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► **To cite this version:**

Yves Aubry, Marc Perret. A Weil theorem for singular curves. Contemporary mathematics, 1996, Walter de Gruyter, pp.1–8. 10.1515/9783110811056.1 . hal-00976485

**HAL Id: hal-00976485**

**<https://hal.science/hal-00976485>**

Submitted on 11 Apr 2014

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# A Weil theorem for singular curves

*Yves Aubry and Marc Perret*

**Abstract.** We generalize Weil's theorem on the number of rational points of smooth curves over a finite field to singular ones.

1991 Mathematics Subject Classification: 14G10, 14H20.

## 1. Introduction

Throughout this paper a *curve* stands for a reduced absolutely irreducible projective algebraic curve defined over the finite field  $\mathbb{F}_q$  with  $q$  elements. André Weil [6] proved that the number  $\sharp X(\mathbb{F}_q)$  of rational points over  $\mathbb{F}_q$  of any smooth curve  $X$  satisfies

$$|\sharp X(\mathbb{F}_q) - (q + 1)| \leq 2g\sqrt{q}$$

where  $g$  is the genus of  $X$ . Moreover, the zeta function  $Z_X$  of  $X$  is a rational function  $\frac{P_X(T)}{(1-T)(1-qT)}$  where  $P_X(T)$  is a polynomial of degree  $2g$ , with integer coefficients and whose inverse roots have modulus  $\sqrt{q}$  (this is the so called "Riemann Hypothesis").

In this paper, we first give an explicit form of the zeta function of any singular curve  $X$ . We know by Dwork's theorem that it is a rational function. Here again, we show that  $Z_X(T) = \frac{P_X(T)}{(1-T)(1-qT)}$ , where  $P_X(T)$  is a polynomial with integer coefficients which is the product of the numerator of the zeta function of the normalization  $\tilde{X}$  of  $X$  with an explicit product of cyclotomic polynomials.

Then, the study of the difference between the number of rational points of  $\tilde{X}$  and of  $X$  will enable us to show that the Weil inequality still holds for  $X$ , provided that we replace the geometric genus  $g$  of  $X$  by its arithmetic genus  $\pi$  (in fact, we give a better estimate).

Finally, we give some applications to permutation polynomials and explicit formulas.

## 2. The zeta function of a singular curve

The zeta function

$$Z_X(T) = \exp \left( \sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right)$$

of a curve  $X$  can be written as

$$Z_X(T) = \frac{\det(1 - TF \mid H_c^1(X, \mathbb{Q}_\ell))}{(1 - T)(1 - qT)},$$

where  $F$  is the Frobenius morphism on the first  $\ell$ -adic cohomology group with compact support  $H_c^1(X, \mathbb{Q}_\ell)$  of  $X$ , and the eigenvalues of the Frobenius have modulus  $\sqrt{q}$  or 1 (see [1]). The purpose of this section is to give a more precise statement, using only elementary methods.

We denote by  $\tilde{X}$  the normalization of  $X$  and by  $\nu: \tilde{X} \rightarrow X$  the normalization map. If  $P$  is a point, we denote by  $d_P = [\mathbb{F}_q(P) : \mathbb{F}_q]$  its degree over  $\mathbb{F}_q$  (i.e. the degree of the extension of the residue field of  $P$  over  $\mathbb{F}_q$ ).

**Theorem 2.1.** *Let  $S$  be the (finite) set of singular points of  $X$ . Then*

$$Z_X(T) = \frac{P_X(T)}{(1 - T)(1 - qT)}$$

where

$$P_X(T) = P_{\tilde{X}}(T) \prod_{P \in S} \left( \frac{\prod_{Q \in \nu^{-1}(P)} (1 - T^{d_Q})}{1 - T^{d_P}} \right),$$

and where  $P_{\tilde{X}}$  is the numerator of the zeta function  $Z_{\tilde{X}}$  of  $\tilde{X}$ .

*Proof.* We have

$$\begin{aligned} \frac{Z_X(T)}{Z_{\tilde{X}}(T)} &= \exp \left( - \sum_{n=1}^{\infty} (\#\tilde{X}(\mathbb{F}_{q^n}) - \#X(\mathbb{F}_{q^n})) \frac{T^n}{n} \right) \\ &= \prod_{P \in S} \exp \left( - \sum_{n=1}^{\infty} (\alpha_P(n) - d_P) \delta_{d_P|n} \frac{T^n}{n} \right), \end{aligned}$$

where  $\delta_{m|n} = 1$  if  $m \mid n$ , else  $\delta_{m|n} = 0$ , and where  $\alpha_P(n)$  is the number of points of  $\tilde{X}$  lying over  $P$ , that are rational over  $\mathbb{F}_{q^n}$ .

Thus,

$$\frac{Z_X(T)}{Z_{\tilde{X}}(T)} = \prod_{P \in S} \exp \left( - \sum_{m=1}^{\infty} (\alpha_P(md_P) - d_P) \frac{T^{md_P}}{md_P} \right).$$

But

$$\alpha_P(md_P) = \sum_{Q \in \nu^{-1}(P)} d_Q \delta_{d_Q|md_P},$$

hence

$$\begin{aligned} \frac{Z_X(T)}{Z_{\tilde{X}}(T)} &= \prod_{P \in S} \left( \frac{\prod_{Q \in v^{-1}(P)} \exp\left(-\sum_{m=1}^{\infty} d_Q \delta_d Q |m d_P \frac{T^{m d_P}}{m d_P}\right)}{1 - T^{d_P}} \right) \\ &= \prod_{P \in S} \left( \frac{\prod_{Q \in v^{-1}(P)} \exp\left(-\sum_{\ell=1}^{\infty} \frac{T^{\ell d_Q}}{\ell}\right)}{1 - T^{d_P}} \right) \\ &= \prod_{P \in S} \left( \frac{\prod_{Q \in v^{-1}(P)} (1 - T^{d_Q})}{1 - T^{d_P}} \right) \end{aligned}$$

and the theorem is proved.  $\square$

The zeta function of  $X$  is thus the product of a polynomial of degree  $2g_X$  by a polynomial of degree

$$\Delta_X = \#\{\tilde{X}(\overline{\mathbb{F}}_q) \setminus X(\overline{\mathbb{F}}_q)\},$$

where  $\overline{\mathbb{F}}_q$  is an algebraic closure of  $\mathbb{F}_q$ . Note that the quantity  $\Delta_X$  is well-defined, since the set of singular points is a finite subset of  $X$ . We now have to evaluate  $\Delta_X$  (which can be seen as the dimension of the toric component of the generalized Jacobian of  $X$  (see [4])).

As in the proof of Theorem 2.1, if  $P \in X(\overline{\mathbb{F}}_q)$  let  $\alpha_P(1) = \alpha$  (respectively  $\alpha_P(\infty)$ ) be the number of rational points over  $\mathbb{F}_q$  (respectively over  $\tilde{\mathbb{F}}_q$ ) of  $\tilde{X}$ , lying over  $P$ . Let  $\mathcal{O}_P$  be the local ring of  $P$  on  $X$  and  $\overline{\mathcal{O}}_P$  its integral closure in the function field  $\mathbb{F}_q(X)$  of the curve  $X$ . The quotient  $\overline{\mathcal{O}}_P/\mathcal{O}_P$  is a finite dimensional  $\mathbb{F}_q$ -vector space whose dimension is denoted by  $\delta_P$ .

We get the following lemma.

**Lemma 2.2.** *Let  $P$  be an  $\mathbb{F}_q$ -rational point of  $X$ . Then*

$$\alpha_P(1) - 1 \leq \delta_P.$$

*Proof.* Let  $Q_1, \dots, Q_{\alpha_P(\infty)}$  be the points of  $\tilde{X}(\overline{\mathbb{F}}_q)$  lying over  $P$  (where the first  $\alpha$  points are the  $\mathbb{F}_q$ -rational ones), and  $\phi$  the linear map of  $\mathbb{F}_q$ -vector spaces

$$\begin{aligned} \phi: \overline{\mathcal{O}}_P &\longrightarrow \mathbb{F}_q^\alpha \\ f &\longmapsto (f(Q_i))_{1 \leq i \leq \alpha} \end{aligned}$$

Let us show that  $\phi$  is surjective. Let  $(x_1, \dots, x_\alpha) \in \mathbb{F}_q^\alpha$  and  $f_i = x_i \in \mathbb{F}_q \subset \mathbb{F}_q(X)$  for  $i \leq \alpha$ . For  $i \geq \alpha + 1$ , we set  $f_i = 0$ . By the weak approximation theorem, there exists  $g \in \mathbb{F}_q(X)$  such that  $v_{Q_i}(g - f_i) \geq 1$  for  $1 \leq i \leq \alpha_P(\infty)$ . Hence,  $\phi(g) = (x_1, \dots, x_\alpha)$  and

$$g \in \bigcap_{1 \leq i \leq \alpha_P(\infty)} \mathcal{O}_{Q_i} = \overline{\mathcal{O}}_P.$$

Since  $f(Q_1) = \dots = f(Q_\alpha)$  for  $f \in \mathcal{O}_P$ , we have that  $\phi(\mathcal{O}_P)$  is contained in a one dimensional vector space  $L \subset \mathbb{F}_q^\alpha$ . We obtain a surjective linear map

$$\tilde{\phi} : \overline{\mathcal{O}_P} / \mathcal{O}_P \longrightarrow \mathbb{F}_q^\alpha / L,$$

and the lemma is proved.  $\square$

**Proposition 2.3.**

$$|\#\tilde{X}(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq \pi - g.$$

*Proof.* Let  $P$  be a point of  $X$  and  $Q$  be a point of  $\tilde{X}$  lying over  $P$ . Then  $P$  is rational over  $\mathbb{F}_q$  if  $Q$  is. With the previous notations, we get by Lemma 2.2

$$\#\tilde{X}(\mathbb{F}_q) - \#X(\mathbb{F}_q) = \sum_{P \in \text{Sing } X(\mathbb{F}_q)} (\alpha_P(1) - 1) \leq \sum_{P \in \text{Sing } X(\mathbb{F}_q)} \delta_P \leq \pi - g$$

since

$$\pi - g = \sum_{P \in \text{Sing } X(\overline{\mathbb{F}_q})} \delta_P.$$

On the other hand,

$$\sum_{P \in \text{Sing } X(\mathbb{F}_q)} (\alpha_P(1) - 1) \geq -\#\text{Sing } X(\mathbb{F}_q) \geq -\#\text{Sing } X(\overline{\mathbb{F}_q}) \geq -(\pi - g),$$

which concludes the proof.  $\square$

Thus, the numerator of the zeta function of  $X$  is a polynomial with integer coefficients of degree  $2g + \Delta_X \in \{2g, \dots, \pi + g\}$ , where  $\Delta_X = \#\{\tilde{X}(\overline{\mathbb{F}_q}) \setminus X(\overline{\mathbb{F}_q})\}$ , whose inverse roots have either modulus  $\sqrt{q}$  (for  $2g$  of them) or modulus 1 (for  $\Delta_X$  of them).

**Corollary 2.4.** *Let  $\omega_1, \dots, \omega_{2g}$  be the inverse roots of  $P_{\tilde{X}}$ , and  $\beta_1, \dots, \beta_{\Delta_X}$  be the inverse roots of the cyclotomic part of  $P_X$ . Then, for all  $n \geq 1$ ,*

$$\#X(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n - \sum_{j=1}^{\Delta_X} \beta_j^n.$$

*In particular,*

$$|\#X(\mathbb{F}_q) - (q + 1)| \leq 2g\sqrt{q} + \Delta_X \leq 2g\sqrt{q} + \pi - g \leq 2\pi\sqrt{q},$$

and

$$\dim_{\mathbb{Q}_\ell} H_c^1(X, \mathbb{Q}_\ell) = 2g + \Delta_X. \quad \square$$

Using the last inequality, we get:

**Corollary 2.5.** *If  $X$  is an absolutely irreducible curve which is a complete intersection in  $\mathbb{P}^{n+1}$  of  $n$  hypersurfaces of degree  $d_1, \dots, d_n$ , and if we set  $d = \prod_{i=1}^n d_i$ , then:*

$$|\sharp X(\mathbb{F}_q) - (q+1)| \leq (d-1)(d-2)\sqrt{q}.$$

*In particular, this inequality holds for any absolutely irreducible plane curve of degree  $d$ .*

*Proof.* The arithmetic genus  $\pi_X$  of a complete intersection which is given by  $2\pi_X = d(\sum_{i=1}^n d_i - n - 2) + 2$  (see [5 p.73]) is clearly at most equal to  $(d-1)(d-2)$ . The second assertion is obviously a particular case of the first one. Observe that the arithmetic genus of a plane curve of degree  $d$  is equal to  $\frac{(d-1)(d-2)}{2}$ .  $\square$

### 3. Applications

#### 3.1. Explicit formulas

The explicit formulas given by J.-P. Serre in [5] related to the function field of a curve over a finite field still hold in the singular case, provided that we replace the geometric genus  $g$  of the curve by its arithmetic genus  $\pi$ . Furthermore, we can replace (and it is better)  $g$  by  $g + \frac{\Delta_X}{2}$ .

Consider a function  $f(\theta) = 1 + 2 \sum_{n \geq 1} c_n \cos n\theta$  which satisfies  $f \gg 0$  (i.e.  $f(\theta) \geq 0$  for all  $\theta \in \mathbb{R}$  and  $c_n \geq 0$  for all  $n \geq 1$ ). The formula of Corollary 2.4 gives  $\sharp X(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n - \sum_{j=1}^{\Delta_X} \beta_j^n$ . Arguing as in [5], we get

$$\sharp X(\mathbb{F}_q) \leq a_f \left( g + \frac{\Delta_X}{2} \right) + b_f \leq a_f \pi + b_f \quad (*)$$

with  $a_f = 1/\Psi(q^{-1/2})$  and  $b_f = 1 + (\Psi(q^{1/2})/\Psi(q^{-1/2}))$ , where  $\Psi(t) = \sum_{n \geq 1} c_n t^n$ .

Furthermore, we obtain the same estimations as the ones in [5] of the maximum number of rational points over  $\mathbb{F}_2$  of a curve for  $g + \frac{\Delta_X}{2}$  fixed. For example, a curve with  $g + \frac{\Delta_X}{2} \leq 6$  has at most 10 rational points. These remarks have been communicated to the authors by J.-P. Serre.

Another application of this results is the following one. Consider the number

$$N_q(\pi) = \max_X \sharp X(\mathbb{F}_q)$$

where  $X$  runs over the curves (possibly singular!) over  $\mathbb{F}_q$  of arithmetic genus  $\pi$ . It is of interest to study the behavior of  $N_q(\pi)$  for  $\pi \rightarrow \infty$ . We define, in analogy with the quantity  $A(q)$  (see [5] for example), the number  $A'(q)$  by

$$A'(q) = \limsup_{\pi \rightarrow \infty} \frac{N_q(\pi)}{\pi}.$$

Corollary 2.5 readily implies  $A'(q) \leq 2\sqrt{q}$ . In fact, using the previous explicit formulas we get the following bound which is the same as that for  $A(q)$  of Drinfeld and Vlăduț (see [2]). Note that we obviously have  $A(q) \leq A'(q)$ .

**Proposition 3.1.**

$$A'(q) \leq \sqrt{q} - 1.$$

*Proof.* Take

$$f_m(\theta) = 1 + 2 \sum_{n=1}^m \left(1 - \frac{n}{m}\right) \cos n\theta = \frac{1}{m} \left| \sum_{k=1}^m e^{ik\theta} \right|^2.$$

Thus  $f_m \gg 0$ . Now, let

$$\Psi_m(t) = \sum_{n=1}^m \left(1 - \frac{n}{m}\right) t^n.$$

Thus, (\*) gives

$$\frac{\#X(\mathbb{F}_q)}{\pi} \leq 1/\Psi_m(q^{-1/2}) + \frac{1}{\pi} \left(1 + (\Psi_m(q^{1/2})/\Psi_m(q^{-1/2}))\right)$$

Since  $\Psi_m(t) \rightarrow t/(1-t)$  for  $m \rightarrow \infty$  and  $|t| < 1$ , we get

$$\Psi_m(q^{-1/2}) \rightarrow 1/(\sqrt{q} - 1) \quad \text{for } m \rightarrow \infty.$$

Hence, for any  $\epsilon$  there exists  $m_0$  such that  $m \geq m_0$  implies

$$1/\Psi_m(q^{-1/2}) < \sqrt{q} - 1 + \frac{\epsilon}{2}.$$

For  $\pi$  large enough, the second term of the right hand side of the inequality is less than  $\frac{\epsilon}{2}$ , and this concludes the proof.  $\square$

### 3.2. Permutation polynomials and exceptional polynomials

Corollary 2.5 gives us the following explicit form of the Lemma 7.28 of [3]. This result enables us to precise the relationship between permutation polynomials and exceptional polynomials over  $\mathbb{F}_q$  (see [3]).

**Lemma 3.2.** *Let  $\phi \in \mathbb{F}_q[x, y]$  be an absolutely irreducible polynomial of degree  $d$  and  $C_\phi$  the affine curve of equation  $\phi(x, y) = 0$ . Set*

$$k_d = \frac{1}{4} \left( (d-1)(d-2) + \sqrt{d^2 + 5d - 2} \right)^2.$$

*If  $q \geq k_d$ , then either  $C_\phi$  has a rational point  $(a, b)$  with  $a \neq b$  or  $\phi$  is of the form  $c(x - y)$  for some  $c \in \mathbb{F}_q$ .*

*Proof.* According to the affine version of Corollary 2.5, the number  $N$  of rational points over  $\mathbb{F}_q$  of the (affine) curve  $C_\phi$  satisfies

$$|N - q| \leq (d-1)(d-2)\sqrt{q} + d - 1$$

Arguing as in the proof of Lemma 7.28 of [3], the result holds with any  $k_d$  such that  $q - (d-1)(d-2)\sqrt{q} - 2d + 1 > 0$  for all  $q \geq k_d$ .  $\square$

Hence, this gives explicit forms for the Propositions 7.29 until 7.33 of [3]. For example, any permutation polynomial of degree 2 is exceptional over  $\mathbb{F}_q$  if  $q$  is odd.

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