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## Coverings of Singular Curves over Finite Fields

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We prove that if  $f : Y \rightarrow X$  is a finite flat morphism between two reduced absolutely irreducible algebraic projective curves defined over the finite field  $\mathbb{F}_q$ , then

$$|\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq 2(\pi_Y - \pi_X)\sqrt{q},$$

where  $\pi_C$  is the arithmetic genus of a curve  $C$ . As application, we give some character sum estimation on singular curves.

In this paper, the word *curve* stands for a reduced absolutely irreducible algebraic projective curve defined over the finite field  $\mathbb{F}_q$  with  $q$  elements. If  $X$  is a smooth curve, it has been shown by Weil in [6] that the number of  $\mathbb{F}_q$ -rational points of  $X$ , denoted by  $\#X(\mathbb{F}_q)$ , is related to the geometric genus  $g_X$  by :

$$|\#X(\mathbb{F}_q) - (q + 1)| \leq 2g_X\sqrt{q} \quad (1)$$

(with an improvement of Serre using the integral part, see [5]). In fact, Weil's statement, involving the zeta function of  $X$ , is more precise. It implies that if there is a finite morphism  $f : Y \rightarrow X$  between two smooth curves  $X$  and  $Y$  having (geometric) genus  $g_X$  and  $g_Y$  respectively, then (see for instance [2], proposition 6)

$$|\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq 2(g_Y - g_X)\sqrt{q}. \quad (2)$$

When  $X$  is the projective line, this is exactly Weil's bound for  $Y$ .

On the other hand, the authors proved in [1] that if  $X$  is a singular curve, then Weil's inequality (1) holds if one replaces the geometric genus  $g_X$  of  $X$  by its arithmetic genus  $\pi_X$ . The aim of this paper is to give a generalization of both (2) and (1) for singular curves. Namely, if  $f : Y \rightarrow X$  is a finite flat morphism between two singular curves, then

$$|\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq 2(\pi_Y - \pi_X)\sqrt{q} \quad (3)$$

holds.

We will prove (3) in the third section. The proof goes as follows. Inequality (2) can be applied to the finite morphism  $\tilde{f} : \tilde{Y} \rightarrow \tilde{X}$  induced by  $f$  on the smooth models of  $X$  and  $Y$  respectively. Furthermore the number of  $\mathbb{F}_q$ -rational points of a curve is related to the number of  $\mathbb{F}_q$ -rational points of its smooth model (first section). Unfortunately, this is not sufficient to prove (3). One has to introduce (in the second section) the auxiliary curve  $Z = \tilde{X} \times_X Y$ , birational to  $Y$ .

Finally, we apply this result in a fifth section to obtain some character sum estimations.

### 1. A lemma

The following lemma relates the number of  $\mathbb{F}_q$ -rational points of a curve and that of its smooth model. It is given in [1], but in order to be self contained, we give here its short proof. If  $P$  is a  $\mathbb{F}_q$ -rational point of a curve  $X$ , we denote by  $\alpha_P$  (respectively  $\alpha_P(\infty)$ ) the number of  $\mathbb{F}_q$ -rational points (respectively of  $\overline{\mathbb{F}}_q$ -rational points, where  $\overline{\mathbb{F}}_q$  stands for an algebraic closure of  $\mathbb{F}_q$ ) of  $\tilde{X}$ , lying over  $P$  in the normalization map  $\nu_X : \tilde{X} \rightarrow X$ . Let  $\mathcal{O}_P$  be the local ring of  $X$  at  $P$ , and  $\overline{\mathcal{O}}_P$  its integral closure in the function field  $\mathbb{F}_q(X)$  of  $X$ . The quotient  $\overline{\mathcal{O}}_P/\mathcal{O}_P$  is a  $\mathbb{F}_q$ -vector space of finite length : let  $\delta_P$  be its dimension.

**Lemma 1.** *Let  $X$  be a reduced absolutely irreducible projective algebraic curve defined over  $\mathbb{F}_q$ . Then*

$$|\#\tilde{X}(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq \sum_{P \in X(\mathbb{F}_q)} |\alpha_P - 1| \leq \pi_X - g_X.$$

*Proof.* Let us first prove that if  $P$  is a  $\mathbb{F}_q$ -rational singular point of  $X$ , then  $\alpha_P - 1 \leq \delta_P$ . Let  $Q_1, \dots, Q_{\alpha_P(\infty)}$  be the  $\overline{\mathbb{F}}_q$ -rational points of  $\tilde{X}$  lying over  $P$  (the  $\alpha_P$  first being the  $\mathbb{F}_q$ -rational ones), and  $\phi$  the  $\mathbb{F}_q$ -linear map

$$\begin{aligned} \phi : \overline{\mathcal{O}}_P &\longrightarrow \mathbb{F}_q^{\alpha_P} \\ f &\longmapsto (f(Q_i))_{1 \leq i \leq \alpha_P} \end{aligned}$$

We prove that  $\phi$  is onto : let  $(x_1, \dots, x_{\alpha_P}) \in \mathbb{F}_q^{\alpha_P}$  and  $f_i = x_i \in \mathbb{F}_q \subset \mathbb{F}_q(X)$  if  $i \leq \alpha_P$ . For  $i \geq \alpha_P + 1$ , let  $f_i = 0$ . Then by the weak approximation theorem, there exists  $g \in \mathbb{F}_q(X)$  such that  $v_{Q_i}(g - f_i) \geq 1$  for  $1 \leq i \leq \alpha_P(\infty)$ . Hence,  $\phi(g) = (x_1, \dots, x_{\alpha_P})$  and

$$g \in \bigcap_{1 \leq i \leq \alpha_P(\infty)} \mathcal{O}_{Q_i} = \overline{\mathcal{O}}_P.$$

Since  $f(Q_1) = \dots = f(Q_{\alpha_P})$  for  $f \in \mathcal{O}_P$ , it follows that  $\phi(\mathcal{O}_P)$  is contained in the vector-line  $L \subset \mathbb{F}_q^{\alpha_P}$  spanned by  $(1, \dots, 1)$ . One obtains a surjective linear map

$$\tilde{\phi} : \overline{\mathcal{O}_P} / \mathcal{O}_P \longrightarrow \mathbb{F}_q^{\alpha_P} / L.$$

Taking dimensions, we obtain that  $\alpha_P - 1 \leq \delta_P$ .

Now, Lemma 1 follows from the formulas

$$\pi_X - g_X = \sum_{P \in \text{Sing } X(\overline{\mathbb{F}}_q)} \delta_P$$

and

$$\#\tilde{X}(\mathbb{F}_q) - \#X(\mathbb{F}_q) = \sum_{P \in X(\mathbb{F}_q)} (\alpha_P - 1).$$

□

## 2. An auxiliary curve

Let  $X$  and  $Y$  be two curves, and  $f : Y \longrightarrow X$  be a finite flat morphism. In order to give an estimate for the difference between the numbers of  $\mathbb{F}_q$ -rational points of  $X$  and  $Y$ , it is convenient to consider the fibre product  $Z = \tilde{X} \times_X Y$ .

**Lemma 2.**  *$Z$  is a reduced absolutely irreducible projective curve.*

*Proof.* The map  $f$  being finite, the map  $Z \longrightarrow \tilde{X}$  is finite and onto, which implies that  $\dim Z = \dim \tilde{X} = 1$ , and  $Z$  is a curve.

In order to prove that  $Z$  is absolutely integral, one has to prove that given an affine open set  $\text{Spec } A(X_i)$  of  $X$ , the ring  $A = \overline{A(X_i)} \otimes_{A(X_i)} A(Y_i)$  is an integral domain, where  $\overline{A}$  stands for the integral closure of a ring  $A$ , and  $A(Y_i)$  is defined by  $\text{Spec } A(Y_i) = f^{-1}(\text{Spec } A(X_i))$ . Denote by  $\text{Frac}(A)$  the quotient field of a domain  $A$ . By the flatness hypothesis,

$$0 \longrightarrow \overline{A(X_i)} \longrightarrow \text{Frac}(A(X_i))$$

induce

$$0 \longrightarrow A \longrightarrow \text{Frac}(A(X_i)) \otimes_{A(X_i)} A(Y_i).$$

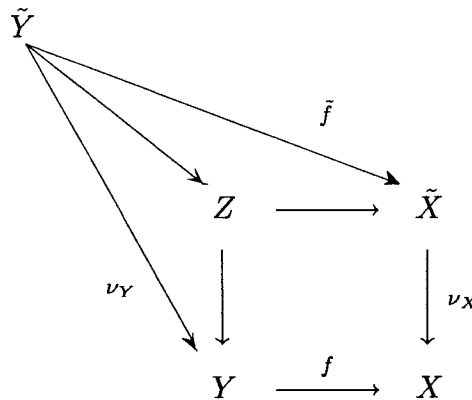
Then, the injective map

$$\text{Frac}(A(X_i)) \otimes_{A(X_i)} A(Y_i) \hookrightarrow \text{Frac}(A(Y_i))$$

proves that  $A$  is an integral domain, as a subring of a field.

Finally,  $Z$  is projective. Indeed,  $f$  is proper since it is finite, and  $Z \rightarrow \tilde{X}$  is also proper, so that the composite morphism  $Z \rightarrow \tilde{X} \rightarrow \text{Spec } \mathbb{F}_q$  is proper since  $\tilde{X}$  is complete. Hence,  $Z$  is projective as a complete curve.  $\square$

By the universal properties of the fibre product and of normalization maps, one can write the following diagram, where all triangles and squares are commutative, and all morphisms are finite :



The sheaf of  $\mathcal{O}_X$ -modules  $f_*\mathcal{O}_Y$  is coherent because  $f$  is finite. It is then locally free since  $f$  is flat. After localization, the stalks  $(f_*\mathcal{O}_Y)_P$  are  $\mathcal{O}_{X,P}$ -module of finite type, whose rank doesn't depend on  $P$  since  $X$  is connected. Let

$$\begin{aligned}
 r &= \dim_{\mathbb{F}_q(P)}(f_*\mathcal{O}_Y)_P \otimes_{\mathcal{O}_{X,P}} \mathbb{F}_q(P) \\
 &= \sum_{Q \in f^{-1}(P)} \dim_{\mathbb{F}_q(P)}(\mathcal{O}_{Y,Q} \otimes_{\mathcal{O}_{X,P}} \mathbb{F}_q(P))
 \end{aligned}$$

be this rank, where  $\mathbb{F}_q(P)$  denotes the residue field of the point  $P$ . Note that if  $r = 1$ , then  $f^{-1}(P) = \{Q\}$  contains only one element for any  $P \in X$ , and  $f$  induces a morphism of local rings  $\mathcal{O}_{Y,Q} \rightarrow \mathcal{O}_{X,P}$ , so that  $f$  is an isomorphism. Hence, one can suppose  $r \geq 2$ .

**Proposition 3.** (i)  $Z$  is birational to  $Y$ . In particular,  $g_Z = g_Y$ .

(ii) The arithmetic genus of  $Z$  is given by

$$\pi_Z = \pi_Y - r(\pi_X - g_X).$$

*Proof.* (i) This is trivial since there are dominant morphisms  $\tilde{Y} \rightarrow Z$  and  $Z \rightarrow Y$ .

(ii) Since both arithmetic and geometric genus of a curve  $C$  and of  $C \times_{\mathbb{F}_q} \overline{\mathbb{F}_q} = \overline{C}$  are the same, it is sufficient to compute  $\pi_{\overline{Z}}$ . To simplify notations, we continue to denote by  $X, Y$  and  $Z$  the curves  $\overline{X}, \overline{Y}$  and  $\overline{Z}$  respectively. The exact sequence of  $\mathcal{O}_X$ -module sheaves

$$0 \rightarrow \mathcal{O}_X \rightarrow \nu_{X,*}\mathcal{O}_{\tilde{X}} \rightarrow (\nu_{X,*}\mathcal{O}_{\tilde{X}})/\mathcal{O}_X \rightarrow 0$$

tensorized by the flat  $\mathcal{O}_X$ -module  $f_*\mathcal{O}_Y$ , gives the exact sequence

$$\begin{aligned}
 0 \rightarrow f_*\mathcal{O}_Y \rightarrow (\nu_{X,*}\mathcal{O}_{\tilde{X}}) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \rightarrow \\
 \rightarrow ((\nu_{X,*}\mathcal{O}_{\tilde{X}})/\mathcal{O}_X) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \rightarrow 0
 \end{aligned} \tag{4}$$

from which we deduce a long exact sequence of cohomology. Let us scrutinize its different terms.

Since  $H^i(X, f_*\mathcal{O}_Y) \cong H^i(Y, \mathcal{O}_Y)$  and  $Y$  is projective, then we have  $\dim_{\overline{\mathbb{F}}_q} H^0(X, f_*\mathcal{O}_Y) = 1$  and  $\dim_{\overline{\mathbb{F}}_q} H^1(X, f_*\mathcal{O}_Y) = \pi_Y$ . In the same way,

$$\dim_{\overline{\mathbb{F}}_q} H^0(X, (\nu_{X,*}\mathcal{O}_{\tilde{X}}) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y) = 1$$

and

$$\dim_{\overline{\mathbb{F}}_q} H^1(X, (\nu_{X,*}\mathcal{O}_{\tilde{X}}) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y) = \pi_Z.$$

Furthermore, the sheaf  $(\nu_{X,*}\mathcal{O}_{\tilde{X}})/\mathcal{O}_X$  is a sum of skyscraper sheaves on the singular points of  $X$ . Hence, this is also the case for  $((\nu_{X,*}\mathcal{O}_{\tilde{X}})/\mathcal{O}_X) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y$ , and this prove the vanishing of its  $H^1$ . Finally,

$$\begin{aligned} & \dim_{\overline{\mathbb{F}}_q} H^0(X, ((\nu_{X,*}\mathcal{O}_{\tilde{X}})/\mathcal{O}_X) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y) \\ &= \sum_{P \in \text{Sing } X(\overline{\mathbb{F}}_q)} \dim_{\overline{\mathbb{F}}_q} ((\overline{\mathcal{O}_{X,P}}/\mathcal{O}_{X,P}) \otimes_{\mathcal{O}_{X,P}} (f_*\mathcal{O}_Y)_P) = \\ & \sum_{P \in \text{Sing } X(\overline{\mathbb{F}}_q)} \dim_{\overline{\mathbb{F}}_q} ((\overline{\mathcal{O}_{X,P}}/\mathcal{O}_{X,P}) \otimes_{\overline{\mathbb{F}}_q} (\overline{\mathbb{F}}_q \otimes_{\mathcal{O}_{X,P}} (f_*\mathcal{O}_Y)_P)) \\ &= \sum_{P \in \text{Sing } X(\overline{\mathbb{F}}_q)} \dim_{\overline{\mathbb{F}}_q} ((\overline{\mathcal{O}_{X,P}}/\mathcal{O}_{X,P}) \otimes_{\overline{\mathbb{F}}_q} (\overline{\mathbb{F}}_q)^r) \\ &= \sum_{P \in \text{Sing } X(\overline{\mathbb{F}}_q)} \dim_{\overline{\mathbb{F}}_q} (\overline{\mathcal{O}_{X,P}}/\mathcal{O}_{X,P})^r \\ &= r(\pi_X - g_X). \end{aligned}$$

The nullity of the alternating sum of the  $\overline{\mathbb{F}}_q$ -dimensions of the different terms of the long exact sequence of cohomology given by (4) gives :

$$1 - 1 + r(\pi_X - g_X) - \pi_Y + \pi_Z - 0 = 0,$$

which was to be proved. □



### 3. The main theorem

**Theorem 4.** *Let  $X$  and  $Y$  be two reduced absolutely irreducible projective algebraic curves defined over  $\mathbb{F}_q$ , with respective arithmetic genus  $\pi_X$  and  $\pi_Y$ , and let  $f : Y \rightarrow X$  be a finite flat morphism defined over  $\mathbb{F}_q$ . Then*

$$|\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq 2(\pi_Y - \pi_X)\sqrt{q}.$$

The proof depends on the following lemma.

**Lemma 5.**

$$|(\#Z(\mathbb{F}_q) - \#\tilde{X}(\mathbb{F}_q)) - (\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q))| \leq (r - 1)(\pi_X - g_X)$$

(recall that  $r = \dim_{\mathbb{F}_q(P)}(f_*\mathcal{O}_Y)_P \otimes_{\mathcal{O}_{X,P}} \mathbb{F}_q(P)$ ).

*Proof.* If  $P \in X(\mathbb{F}_q)$ , let

$$\alpha_P = \#\{\tilde{P} \in \tilde{X}(\mathbb{F}_q) \mid \nu_X(\tilde{P}) = P\}$$

and

$$\beta_P = \#\{Q \in Y(\mathbb{F}_q) \mid f(Q) = P\}.$$

Then

$$\beta_P \leq \sum_{Q \in f^{-1}(P)} 1 \leq \sum_{Q \in f^{-1}(P)} \dim_{\mathbb{F}_q(P)}(\mathcal{O}_{Y,Q} \otimes_{\mathcal{O}_{X,P}} \mathbb{F}_q(P)) = r$$

Moreover,

$$Z(\mathbb{F}_q) = \{(\tilde{P}, Q) \in \tilde{X}(\mathbb{F}_q) \times Y(\mathbb{F}_q) \mid \nu_X(\tilde{P}) = f(Q)\}$$

hence

$$\begin{aligned} \#Z(\mathbb{F}_q) - \#\tilde{X}(\mathbb{F}_q) &= \sum_{P \in X(\mathbb{F}_q)} \alpha_P \beta_P - \sum_{P \in X(\mathbb{F}_q)} \alpha_P \\ &= \sum_{P \in X(\mathbb{F}_q)} \alpha_P(\beta_P - 1), \end{aligned}$$

and

$$\begin{aligned} \#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q) &= \sum_{P \in X(\mathbb{F}_q)} \beta_P - \sum_{P \in X(\mathbb{F}_q)} 1 \\ &= \sum_{P \in X(\mathbb{F}_q)} (\beta_P - 1). \end{aligned}$$

By lemma 1, and since  $r \geq 2$ , we obtain

$$\begin{aligned} |(\#Z(\mathbb{F}_q) - \#\tilde{X}(\mathbb{F}_q)) - (\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q))| &\leq \\ &\leq (r - 1) \sum_{P \in X(\mathbb{F}_q)} |\alpha_P - 1| \\ &\leq (r - 1)(\pi_X - g_X). \end{aligned}$$

□

The theorem follows easily from the preceding and the triangular inequality

$$\begin{aligned} | \#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q) | &\leq | \#\tilde{Y}(\mathbb{F}_q) - \#Z(\mathbb{F}_q) | \\ &\quad + | (\#Z(\mathbb{F}_q) - \#\tilde{X}(\mathbb{F}_q)) - (\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q)) | \\ &\quad + | \#\tilde{X}(\mathbb{F}_q) - \#\tilde{Y}(\mathbb{F}_q) | \end{aligned}$$

#### 4. Remarks

**4.1.** The proof of theorem 4 gives a better upper bound, namely

$$| \#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q) | \leq (\pi_Y - g_Y) - (\pi_X - g_X) + (g_Y - g_X)[2\sqrt{q}].$$

See lemma 4.1 of [3] for the reason of the integer part  $[2\sqrt{q}]$  of  $2\sqrt{q}$ .

**4.2.** It couldn't be expected theorem 4 to be true for any finite morphism  $f : Y \rightarrow X$ . For instance, this is false for the normalization map of a singular curve. Here is another example : let  $X$  be the plane curve  $Y^2Z = X^3$ . This is a singular cubic curve, its arithmetic genus is  $\pi_X = 1$ , and for any integer  $n$ , we have  $\#X(\mathbb{F}_{2^n}) = 2^n + 1$ . Let us consider the morphism

$$\begin{aligned} \pi : \mathbb{P}^1 &\longrightarrow X \\ (u : v) &\longmapsto (u^2v : u^3 : v^3) \end{aligned}$$

Suppose theorem 4 would be true for any finite morphism  $f$ , and let  $C$  be any smooth curve of genus  $g_C$ , defined over  $\mathbb{F}_2$ .

There is a finite morphism  $\pi' : C \longrightarrow \mathbb{P}^1$ , hence by composition a finite morphism  $f = \pi \circ \pi' : C \longrightarrow X$ , and theorem 4 would imply that for  $q = 2^n$ ,

$$|\#C(\mathbb{F}_q) - (q + 1)| \leq 2(g_C - 1)\sqrt{q},$$

which is false : one can consider for instance the smooth curve  $C$  given by  $X^3 + Y^3 + Z^3 = 0$ , which has 9 rational points over  $\mathbb{F}_4$  and genus  $g_C = 1$ .

**4.3.** If  $X$  is smooth (in particular if  $X$  is the projective line), then any finite morphism  $Y \longrightarrow X$  is flat. Indeed, if  $f(Q) = P$ , then the local ring  $\mathcal{O}_{Y,Q}$  is a finitely generated  $\mathcal{O}_{X,P}$ -module without torsion element. But  $\mathcal{O}_{X,P}$  is a principal ideal domain since  $X$  is supposed to be smooth, so that  $\mathcal{O}_{Y,Q}$  is a flat  $\mathcal{O}_{X,P}$ -module. This shows that theorem 4 contains all known results (1) and (2) of the introduction.

Note that there are many other situations where  $Y \longrightarrow X$  is flat. This is the case for instance if  $X = Y/G$ , where  $G$  is a finite group of automorphisms of  $Y$  (see [4]), or (by stability of flatness by base change) if  $f$  is the reduction modulo a prime ideal of a flat morphism between two curves defined over a number field, or (also by base change, see section 5 below) if  $f$  is a Kummer, or an Artin-Schreier morphism.

## 5. Application to exponential sums

Let  $X$  be a curve over a finite field  $\mathbb{F}_q$  of odd characteristic and  $f \in \mathbb{F}_q(X)$  a function on  $X$ . When  $X$  is smooth, there is an extensive literature on estimations of the character sums  $\sum_{P \in X(\mathbb{F}_q)} \left(\frac{f(P)}{q}\right)$ , where  $\left(\frac{\cdot}{q}\right)$  is the Legendre symbol on  $\mathbb{F}_q$  (with the convention that  $\left(\frac{f(P)}{q}\right) = 0$  if  $P$  is a zero or a pole of  $f$ ). In fact, this character sum equals to  $\#Y(\mathbb{F}_q) - \#X(\mathbb{F}_q)$ , where  $Y$  is the Zariski closure of the curve  $\{(P, y) \in X \times \mathbb{P}^1 \mid y^2 = f(P)\}$  in the surface  $X \times \mathbb{P}^1$ . Indeed, there are exactly  $1 + \left(\frac{f(P)}{q}\right)$  points in  $Y(\mathbb{F}_q)$  above a given point  $P \in X(\mathbb{F}_q)$ .

The geometric genus  $g_Y$  of  $Y$  is given from  $g_X$  by Hurwitz formula, and its arithmetic genus is given by the following lemma, whose proof is a standard and tedious calculation using Čech cohomology.

**Lemma 6.** *Let  $Y$  be the Zariski closure of  $\{(P, y) \in X \times \mathbb{P}^1 \mid y^n = f(P)\}$ . Then,  $\pi_Y = n\pi_X + (n - 1)(\deg(f)_0 - 1)$ , where  $(f)_0$  denotes the divisors of zeroes of  $f$ .*

Since the squaring map  $\mathbb{P}^1 \rightarrow \mathbb{P}^1, y \mapsto y^2$  is flat, then by base change  $Y = X \times_{\mathbb{P}^1} \mathbb{P}^1 \rightarrow X$  is also flat. Hence, theorem 4 applies, so that the following holds :

**Theorem 7.**

$$\left| \sum_{P \in X(\mathbb{F}_q)} \left( \frac{f(P)}{q} \right) \right| \leq 2(n - 1)(\pi_X + \deg(f)_0 - 1)\sqrt{q}.$$

Of course, one can give a better upper bound using remark 4.1., and one can study additive character sums via Artin-Schreier coverings.

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