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A Weil theorem for singular curves

Yves Aubry and Marc Perret

Abstract. We generalize Weil's theorem on the number of rational points of smooth curves over a finite field to singular ones.

1991 Mathematics Subject Classification: 14G10, 14H20.

1. Introduction

Throughout this paper a *curve* stands for a reduced absolutely irreducible projective algebraic curve defined over the finite field \mathbb{F}_q with q elements. André Weil [6] proved that the number $\sharp X(\mathbb{F}_q)$ of rational points over \mathbb{F}_q of any smooth curve X satisfies

$$|\sharp X(\mathbb{F}_q) - (q + 1)| \leq 2g\sqrt{q}$$

where g is the genus of X . Moreover, the zeta function Z_X of X is a rational function $\frac{P_X(T)}{(1-T)(1-qT)}$ where $P_X(T)$ is a polynomial of degree $2g$, with integer coefficients and whose inverse roots have modulus \sqrt{q} (this is the so called "Riemann Hypothesis").

In this paper, we first give an explicit form of the zeta function of any singular curve X . We know by Dwork's theorem that it is a rational function. Here again, we show that $Z_X(T) = \frac{P_X(T)}{(1-T)(1-qT)}$, where $P_X(T)$ is a polynomial with integer coefficients which is the product of the numerator of the zeta function of the normalization \tilde{X} of X with an explicit product of cyclotomic polynomials.

Then, the study of the difference between the number of rational points of \tilde{X} and of X will enable us to show that the Weil inequality still holds for X , provided that we replace the geometric genus g of X by its arithmetic genus π (in fact, we give a better estimate).

Finally, we give some applications to permutation polynomials and explicit formulas.

2. The zeta function of a singular curve

The zeta function

$$Z_X(T) = \exp \left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right)$$

of a curve X can be written as

$$Z_X(T) = \frac{\det(1 - TF \mid H_c^1(X, \mathbb{Q}_\ell))}{(1 - T)(1 - qT)},$$

where F is the Frobenius morphism on the first ℓ -adic cohomology group with compact support $H_c^1(X, \mathbb{Q}_\ell)$ of X , and the eigenvalues of the Frobenius have modulus \sqrt{q} or 1 (see [1]). The purpose of this section is to give a more precise statement, using only elementary methods.

We denote by \tilde{X} the normalization of X and by $\nu: \tilde{X} \rightarrow X$ the normalization map. If P is a point, we denote by $d_P = [\mathbb{F}_q(P) : \mathbb{F}_q]$ its degree over \mathbb{F}_q (i.e. the degree of the extension of the residue field of P over \mathbb{F}_q).

Theorem 2.1. *Let S be the (finite) set of singular points of X . Then*

$$Z_X(T) = \frac{P_X(T)}{(1 - T)(1 - qT)}$$

where

$$P_X(T) = P_{\tilde{X}}(T) \prod_{P \in S} \left(\frac{\prod_{Q \in \nu^{-1}(P)} (1 - T^{d_Q})}{1 - T^{d_P}} \right),$$

and where $P_{\tilde{X}}$ is the numerator of the zeta function $Z_{\tilde{X}}$ of \tilde{X} .

Proof. We have

$$\begin{aligned} \frac{Z_X(T)}{Z_{\tilde{X}}(T)} &= \exp \left(- \sum_{n=1}^{\infty} (\#\tilde{X}(\mathbb{F}_{q^n}) - \#X(\mathbb{F}_{q^n})) \frac{T^n}{n} \right) \\ &= \prod_{P \in S} \exp \left(- \sum_{n=1}^{\infty} (\alpha_P(n) - d_P) \delta_{d_P|n} \frac{T^n}{n} \right), \end{aligned}$$

where $\delta_{m|n} = 1$ if $m \mid n$, else $\delta_{m|n} = 0$, and where $\alpha_P(n)$ is the number of points of \tilde{X} lying over P , that are rational over \mathbb{F}_{q^n} .

Thus,

$$\frac{Z_X(T)}{Z_{\tilde{X}}(T)} = \prod_{P \in S} \exp \left(- \sum_{m=1}^{\infty} (\alpha_P(md_P) - d_P) \frac{T^{md_P}}{md_P} \right).$$

But

$$\alpha_P(md_P) = \sum_{Q \in \nu^{-1}(P)} d_Q \delta_{d_Q|md_P},$$

hence

$$\begin{aligned} \frac{Z_X(T)}{Z_{\tilde{X}}(T)} &= \prod_{P \in S} \left(\frac{\prod_{Q \in v^{-1}(P)} \exp\left(-\sum_{m=1}^{\infty} d_Q \delta_{d_Q} \frac{T^{md_P}}{md_P}\right)}{1-T^{d_P}} \right) \\ &= \prod_{P \in S} \left(\frac{\prod_{Q \in v^{-1}(P)} \exp\left(-\sum_{\ell=1}^{\infty} \frac{T^{\ell d_Q}}{\ell}\right)}{1-T^{d_P}} \right) \\ &= \prod_{P \in S} \left(\frac{\prod_{Q \in v^{-1}(P)} (1-T^{d_Q})}{1-T^{d_P}} \right) \end{aligned}$$

and the theorem is proved. \square

The zeta function of X is thus the product of a polynomial of degree $2g_X$ by a polynomial of degree

$$\Delta_X = \#\{\tilde{X}(\overline{\mathbb{F}}_q) \setminus X(\overline{\mathbb{F}}_q)\},$$

where $\overline{\mathbb{F}}_q$ is an algebraic closure of \mathbb{F}_q . Note that the quantity Δ_X is well-defined, since the set of singular points is a finite subset of X . We now have to evaluate Δ_X (which can be seen as the dimension of the toric component of the generalized Jacobian of X (see [4])).

As in the proof of Theorem 2.1, if $P \in X(\overline{\mathbb{F}}_q)$ let $\alpha_P(1) = \alpha$ (respectively $\alpha_P(\infty)$) be the number of rational points over \mathbb{F}_q (respectively over $\tilde{\mathbb{F}}_q$) of \tilde{X} , lying over P . Let \mathcal{O}_P be the local ring of P on X and $\overline{\mathcal{O}}_P$ its integral closure in the function field $\mathbb{F}_q(X)$ of the curve X . The quotient $\overline{\mathcal{O}}_P/\mathcal{O}_P$ is a finite dimensional \mathbb{F}_q -vector space whose dimension is denoted by δ_P .

We get the following lemma.

Lemma 2.2. *Let P be an \mathbb{F}_q -rational point of X . Then*

$$\alpha_P(1) - 1 \leq \delta_P.$$

Proof. Let $Q_1, \dots, Q_{\alpha_P(\infty)}$ be the points of $\tilde{X}(\overline{\mathbb{F}}_q)$ lying over P (where the first α points are the \mathbb{F}_q -rational ones), and ϕ the linear map of \mathbb{F}_q -vector spaces

$$\begin{aligned} \phi: \overline{\mathcal{O}}_P &\longrightarrow \mathbb{F}_q^\alpha \\ f &\longmapsto (f(Q_i))_{1 \leq i \leq \alpha} \end{aligned}$$

Let us show that ϕ is surjective. Let $(x_1, \dots, x_\alpha) \in \mathbb{F}_q^\alpha$ and $f_i = x_i \in \mathbb{F}_q \subset \mathbb{F}_q(X)$ for $i \leq \alpha$. For $i \geq \alpha + 1$, we set $f_i = 0$. By the weak approximation theorem, there exists $g \in \mathbb{F}_q(X)$ such that $v_{Q_i}(g - f_i) \geq 1$ for $1 \leq i \leq \alpha_P(\infty)$. Hence, $\phi(g) = (x_1, \dots, x_\alpha)$ and

$$g \in \bigcap_{1 \leq i \leq \alpha_P(\infty)} \mathcal{O}_{Q_i} = \overline{\mathcal{O}}_P.$$

Since $f(Q_1) = \dots = f(Q_\alpha)$ for $f \in \mathcal{O}_P$, we have that $\phi(\mathcal{O}_P)$ is contained in a one dimensional vector space $L \subset \mathbb{F}_q^\alpha$. We obtain a surjective linear map

$$\tilde{\phi} : \overline{\mathcal{O}_P} / \mathcal{O}_P \longrightarrow \mathbb{F}_q^\alpha / L,$$

and the lemma is proved. \square

Proposition 2.3.

$$|\#\tilde{X}(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq \pi - g.$$

Proof. Let P be a point of X and Q be a point of \tilde{X} lying over P . Then P is rational over \mathbb{F}_q if Q is. With the previous notations, we get by Lemma 2.2

$$\#\tilde{X}(\mathbb{F}_q) - \#X(\mathbb{F}_q) = \sum_{P \in \text{Sing } X(\mathbb{F}_q)} (\alpha_P(1) - 1) \leq \sum_{P \in \text{Sing } X(\mathbb{F}_q)} \delta_P \leq \pi - g$$

since

$$\pi - g = \sum_{P \in \text{Sing } X(\overline{\mathbb{F}_q})} \delta_P.$$

On the other hand,

$$\sum_{P \in \text{Sing } X(\mathbb{F}_q)} (\alpha_P(1) - 1) \geq -\#\text{Sing } X(\mathbb{F}_q) \geq -\#\text{Sing } X(\overline{\mathbb{F}_q}) \geq -(\pi - g),$$

which concludes the proof. \square

Thus, the numerator of the zeta function of X is a polynomial with integer coefficients of degree $2g + \Delta_X \in \{2g, \dots, \pi + g\}$, where $\Delta_X = \#\{\tilde{X}(\overline{\mathbb{F}_q}) \setminus X(\overline{\mathbb{F}_q})\}$, whose inverse roots have either modulus \sqrt{q} (for $2g$ of them) or modulus 1 (for Δ_X of them).

Corollary 2.4. *Let $\omega_1, \dots, \omega_{2g}$ be the inverse roots of $P_{\tilde{X}}$, and $\beta_1, \dots, \beta_{\Delta_X}$ be the inverse roots of the cyclotomic part of P_X . Then, for all $n \geq 1$,*

$$\#X(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n - \sum_{j=1}^{\Delta_X} \beta_j^n.$$

In particular,

$$|\#X(\mathbb{F}_q) - (q + 1)| \leq 2g\sqrt{q} + \Delta_X \leq 2g\sqrt{q} + \pi - g \leq 2\pi\sqrt{q},$$

and

$$\dim_{\mathbb{Q}_\ell} H_c^1(X, \mathbb{Q}_\ell) = 2g + \Delta_X. \quad \square$$

Using the last inequality, we get:

Corollary 2.5. *If X is an absolutely irreducible curve which is a complete intersection in \mathbb{P}^{n+1} of n hypersurfaces of degree d_1, \dots, d_n , and if we set $d = \prod_{i=1}^n d_i$, then:*

$$|\sharp X(\mathbb{F}_q) - (q+1)| \leq (d-1)(d-2)\sqrt{q}.$$

In particular, this inequality holds for any absolutely irreducible plane curve of degree d .

Proof. The arithmetic genus π_X of a complete intersection which is given by $2\pi_X = d(\sum_{i=1}^n d_i - n - 2) + 2$ (see [5 p.73]) is clearly at most equal to $(d-1)(d-2)$. The second assertion is obviously a particular case of the first one. Observe that the arithmetic genus of a plane curve of degree d is equal to $\frac{(d-1)(d-2)}{2}$. \square

3. Applications

3.1. Explicit formulas

The explicit formulas given by J.-P. Serre in [5] related to the function field of a curve over a finite field still hold in the singular case, provided that we replace the geometric genus g of the curve by its arithmetic genus π . Furthermore, we can replace (and it is better) g by $g + \frac{\Delta_X}{2}$.

Consider a function $f(\theta) = 1 + 2 \sum_{n \geq 1} c_n \cos n\theta$ which satisfies $f \gg 0$ (i.e. $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$ and $c_n \geq 0$ for all $n \geq 1$). The formula of Corollary 2.4 gives $\sharp X(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n - \sum_{j=1}^{\Delta_X} \beta_j^n$. Arguing as in [5], we get

$$\sharp X(\mathbb{F}_q) \leq a_f \left(g + \frac{\Delta_X}{2} \right) + b_f \leq a_f \pi + b_f \quad (*)$$

with $a_f = 1/\Psi(q^{-1/2})$ and $b_f = 1 + (\Psi(q^{1/2})/\Psi(q^{-1/2}))$, where $\Psi(t) = \sum_{n \geq 1} c_n t^n$.

Furthermore, we obtain the same estimations as the ones in [5] of the maximum number of rational points over \mathbb{F}_2 of a curve for $g + \frac{\Delta_X}{2}$ fixed. For example, a curve with $g + \frac{\Delta_X}{2} \leq 6$ has at most 10 rational points. These remarks have been communicated to the authors by J.-P. Serre.

Another application of this results is the following one. Consider the number

$$N_q(\pi) = \max_X \sharp X(\mathbb{F}_q)$$

where X runs over the curves (possibly singular!) over \mathbb{F}_q of arithmetic genus π . It is of interest to study the behavior of $N_q(\pi)$ for $\pi \rightarrow \infty$. We define, in analogy with the quantity $A(q)$ (see [5] for example), the number $A'(q)$ by

$$A'(q) = \limsup_{\pi \rightarrow \infty} \frac{N_q(\pi)}{\pi}.$$

Corollary 2.5 readily implies $A'(q) \leq 2\sqrt{q}$. In fact, using the previous explicit formulas we get the following bound which is the same as that for $A(q)$ of Drinfeld and Vlăduț (see [2]). Note that we obviously have $A(q) \leq A'(q)$.

Proposition 3.1.

$$A'(q) \leq \sqrt{q} - 1.$$

Proof. Take

$$f_m(\theta) = 1 + 2 \sum_{n=1}^m \left(1 - \frac{n}{m}\right) \cos n\theta = \frac{1}{m} \left| \sum_{k=1}^m e^{ik\theta} \right|^2.$$

Thus $f_m \gg 0$. Now, let

$$\Psi_m(t) = \sum_{n=1}^m \left(1 - \frac{n}{m}\right) t^n.$$

Thus, (*) gives

$$\frac{\#X(\mathbb{F}_q)}{\pi} \leq 1/\Psi_m(q^{-1/2}) + \frac{1}{\pi} \left(1 + (\Psi_m(q^{1/2})/\Psi_m(q^{-1/2}))\right)$$

Since $\Psi_m(t) \rightarrow t/(1-t)$ for $m \rightarrow \infty$ and $|t| < 1$, we get

$$\Psi_m(q^{-1/2}) \rightarrow 1/(\sqrt{q} - 1) \quad \text{for } m \rightarrow \infty.$$

Hence, for any ϵ there exists m_0 such that $m \geq m_0$ implies

$$1/\Psi_m(q^{-1/2}) < \sqrt{q} - 1 + \frac{\epsilon}{2}.$$

For π large enough, the second term of the right hand side of the inequality is less than $\frac{\epsilon}{2}$, and this concludes the proof. \square

3.2. Permutation polynomials and exceptional polynomials

Corollary 2.5 gives us the following explicit form of the Lemma 7.28 of [3]. This result enables us to precise the relationship between permutation polynomials and exceptional polynomials over \mathbb{F}_q (see [3]).

Lemma 3.2. *Let $\phi \in \mathbb{F}_q[x, y]$ be an absolutely irreducible polynomial of degree d and C_ϕ the affine curve of equation $\phi(x, y) = 0$. Set*

$$k_d = \frac{1}{4} \left((d-1)(d-2) + \sqrt{d^2 + 5d - 2} \right)^2.$$

If $q \geq k_d$, then either C_ϕ has a rational point (a, b) with $a \neq b$ or ϕ is of the form $c(x - y)$ for some $c \in \mathbb{F}_q$.

Proof. According to the affine version of Corollary 2.5, the number N of rational points over \mathbb{F}_q of the (affine) curve C_ϕ satisfies

$$|N - q| \leq (d-1)(d-2)\sqrt{q} + d - 1$$

Arguing as in the proof of Lemma 7.28 of [3], the result holds with any k_d such that $q - (d-1)(d-2)\sqrt{q} - 2d + 1 > 0$ for all $q \geq k_d$. \square

Hence, this gives explicit forms for the Propositions 7.29 until 7.33 of [3]. For example, any permutation polynomial of degree 2 is exceptional over \mathbb{F}_q if q is odd.

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