed, by dominated convergence and Lemma 9.1, we can differentiate the first formula in (73) with respect to t with $\phi_0 = \mathbb{P}_{B^*D} \varphi_0$ as above. With the same arguments, we obtain the second line replacing $F_{t+\frac{\tau}{2}}$ by $\partial_t F_{t+\frac{\tau}{2}}$. After this, we reexpress the inner product inside in terms of φ_0 , so that we have obtained

$$\langle \varphi_0, \partial_t f_t \rangle = c \int_0^\infty \langle \varphi_0, \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} \tau^N \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}.$$

Now, again with the L^2 theory,

$$\begin{aligned} \partial_\tau^{M+N} \partial_\tau F_{t+\tau} &= \partial_t^{M+N+1} F_{t+\tau} = e^{-\frac{\tau}{2}DB} \chi^+(DB) (-DB)^M \partial_t^N \partial_t F_{t+\frac{\tau}{2}} \\ &= (-1)^M 2^N (DB)^M e^{-\frac{\tau}{2}DB} \chi^+(DB) \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}} = (-1)^M 2^N (DB)^M e^{-\frac{\tau}{2}|DB|} \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}}. \end{aligned}$$

Hence, with $c = \frac{(-2)^N}{(N+M-1)!}$, we have

$$\begin{aligned} \langle \varphi_0, \partial_t f_t \rangle &= \frac{(-1)^{N+M}}{(N+M-1)!} \int_0^\infty \langle \varphi_0, \partial_\tau^{N+M} \partial_\tau F_{t+\tau} \rangle \tau^{M+N-1} d\tau \\ &= \frac{(-1)^{N+M-2}}{(N+M-1)!} \int_t^\infty \langle \varphi_0, \partial_\tau^{N+M} (\partial_\tau F_\tau) \rangle (\tau-t)^{M+N-1} d\tau. \end{aligned}$$

It follows from (72) and theory of analytic semigroups that $\tau^k \partial_\tau^k (\partial_\tau F_\tau)$ converges to 0 in $\mathbb{H}_{DB}^2 = \overline{\mathbb{R}_2(DB)}$ for all $k \geq 0$ as $\tau \rightarrow \infty$. As $\mathbb{H}_{DB}^2 = \mathbb{H}_D^2$, this implies convergence in \mathcal{S}' . Thus, we can apply the following elementary lemma (whose proof is left to the reader) to $g(\tau) := \langle \varphi_0, \partial_\tau F_\tau \rangle$ with $k = M+N$, and we see that the previous integral is equal to $\langle \varphi_0, \partial_t F_t \rangle$.

Lemma 9.2. *Let $k \in \mathbb{N}, k \geq 1$ and $g : \mathbb{R}^+ \rightarrow \mathbb{C}$ be a function of class C^k with $x^j g^{(j)}(x) \rightarrow 0$ as $x \rightarrow \infty$ when $j = 0, 1, \dots, k-1$. Then*

$$g(x) = \frac{(-1)^{k-2}}{(k-1)!} \int_x^\infty g^{(k)}(y) (y-x)^{k-1} dy.$$

Conclusion: we show that $f_t = F_t$ for all $t > 0$. From what precedes, we have that $F_t = f_t + G$ where $G \in \mathcal{S}'$ is a constant distribution. We assumed that $F_t \rightarrow 0$ in \mathcal{D}' as $t \rightarrow \infty$. By the properties of $S_p(t)$, we have that $f_t \rightarrow 0$ in H_{DB}^p when $t \rightarrow \infty$ (See [AS], Section 6) thus in \mathcal{D}' , as $H_{DB}^p = H_D^p$. It follows that $G = 0$ and we are done.

Case $p > 2$. Again $p > 2$ means here $2 < p < p_+(DB)$. The beginning is exactly the same as the similar case when proving (i) implies (iii) with $\partial_t F_t$ replacing F_t and we obtain (compare (64))

$$(74) \quad \langle \varphi_0, \partial_t^2 F_{t+\tau} \rangle = \langle e^{-\tau B^* D} \chi^+(B^* D) \phi_0, \partial_t^2 F_t \rangle$$

for all $\varphi_0 \in \mathcal{S}$, $t > 0$ and $\tau \geq 0$, with $\phi_0 = \mathbb{P}_{B^* D} \varphi_0$. As before, taking $t = \tau$ leads to

$$\begin{aligned} |\langle \varphi_0, \partial_t^2 F_{2t} \rangle| &\leq \|e^{-t B^* D} \chi^+(B^* D) \phi_0\|_{E_t^{p'}} \|\partial_t^2 F_t\|_{E_t^p} \\ &\lesssim \|\chi^+(B^* D) \phi_0\|_{p'} \|\partial_t^2 F_t\|_{E_t^p} \\ &\lesssim \|\varphi_0\|_{p'} t^{-2}. \end{aligned}$$

In the last line we used (44) with $\partial_t F_t$ being the conormal gradient of a solution and then Corollary 5.4:

$$\|t^2 \partial_t^2 F_t\|_{E_t^p} \lesssim \|t^2 \partial_t^2 F\|_{T_2^p}.$$

Following the same line of argument as before, we conclude that $\partial_t^2 F_t \in H_{DB}^{p,+}$ and for all $t > 0$ and $\tau \geq 0$,

$$\partial_t^2 F_{t+\tau} = S_p^+(\tau) \partial_t^2 F_t = S_p(\tau) \partial_t^2 F_t.$$

As in step 2 of the case $p \leq 2$ above, we exhibit for all $t \geq 0$ a function $f_t \in H_{DB}^{p,+}$ such that

$$f_t = S_p^+(t) f_0 = S_p(t) f_0$$

and such that $\partial_t^2 f_t = \partial_t^2 F_t$ in \mathcal{S}' when $t > 0$. We give details below as the justifications are not the same. Admitting this, we have $F_t = f_t + G + tH$ for two distributions G, H in \mathcal{S}' . Using again that $F_t \rightarrow 0$ in \mathcal{D}' by hypothesis and $f_t \rightarrow 0$

in H_{DB}^p , hence in \mathcal{D}' because of the value of p , we conclude that $G = H = 0$ and $F_t = f_t$ as desired.

It remains to justify our claims on f_t . We construct $f_{t,k}$ similarly by

$$\begin{aligned} f_{t,k} &= c \int_0^\infty \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} (\chi_k \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}}) \frac{d\tau}{\tau} \\ &= \mathbb{S}_{\varphi, DB}(\chi_k \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}}). \end{aligned}$$

Because $p > 2$, the [AS, Corollary 4.4] for DB shows that if $M > 0$ then $f_{t,k} \in \mathbb{H}_{DB}^p$ and converges in H_{DB}^p to f_t . Now if $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$, then we obtain

$$\begin{aligned} (75) \quad \langle \phi_0, f_t \rangle &= c \iint_{\mathbb{R}_+^{1+n}} (\tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0(x) \cdot (\tau^N \partial_\tau^N F_{t+\frac{\tau}{2}})(x)) \frac{d\tau dx}{\tau} \\ &= c \int_0^\infty \langle \tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0, \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}. \end{aligned}$$

Here, the pairings hold in the sense of $L^{p'}$, L^p duality if $N \geq 2$ (because all derivatives of F_t with order exceeding 2 are in L^p so far). Next, the argument that f_t satisfies the semigroup equation is the same argument involving $L^{p'}$, L^p duality pairings.

Then, one can replace such ϕ_0 in the second line of (75) by any $\varphi_0 \in \mathcal{S}$ (argue with $\varphi_0 - \phi_0 \in \mathbf{N}_{p'}(D)$ when $\phi_0 = \mathbb{P}_{B^*D}\varphi_0$ and recall that \mathbb{P}_{B^*D} is bounded on $L^{p'}$ for $p_-(B^*D) < p' < 2$) and we have

$$\langle \varphi_0, f_t \rangle = c \int_0^\infty \langle \tau^M (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \varphi_0, \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}.$$

The rest of the argument is similar, choosing the same constant c , and we have to differentiate twice here, systematically computing with the $L^{p'}$, L^p duality, using that $\tau^k \partial_\tau^k (\partial_\tau^2 F_\tau)$ converges to 0 in H_{DB}^p for all $k \geq 0$ as $\tau \rightarrow \infty$. As $H_{DB}^p = \overline{\mathbf{R}_p(D)}$, this implies convergence in \mathcal{S}' . Thus, by Lemma 9.2 applied to $g(\tau) = \langle \varphi_0, \partial_\tau^2 F_\tau \rangle$ with $k = M + N$, we see that $\langle \varphi_0, \partial_t^2 f_t \rangle = \langle \varphi_0, \partial_t^2 F_t \rangle$.

10. PROOF OF THEOREM 1.3

Recall that $q \in I_{L^*}$, that is, $H_{D\tilde{B}}^q = H_D^q$. This is the same as $H_{DB^*}^q = H_D^q$ (See [AS], Section 12.2). We distinguish the two cases $q > 1$ or $q \leq 1$ of the statement.

Case $q > 1$: $(\alpha) \implies (\beta)$. We set $p = q'$. Thus we assume that $tF \in T_2^p$, where F is a solution of (45) in \mathbb{R}_+^{1+n} , that is, $F = \nabla_A u$. We also assume that $u(t, \cdot)$ converges to 0 in \mathcal{D}' modulo constants as $t \rightarrow \infty$. We will show at the end how this assumption can be removed in some cases. In terms of F , this means that the tangential part of F_t converges to 0 in \mathcal{D}' as $t \rightarrow \infty$. Indeed, if φ_0 is a test function, then

$$\langle \varphi_0, F_{t\parallel} \rangle = \langle \varphi_0, \nabla_x u(t, \cdot) \rangle = -\langle \operatorname{div} \varphi_0, u(t, \cdot) \rangle.$$

We follow the same paths as before. We point out the differences when necessary.

Step 1. Finding the semigroup equation.

This is achieved by taking limits in (49) by selecting χ , η and ϕ_0 in the definition of φ_s in (48).

An observation is in order. To define φ_s we took $\phi_0 \in \mathbb{H}_{B^*D}^2 = \overline{\mathbb{R}_2(B^*D)}$. But equation (49) involves only $B^*D\varphi_s$ and $\partial_s\varphi_s$ which are equal. Now, if we further assume $\phi_0 \in \mathbb{D}_2(B^*D) = \mathbb{D}_2(D)$, then one can use the similarity relation between the functional calculi of DB^* and B^*D via D , so that

$$(76) \quad \eta(s)B^*D\varphi_s = \begin{cases} \eta(s)B^*e^{-(t-s)DB^*}\chi^+(DB^*)D\phi_0 & \text{if } s < t, \\ -\eta(s)B^*e^{-(s-t)DB^*}\chi^-(DB^*)D\phi_0 & \text{if } t < s. \end{cases}$$

But these expressions are defined for all $\phi_0 \in \mathbb{D}_2(D)$ and, as D annihilates $\mathbb{N}_2(B^*D)$, this means that (49) is valid for any $\phi_0 \in \mathbb{D}_2(D)$ (up to changing ϕ_0 to $\mathbb{P}_{B^*D}\phi_0$ in the definition of φ_s).

Step 1a. Limit in space. We show that if $\phi_0 \in \mathbb{D}_2(D)$ with $D\phi_0 \in \mathbb{H}_D^q$, then

$$(77) \quad \iint_{\mathbb{R}_+^{1+n}} (\eta'(s)B^*(x)D\varphi_s(x) \cdot F(s, x)) dsdx = 0$$

with $\eta'(s)B^*(x)D\varphi_s(x)$ as in (76) and the integral is defined in the Lebesgue sense.

We replace χ in (49) by χ_R with $\chi_R(x) = \chi(x/R)$ where $\chi \equiv 1$ in the unit ball $B(0, 1)$, has compact support in the ball $B(0, 2)$ and let $R \rightarrow \infty$. As χ_R tends to 1 and $D\chi_R$ to 0, it suffices by dominated convergence to show that $|\eta'(s)B^*D\varphi_s F|$ and $|\eta(s)\partial_s\varphi_s F|$ are integrable on \mathbb{R}_+^{1+n} . As $sF \in T_2^p$, it is enough to have that $\eta'(s)s^{-1}B^*D\varphi_s$ and $\eta(s)s^{-1}\partial_s\varphi_s$ belong to $(T_2^p)' = T_2^q$. As $\partial_s\varphi_s = B^*D\varphi_s$ on $\text{supp } \eta$, it suffices to invoke the following lemma.

Lemma 10.1. *Let q with $H_{DB^*}^q = H_D^q$. Assume $\phi_0 \in \mathbb{D}_2(D)$ and $D\phi_0 \in \mathbb{H}_{DB^*}^q$. Then $\eta(s)B^*D\varphi_s \in T_2^q$ for all η bounded and compactly supported away from t .*

Proof. We use (76). By geometric considerations as $t-s$ is bounded away from 0, we see that if (s, y) belongs to a cone $\Gamma(x)$ with $s \in \text{supp } (\eta) \cap (0, t)$ then $(t-s, y)$ belongs to a cone $\tilde{\Gamma}(x)$ with (bad) finite aperture depending on the support of η . Thus, setting $t-s = \sigma$ and using also that s is bounded below and $t-s$ bounded above on $\text{supp } (\eta)$, we obtain from elementary estimates based on these considerations and L^∞ boundedness of B^* , and adjusting the parameters c_0, c_1 in \tilde{N}_* ,

$$\iint_{\Gamma(x), s < t} |\eta(s)B^*(y)D\varphi_s(y)|^2 \frac{dsdy}{s^{n+1}} \lesssim_\eta \tilde{N}_*(e^{-\sigma DB^*}\chi^+(DB^*)D\phi_0)^2(x).$$

By Theorem 9.1 and Theorem 4.19 of [AS], we have for q in our range,

$$\|\tilde{N}_*(e^{-\sigma DB^*}\chi^+(DB^*)D\phi_0)\|_q \lesssim \|\chi^+(DB^*)D\phi_0\|_{H^q} \lesssim \|D\phi_0\|_{H^q}.$$

The argument is the same when $t < s$. □

Step 1b. Limit in time.

For fixed $t > 0$, $0 < \varepsilon < \inf(t/4, 1/4, 1/t)$, we obtain by making the same choices of η as the proof of (i) implies (iii) of Theorem 1.1,

$$(78) \quad \begin{aligned} \frac{1}{\varepsilon} \iint_{[t+\varepsilon, t+2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) ds dx \\ = 2\varepsilon \iint_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) ds dx \end{aligned}$$

and

$$(79) \quad \begin{aligned} \frac{1}{\varepsilon} \iint_{[t-2\varepsilon, t-\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) ds dx \\ = \frac{1}{\varepsilon} \iint_{[\varepsilon, 2\varepsilon] \times \mathbb{R}^n} (B^*(x)D\varphi_s(x) \cdot F(s, x)) ds dx. \end{aligned}$$

The second integral in (78) converges to 0 as $\varepsilon \rightarrow 0$ for fixed t . Indeed, let

$$\begin{aligned} G(s, x) &= G_s(x) = 1_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}]}(s) B^*(x) D\varphi_s(x) \\ &= 1_{[t+\frac{1}{2\varepsilon}, t+\frac{1}{\varepsilon}]}(s) (B^* e^{-(s-t)DB^*} \chi^-(DB^*) D\phi_0)(x), \end{aligned}$$

under the conditions on ϕ_0 in Step 1a. This integral is bounded by $2\varepsilon \|sF\|_{T_2^p} \|G\|_{T_2^q}$. As $t\varepsilon < 1$ and $s \in [t + \frac{1}{2\varepsilon}, t + \frac{1}{\varepsilon}]$, we have $\sigma = s - t \in [\frac{1}{2\varepsilon}, \frac{1}{\varepsilon}]$, so that as in the proof of the lemma above,

$$\|G\|_{T_2^q} \lesssim \|\tilde{N}_*(e^{-\sigma|DB^*|} \chi^-(DB^*) D\phi_0)\|_q$$

and examination shows uniformity with respect to ε .

Next, to obtain the expression of the limits of the other terms in (78) and (79), we shall impose further that $\chi^\pm(DB^*)D\phi_0 \in E_\delta^q$ for some (any) $\delta > 0$. It is convenient to introduce \mathbb{D}_q as the space of $\phi_0 \in \mathbb{D}_2(D)$ with $D\phi_0 \in \mathbb{H}_D^q$ and $\chi^\pm(DB^*)D\phi_0 \in E_\delta^q$ for some (any) $\delta > 0$. Here this definition makes sense for any $q \in I_{L^*}$ since in that case $\mathbb{H}_D^q = \mathbb{H}_{DB^*}^q$ and $\chi^\pm(DB^*)$ are bounded operators on $\mathbb{H}_{DB^*}^q$. The following lemmas hold for any such q .

Lemma 10.2. *Let $q \in I_{L^*}$. Let $h \in \mathbb{H}_{DB^*}^{q, \pm}$. For all $\delta > 0$, $e^{-\delta|DB^*|}h \in E_\delta^q$ with uniform bound with respect to δ . More precisely, $\sup_{\delta > 0} \|e^{-\delta|DB^*|}h\|_{E_\delta^q} \lesssim \|h\|_q$.*

Proof. Direct consequence of Theorem 9.1 of [AS] together with the method of proof of Lemma 8.8. \square

Lemma 10.3. *Let $q \in I_{L^*}$. If $\phi_0 \in \mathbb{D}_2(D)$ with $D\phi_0 \in \mathbb{H}_{DB^*}^q$, then for all $\delta > 0$ and $M \in \mathbb{N}$, $(B^*D)^M e^{-\delta|B^*D|} \chi^\pm(B^*D)\phi_0 \in \mathbb{D}_q$.*

Proof. For this proof, set $h = (B^*D)^M e^{-\delta|B^*D|} \chi^\pm(B^*D)\phi_0$. As $\phi_0 \in \mathbb{D}_2(D)$, then $h \in \mathbb{D}_2(D)$ and

$$Dh = (DB^*)^M e^{-\delta|DB^*|} \chi^\pm(DB^*) D\phi_0.$$

As $(DB^*)^M e^{-\delta|DB^*|} \chi^\pm(DB^*)$ is bounded on $\mathbb{H}_{DB^*}^q = \mathbb{H}_D^q$ by $q \in I_{L^*}$, we have $Dh \in \mathbb{H}_{DB^*}^q$.

Finally, $\chi^\pm(DB^*)Dh = (DB^*)^M e^{-\delta|DB^*|} \chi^\pm(DB^*) (\chi^\pm(DB^*) D\phi_0)$ which belongs to E_δ^q . For $M = 0$, this is in Lemma 10.2. For $M > 0$, one can take derivatives and apply Corollaries 5.4 and 5.5. We skip details. \square

Lemma 10.4. Fix $\delta > 0$. Let $q \in I_{L^*}$. Let $h \in \mathbb{H}_{DB^*}^{q,\pm} \cap E_\delta^q$. Then $e^{-s|DB^*|}h$ converges to h in E_δ^q when $s \rightarrow 0$.

Proof. The proof is exactly as that of Lemma 8.9. \square

Lemma 10.5. Let $\varphi_0 \in \mathcal{S}$. Then $\varphi_0 \in \mathbb{D}_q$ for any $q \in I_{L^*}$.

We prove this lemma in Section 15.

Armed with these lemmas and reexpressing (78) and (79) with the E_t^q, E_t^p duality pairings and taking limits we obtain exactly as in the previous arguments for all $\phi_0 \in \mathbb{D}_q$, $t > 0$ and $\tau \geq 0$,

$$(80) \quad \langle B^* \chi^-(DB^*)D\phi_0, F_t \rangle = 0,$$

$$(81) \quad \langle B^* e^{-\tau DB^*} \chi^+(DB^*)D\phi_0, F_t \rangle = \langle B^* \chi^+(DB^*)D\phi_0, F_{t+\tau} \rangle,$$

so that summing with the first equation at $t + \tau$, we get

$$(82) \quad \langle B^* D\phi_0, F_{t+\tau} \rangle = \langle B^* e^{-\tau DB^*} \chi^+(B^* D)D\phi_0, F_t \rangle.$$

Step 2. We show that $\partial_t^k F_t \in \dot{W}^{-1,p}$, for $k \geq 1$. By Lemma 10.5, any $\varphi_0 \in \mathcal{S}$ has the required properties to apply (82) and using integration by parts in slice-spaces (Lemma 3.8), we have obtained

$$(83) \quad -\langle \varphi_0, \partial_t F_{t+\tau} \rangle = \langle B^* e^{-\tau DB^*} \chi^+(DB^*)D\varphi_0, F_t \rangle,$$

where both pairings can be expressed as Lebesgue integrals.

Now, using this equality with $\tau = t$, the E_t^q, E_t^p duality and Lemma 10.2, we have

$$\begin{aligned} |\langle \varphi_0, \partial_t F_{2t} \rangle| &\leq \|B^* e^{-t DB^*} \chi^+(DB^*)D\varphi_0\|_{E_t^q} \|F_t\|_{E_t^p} \\ &\lesssim \|D\varphi_0\|_q \|F_t\|_{E_t^p} \\ &\lesssim \|\varphi_0\|_{\dot{W}^{1,q}} t^{-1}. \end{aligned}$$

As this is true for all $\varphi_0 \in \mathcal{S}$, this shows that $\partial_t F_{2t}$ is an element in $\dot{W}^{-1,p}$ with norm controlled by t^{-1} . As usual, as $\partial_t F_t$ is a conormal gradient, this automatically implies that it belongs to $\dot{W}_D^{-1,p}$. Remark that one can differentiate (83) with respect to t (using dominated convergence) and then make $\tau = t$ to obtain

$$|\langle \varphi_0, \partial_t^k F_{2t} \rangle| \lesssim \|\varphi_0\|_{\dot{W}^{1,q}} t^{-k}$$

for all integer $k \geq 1$.

Step 3. Now, for all $t > 0$, we create an element $f_t \in \dot{W}_{DB}^{-1,p,+}$ which, in the end, will be F_t . Recall that $p = q'$. We require the following lemma.

Lemma 10.6. For each $t > 0$ and $N > \frac{n}{2}$, we have $(s, x) \mapsto s^{N+1} \partial_s^N F_{t+s}(x)$ belongs to T_2^p with uniform bound with respect to t . Moreover, it is C^∞ as a function of t valued in T_2^p . If $t \rightarrow 0$, then it converges to $(s, x) \mapsto s^{N+1} \partial_s^N F_s(x)$ in T_2^p and, if $t \rightarrow \infty$, it converges to 0 in T_2^p .

The proof is identical to that of Lemma 9.1.

We need a little bit of Sobolev theory for DB (only evoked at the end of [AS] but working the same way as the Hardy space theory). For $1 < p < \infty$, let $\dot{W}_{DB}^{-1,p}$ be the space of functions of the form $\mathbb{S}_{\psi, DB} F$ with $F \in T_2^p$ and $\tau F \in T_2^p$ with norm

$\inf \|\tau F\|_{T_2^p} < \infty$. This space does not depend on the particular choice of ψ among bounded holomorphic functions in bisectors S_μ with enough decay at 0 and ∞ . It coincides with the space of functions $h \in \overline{\mathbb{R}_2(DB)} = \mathbb{H}_{DB}^2$ such that $\|\tau \mathbb{Q}_{\psi, DB} h\|_{T_2^p}$ again for any choice of ψ as above. We let $\dot{W}_{DB}^{-1,p}$ be its completion.

Also for $1 < q < \infty$, let $\dot{W}_{B^*D}^{1,q}$ be the set of functions $h \in \overline{\mathbb{R}_2(B^*D)} = \mathbb{H}_{B^*D}^2$ with $\tau^{-1} \mathbb{Q}_{\psi, B^*D} h \in T_2^q$, equipped with norm $\|\tau^{-1} \mathbb{Q}_{\psi, B^*D} h\|_{T_2^q}$. Again, this space does not depend on the particular choice of ψ as above and can be characterized as well by the \mathbb{S}_{ψ, B^*D} maps. Let $\dot{W}_{B^*D}^{1,q}$ be its completion.

When $q = p'$, both spaces $\dot{W}_{B^*D}^{1,q}$, $\dot{W}_{DB}^{-1,p}$ are in duality for the L^2 duality. This duality extends to the completed spaces.

For $q \in I_{L^*}$, $q > 1$ and $p' = q$, we have that $\dot{W}_{DB}^{-1,p} = \dot{W}_D^{-1,p}$ with equivalence of norms and \mathbb{P} extends to an isomorphism from $\dot{W}_{B^*D}^{1,q}$ onto $\dot{W}_D^{1,q}$.

Lemma 10.7. *Let $q \in I_{L^*}$ with $q > 1$. Then $\mathbb{D}_q \cap \mathbb{H}_{B^*D}^2$ is a dense subspace of $\dot{W}_{B^*D}^{1,q}$.*

Proof. See Section 15. □

Let N be as above with $N \geq 1$, and M be another integer also chosen large. Let c be some constant to be chosen later and $\varphi(z) = cz^M e^{-\frac{1}{2}|z|}$. For all integers k and $t \geq 0$, set

$$\begin{aligned} f_{t,k} &= c \int_0^\infty \tau^M (DB)^M e^{-\frac{\tau}{2}|DB|} (\chi_k \tau^N \partial_\tau^N F_{t+\frac{\tau}{2}}) \frac{d\tau}{\tau} \\ &= \mathbb{S}_{\varphi, DB} \left(\frac{1}{\tau} \chi_k \tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}} \right), \end{aligned}$$

with χ_k being the indicator function of $[\frac{1}{k}, k] \times B(0, k)$ as before. The $\dot{W}_{DB}^{-1,p}$ theory shows that if M is large enough, then $f_{t,k} \in \dot{W}_{DB}^{-1,p}$ with $\|f_{t,k}\|_{\dot{W}_{DB}^{-1,p}} \lesssim \|\chi_k \tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}}\|_{T_2^p}$ uniformly in t and k , and, by Lemma 10.6, $f_{t,k}$ converges to some $f_t \in \dot{W}_{DB}^{-1,p}$ as $k \rightarrow \infty$. In particular, for all $\phi_0 \in \mathbb{D}_q \cap \mathbb{H}_{B^*D}^2$, we have

$$\begin{aligned} \langle \phi_0, f_t \rangle &= c \iint_{\mathbb{R}_+^{1+n}} (\tau^{M-1} (B^*D)^M e^{-\frac{\tau}{2}|B^*D|} \phi_0(x) \cdot (\tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}})(x)) \frac{d\tau dx}{\tau} \\ &= c \int_0^\infty \langle \tau^{M-1} B^* (DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\phi_0, \tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}. \end{aligned}$$

The first integral converges in the Lebesgue sense and $\langle \phi_0, f_t \rangle$ is interpreted using the duality of the spaces $\dot{W}_{B^*D}^{1,q}$, $\dot{W}_{DB}^{-1,p}$ extending the L^2 duality on $\dot{W}_{B^*D}^{1,q}$, $\dot{W}_{DB}^{-1,p}$ as explained above. One can see that the integral with respect to x can be interpreted in the E_δ^q, E_δ^p duality for each τ and the second equality follows from Fubini's theorem.

This interpretation allows us to show that

$$(84) \quad f_t = \tilde{S}_p^+(t) f_0 = \tilde{S}_p(t) f_0,$$

where $\tilde{S}_p^+(\tau)$ is the extension of $e^{-\tau DB} \chi^+(DB)$ on $\dot{W}_D^{-1,p}$ described in Lemma 14.4 of [AS], which agrees with the extension $\tilde{S}_p(\tau)$ of $e^{-\tau|DB|}$ on $\dot{W}_{DB}^{-1,p,+}$. Indeed, by Lemma 10.7, it suffices to test against some $\phi_0 \in \mathbb{D}_q \cap \mathbb{H}_{B^*D}^2$. First, note that we have $\phi_t = e^{-tB^*D} \chi^+(B^*D) \phi_0 \in \mathbb{D}_q \cap \mathbb{H}_{B^*D}^2$ as well, hence $\langle \phi_0, \tilde{S}_p^+(t) f_0 \rangle = \langle \phi_t, f_0 \rangle$

by definition of the semigroup $\widetilde{S}_p^+(t)$. Secondly, we may compute $\langle \phi_t, f_0 \rangle$ using the defining representation of f_0 with ϕ_t replacing ϕ_0 and obtain

$$\langle \phi_t, f_0 \rangle = c \int_0^\infty \langle \tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\phi_t, \tau^{N+1} \partial_\tau^N F_{\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}.$$

Observe that

$$B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\phi_t = B^* e^{-tDB^*} \chi^+(DB^*) D(B^*D)^{M-1} e^{-\frac{\tau}{2}|B^*D|} \phi_0,$$

so by (81) with $(B^*D)^{M-1} e^{-\frac{\tau}{2}|B^*D|} \phi_0 \in \mathbb{D}_q$ replacing ϕ_0 and $M \geq 1$,

$$\begin{aligned} \langle \tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\phi_t, \tau^{N+1} \partial_\tau^N F_{\frac{\tau}{2}} \rangle \\ = \langle \tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\phi_0, \tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle. \end{aligned}$$

Inserting this equality in the τ -integral, we find $\langle \phi_t, f_0 \rangle = \langle \phi_0, f_t \rangle$. This and a density argument prove (84).

Step 4. Now, we connect this to the calculus in \mathcal{S}' and show that $\partial_t f_t = \partial_t F_t$ in \mathcal{S}' if f_t is appropriately normalized. Since $\dot{W}_{DB}^{-1,p} = \dot{W}_D^{-1,p}$ with equivalence of norms and $f_{t,k} \rightarrow f_t$ in $\dot{W}_{DB}^{-1,p}$, we have for all $\varphi_0 \in \mathcal{S}$, $\langle \varphi_0, f_t \rangle = \lim_k \langle \varphi_0, f_{t,k} \rangle$. For each k , we have by the L^2 theory, we can express $\langle \varphi_0, f_{t,k} \rangle$ as

$$\langle \varphi_0, f_{t,k} \rangle = c \iint_{\mathbb{R}_+^{1+n}} (\tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\varphi_0(x) \cdot (\chi_k \tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}})(x)) \frac{d\tau dx}{\tau}.$$

By our assumption $H_{DB^*}^q = H_D^q$ implies

$$\|\tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\varphi_0\|_{T_2^q} \lesssim \|D\varphi_0\|_q \lesssim \|\varphi_0\|_{\dot{W}^{1,q}}.$$

We can take the limit in the integral above as $k \rightarrow \infty$ and obtain first the integral on \mathbb{R}_+^{1+n} and then by Fubini's theorem,

$$\langle \varphi_0, f_t \rangle = c \int_0^\infty \langle \tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\varphi_0, \tau^{N+1} \partial_\tau^N F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}.$$

The same argument having first differentiated $\langle \varphi_0, f_t \rangle$ in t using Lemma 10.6 and the double integral representation leads to

$$\langle \varphi_0, \partial_t f_t \rangle = c \int_0^\infty \langle \tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\varphi_0, \tau^{N+1} \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}} \rangle \frac{d\tau}{\tau}.$$

The meaning of the pairing for the integrand is as before with respect to E_δ^q, E_δ^p duality. We have,

$$\begin{aligned} & \langle \tau^{M-1} B^*(DB^*)^{M-1} e^{-\frac{\tau}{2}|DB^*|} D\varphi_0, \tau^{N+1} \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}} \rangle \\ &= \langle \tau^{M-1} B^* D[(B^*D)^{M-1} e^{-\frac{\tau}{2}|B^*D|} \varphi_0], \tau^{N+1} \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}} \rangle \\ &= \langle \tau^{M-1} (B^*D)^{M-1} B^* e^{-\frac{\tau}{2}|DB^*|} \chi^+(DB^*) D\varphi_0, \tau^{N+1} \partial_\tau^N \partial_t F_{t+\frac{\tau}{2}} \rangle \\ &= \tau^{M+N} 2^{-N} \langle B^* e^{-\frac{\tau}{2}|DB^*|} \chi^+(DB^*) D\varphi_0, (DB)^{M-1} \partial_t^N \partial_t F_{t+\frac{\tau}{2}} \rangle \\ &= \tau^{M+N} 2^{-N} (-1)^{M-1} \langle B^* e^{-\frac{\tau}{2}|DB^*|} \chi^+(DB^*) D\varphi_0, \partial_t^{N+M-1} \partial_t F_{t+\frac{\tau}{2}} \rangle \\ &= \tau^{M+N} 2^{-N} (-1)^{M-1} \langle B^* D\varphi_0, \partial_t^{N+M-1} \partial_t F_{t+\tau} \rangle \\ &= \tau^{M+N} 2^{-N} (-1)^M \langle \varphi_0, \partial_\tau^{N+M} \partial_\tau F_{t+\tau} \rangle. \end{aligned}$$

In the first equality, we used that $\varphi_0 \in D_2(D)$. In the second equality, we used (80) to insert $\chi^+(DB^*)$ as $(B^*D)^{M-1}e^{-\frac{\tau}{2}|B^*D|}\varphi_0 \in E_\delta^q$ combining Lemma 10.3 and Lemma 10.5. In the third equality, we used the integration by parts iteratively in slice-spaces (Lemma 3.8). In the fourth we used the differential equation $\partial_t F_t = -DBF_t$ repeatedly, in the fifth (82) and in the sixth, integration by parts in slice-spaces again together with the differential equation. Thus choosing $c = \frac{(-2)^N}{(N+M-1)!}$ as before we obtain

$$\begin{aligned} \langle \varphi_0, \partial_t f_t \rangle &= \frac{(-1)^{N+M}}{(N+M-1)!} \int_0^\infty \langle \varphi_0, \partial_\tau^{N+M} \partial_\tau F_{t+\tau} \rangle \tau^{M+N-1} d\tau \\ &= \frac{(-1)^{N+M-2}}{(N+M-1)!} \int_t^\infty \langle \varphi_0, \partial_\tau^{N+M} (\partial_\tau F_\tau) \rangle (\tau-t)^{M+N-1} d\tau. \end{aligned}$$

Using again Lemma 9.2 applied to $g(\tau) = \langle \varphi_0, \partial_\tau F_\tau \rangle$ since $\tau^k g^{(k)}(\tau)$ is controlled by τ^{-1} for all $k \geq 0$, we conclude that $\langle \varphi_0, \partial_t f_t \rangle = \langle \varphi_0, \partial_t F_t \rangle$.

Step 5. Conclusion: $f_t = F_t$ in \mathcal{S}' . From Step 4, there exists $G \in \mathcal{S}'$, such that $F_t = f_t + G$ for all $t > 0$. Recall that $F_{t,\parallel} \rightarrow 0$ in \mathcal{D}' by our assumption. Also, $f_t \rightarrow 0$ in $\dot{W}_D^{-1,p}$ from the semigroup equation, hence in \mathcal{D}' . Thus we have $G_\parallel = 0$. As $G = \nabla_A w$ for some solution, we have $G_\parallel = \nabla_x w$ and $G_\perp = a\partial_t w + b \cdot \nabla_x w$. It follows that $w(t, x) = w(t)$ and, as G is independent of t , $w'(t) = \alpha$ is a constant in \mathbb{C}^m . Here, a is the first diagonal block of A in (9). In the construction of the matrix $B = \hat{A}$, it is proved (and used) that a is invertible in L^∞ with $\lambda|\alpha| \leq |a(x)\alpha|$ almost everywhere, where λ is the ellipticity constant for A . But at the same time we must have $tG \in E_t^p$ for all $t > 0$. This implies that $a\alpha \in E_1^p$ and the only possibility is $\alpha = 0$.

Step 6. Elimination of the condition $u(t, \cdot) \rightarrow 0$ as $t \rightarrow \infty$ in \mathcal{D}' modulo constants when $p < \frac{2n}{n-2}$. Recall that we have used the consequence that $F_{t,\parallel} \rightarrow 0$ as $t \rightarrow \infty$ in \mathcal{D}' when $F = \nabla_A u$. [Actually, the converse holds using that any test function with mean value 0 is the divergence of some test function.] However this limit always hold when $tF \in T_2^p$ and $p < \frac{2n}{n-2}$ [which means that $u(t, \cdot) \rightarrow 0$ as $t \rightarrow \infty$ in \mathcal{D}' modulo constants when $t\nabla u \in T_2^p$]. Indeed, if $p \leq 2$, then we know that $tF_t \in E_t^p$ with uniform bound thus $tF_t \in L^p$ uniformly. This gives the desired limit. If $2 < p$, then that $tF_t \in E_t^p$ uniformly implies $t^{n/p-n/2} \|F_t\|_{L^2(B_{t/2})} \leq C$ uniformly as a simple consequence of Hölder inequality (we leave the proof to the reader). In particular on any fixed ball B we obtain $\|F_t\|_{L^2(B)} \rightarrow 0$ as $t \rightarrow \infty$ when $p < \frac{2n}{n-2}$ and we are done. [Remark that using Meyers' improvement of local L^2 estimates for the gradient of u to local L^{p_0} estimates for some $p_0 > 2$, one can improve the upper bound to $p < \frac{p_0 n}{n-p_0}$.]

Case $q \leq 1$: (a) \implies (b). We assume $q \in I_{L^*}$ and $q \leq 1$. For $\alpha = n(\frac{1}{q} - 1)$, we assume that $tF \in T_{2,\alpha}^\infty$, where F is a solution of (45) in \mathbb{R}_+^{1+n} , that is, $F = \nabla_A u$. We also assume that $u(t, \cdot)$ converges to 0 in \mathcal{D}' modulo constants as $t \rightarrow \infty$. In terms of F , this means that the tangential part of F_t converges to 0 in \mathcal{D}' as $t \rightarrow \infty$.

The proof is almost identical to the previous one: Step 1a is valid for $q \leq 1$. Next, the stated series of lemmas in Step 1b is valid for $q \in I_{L^*}$, $q \leq 1$, and the conclusion is the same. In Step 2, we deduce that $\partial_t^k F_t \in \dot{\Lambda}^{\alpha-1}$ for $k \geq 1$. In step

3, Lemma 10.6 is replaced by Lemma 10.8 below and Lemma 10.7 holds for $q \leq 1$ with appropriate replacement of $\dot{\mathbb{W}}_{B^*D}^{1,q}$ explained below. This allows us to define a candidate $f_t \in \dot{\Lambda}^{\alpha-1}$ which enjoys the desired semigroup formula and which will be F_t in the end. We shall give some detail of the steps 4 and 5 to show this is the case.

Lemma 10.8. *For each $t > 0$ and $2(N+1) - 1 > n + 2\alpha$, we have $(s, x) \mapsto s^{N+1} \partial_s^N F_{t+s}(x)$ belongs to $T_{2,\alpha}^\infty$ with uniform bound with respect to t . Moreover, it is C^∞ as a function of t valued in $T_{2,\alpha}^\infty$ equipped with the weak-star topology. If $t \rightarrow 0$, then it converges for the weak-star topology to $(s, x) \mapsto s^{N+1} \partial_s^N F_s(x)$ in $T_{2,\alpha}^\infty$.*

Proof. Taking derivatives, $\|t^{N+1} \partial_t^N F\|_{T_{2,\alpha}^\infty} < \infty$ for each integer $N \geq 1$ by the assumption for $N = 0$ and Corollary 5.4.

We take a Carleson box $(0, R) \times B_R$ where B_R is a ball of radius R in \mathbb{R}^n . For $t \geq 0$, let $I_{t,R} = \int_{B_R} \int_0^R |s^{N+1} \partial_s^N F_{t+s}(x)|^2 \frac{dsdx}{s}$.

If $t \leq R$, we write

$$\begin{aligned} I_{t,R} &= \int_{B_R} \int_0^R s^{2(N+1)-1} |\partial_s^N F_{t+s}(x)|^2 dsdx \\ &\leq \int_{B_R} \int_0^R (t+s)^{2(N+1)-1} |\partial_s^N F_{t+s}(x)|^2 dsdx \\ &= \int_{B_R} \int_t^{t+R} s^{2(N+1)-1} |\partial_s^N F_s(x)|^2 dsdx \\ &\leq \int_{B_{2R}} \int_0^{2R} s^{2(N+1)-1} |\partial_s^N F_s(x)|^2 dsdx \\ &\leq (2R)^{n+2\alpha} \|t^{N+1} \partial_t^N F\|_{T_{2,\alpha}^\infty}^2. \end{aligned}$$

If $t \geq R$,

$$\begin{aligned} I_{t,R} &= \int_{B_R} \int_t^{t+R} (s-t)^{2(N+1)-1} |\partial_s^N F_s(x)|^2 dsdx \\ &\leq \frac{R^{2(N+1)-1}}{t^{2(N+1)-1}} \int_{B_R} \int_t^{t+R} s^{2(N+1)-1} |\partial_s^N F_s(x)|^2 dsdx \\ &\leq (2t)^{n+2\alpha} \frac{R^{2(N+1)-1}}{t^{2(N+1)-1}} \|t^{N+1} \partial_t^N F\|_{T_{2,\alpha}^\infty}^2 \\ &\leq (2R)^{n+2\alpha} \|t^{N+1} \partial_t^N F\|_{T_{2,\alpha}^\infty}^2, \end{aligned}$$

where the condition on N is used.

To show the weak-star continuity and convergence, it suffices to test against L^2 functions H supported in compacta of \mathbb{R}_+^{1+n} , as such functions form a dense subspace of T_2^q . Then, as a function of $t \geq 0$, the integral

$$\iint_K (H(s, x) \cdot s^{N+1} \partial_s^N F_{t+s}(x)) \frac{dsdx}{s},$$

is clearly continuous and even C^∞ because $F \in C^\infty(0, \infty; L_{loc}^2)$ for example. \square

The second tool we need is the adapted Hölder theory to define f_t . For $q \leq 1$, we begin with the space $\mathbb{H}_{B^*D}^{1,q}$ as the set of functions $h \in \overline{\mathbb{R}_2(B^*D)} = \mathbb{H}_{B^*D}^2$ with $\tau^{-1} \mathbb{Q}_{\psi, B^*D} h \in T_2^q$, equipped with quasi-norm $\|\tau^{-1} \mathbb{Q}_{\psi, B^*D} h\|_{T_2^q}$. Again, this space

does not depend on the particular choice of ψ among bounded holomorphic functions in bisectors S_μ with enough decay at 0 and ∞ and can be characterized as well by the \mathbb{S}_{ψ, B^*D} maps.

Then, we define $\dot{\Lambda}_{DB}^{\alpha-1}$ as the dual space of $\dot{\mathbb{H}}_{B^*D}^{1,q}$. Arguing as in Proposition 4.7 of [AS] and using $(T_2^q)' = T_{2,\alpha}^\infty$, we see that given any ψ as above, any linear functional ℓ on $\dot{\mathbb{H}}_{B^*D}^{1,q}$ can be written as

$$\ell(\phi_0) = \iint_{\mathbb{R}_+^{1+n}} (\psi(sB^*D)\phi_0(x) \cdot G(s, x)) \frac{dsdx}{s}$$

for some G with $sG \in T_{2,\alpha}^\infty$ and $\|\ell\| \sim \|sG\|_{T_{2,\alpha}^\infty}$. Letting χ_k be the cut-off function as before, replacing G by $\chi_k G$ leads to a T_2^2 function, hence

$$\iint_{\mathbb{R}_+^{1+n}} (\psi(sB^*D)\phi_0(x) \cdot \chi_k(s, x)G(s, x)) \frac{dsdx}{s} = \langle \phi_0, h_k \rangle$$

for some $h_k \in \mathbb{H}_{DB}^2$. Arguing as in [AS], we see that h_k belongs in fact to $\dot{\mathbb{L}}_{DB}^{\alpha-1}$, the pre-Hölder space whose weak-star completion is $\dot{\Lambda}_{DB}^{\alpha-1}$, and that $\dot{\mathbb{L}}_{DB}^{\alpha-1} = \dot{\mathbb{L}}_D^{\alpha-1}$ with equivalent topology because of the assumption on q . As $s\chi_k G$ weakly-star converges to sG in $T_{2,\alpha}^\infty$, this means that h_k weakly-star converges to some $h \in \dot{\Lambda}_D^{\alpha-1}$. Thus we may write $\ell(\phi_0) = \langle \phi_0, h \rangle$ and observe that one can replace ϕ_0 by any $\varphi_0 \in \mathcal{S}$ in the formula.

With this in hand, we can define f_t , and using the $E_\delta^q, E_\delta^{\infty,\alpha}$ duality, run the same computation as above to prove the semigroup formula for f_t in $\dot{\Lambda}_D^{\alpha-1}$ and $\partial_t f_t = \partial_t F_t$ in \mathcal{S}' .

Thus, $F_t = f_t + G$ for some distribution $G \in \mathcal{S}'$. As before, we have $F_{t,\parallel} \rightarrow 0$ in \mathcal{D}' . Now if $\varphi_0 \in \mathcal{S}$ and $\phi_0 = \mathbb{P}_{B^*D}\varphi_0$, the proof of the semigroup formula contains the equalities

$$\langle \varphi_0, f_t \rangle = \langle \phi_0, f_t \rangle = \langle e^{-tB^*D}\chi^+(B^*D)\phi_0, f_0 \rangle.$$

The left entry of the last pairing goes to 0 in $\mathbb{H}_{B^*D}^{1,q}$ (the proof is completely analogous to Proposition 4.6 of [AS] using the square function characterization of the space $\mathbb{H}_{B^*D}^{1,q}$). Thus, $\langle \varphi_0, f_t \rangle$ must tend to 0 as $t \rightarrow \infty$.

We obtain $G_\parallel = 0$ and, as before, $G_\perp = a\sigma$, where σ is some constant in \mathbb{C}^m . Note that since $f_0 \in \dot{\Lambda}_{DB}^{\alpha-1,+}$ we have $tf_0 \in T_{2,\alpha}^\infty$ (see [AS], Lemma 14.4 with \tilde{B} replaced by B) where we use the assumption on q . Using $sG = s(F_s - f_s) \in T_{2,\alpha}^\infty$, we have

$$\int_{B_R} \int_0^R |sG(x)|^2 \frac{dsdx}{s} \leq CR^{n+2\alpha}.$$

But using $\lambda|\sigma| \leq |a(x)\sigma|$ almost everywhere, hence $|G(x)| \geq \lambda|\sigma|$,

$$\int_{B_R} \int_0^R |sG(x)|^2 \frac{dsdx}{s} \geq \lambda^2|\sigma|^2 \frac{R^{n+2}}{2}.$$

It follows that $\lambda^2|\sigma|^2 R^{2-2\alpha} \leq 2C$. Taking $R \rightarrow \infty$ forces $\sigma = 0$ as $\alpha < 1$, and this yields $G = 0$ as desired.

11. PROOF OF COROLLARY 1.2

Let u be a weak solution to $Lu = 0$ on \mathbb{R}_+^{1+n} with $\|\tilde{N}_*(\nabla u)\|_p < \infty$. By (iii) in Theorem 1.1, we have $\nabla_A u(t, \cdot) \in H_D^p$ and $\nabla_A u(t, \cdot) = S_p(t)(\nabla_A u|_{t=0})$ for all $t \geq 0$. As $S_p(t)$ is a continuous semigroup on H_D^p , we obtain the continuity for $t \geq 0$. The

limit at ∞ can be seen from Proposition 4.5 in [AS] and taking bounded extension. The C^∞ regularity can be obtained as in this same proposition. We skip details.

As in the proof of Theorem 1.3, $D : \dot{\mathbb{H}}_{BD}^{1,p} \rightarrow \mathbb{H}_{DB}^p$, where $\dot{\mathbb{H}}_{BD}^{1,p} = \dot{\mathbb{W}}_{BD}^{1,p}$ is $p > 1$, is an isomorphism. Thus it extends to an isomorphism, still denoted by D , $D : \dot{H}_{BD}^{1,p} \rightarrow H_{DB}^p$. We also know that $\mathbb{P} : \dot{\mathbb{H}}_{BD}^{1,p} \rightarrow \dot{\mathbb{H}}_D^{1,p}$ is an isomorphism for $p \in I_L$, thus its extension to the completed spaces is also an isomorphism. Set $v(t, \cdot) = -D^{-1}\nabla_A u(t, \cdot) \in \dot{H}_{BD}^{1,p,+}$ for all $t \geq 0$. Then $t \mapsto v(t, \cdot)$ is uniformly bounded in $\dot{H}_{BD}^{1,p,+}$. Applying \mathbb{P} ,

$$\sup_{t \geq 0} \|\mathbb{P}v(t, \cdot)\|_{\dot{H}_D^{1,p}} \sim \|\mathbb{P}v(0, \cdot)\|_{\dot{H}_D^{1,p}} \sim \|\nabla_A u|_{t=0}\|_{H^p}.$$

In particular, $v_\perp(t, \cdot) \in \dot{H}^{1,p}$ and $\nabla_x v_\perp(t, \cdot) = \nabla_x u(t, \cdot)$, which gives a meaning to $u(t, \cdot) \in \dot{H}^{1,p}$ for all $t \geq 0$ with the above estimate. Note also that $\mathbb{P}v(t, \cdot) \rightarrow \mathbb{P}v(0, \cdot)$ as $t \rightarrow 0$ in $\dot{H}_D^{1,p}$. In particular, $u(t, \cdot) \rightarrow u(0, \cdot)$ in $\dot{H}^{1,p}$.

We next show if $p < n$, that one can select a constant such that $u(t, \cdot) - c \in L^{p^*}$ for all $t \geq 0$. One can write $u|_{t=0} = f + c \in L^{p^*} + \mathbb{C}^m$. At the same time, it is shown in [KP], Theorem 3.2, p.462, that

$$(85) \quad \left| \iint_{W(t,x)} u(s, y) dsdy - u(0, x) \right| \lesssim t\tilde{N}_*(\nabla u)(x)$$

almost everywhere (the proof done for $1 < p$ extends without change to $\frac{n}{n+1} < p$ and it does not use the specificity of real symmetric equations if we replace pointwise values of $u(t, x)$ by Whitney averages as here). Finally, there exists $\tilde{u}(t, \cdot) \in L^{p^*}$ and $c(t) \in \mathbb{C}^m$ such that $u(t, \cdot) = \tilde{u}(t, \cdot) + c(t)$ almost everywhere (t fixed). Thus, we obtain

$$\begin{aligned} \left| \int_{[c_0^{-1}t, c_0t]} c(s) ds - c \right| &= \left| \iint_{W(t,x)} c(s) dsdy - c \right| \\ &\lesssim \left| \iint_{W(t,x)} \tilde{u}(s, y) dsdy - f(x) \right| + t\tilde{N}_*(\nabla u)(x) \end{aligned}$$

almost everywhere. As the right hand side belongs to $L^{p^*} + L^p$, this implies that $\int_{[c_0^{-1}t, c_0t]} c(s) ds - c = 0$ for all $t > 0$, hence $c(t) = c$ for $t > 0$. Having this at hand we have

$$\sup_{t \geq 0} \|\tilde{u}(t, \cdot)\|_{p^*} \lesssim \sup_{t \geq 0} \|\nabla_x u(t, \cdot)\|_{H^p} \lesssim \|\nabla_A u|_{t=0}\|_{H^p}.$$

(Again, $H^p = L^p$ if $p > 1$). On the other hand, as $\tilde{u}(t, \cdot)$ belong to L^{p^*} for $t \geq 0$, Sobolev inequality implies

$$\|\tilde{u}(t, \cdot) - \tilde{u}(s, \cdot)\|_{p^*} \lesssim \|\nabla_x u(t, \cdot) - \nabla_x u(s, \cdot)\|_{H^p}$$

which shows continuity on $t \geq 0$ in L^{p^*} topology. That the limit is 0 when $t \rightarrow \infty$ follows from

$$\|\tilde{u}(t, \cdot)\|_{p^*} \lesssim \|\nabla_x u(t, \cdot)\|_{H^p}.$$

The C^∞ regularity follows from repeated use of the Sobolev inequality for t -derivatives of $\partial_t \tilde{u}$. We have proved all the stated properties for u in the case $p < n$.

In the case $p \geq n$, we do not have to wonder about the constant and barely use the Sobolev embedding $\dot{H}^{1,p} \subset \dot{\Lambda}^s$. Thus the topology on $\dot{\Lambda}^s$ in this argument is the strong topology. Details are easier and left to the reader.

To conclude this proof, we turn to almost everywhere convergence. We begin with the one for u , namely (16). That

$$\lim_{t \rightarrow 0} \iint_{W(t,x)} u(s, y) ds dy = u|_{t=0}(x)$$

follows directly from (85). As

$$\left| \int_{B(x, c_1 t)} u(t, y) dy - \iint_{W(t,x)} u(s, y) ds dy \right| \lesssim t \tilde{N}_*(\nabla u)(x),$$

we also obtain the almost everywhere convergence of slice averages.

We turn to the almost everywhere convergence result for the conormal gradient, namely (15). It is shown in [AS], Theorem 9.9, that for all $h \in \overline{\mathbb{R}_2(DB)} = H_D^2$, for almost every $x_0 \in \mathbb{R}^n$,

$$(86) \quad \lim_{t \rightarrow 0} \iint_{W(t, x_0)} |e^{-s|DB|} h(y) - h(x_0)|^2 ds dy = 0.$$

This holds in particular for any $h \in \mathbb{H}_{DB}^{p,+}$, which is by construction a dense class in $H_{DB}^{p,+}$. Next, we know that $h \mapsto \tilde{N}_*(e^{-s|DB|} h)$ is bounded from $\mathbb{H}_{DB}^{p,+}$ into L^p when $p \in I_L$ by [AS, Theorem 9.1] and this extends by density. Remark that for $p \geq 1$, H^p embeds in L^p , thus the elements in $H_{DB}^{p,+} \subset H_D^p$ are measurable L^p functions. Hence, the classical density argument for almost everywhere convergence allows us to show that (86) extends to all $h \in H_{DB}^{p,+}$ provided we replace $e^{-s|DB|}$ by its extension $S_p(s)$ as usual. This yields (15) applied to $h = \nabla_A u|_{t=0} \in H_{DB}^{p,+}$.

We have the same argument for the slice averages. We skip further details.

Remark 11.1. When $p < 1$, starting from $h \in H_{DB}^p$, the above argument shows that almost everywhere limit of Whitney averages of $S_p(s)h$ exists and defines a measurable function h_0 at almost every $x_0 \in \mathbb{R}^n$. However, this function could be not related to the distribution $h \in H_{DB}^p$. This fact is well known in classical Hardy space theory (see [Ste], p. 127).

12. PROOF OF COROLLARY 1.4

First assume $p = q'$ with $q \in I_{L^*}$ and $q > 1$. The regularity properties and (19) are a consequence of semigroup theory on Banach spaces. We skip details.

Next, we want to show $u = \tilde{u} + c$ on \mathbb{R}_+^{1+n} with $t \mapsto \tilde{u}(t, \cdot) \in C_0([0, \infty); L^p) \cap C^\infty(0, \infty; L^p)$ and $c \in \mathbb{C}^m$. Let $h = \nabla_A u|_{t=0} \in \dot{W}_{DB}^{-1,p,+}$. There exists $\tilde{h} \in H_{BD}^{p,+}$ such that $D\tilde{h} = h$. Here again, we use the extension of the isomorphism $D : \mathbb{H}_{BD}^p \rightarrow \dot{W}_{DB}^{-1,p}$. Then, $v(t, \cdot) := S_{p,BD}(t)\tilde{h}$, where $S_{p,BD}(t)$ is the extension of the semigroup $e^{-t|BD|}$ from \mathbb{H}_{BD}^p to H_{BD}^p , satisfies $Dv(t, \cdot) = \nabla_A u(t, \cdot)$. As $\mathbb{P} : H_{BD}^p \rightarrow H_D^p$ is an isomorphism ([AS], Theorem 4.20) we have that $t \mapsto v_\perp(t, \cdot) \in C_0([0, \infty); L^p) \cap C^\infty(0, \infty; L^p)$. Following the proof of Theorem 9.3 in [AA], we obtain that $u + v_\perp$ is constant on \mathbb{R}_+^{1+n} . So our claim holds with $\tilde{u} = -v_\perp$ (Another possible argument is sketched in [AS], Section 14.1).

Let us see the non-tangential maximal estimate (21) for \tilde{u} . In fact, let $\tilde{h} \in H_{BD}^p$ and approximate by $\tilde{h}_k \in \mathbb{H}_{BD}^p$. This implies $\mathbb{P}\tilde{h}_k \rightarrow \mathbb{P}\tilde{h}$ in L^p by the isomorphism property above. But

$$(e^{-t|BD|}\tilde{h}_k)_\perp = (\mathbb{P}e^{-t|BD|}\tilde{h}_k)_\perp = (\mathbb{P}e^{-t|BD|}\mathbb{P}\tilde{h}_k)_\perp = (e^{-t|BD|}\mathbb{P}\tilde{h}_k)_\perp.$$

Thus [AS], Theorem 9.3, yields in our range of p ,

$$\|\tilde{N}_*(e^{-t|BD|}\tilde{h}_k)_\perp\|_p \lesssim \|\mathbb{P}\tilde{h}_k\|_p.$$

But $e^{-t|BD|}\tilde{h}_k \rightarrow S_{p,BD}(t)\tilde{h}$ in $L^2_{loc}(\mathbb{R}_+^{1+n})$ and $\|\mathbb{P}\tilde{h}_k\|_p \rightarrow \|\mathbb{P}\tilde{h}\|_p$. Thus using Fatou's lemma for arbitrary linearisations of the left hand side, we obtain

$$\|\tilde{N}_*(S_{p,BD}(t)\tilde{h})_\perp\|_p \lesssim \|\mathbb{P}\tilde{h}\|_p.$$

Now taking the element $\tilde{h} \in H_{BD}^{p,+}$ associated to the solution u as above, we have $(S_{p,BD}(t)\tilde{h})_\perp = -\tilde{u}$ and

$$\|\tilde{N}_*\tilde{u}\|_p \lesssim \|\mathbb{P}\tilde{h}\|_p \sim \|\nabla_A u(t, \cdot)\|_{\dot{W}^{-1,p}} \sim \|S(t\nabla u)\|_p$$

where the last comparison is Theorem 1.1. Thus, (21) is proved.

Next, let us see the almost everywhere convergence (22). It suffices to show it for the solution \tilde{u} that we exhibited. By Theorem 9.9 of [AS], we have for $\tilde{h} \in \mathbb{H}_{BD}^2$ and almost every $x_0 \in \mathbb{R}^n$,

$$(87) \quad \lim_{t \rightarrow 0} \iint_{W(t,x_0)} |(e^{-s|BD|}\tilde{h} - \tilde{h})_\perp(x_0)|^2 = 0.$$

As we have the non-tangential maximal estimate $\|\tilde{N}_*(S_{p,BD}(t)\tilde{h})_\perp\|_p \lesssim \|\mathbb{P}\tilde{h}\|_p$ the usual density argument gives the almost everywhere convergence (87) for any $\tilde{h} \in H_{BD}^p$. For slice averages, we also have the maximal estimate and the almost everywhere convergence on a dense class (see Remark 9.7 in [AS]) and we conclude as above. Specializing again to $\tilde{h} \in H_{BD}^{p,+}$ we obtain (22).

We turn to the case $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1) \in [0, 1)$. The regularity for $t \mapsto \nabla_A u(t, \cdot)$ follows from semigroup theory and Theorem 1.3, where the topology on $\dot{\Lambda}^{\alpha-1}$ is the weak-star topology of dual of $\dot{H}^{1,q}$.

Next, in particular, we have $t \mapsto \nabla_x u(t, \cdot) \in C_0([0, \infty); \dot{\Lambda}^{\alpha-1}) \cap C^\infty(0, \infty; \dot{\Lambda}^{\alpha-1})$, thus $t \mapsto u(t, \cdot) \in C_0([0, \infty); \dot{\Lambda}^\alpha) \cap C^\infty(0, \infty; \dot{\Lambda}^\alpha)$, with continuity in the sense of the weak-star topology of dual of H^q and (24) holds.

We finish with the proof of (25). Let u be a solution with $t\nabla u \in T_{2,\alpha}^\infty$ and $u(t, \cdot)$ converges to 0 in \mathcal{D}' modulo constant as $t \rightarrow \infty$. Let $R > 0$ and B_R a ball of radius R in \mathbb{R}^n . Set $T_R = (0, R] \times B_R$. Fix $t \geq 0$. Applying the classical inequality (see [BS])

$$\iint_{T_R} \left| f - \iint_{T_R} f \right|^2 dsdy \leq C \iint_{B_R} \int_0^R |\nabla f(s, y)|^2 dsdy$$

to $f(s, y) = u(t + s, y)$ and using translation invariance, we have that

$$\iint_{(t,0)+T_R} \left| u - \iint_{(t,0)+T_R} u \right|^2 dsdy \leq C \iint_{B_R} \int_0^R |\nabla u(t + s, y)|^2 dsdy.$$

By (18) for $(s, y) \mapsto \nabla_A u(t + s, y)$ and the uniform boundedness of the semigroup $\nabla_A u|_{t=0} \rightarrow \nabla_A u(t, \cdot)$, we have

$$\iint_{B_R} \int_0^R |\nabla u(t + s, y)|^2 dsdy \lesssim R^{2\alpha} \|\nabla_A u(t, \cdot)\|_{\dot{\Lambda}^{\alpha-1}}^2 \lesssim R^{2\alpha} \|\nabla_A u|_{t=0}\|_{\dot{\Lambda}^{\alpha-1}}^2.$$

Hence, we have obtained that

$$\iint_{(t,0)+T_R} \left| u - \iint_{(t,0)+T_R} u \right|^2 dsdy \lesssim R^{2\alpha} \|\nabla_A u|_{t=0}\|_{\dot{\Lambda}^{\alpha-1}}^2.$$

As t and T_R are arbitrary, this is the $BMO(\mathbb{R}_+^{1+n})$ property for u if $\alpha = 0$ and the $\dot{\Lambda}^\alpha(\overline{\mathbb{R}_+^{1+n}})$ property for u by the Morrey-Campanato characterization of $\dot{\Lambda}^\alpha(\overline{\mathbb{R}_+^{1+n}})$ and (25) is proved.

13. SOLVABILITY AND WELL-POSEDNESS RESULTS

To make the game a little more symmetric, we make precise the formal fact that Dirichlet and Regularity problems are the same with different topologies.

Lemma 13.1. *We have $(D)_Y^{L^*} = (R)_{Y^{-1}}^{L^*}$, where the regularity problem $(R)_{Y^{-1}}^{L^*}$ means $L^*u = 0$, $\nabla_x u|_{t=0} \in \dot{Y}^{-1}$, $t\nabla u \in \tilde{\mathcal{T}}$.*

Proof. This is an easy consequence of Corollary 1.4. Any solution in $\tilde{\mathcal{T}}$ satisfies $u = \tilde{u} + c$, where $t \mapsto \tilde{u}(t, \cdot) \in C_0([0, \infty); L^p)$ and $c \in \mathbb{C}^m$. Assume $(R)_{Y^{-1}}^{L^*}$ is well-posed and let $f \in L^p$. Then we can solve $(R)_{Y^{-1}}^{L^*}$ for $g = \nabla_x f \in \dot{W}^{-1,p}$ as boundary data. Then, choose the solution that belongs to $C_0([0, \infty); L^p)$. It solves the Dirichlet problem with convergence in L^p to f . Conversely, assume $(D)_Y^{L^*}$ is well-posed and let $g \in \dot{W}^{-1,p}$. Then, there exists $f \in L^p$ such that $g = \nabla_x f$ in \mathcal{S}' . Solve the Dirichlet problem $(D)_Y^{L^*}$ with data f . Then $\nabla_x u(t, \cdot)$ converges to g in $\dot{W}^{-1,p}$. \square

Proof of Theorem 1.5. We prove the first line as the others are the same. By the lemma above, $(D)_Y^{L^*}$ is well-posed if and only if the map $\tilde{\mathcal{T}}/\mathbb{C}^m$ to $\dot{Y}_\parallel^{-1} : [u + c] \mapsto \nabla_x u|_{t=0}$ is an isomorphism (Here, we use the abuse of notation $u \in \tilde{\mathcal{T}}$ to mean $t\nabla u \in \tilde{\mathcal{T}}$ of the introduction). But this map is composed of $\tilde{\mathcal{T}}/\mathbb{C}^m$ onto $\dot{Y}_{D\tilde{B}}^{-1,+}$ given by $[u + c] \mapsto \nabla_{A^*} u|_{t=0}$, which is an isomorphism by Theorem 1.3, followed by $N_\parallel : \dot{Y}_{D\tilde{B}}^{-1,+} \rightarrow \dot{Y}_\parallel^{-1}$. The equivalence follows: existence is the same as ontoeness of N_\parallel and uniqueness is the same as injectivity of N_\parallel . \square

The following is an extension of a result in [AAH] for pairs of projections in a non-Hilbert context.

Lemma 13.2. *Assume that E is a quasi-Banach space having two pairs on complementary bounded projections P_1^\pm and P_2^\pm . Call E_i^\pm the images of E under P_i^\pm . Then*

- (1) $P_1^- : E_2^+ \rightarrow E_1^-$ has a priori estimates if and only if $P_2^- : E_1^+ \rightarrow E_2^-$ has a priori estimates.
- (2) $P_1^- : E_2^+ \rightarrow E_1^-$ is onto if and only if $P_2^- : E_1^+ \rightarrow E_2^-$ is onto.
- (3) $P_1^- : E_2^+ \rightarrow E_1^-$ is an isomorphism if and only if $P_2^- : E_1^+ \rightarrow E_2^-$ is an isomorphism.

A bounded operator T has a priori estimates if $\|Tf\| \gtrsim \|f\|$ for all f , in other words, it is injective with closed range. One can change (P_2^-, P_1^-) to the other pair (P_2^+, P_1^+) .

In the proof and later on, it will be convenient to consider that if P, Q are two projectors then PQ is automatically restricted to the range of Q and maps into the

range of P : this is how we mean *a priori* estimates, ontoness and isomorphism. With this convention, $P : \mathbf{R}(Q) \rightarrow \mathbf{R}(P)$ is an isomorphism if and only if PQ is an isomorphism.

Proof. The proof is identical to the one in [AAH], Proposition 2.52. We give it for completeness. The final claim is just a symmetry observation. We are left with proving the equivalences. We begin with (1). In fact, $P_1^- : E_2^+ \rightarrow E_1^-$ with *a priori* estimates is equivalent to the transversality of the spaces E_1^+ and E_2^+ , that is $\|x_1 + x_2\| \gtrsim \|x_1\| + \|x_2\|$ when $x_i \in E_i^+$, which is symmetric in the indices $i = 1, 2$. Indeed, assume $P_1^- : E_2^+ \rightarrow E_1^-$ has *a priori* estimates. If $x_2 \in E_2^+$, then

$$\|x_2\| \lesssim \|P_1^- x_2\| = \|P_1^-(x_2 + x_1)\| \lesssim \|x_1 + x_2\|$$

for any $x_1 \in E_1^+$. The transversality follows from the quasi-triangle inequality. Conversely, assume E_1^+ and E_2^+ are transversal. Let $x_2 \in E_2^+$. Set $x_1 = x_2 - P_1^- x_2 \in E_1^+$. Thus

$$\|P_1^- x_2\| = \|x_2 - x_1\| \gtrsim \|x_2\| + \|x_1\| \geq \|x_2\|.$$

This proves the first equivalence.

For (2), by symmetry again, it is enough to show that ontoness of $P_1^- : E_2^+ \rightarrow E_1^-$ is equivalent to $E_1^+ + E_2^+ = E$. Let us see that. Assume $P_1^- : E_2^+ \rightarrow E_1^-$ onto. Let $x \in E$. There exists $x_2 \in E_2^+$ such that $P_1^- x_2 = P_1^- x$. Hence,

$$P_1^+(x - x_2) = P_1^+ x - P_1^+ x_2 = x - P_1^- x - P_1^+ x_2 = x - P_1^- x_2 - P_1^+ x_2 = x - x_2.$$

Thus $x - x_2 \in E_1^+$. Conversely, assume $E_1^+ + E_2^+ = E$. Let $x \in E_1^-$. Decompose $x = x_1 + x_2$ with $x_i \in E_i^+$. Then

$$x = P_1^- x = P_1^- x_1 + P_1^- x_2 = P_1^- x_2.$$

The proof of (3) is the conjunction of (1) and (2). \square

Lemma 13.3. [*Simultaneous duality*] Assume $q \in I_L$. There is a pairing between the spaces X_D and \dot{Y}_D^{-1} for which $\dot{Y}_{D\tilde{B}}^{-1, \mp}$ realizes as the dual space of $X_{D\tilde{B}}^\pm$, \dot{Y}_\perp^{-1} as the dual of X_\parallel and \dot{Y}_\parallel^{-1} as the dual of X_\perp . If $q > 1$, the situation is reflexive so that the pairing is a duality.

Proof. In this proof, $q \in I_L$ all the time. We identify X_\perp with $[X_\perp, 0]^T$ and X_\parallel with $[0, X_\parallel]^T$ so that they becomes subspaces of X_D and form a splitting of X_D . Do the same for \dot{Y}_D^{-1} . So the goal is to build this simultaneous decomposition.

Recall that \tilde{B} is the matrix corresponding to A^* . It is given by $\tilde{B} = NB^*N$ with $N = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = N_\perp - N_\parallel$ according to the identifications. We have to use it here if we want to put in duality the second order operators L and L^* . Hence N will appear in the pairing.

The pairing is given by the following scheme: the article [AS] defines spaces $H_{\tilde{B}D}^{q'}$ and $\dot{\Lambda}_{\tilde{B}D}^\alpha$ and a consequence of the theory there is that D (restricted to $\mathbb{H}_{\tilde{B}D}^2$ as an unbounded operator) extends to isomorphisms $H_{\tilde{B}D}^{q'} \rightarrow \dot{W}_{D\tilde{B}}^{-1, q'} = \dot{W}_D^{-1, q'}$ and $\dot{\Lambda}_{\tilde{B}D}^\alpha \rightarrow \dot{\Lambda}_{D\tilde{B}}^{\alpha-1} = \dot{\Lambda}_D^{\alpha-1}$. Let us keep calling D these maps and write $D^{-1} : \dot{Y}_D^{-1} = \dot{Y}_{D\tilde{B}}^{-1} \rightarrow \dot{Y}_{\tilde{B}D}$. Recall also ([AS], Section 12.2) that $H_{\tilde{B}D}^{q'}$ is the reflexive dual space of $H_{D\tilde{B}}^q = H_D^q$ when $q > 1$ and $\dot{\Lambda}_{\tilde{B}D}^\alpha$ is the dual space of $H_{D\tilde{B}}^q = H_D^q$ when $q \leq 1$ for the duality pairing $\langle f, g \rangle_N = \langle f, Ng \rangle$ where the pairing $\langle f, g \rangle$ extends the standard L^2 inner

product. We can define the desired pairing on X_D, \dot{Y}_D^{-1} by $\langle h, D^{-1}g \rangle_N$ where D^{-1} is defined just above. Let us check the duality statements in the lemma for this pairing.

Call χ_X^\pm the extension of $\chi^\pm(DB)$ on X_D and $\chi_{Y^{-1}}^\pm$ the extension of $\chi^\pm(D\tilde{B})$ on \dot{Y}_D^{-1} . We claim that $\chi_{Y^{-1}}^\mp$ is the adjoint of χ_X^\pm in this pairing. Indeed, working with appropriate functions h, g in dense classes,

$$\begin{aligned} \langle \chi^\pm(DB)h, D^{-1}g \rangle_N &= \langle \chi^\pm(DB)h, ND^{-1}g \rangle = \langle h, \chi^\pm(B^*D)ND^{-1}g \rangle = \\ &= \langle h, N\chi^\mp(\tilde{B}D)D^{-1}g \rangle = \langle h, ND^{-1}\chi^\mp(D\tilde{B})g \rangle = \langle h, D^{-1}\chi^\mp(D\tilde{B})g \rangle_N. \end{aligned}$$

The change from \pm to \mp comes from the anti-commutation $ND = -DN$ (see [AS], Section 12.2). It follows that the splitting $\dot{Y}_D^{-1} = \dot{Y}_{D\tilde{B}}^{-1,-} \oplus \dot{Y}_{D\tilde{B}}^{-1,+}$ is adjoint of $X_D = X_{D\tilde{B}}^+ \oplus X_{D\tilde{B}}^-$ for this pairing.

At the same time, it was proved that $\mathbb{P} : \mathbb{H}_{\tilde{B}D}^2 \rightarrow \mathbb{H}_D^2$ (where we recall that \mathbb{P} is the orthogonal projection onto \mathbb{H}_D^2) extends to an isomorphism $\dot{Y}_{\tilde{B}D} \rightarrow \dot{Y}_D$, which we denote again by \mathbb{P} . As (the extension of) \mathbb{P} preserves X_D and commutes with N , we have

$$\langle h, D^{-1}g \rangle_N = \langle h, \mathbb{P}D^{-1}g \rangle_N.$$

As N preserves scalar and tangential spaces, while D swaps them, using the pairing $\langle h, \mathbb{P}D^{-1}g \rangle_N$, it is easy to see \dot{Y}_\perp^{-1} as the dual of X_\parallel and \dot{Y}_\parallel^{-1} as the dual of X_\perp . Also the splitting $\dot{Y}_D^{-1} = \dot{Y}_\parallel^{-1} \oplus \dot{Y}_\perp^{-1}$ is adjoint of $X_D = X_\perp \oplus X_\parallel$ for this pairing. \square

Proof of Theorem 1.6. Let us prove the direction from $(R)_X^L$ is well-posed to $(D)_Y^{L*}$ is well-posed. Because of Theorem 1.5, it is enough to argue on the maps N_\parallel in each context. We have the situation of Lemma 13.2 for $E = X_D$ with $P_2^\pm = \chi_X^\pm$, $P_1^- = N_\parallel$ and $P_1^+ = N_\perp$. We assume here that $P_1^- : E_2^+ \rightarrow E_1^-$ is an isomorphism. Thus, $P_2^- : E_1^+ \rightarrow E_2^-$ is an isomorphism, which concretely means $\chi_X^- : X_\perp \rightarrow \dot{X}_{D\tilde{B}}^-$ is an isomorphism, or equivalently that $\chi_X^- N_\perp$ is an isomorphism (recall that it is understood that this is from the range of N_\perp onto the range of χ_X^-). By taking adjoint for the pairing of the lemma above, we have that $N_\parallel \chi_{Y^{-1}}^+ = (\chi_X^- N_\perp)^*$ is an isomorphism. This precisely means that $N_\parallel : \dot{Y}_{D\tilde{B}}^{-1,+} \rightarrow \dot{Y}_\parallel^{-1}$ is an isomorphism. The other implications all have the same proof. \square

As said in the Introduction, we can recover the Green's formula from such abstract considerations: in the pairing of $\langle f, g \rangle_N$ of Lemma 13.3, one can prove that the polar set of $X_{D\tilde{B}}^+$ is precisely $\dot{Y}_{D\tilde{B}}^{-1,+}$. The orthogonality equation expressing this fact is the Green's formula. It is also possible to prove it directly, which we do for convenience.

Proof of Theorem 1.7. By Theorems 1.1 and 1.3, we know that $h = \nabla_A u|_{t=0} \in H_{D\tilde{B}}^{q,+} \subset H_D^q$ and $g = \nabla_{A^*} w|_{t=0} \in \dot{W}_{D\tilde{B}}^{-1,q',+} \subset \dot{W}_D^{-1,q'}$ or $\dot{\Lambda}_{D\tilde{B}}^{\alpha-1,+} \subset \dot{\Lambda}_D^{\alpha-1}$. Using standard approximation theory, we can approximate h in H^q topology by functions \tilde{h}_k in the Schwartz class with compactly supported Fourier transform away from the origin. Then, applying the bounded projection \mathbb{P} , $\mathbb{P}\tilde{h}_k \in H_D^q$ approximate $h = \mathbb{P}h$ as well. Note that $\mathbb{P}\tilde{h}_k \in \mathbb{H}_D^2$ as well as $\dot{\mathcal{H}}_D^{-1/2}$. Applying the projection $\chi^+(DB)$, we have obtained an approximation $h_k \in \mathbb{H}_{D\tilde{B}}^{q,+} \cap \dot{\mathcal{H}}_{D\tilde{B}}^{-1/2,+}$, with $\|h_k - h\|_{H^q} \rightarrow 0$.

Similarly, if $q > 1$, we can find an approximation $g_k \in \dot{W}_{D\tilde{B}}^{-1,q',+} \cap \dot{\mathcal{H}}_{D\tilde{B}}^{-1/2,+}$ with $\|g_k - g\|_{W^{-1,q'}} \rightarrow 0$. If $q \leq 1$, $g_k \in \dot{\Lambda}_{D\tilde{B}}^{\alpha-1,+} \cap \dot{\mathcal{H}}_{D\tilde{B}}^{-1/2,+}$ with $g_k \rightarrow g$ for the weak-star

topology on $\dot{\Lambda}^{\alpha-1}$. Remark that $h_k = \nabla_A u_k|_{t=0}$ where u_k is an energy solution of L and $g_k = \nabla_{A^*} w_k|_{t=0}$ where w_k is an energy solution of L^* . Thus, we have the Green's formula for energy solutions (see [AM], Lemma 2.1, in this context),

$$\langle \partial_{\nu_A} u_k|_{t=0}, w_k|_{t=0} \rangle = \langle u_k|_{t=0}, \partial_{\nu_{A^*}} w_k|_{t=0} \rangle.$$

The pairings are for the homogeneous Sobolev spaces $\dot{H}^{-1/2}$, $\dot{H}^{1/2}$ and $\dot{H}^{1/2}$, $\dot{H}^{-1/2}$. But as said, the different notions of conormal gradients are compatible. Assume $q > 1$ first. We have

$$\begin{aligned} \partial_{\nu_A} u_k|_{t=0} &\rightarrow \partial_{\nu_A} u|_{t=0} \text{ in } L^q, \\ \nabla_x w_k|_{t=0} &\rightarrow \nabla_x w|_{t=0} \text{ in } \dot{W}^{-1,p}. \end{aligned}$$

We can arrange (see Corollary 1.4) that both $w_k|_{t=0}$ and $w|_{t=0}$ are in L^p and it follows that

$$w_k|_{t=0} \rightarrow w|_{t=0} \text{ in } L^p.$$

Having fixed the constant, the first pairing can be reinterpreted in the L^q, L^p duality and converges to $\langle \partial_{\nu_A} u|_{t=0}, w|_{t=0} \rangle$. For the second one, we have

$$\begin{aligned} u_k|_{t=0} &\rightarrow u|_{t=0} \text{ in } \dot{W}^{1,q}, \\ \partial_{\nu_{A^*}} w_k|_{t=0} &\rightarrow \partial_{\nu_{A^*}} w|_{t=0} \text{ in } \dot{W}^{-1,p}, \end{aligned}$$

and the pairing, reinterpreted in the $\dot{W}^{1,q}, \dot{W}^{-1,p}$ duality, converges to $\langle u|_{t=0}, \partial_{\nu_{A^*}} w|_{t=0} \rangle$.

If $q \leq 1$, then

$$\begin{aligned} \partial_{\nu_A} u_k|_{t=0} &\rightarrow \partial_{\nu_A} u|_{t=0} \text{ in } H^q, \\ w_k|_{t=0} &\rightarrow w|_{t=0} \text{ weakly - star in } \dot{\Lambda}^\alpha. \end{aligned}$$

(see Corollary 1.4.)

Thus the first pairing can be reinterpreted in the $H^q, \dot{\Lambda}^\alpha$ duality and converges to $\langle \partial_{\nu_A} u|_{t=0}, w|_{t=0} \rangle$. For the second one, we have

$$\begin{aligned} \nabla_x u_k|_{t=0} &\rightarrow \nabla_x u|_{t=0} \text{ in } H^q, \\ \partial_{\nu_{A^*}} w_k|_{t=0} &\rightarrow \partial_{\nu_{A^*}} w|_{t=0} \text{ weakly - star in } \dot{\Lambda}^{\alpha-1}, \end{aligned}$$

and the pairing, reinterpreted in the $\dot{H}^{1,q}, \dot{\Lambda}^{\alpha-1}$ duality, converges to $\langle u|_{t=0}, \partial_{\nu_{A^*}} w|_{t=0} \rangle$. \square

Proof of Theorem 1.8. It is shown in [AS] (Lemma 14.2, 14.3, 14.5, 14.6) that for each of the four problems, solvability for the energy class implies that the attached map N_{\parallel} or N_{\perp} is an isomorphism from the corresponding trace space onto the space of boundary data. By Theorem 1.5, this implies well-posedness. When the data is an ‘‘energy data’’, our assumption is that the energy solution also solves the same problem. By the just obtained uniqueness, it is the only one. This yields compatible well-posedness. \square

Proof of Theorem 1.9. Let us prove for example the second item as any other one is the same. Assume $(D)_{\dot{Y}}^{L^*}$ is solvable for the energy class.

We first show solvability for the energy class of $(R)_{\dot{X}}^L$. Let $g \in X_{\parallel}^+ \cap \dot{\mathcal{H}}_{\parallel}^{-1/2}$ and consider the energy solution u attached to g (Dirichlet problem). We want to show that $\|\partial_{\nu_A} u|_{t=0}\|_{X_{\perp}} \lesssim \|g\|_{X_{\parallel}}$. Consider a test function $\varphi \in \mathcal{S}$ and the energy solution of the Dirichlet problem $(D)_{\dot{Y}}^{L^*}$ for this data. This means that if w denotes this

energy solution, we have $\|\partial_{\nu_{A^*}} w|_{t=0}\|_{\dot{Y}_\perp^{-1}} \lesssim \|\nabla \varphi\|_{\dot{Y}_\parallel^{-1}} \lesssim \|\varphi\|_{\dot{Y}_\perp}$. Applying (26), we have

$$\langle \partial_{\nu_A} u|_{t=0}, \varphi \rangle = \langle \partial_{\nu_A} u|_{t=0}, w|_{t=0} \rangle = \langle u|_{t=0}, \partial_{\nu_{A^*}} w|_{t=0} \rangle$$

hence

$$|\langle \partial_{\nu_A} u|_{t=0}, \varphi \rangle| \leq \|g\|_{X_\parallel} \|\partial_{\nu_{A^*}} w|_{t=0}\|_{\dot{Y}_\perp^{-1}} \lesssim \|g\|_{X_\parallel} \|\varphi\|_{\dot{Y}_\perp}.$$

Thus, by density of test functions in $L^{q'}$ if $q > 1$, and in VMO (the closure of C_0^∞ for the BMO norm) if $q = 1$, and dualities $L^{q'} - L^q$ and $VMO - H^1$, we have $\partial_{\nu_A} u|_{t=0} \in X_\perp$ as desired.

Let us now prove the uniqueness of $(R)_{X_\perp}^L$. Assume u is any solution with $\nabla u \in \mathcal{N}$ and $\nabla_x u|_{t=0} = 0$. We know that $\partial_{\nu_A} u|_{t=0} \in X_\perp$ and want to show it is 0, so that $u = 0$. We may apply (26) against the same w as before. This time, we obtain $\langle \partial_{\nu_A} u|_{t=0}, \varphi \rangle = 0$ for all $\varphi \in \mathcal{S}$, hence $\partial_{\nu_A} u|_{t=0} = 0$. \square

Remark 13.4. In the previous argument, if $q < 1$, then $\partial_{\nu_A} u|_{t=0} \in \dot{\mathcal{H}}^{-1/2}$, hence it is a Schwartz distribution and it belongs to the dual of the closure of test functions in $\dot{\Lambda}^s$.

Remark 13.5. As in [AM] or [HKMP2], this argument only uses solvability for the energy class **with compactly supported smooth data**. Thus combining (1) and (2) of Theorem 1.9 when $q = 1$ shows that if $(D)_{Y_\perp}^{L^*}$ is solvable for the energy class for any VMO data implies compatible well-posedness (and in particular existence of a solution) for any BMO data.

We continue with the

Proof of Corollary 1.10. As mentioned in Introduction, [HKMP1] implies $(D)_{L^p}^{L^*}$ is solvable for the energy class for some $p \geq 2$ when the coefficients are real. Thus $(R)_{L^{p'}}^L$ is solvable for the energy class. In the case of real equations, we have the De Giorgi-Nash-Moser assumptions on L and its adjoint and also on the reflected operator and its adjoint across \mathbb{R}^n because all four have real coefficients. Thus we may use Theorem 10.1 in [AM] so that $(R)_{L^q}^L$ for $1 < q < p'$ and $(R)_{H^q}^L$ are solvable for the energy class and the lower bound on q is determined by the common De Giorgi exponent of all four operators. Comparison of assumptions show that this exponent is larger than or equal to the exponent p_\parallel found in [AS], Section 13. In particular, all the exponents q here belong to I_L . Thus, we may apply Theorem 1.8 and compatible well-posedness follows.

Applying now Theorem 1.9 settles the Dirichlet problem for L .

When we perturb this situation, we may apply Theorem 14.8 in [AS], whose proof gives in fact invertibility of the N_\parallel map and Theorem 1.5, (2), yields well-posedness of the regularity problem for L^* in the same range of exponents q . Applying Theorem 1.6 gives us the dual range of the Dirichlet problem for L . \square

Let us finish with the proof of Theorem 1.11. There is a standard argument that uses jump relations that we have here. For example, adapt the proof of [HKMP2]. Instead, we exploit the pairs of projections with a complementary result.

Lemma 13.6. *Assume that E is a quasi-Banach space having two pairs of complementary bounded projections P_1^\pm and P_2^\pm . Then $P_1^- P_2^+ P_1^+$ is an isomorphism if and only if the two operators $P_1^- P_2^+$ and $P_1^- P_2^-$ are isomorphisms.*

We adopt here as well the convention that product PQR of projectors are restricted to the range of R into the range of P .

Proof. Remark that $P_1^- P_2^- P_1^+ = P_1^- (I - P_2^+) P_1^+ = -P_1^- P_2^+ P_1^+$. So the first assertion is equivalent to the two operators $P_1^- P_2^\pm P_1^+$ are isomorphisms.

Assume that the two operators $P_1^- P_2^\pm P_1^+$ are isomorphisms. First $P_1^- P_2^+$ is onto and $P_2^- P_1^+$ has *a priori* estimates. By Lemma 13.2, (a), this means that $P_1^- P_2^+$ also has *a priori* estimates. Thus, it is an isomorphism. One does analogously with $P_1^- P_2^-$.

Conversely, assume the two operators $P_1^- P_2^+$ and $P_1^- P_2^-$ are isomorphisms. By Lemma 13.2, (c), $P_2^+ P_1^+$ is also an isomorphism. Thus, $P_1^- P_2^+ P_1^+ = P_1^- P_2^+ P_2^+ P_1^+$ is an isomorphism. \square

Proof of Theorem 1.11. Let us prove (1). Exactly as for the upper half-space, well-posedness of $(R)_X^L$ on the lower half-space is equivalent to $N_\parallel : X_{DB}^- \rightarrow X_\parallel$ is an isomorphism. Thus simultaneous well-posedness is equivalent to both $N_\parallel : X_{DB}^\pm \rightarrow X_\parallel$ being isomorphisms, thus to $N_\parallel \chi_X^\pm$ being isomorphisms (identifying again the \perp and \parallel spaces as subspaces and using our convention for product of projections), where χ_X^\pm are the continuous extensions of $\chi^\pm(DB)$ to X_{DB} . By the previous lemma, this is equivalent to $N_\parallel \chi_X^\pm N_\perp$ being an isomorphism. We claim this is exactly saying that the single layer potential \mathcal{S}_0^L is invertible from X onto \dot{X}^1 .

Indeed, the single layer $\mathcal{S}_{0\pm}^L f$ using the DB method is defined in [R1] and is shown to coincide with the usual definition on appropriate functions f and when $Lu = 0$ is a real equation for instance. See [AS], Section 12.3. In fact, by density (see Theorem 2.6 in [AS]), one can see that for $f \in X$, as $\begin{bmatrix} f \\ 0 \end{bmatrix} \in X_D = X_{DB}$,

$$\nabla_A \mathcal{S}_{0\pm}^L f = \pm \chi_X^\pm \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 0 \\ \nabla_x \mathcal{S}_{0\pm}^L f \end{bmatrix} = \pm N_\parallel \chi_X^\pm N_\perp h$$

for any $h \in X_{DB}$ such that $h_\perp = f$, for example $\begin{bmatrix} f \\ 0 \end{bmatrix}$. Note that the coincidence of $\nabla_x \mathcal{S}_{0+}^L$ and $\nabla_x \mathcal{S}_{0-}^L$ is exactly the formula $N_\parallel \chi_X^+ N_\perp = -N_\parallel \chi_X^- N_\perp$, so that we may remove the signs symbols.

Thus $N_\parallel \chi_X^+ N_\perp$ being an isomorphism is equivalent to $\nabla_x \mathcal{S}_0^L$ being an isomorphism from X_\perp onto X_\parallel , that is, \mathcal{S}_0^L is an isomorphism from X onto \dot{X}^1 .

Let us turn to the proof of (2). As above, the well-posedness of $(D)_Y^{L*}$ is equivalent to $N_\parallel \chi_Y^+ N_\perp$ being an isomorphism, with obvious notation. As $N_\parallel \chi_Y^+ N_\perp = -N_\parallel \chi_Y^- N_\perp$ (jump relation), this is equivalent to $N_\parallel \chi_Y^- N_\perp$ being an isomorphism. We claim this is exactly saying that the single layer potential \mathcal{S}_0^{L*} is invertible from \dot{Y}^{-1} onto Y .

In fact, by density from Lemma 8.1 in [AM], one can see that for $f \in \dot{Y}^{-1}$, as $\begin{bmatrix} f \\ 0 \end{bmatrix} \in \dot{Y}_D^{-1} = \dot{Y}_{DB}^{-1}$,

$$\nabla_{A^*} \mathcal{S}_{0\pm}^{L*} f = \pm \chi_Y^\pm \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 0 \\ \nabla_x \mathcal{S}_{0\pm}^{L^*} f \end{bmatrix} = \pm N_{\parallel} \chi_{Y^{-1}}^{\pm} N_{\perp} h$$

for any $h \in \dot{Y}_{DB}^{-1}$ such that $h_{\perp} = f$, for example $\begin{bmatrix} f \\ 0 \end{bmatrix}$. Again, $\nabla_x \mathcal{S}_{0+}^{L^*}$ and $\nabla_x \mathcal{S}_{0-}^{L^*}$ agree, and $N_{\parallel} \chi_{Y^{-1}}^+ N_{\perp}$ being an isomorphism is equivalent to $\mathcal{S}_0^{L^*}$ being an isomorphism from \dot{Y}^{-1} onto Y .

We continue with the proof of (3). As for (1), well-posedness of $(N)_{\dot{X}}^L$ on the lower half-space is equivalent to $N_{\perp} : X_{DB}^- \rightarrow X_{\perp}$ being an isomorphism. Thus simultaneous well-posedness on both half-spaces is equivalent to $N_{\perp} \chi_X^+ N_{\parallel}$ being an isomorphism. We claim this is exactly saying that the operator $\partial_{\nu_A} \mathcal{D}_0^L$ is invertible from \dot{X}^1 onto X .

In fact, by density from Lemma 8.1 in [AM], one can see that for $f \in \dot{X}^1$, as $\begin{bmatrix} 0 \\ \nabla_x f \end{bmatrix} \in X_D = X_{DB}$,

$$\nabla_A \mathcal{D}_{0\pm}^L f = \mp \chi_X^{\pm} \begin{bmatrix} 0 \\ \nabla_x f \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} \partial_{\nu_A} \mathcal{D}_{0\pm}^L f \\ 0 \end{bmatrix} = \mp N_{\perp} \chi_X^{\pm} N_{\parallel} h$$

for any $h \in X_{DB}$ such that $h_{\perp} = f$, for example $\begin{bmatrix} 0 \\ \nabla_x f \end{bmatrix}$. Note that the coincidence of $\partial_{\nu_A} \mathcal{D}_{0+}^L$ and $\partial_{\nu_A} \mathcal{D}_{0-}^L$ is exactly the formula $N_{\perp} \chi_X^+ N_{\parallel} = -N_{\perp} \chi_X^- N_{\parallel}$, so that we may remove the signs symbols.

Thus $N_{\perp} \chi_X^+ N_{\parallel}$ being an isomorphism is equivalent to $\partial_{\nu_A} \mathcal{D}_0^L$ being an isomorphism from \dot{X}^1 onto X .

The proof of (4) is exactly similar to the other ones and we skip it. \square

14. SPECIFIC SITUATIONS

We want to comment and illustrate our results in two cases.

14.1. Constant coefficients. We assume that L has constant coefficients given by A . We observe that there are many different choices for A but this does not affect the results in this section. We know from [AS] that $I_L = (\frac{n}{n+1}, \infty)$. A reformulation in the current setup of the results in [AAMc], Section 4.3, establishes the invertibility of the four maps of Theorem 1.5 when $X = L^2$ and $Y = L^2$. Thus, Dirichlet problem, regularity problem and Neumann problem (with any given choice of A representing L) are well-posed (with our current understanding of it) in L^2 . Applying this result using (2) and then (4) in Theorem 1.5, we have that the map $N_{\perp} N_{\parallel}^{-1} : L_{\parallel}^2 \rightarrow L_{\perp}^2$ is bounded and invertible. But by taking the Fourier transform, this map and its inverse are Fourier multipliers with homogeneous symbols of degree 0, smooth away from the origin. Thus they are classical singular integral operators and, as a consequence, bounded on any of the spaces X or \dot{Y}^{-1} corresponding to I_L , and also for the space $\dot{W}^{-1/2,2}$ corresponding to energy solutions in which they coincide with the Dirichlet to Neumann map and its inverse (another way is to observe that examination of the proof in [AAMc] shows that directly). This implies from

our general results (Theorem 1.8) that all possible problems are compatibly well-posed on the upper half-space (and, of course, on the lower half-space by a similar argument).

14.2. **Block case.** In the block case, we have $L = -\operatorname{div}A\nabla$ with

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

that is, A is block diagonal. In this case, B is also block diagonal with

$$B = \begin{bmatrix} a^{-1} & 0 \\ 0 & d \end{bmatrix}.$$

The Hardy space theory for DB was described in [AS] in terms of the operator $E = -\operatorname{div}d\nabla a^{-1}$ (we change notation because L is here our second order operator) and it is proved that $\|\sqrt{E}(af)\|_{H^p} \sim \|\nabla_x f\|_{H^p}$ for any $p \in I_L$ and $f \in \dot{W}^{1,2}$, where $H^p = L^p$ if $p > 1$. The case $p = 2$, is the consequence of [AKMc]. A little more work gives the description of the spectral Hardy spaces by

$$H_{DB}^{2,\pm} = \{[\mp\sqrt{E}(af), \nabla_x f]^T; f \in \dot{W}^{1,2}\}.$$

In this case, we thus know the *a priori* inequalities for the energy solution which correspond to $\nabla_x f \in \dot{W}^{-1/2,2}$ or $\sqrt{E}(af) \in \dot{W}^{-1/2,2}$. These two conditions are equivalent because one can interpolate the isomorphisms $\sqrt{E}a : \dot{W}^{1,2} \rightarrow L^2$ and $\sqrt{E}a : L^2 \rightarrow \dot{W}^{-1,2}$, where the second is the dual statement of the first one for E^* and similarity. Thus, for all energy solutions u , we have *a priori* in the range of p above,

$$\|\partial_{\nu_A} u|_{t=0}\|_p \sim \|\nabla_x u|_{t=0}\|_p.$$

This means that the Neumann and regularity problems are solvable in L^p/H^p for the energy class for any $p \in I_L$ (see [AM] for statements), thus compatibly well-posed in this range by Theorem 1.8. By Theorem 1.9, we have the dual compatibly well-posed Dirichlet and Neumann problems for L^* . We leave details to the reader.

Remark 14.1. An interesting observation concerns Theorem 4.1 in [Ma]. There, it is assumed that $a = 1$ and it is proved a non-tangential maximal estimate for $u(t, \cdot) = e^{-t\sqrt{E}}f$ with $f \in \dot{W}^{1,2}$,

$$\|\tilde{N}_*(\nabla u)\|_p \lesssim \|\nabla_x f\|_p,$$

for values of p smaller than the lower bound of I_L in case $\inf I_L > 1$. Looking at the algebra, $\nabla_A u = \nabla u = e^{-t|DB|}h$ with $h = [\mp\sqrt{E}f, \nabla_x f]^T$ so this agrees with our way of constructing solutions. The maximal estimate uses the fact that $\|\sqrt{E}f\|_p \lesssim \|\nabla_x f\|_p$ may hold whereas the opposite inequality fails ([A]). By the results of [KP], one knows at least weak convergence in L^p of averages in time for ∇u towards h under the non-tangential maximal control. So this is a solution of a slightly modified regularity problem in L^p for these values of p .

15. PROOFS OF TECHNICAL LEMMAS

The following is the key lemma.

Lemma 15.1. *Let T be any operator of the form DB or BD . For each integer N , there exists $\phi^\pm \in H^\infty(S_\mu)$ with the following properties:*

- (1) $\phi^\pm(sT)$ is uniformly bounded on L^2 .
- (2) $\phi^\pm(sT)$ coincides with $e^{-s|T|}$ on $\mathbb{H}_T^{2,\pm}$.
- (3) $(\phi^\pm(sT))_{s>0}$ has L^2 off-diagonal decay of order N .
- (4) $\phi^\pm(sT) \rightarrow I$ strongly in L^2 as $s \rightarrow 0$.

Here, S_μ is an open bisector with angle $\omega < \mu < \pi/2$.

Proof. As in Section 5 in [AS], one can find coefficients c_m such that

$$\psi^+(z) := e^{-z} - \sum_{m=1}^N c_m (1 + imz)^{-k} = O(z^N)$$

near 0. For $z \in \mathbb{C}$, set

$$\phi^+(z) := \sum_{m=1}^M c_m (1 + imz)^{-k} + \psi^+(z)\chi^+(z).$$

One can easily see that $\phi^+ \in H^\infty(S_\mu)$.

Next, the resolvents $(I + ismT)^{-1}$ are uniformly bounded on L^2 with respect to s , while $\psi^+\chi^+ \in \Psi(S_\mu)$, hence $(\psi^+\chi^+)(sT)$ is defined on L^2 and its operator norm is bounded by $\|\psi^+\chi^+\|_\infty$ by the H^∞ -calculus of T on L^2 . This proves (1).

To see (2), it is enough to show that $\phi^+(z)\chi^+(z) = e^{-z}\chi^+(z)$. But this is immediate by construction.

Next, the resolvent of T has L^2 off-diagonal decay of any order ([AS], Lemma 2.3), while $\psi^+\chi^+ \in \Psi_N^1(S_\mu)$, hence it has L^2 off-diagonal decay of order N ([AS], Proposition 3.13). Thus (3) holds.

Finally, $\phi^+(sz)$ converges uniformly to 1 on compact subsets of S_μ as $s \rightarrow 0$. By the H^∞ -calculus on $\overline{\mathbb{R}_2(T)}$, this implies that $\phi^+(sT)$ converges strongly to I as $s \rightarrow 0$ on $\overline{\mathbb{R}_2(T)}$. On $\mathbb{N}_2(T)$, we have $(\psi^+\chi^+)(sT) = 0$, hence $\phi^+(sT) = (\sum c_m)I = I$. Thus (4) is proved.

The proof is the same to get ϕ^- , picking (different) coefficients such that

$$\psi^-(z) = e^z - \sum_{m=1}^N c'_m (1 + imz)^{-k} = O(z^N)$$

near 0. □

Proof of Lemma 8.9. Let $h \in \mathbb{H}_{B^*D}^{p',+} \cap E_\delta^{p'}$. Fix $\delta > 0$. Then $e^{-s|B^*D|h} \in E_\delta^{p'}$ using Lemma 8.8. It follows from Lemma 15.1 that $e^{-s|B^*D|h} = T_s h$, where $T_s = \phi^+(sB^*D)$ has L^2 off-diagonal decay of order N with N at our disposal and $T_s \rightarrow I$ in L^2 strongly as $s \rightarrow 0$. By Proposition 4.4, we conclude if $N > \inf(n/p', n/2)$ that $e^{-s|B^*D|h}$ converges to h in $E_\delta^{p'}$.

The argument for $h \in \mathbb{H}_{B^*D}^{p',-} \cap E_\delta^{p'}$ is the same. □

Proof of Lemma 8.10. If $\varphi_0 \in \mathcal{S}$, then $\phi_0 = \mathbb{P}_{B^*D}\varphi_0 \in \mathbb{H}_{B^*D}^2$. Also $\mathbb{P}\phi_0 = \mathbb{P}\varphi_0 \in L^{p'}$. Thus $\phi_0 \in \mathbb{H}_{B^*D}^{p'}$ by [AS], Theorem 4.20 and our range for p' . Since $\phi_0 - \varphi_0 \in \mathbb{N}_2(D)$, $D\phi_0 = D\varphi_0$. Thus $B^*D\phi_0 = B^*D\varphi_0$ and the latter is of course in $\mathbb{H}_{B^*D}^2 = \overline{\mathbb{R}_2(B^*D)}$ and in $L^{p'}$, thus $\mathbb{P}B^*D\varphi_0 \in L^{p'}$ and, again, $B^*D\varphi_0 \in \mathbb{H}_{B^*D}^{p'}$. Thus, $\chi^\pm(B^*D)B^*D\phi_0 \in \mathbb{H}_{B^*D}^{p',\pm}$ by the H^∞ -calculus on this space for B^*D .

It remains to prove that $\chi^\pm(B^*D)B^*D\varphi_0 \in E_\delta^{p'}$. As $B^*D\varphi_0 = \chi^+(B^*D)B^*D\varphi_0 + \chi^-(B^*D)B^*D\varphi_0$ and the left hand side is clearly in $E_\delta^{p'}$ (for example, $D\varphi_0 \in \mathcal{S}$

so contained in $E_\delta^{p'}$ and multiplication by B^* preserves $E_\delta^{p'}$, it suffices to do it for $\chi^+(B^*D)B^*D\varphi_0$. Pick $\psi \in \Psi_N^N(S_\mu)$ with N large enough and $\int_0^\infty \psi(sz) \frac{ds}{s} = 1$ for all $z \in S_\mu$. Set $\phi(z) = \int_1^\infty \psi(sz) \frac{ds}{s}$. Thus $\phi \in H^\infty(S_\mu)$, $\phi(z) - 1 = O(z^N)$ and $\phi(z) = O(z^{-N})$. Set $h = \chi^+(B^*D)B^*D\varphi_0$. Then by H^∞ -calculus on L^2 , and as $h \in \overline{\mathbb{R}_2(B^*D)}$, we have

$$h = \int_0^\delta (\chi^+\psi)(sB^*D)(B^*D\varphi_0) \frac{ds}{s} + \phi(\delta B^*D)h.$$

As in Lemma 8.8, we have $\phi(\delta B^*D)h \in E_\delta^{p'}$. For the other term, we notice that $\chi^+\psi \in \Psi_N^N(S_\mu)$, so $(\chi^+\psi)(sB^*D)$ has L^2 off-diagonal estimates of order N . Using this with N large, L^2 boundedness of the operator $\int_0^\delta (\chi^+\psi)(sB^*D) \frac{ds}{s}$ and the fact that $B^*D\varphi_0$ satisfies

$$\int_{\mathbb{R}^n} |B^*D\varphi_0(x)|^2 \langle x/\delta \rangle^{-M} dx < \infty$$

for any $M > 0$, it is not hard, though a little tedious, to check that the function $g = \int_0^\delta (\chi^+\psi)(B^*D)(B^*D\varphi_0) \frac{ds}{s}$ satisfies the estimate

$$\int_{\mathbb{R}^n} |g(x)|^2 \langle x/\delta \rangle^{-2N} dx < \infty.$$

It follows that for all $x \in \mathbb{R}^n$,

$$\left(\int_{B(x,\delta)} |g|^2 \right)^{1/2} \lesssim \delta^{-n/2} \langle x/\delta \rangle^{-N},$$

which is $L^{p'}$ as a function of x when $Np' > 1$, in which case $g \in E_\delta^{p'}$. We skip further details. \square

Proof of Lemma 10.5. Recall that $q \in I_{L^*}$. Clearly $\varphi_0 \in \mathcal{S}$ implies $\varphi_0 \in \mathbb{D}_2(D)$ and $D\varphi_0 \in \mathbb{H}_D^q = \mathbb{H}_{DB^*}^q$ given the description of \mathbb{H}_D^q and the value of q . That $\chi^\pm(DB^*)D\varphi_0 \in E_\delta^q$ is trivial if $q \geq 2$. Indeed, $\chi^\pm(DB^*)D\varphi_0 \in L^q \subset E_\delta^q$ in that case. If $q < 2$, then one adapts the argument just given in the proof of Lemma 8.10 above. \square

Proof of Lemma 10.7. Notice that we stated the lemma for $q > 1$ using $\dot{\mathbb{W}}_{B^*D}^{1,q}$ but the lemma and its proof are valid with $q \leq 1$ using the space $\dot{\mathbb{H}}_{B^*D}^{1,q}$ instead. We prove both cases simultaneously. Let us first see the inclusion. Let $\phi_0 \in \mathbb{D}_q \cap \mathbb{H}_{B^*D}^2$. We have to show that $\tau^{-1}\psi(\tau B^*D)\phi_0 \in T_2^q$ where $\psi \in \Psi(S_\mu)$ has sufficiently large decay at 0 and infinity. Writing $\psi(z) = z\tilde{\psi}(z)$, we have

$$\tau^{-1}\psi(\tau B^*D)\phi_0 = B^*\tilde{\psi}(\tau DB^*)D\phi_0.$$

As $D\phi_0 \in \mathbb{H}_D^q = \mathbb{H}_{DB^*}^q$, the tent space characterization of $\mathbb{H}_{DB^*}^q$, yields $\tilde{\psi}(\tau DB^*)D\phi_0 \in T_2^q$. The boundedness of B^* allows us to conclude for the inclusion.

For the density, we use that for any $h \in \dot{\mathbb{W}}_{B^*D}^{1,q}$ (or $\dot{\mathbb{H}}_{B^*D}^{1,q}$), we have $e^{-s|B^*D|}h \rightarrow h$ as $s \rightarrow 0$ in that space. The proof is exactly the same as the one of [AS], Proposition 4.5. Now, by Lemma 10.3, we have that $e^{-s|B^*D|}h \in \mathbb{D}_q$ and it also belongs to $\mathbb{H}_{B^*D}^2$, as h belongs to $\mathbb{H}_{B^*D}^2$. \square

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LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UNIV. PARIS-SUD, CNRS, UNIVERSITÉ PARIS-SACLAY, 91405 ORSAY, FRANCE.

E-mail address: `pascal.auscher@math.u-psud.fr`

INSTITUT DES HAUTES ETUDES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, F-91440 BURES-SUR-YVETTE

E-mail address: `mihalis.mourgoglou@ihes.fr`