The image of the Borel-Serre bordification in algebraic K-theory

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THE IMAGE OF THE BOREL-SERRE BORDIFICATION IN ALGEBRAIC K-THEORY

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Abstract. We give a method for constructing explicit non-trivial elements in the third K-group (modulo torsion) of an imaginary quadratic number field. These arise from the relative homology of the map attaching the Borel–Serre boundary to the orbit space of the SL_2 group over the ring of imaginary quadratic integers on its symmetric space - hyperbolic three-space. We provide an algorithm which produces a chain of matrix quadruples specifying our element of K_3 of the field, modulo torsion. We carry out the algorithm for the Eisenteinian integers as well as for the imaginary quadratic integers of discriminant -7.

1. Introduction

We use the SL_2-group of the ring of integers $\mathcal{O} := \mathcal{O}_{\mathbb{Q}(\sqrt{-m}\,)}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-m}\,)$, where m is a square-free positive integer, in order to obtain explicit elements of infinite order in the algebraic K-group $K_3(\mathcal{O})/\text{torsion} = K_3(k)/\text{torsion}$.

Let Γ be a subgroup of finite index in $SL_2(\mathcal{O})$, with torsion-free image of Γ in $PSL_2(\mathcal{O})$; and let $\Gamma_1, \ldots, \Gamma_s$ denote the stabilisers inside Γ of the cusps of the action on hyperbolic 3-space. Let $P_\bullet$ be a projective resolution for Γ, and denote the homology of the cone of the natural map

$$\oplus_i P_\bullet(\Gamma_i) \otimes_{\mathbb{Z}[\Gamma_i]} \mathbb{Z} \to P_\bullet(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}\,,$$

by $\mathrm{H}_*(\Gamma; \Gamma_1, \ldots, \Gamma_s)$. Then we have a long exact sequence

$$\cdots \to \oplus_i \mathrm{H}_n(\Gamma_i) \to \mathrm{H}_n(\Gamma) \to \mathrm{H}_n(\Gamma; \Gamma_1, \ldots, \Gamma_s) \to \oplus_i \mathrm{H}_{n-1}(\Gamma_i) \to \mathrm{H}_{n-1}(\Gamma) \to \cdots .$$

Let Δ_i be the stabiliser of the ith cusp inside $\Delta = GL_2(\mathbb{Q}(\sqrt{-m}\,))$, so that Γ is a subgroup of Δ and each Γ_i is a subgroup of Δ_i. We can replace Γ with Δ and each Γ_i with Δ_i in the above, obtaining a long exact sequence of relative homology involving Δ and the Δ_i, and the natural homomorphisms give rise to a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \oplus_i \mathrm{H}_n(\Gamma_i) \arrow[r, "\alpha_n"] \arrow[d] & \mathrm{H}_n(\Gamma) \arrow[r] \arrow[d] & \mathrm{H}_n(\Gamma; \Gamma_1, \ldots, \Gamma_s) \arrow[r] \arrow[d] & \oplus_i \mathrm{H}_{n-1}(\Gamma_i) \arrow[r, "\alpha_{n-1}"] \arrow[d] & \ldots \\
\ldots \arrow[r] & \oplus_i \mathrm{H}_n(\Delta_i) \arrow[r] & \mathrm{H}_n(\Delta) \arrow[r] & \mathrm{H}_n(\Delta; \Delta_1, \ldots, \Delta_s) \arrow[r] & \oplus_i \mathrm{H}_{n-1}(\Delta_i) \arrow[r] & \ldots .
\end{tikzcd} \tag{1.1}$$

Using a result of Suslin, one can see that the map $\oplus_i \mathrm{H}_{n-1}(\Gamma_i) \to \oplus_i \mathrm{H}_{n-1}(\Delta_i)$ is trivial for $n \geq 2$ (cf. Lemma 2.10). Therefore the image of a class in $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ lifts to a class in $\mathrm{H}_3(\Delta)$. From this lift, one can obtain an element in $K_3(k)/\text{torsion} \simeq \mathbb{Z}$ by standard methods. However, in order to show that the resulting element is non-trivial we need some more control. In Section 2 we therefore carry out a variation of this process, and apply it to suitable homology classes obtained from triangulations in Section 4. For the Eisenstein integers, our triangulation actually provides an element of $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$, but in the case of $\mathcal{O}$ a non-principal ideal domain, the fact that there are several $SL_2(\mathcal{O})$–orbits of cusps does force us to bypass $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ completely, which we achieve in Section 4, yielding an algorithm that works for all rings of imaginary quadratic integers.

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2. THE RELATION WITH $K_3(k)$

In this section we modify some by now classical constructions in [9] and [3, p. 73] to suit our current needs. We shall use this in Theorem 4.6 below to show that our algorithm in Section 4 can be used to obtain a non-trivial element in $K_3(k)/\text{torsion}$ if k is an imaginary quadratic number field.

In order to be able to take non-trivial torsion in stabilisers into account, and be able to work in $PSL_2(\mathcal{O})$ instead of $SL_2(\mathcal{O})$ if desired, we work in somewhat greater generality.

Let L be a field, and fix two subgroups $\Lambda \subseteq \widetilde{\Lambda}$ of L^*. (Typically, we have in mind $\Lambda = \{1\}$ or $\{\pm 1\}$, and $\widetilde{\Lambda} = \mathcal{O}^*_{\text{torsion}}$ with $\mathcal{O}$ the ring of algebraic integers in a number field.) Let $\Delta = GL_2(L)/\Lambda$. Let $\mathcal{L}$ be the set of orbits for the action of $\widetilde{\Lambda}$ on $L^2 \setminus \{(0,0)\}$ given by scalar multiplication, which has a natural map to $\mathbb{P}^1_L$. For $n \geq 0$ we let $C_n(\mathcal{L}^2)$ be the free Abelian group with as generators $(n+1)$-tuples $(l_0, \ldots, l_n)$ of elements in $\mathcal{L}$ such that if l_{i_1} and l_{i_2} have the same image in $\mathbb{P}^1_L$ then $l_{i_1} = l_{i_2}$. We shall call such a tuple $(l_0, \ldots, l_n)$ with all l_i distinct in $\mathcal{L}$ (or equivalently, in $\mathbb{P}^1_L$) non-degenerate, and we shall call it degenerate otherwise. Then Δ acts on $C_n(\mathcal{L}^2)$ as $\Lambda \subseteq \widetilde{\Lambda}$, and with the usual boundary map $\mathrm{d} : C_n(\mathcal{L}^2) \to C_{n-1}(\mathcal{L}^2)$ for $n \geq 1$ given by $\mathrm{d}(l_0, \ldots, l_n) = \sum_{i=0}^{n} (-1)^i (l_0, \ldots, \widehat{l_i}, \ldots, l_n)$ we get a complex

$$\cdots \xrightarrow{\mathrm{d}} C_4(\mathcal{L}^2) \xrightarrow{\mathrm{d}} C_3(\mathcal{L}^2) \xrightarrow{\mathrm{d}} C_2(\mathcal{L}^2) \xrightarrow{\mathrm{d}} C_1(\mathcal{L}^2) \xrightarrow{\mathrm{d}} C_0(\mathcal{L}^2)$$

of $\mathbb{Z}[\Delta]$-modules.

Let

$$\tilde{\wedge}^2 L^* = \frac{L^* \otimes_{\mathbb{Z}} L^*}{\langle (x) \otimes (-x) \text{ with } x \text{ in } L^* \rangle}.$$

We denote the class of $a \otimes b$ in $\tilde{\wedge}^2 L^*$ with $(a)\tilde{\wedge}(b)$. It is easily verified that $(a)\tilde{\wedge}(b) + (b)\tilde{\wedge}(a)$ is trivial.

For three non-zero points p_0, p_1 and p_2 in L^2 with distinct images in $\mathbb{P}^1_L$, we define $cr_2(p_0, p_1, p_2)$ in $\tilde{\wedge}^2 L^*$ by the rules:

- $cr_2(gp_1, gp_2, gp_3) = cr_2(p_1, p_2, p_3)$ for every g in $GL_2(L)$;
- $cr_2((1,0), (0,1), (a,b)) = (a)\tilde{\wedge}(b)$.

Because $cr_2((0,1), (1,0), (a,b)) = (b)\tilde{\wedge}(a)$ and $cr_2((1,0), (a,b), (0,1)) = (-ab^{-1})\tilde{\wedge}(b^{-1}) = -(a)\tilde{\wedge}(b)$, we see that cr_2 is alternating. It is also clear that if we scale one of the p_i by λ in $\widetilde{\Lambda}$ then $cr_2(p_0, p_1, p_2)$ changes by a term $(\lambda)\tilde{\wedge}(c)$ with c in L^*. Let

$$\tilde{\wedge}^2 L^* / \widetilde{\Lambda} \tilde{\wedge} L^* = \frac{\tilde{\wedge}^2 L^*}{\langle (\lambda)\tilde{\wedge}(c) \text{ with } \lambda \text{ in } \widetilde{\Lambda} \text{ and } c \text{ in } L^* \rangle}. \tag{2.1}$$

We then obtain a group homomorphism $f_{2,L} : C_2(\mathcal{L}^2) \to \tilde{\wedge}^2 L^* / \widetilde{\Lambda} \tilde{\wedge} L^*$ by letting this be trivial on a degenerate generator (l_0, l_1, l_2), and by mapping a non-degenerate generator (l_0, l_1, l_2) to $cr_2(p_0, p_1, p_2)$ with p_i a point in l_i.

We also define a group homomorphism $f_{3,L} : C_3(\mathcal{L}^2) \to \mathbb{Z}[L^\flat]$, with $L^\flat = L \setminus \{0, 1\}$. We define it to be trivial on a degenerate generator (l_0, l_1, l_2, l_3), and we let it map a non-degenerate generator (l_0, l_1, l_2, l_3) to $[cr_3(\overline{l_0}, \overline{l_1}, \overline{l_2}, \overline{l_3})]$, the cross-ratio of the images of the points in $\mathbb{P}^1_L$. Recall that cr_3 is defined by similar rules as those for cr_2:

- $cr_3(g\overline{l_0}, g\overline{l_1}, g\overline{l_2}, g\overline{l_3}) = cr_3(\overline{l_0}, \overline{l_1}, \overline{l_2}, \overline{l_3})$ for every g in $GL_2(L)$;
- $cr_3([1,0], [0,1], [1,1], [x,1]) = x$ for $x \neq 0, 1$.

We define $\delta_{2,L} : \mathbb{Z}[L^\flat] \to \tilde{\wedge}^2 L^* / \widetilde{\Lambda} \tilde{\wedge} L^*$ by mapping a generator $[x]$ to the class of $(1-x)\tilde{\wedge}(x)$. Then the diagram

$$\begin{array}{ccc} C_3(\mathcal{L}^2) & \xrightarrow{\mathrm{d}} & C_2(\mathcal{L}^2) \\ \downarrow{\scriptstyle f_{3,L}} & & \downarrow{\scriptstyle f_{2,L}} \\ \mathbb{Z}[L^\flat] & \xrightarrow{\delta_{2,L}} & \tilde{\wedge}^2 L^* / \widetilde{\Lambda} \tilde{\wedge} L^* \end{array}$$

commutes. For the non-degenerate case this is a direct calculation: using the $GL_2(L)$-invariance of both $f_{3,L}$ and $f_{2,L}$ and the $GL_2(L)$-equivariance of d we can start with a generator (l_0, l_1, l_2, l_3) containing the classes of $(a, 0)$, $(0, b)$, $(1, 1)$ and (xc, c) in $\mathcal{L}$ for some and a, b and c in L^* and x in $L^\flat$, which results in $[x]$ in $\mathbb{Z}[L^\flat]$ under $f_{3,L}$ and the class of $(1-x)\tilde{\wedge}(x)$ under $f_{2,L} \circ \mathrm{d}$. For a degenerate tuple (l_0, l_1, l_2, l_3) the commutativity is obvious if $\{l_0, l_1, l_2, l_3\}$ has at most two elements as then $f_{2,L}$ is trivial on every term in $\mathrm{d}(l_0, l_1, l_2, l_3)$. If it consists of three classes A, B and C with A occurring twice among l_0, l_1, l_2 and l_3, then up to permuting B and C the six possibilities for (l_0, l_1, l_2, l_3) are (A, A, B, C), (A, B, A, C), (A, B, C, A), (B, A, A, C), (B, A, C, A) and (B, C, A, A). After cancellation of identical terms with opposite signs in $\mathrm{d}(l_0, l_1, l_2, l_3)$, we see that commutativity follows because $f_{2,L}$ is alternating.

From the above diagram we now obtain the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{d}"] & C_4(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] \arrow[d] & C_3(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] \arrow[d, "f_{3,L}"] & C_2(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] \arrow[d, "f_{2,L}"] & C_1(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] \arrow[d] & C_0(\mathcal{L}^2) \\
 & 0 \arrow[r] & \overline{B}_2(L) \arrow[r, "\delta_{2,L}"] & \tilde{\wedge}^2 L^*/\tilde{\Lambda}\tilde{\wedge}L^* \arrow[r] & 0 &
\end{tikzcd} \tag{2.2}$$

with

$$\overline{B}_2(L) = \frac{\mathbb{Z}[L^\flat]}{f_{3,L} \circ \mathrm{d}(C_4(\mathcal{L}^2))} \, . \tag{2.3}$$

We observe that we could take $GL_2(L)$-coinvariants in the top row because of the properties of $f_{3,L}$ and $f_{2,L}$. In particular, $f_{3,L}$ induces a homomorphism $H_3(C_\bullet(\mathcal{L}^2)_\Delta) \to \ker(\delta_{2,L})$.

In order to describe $\overline{B}_2(L)$ more explicitly, we note that $f_{3,L} \circ \mathrm{d}$ maps a non-degenerate generator $(l_0, \ldots, l_4)$ to $\sum_{i=1}^5 (-1)^i cr_3(\overline{l_0}, \ldots, \widehat{\overline{l_i}}, \ldots, \overline{l_4})$ with $\overline{l_0}, \ldots, \overline{l_4}$ distinct points in $\mathbb{P}^1_L$. Using the invariance of cr_3 under the action of $GL_2(L)$, we may use the points $[1, 0]$, $[0, 1]$, $[1, 1]$, $[x, 1]$ and $[y, 1]$ in $\mathbb{P}^1_L$ for x and y in $L^\flat$ with $x \neq y$. This then under $f_{3,L} \circ \mathrm{d}$ yields the 5-term relation

$$[x] - [y] + [y/x] - [(1-y)/(1-x)] + [(1-y^{-1})/(1-x^{-1})] \, . \tag{2.4}$$

For a degenerate generator $(l_0, \ldots, l_4)$ we note that $f_{3,L} \circ \mathrm{d}$ is trivial if $\{l_0, \ldots, l_4\}$ has at most three elements because all the terms in $\mathrm{d}(l_0, \ldots, l_4)$ are degenerate. If this set contains four classes, then after cancelling possible identical terms in $\mathrm{d}(l_0, \ldots, l_4)$ and applying $f_{3,L}$ to the result we see that it is of the form $[cr_3(\overline{m_1}, \ldots, \overline{m_4})] - \mathrm{sgn}(\sigma)[cr_3(\overline{m_{\sigma(1)}}, \ldots, \overline{m_{\sigma(4)}})]$ for a permutation σ in S_4 with sign $\mathrm{sgn}(\sigma)$, and four distinct points $\overline{m_i}$ in $\mathbb{P}^1_L$. Those images generate the subgroup

$$\langle [x] + [x^{-1}] \text{ with } x \text{ in } L^\flat \rangle + \langle [y] + [1-y] \text{ with } y \text{ in } L^\flat \rangle \tag{2.5}$$

of $\mathbb{Z}[L^\flat]$.

Remark 2.6. According to (the proofs of) Lemmas 1.2, 1.3 and 1.4 of [9], if L has at least four elements, then the group $\overline{B}_2(L)$ is a quotient of the group

$$\mathfrak{p}(L) = \frac{\mathbb{Z}[L^\flat]}{\langle [x] - [y] + [y/x] + [(1-x)/(1-y)] - [(1-x^{-1})/(1-y^{-1})] \text{ with } x, y \text{ in } L^\flat, x \neq y \rangle}$$

of loc. cit. It is obtained from $\mathfrak{p}(L)$ by dividing out certain elements of exponent dividing 6 (and their exponent divides 2 if $x^2 - x + 1$ has a solution in L).

Recall from loc. cit. that if L is infinite, then there is a short exact sequence

$$0 \to \mathrm{Tor}(L^*, L^*)^\sim \to K_3(L)_{\mathrm{ind}} \to B(L) \to 0 \, ,$$

where $\mathrm{Tor}(L^*, L^*)^\sim$ denotes the unique non-trivial extension of $\mathrm{Tor}(L^*, L^*)$ by $\mathbb{Z}/2\mathbb{Z}$ if L has characteristic different from 2, and $\mathrm{Tor}(L^*, L^*)$ otherwise, $K_3(L)_{\mathrm{ind}} = \mathrm{coker}(K_3^M(L) \to K_3(L))$, and $B(F)$ is the kernel of $\mathfrak{p}(F) \to F^* \otimes_{\mathbb{Z}} F^*/\langle x \otimes y + y \otimes x \rangle$ under the map $[x] \to x \otimes (1-x)$.

By mapping $x\otimes y$ to $(y)\tilde{\wedge}(x)$, we get a map from the complex $0 \to \mathfrak{p}(L) \to L^*\otimes_{\mathbb{Z}} L^*/\langle x\otimes y+y\otimes x\rangle \to 0$ to the complex in the bottom row of (2.2). Since $2(-x)\otimes x = 2(x\otimes x)$ it is clear from the definition of $\tilde{\wedge}^2 L^*/\widetilde{\Lambda}\tilde{\wedge}L^*$ in (2.1) that the map $F^*\otimes F^*/\langle x\otimes y+y\otimes x\rangle \to \tilde{\wedge}^2 L^*/\widetilde{\Lambda}\tilde{\wedge}L^*$ is surjective with kernel of exponent dividing $a = \mathrm{lcm}(2, |\widetilde{\Lambda}|)$ whenever $\widetilde{\Lambda}$ is finite. Multiplying by a then gives a map $\ker(\delta_{2,L})/\text{torsion} \to B(L)/\text{torsion}$, with the latter isomorphic with $K_3(L)_{\text{ind}}/\text{torsion}$ if L is infinite.

Fix a non-zero point p in L^2, and for $n \geq 0$ let $P_n(L) = P_n(L,p,\Lambda)$ be the free Abelian group with as generators the $(n+1)$-tuples $(g_0,\ldots,g_n)$ of elements in Δ such that if $g_{i_1}p$ and $g_{i_2}p$ give the same point in $\mathbb{P}^1_L$ then $g_{i_1}p = g_{i_2}p$ already in $\mathcal{L}$. Clearly, $P_n(L)$ is a projective $\mathbb{Z}[\Delta]$-module, and $P_\bullet(L)$ with the boundary $\mathrm{d} : P_n(L) \to P_{n-1}(L)$ for $n \geq 1$ given by $\mathrm{d}(g_0,\ldots,g_n) = \sum_{i=0}^n (-1)^i (g_0,\ldots,\widehat{g_i},\ldots,g_n)$ is a complex of projective $\mathbb{Z}[\Delta]$-modules. The evaluation homomorphism $P_n(L) \to C_n(\mathcal{L}^2)$ mapping $(g_0,\ldots,g_n)$ to $(g_0p,\ldots,g_np)$ gives the commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "\mathrm{d}"] & P_4(L) \arrow[r, "\mathrm{d}"] \arrow[d] & P_3(L) \arrow[r, "\mathrm{d}"] \arrow[d] & P_2(L) \arrow[r, "\mathrm{d}"] \arrow[d] & P_1(L) \arrow[r, "\mathrm{d}"] \arrow[d] & P_0(L) \arrow[d] \\
\cdots \arrow[r, "\mathrm{d}"] & C_4(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] & C_3(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] & C_2(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] & C_1(\mathcal{L}^2) \arrow[r, "\mathrm{d}"] & C_0(\mathcal{L}^2)
\end{tikzcd}
\tag{2.7}
$$

of $\mathbb{Z}[\Delta]$-modules, and we obtain a homomorphism $H_3(P_\bullet(L)_\Delta) \to H_3(C_\bullet(\mathcal{L}^2)_\Delta)$. Combining it with the map in (2.2), we obtain a homomorphism

$$H_3(P_\bullet(L)_\Delta) \to \ker(\delta_{2,L}) . \tag{2.8}$$

Remark 2.9. There are augmentations $P_\bullet(L) \to \mathbb{Z}$ and $C_\bullet(\mathcal{L}^2) \to \mathbb{Z}$, and if L is infinite those give $\mathbb{Z}[\Delta]$-resolutions of $\mathbb{Z}$, with the first being a projective resolution. Clearly $P_\bullet(L)$ is a subcomplex of the standard resolution for Δ, so we obtain a canonical isomorphism $H_3(P_\bullet(L)_\Delta) \xrightarrow{\simeq} H_3(\Delta,\mathbb{Z})$. In this case the morphism $P_\bullet(L) \to C_\bullet(\mathcal{L}^2)$ is the (up to chain homotopy unique) morphism of $\mathbb{Z}[\Delta]$-resolutions of $\mathbb{Z}$.

Now let p be the point $[1,0]$ in $\mathbb{P}^1_L$ and consider its stabiliser

$$\Delta^\infty = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ with } a,c \text{ in } L^*, b \text{ in } L \right\} / \Lambda\cdot \mathrm{id}_2$$

in Δ. Let $N_1 = \{\left(\begin{smallmatrix} 1 & b \\ 0 & c \end{smallmatrix}\right) \text{ with } b \text{ in } L, c \text{ in } L^*\} \subset GL_2(L)$ and $N_{1,1}$ the subgroup of N_1 where $c=1$. Let Δ^∞_1 be the image of $\widetilde{\Lambda}\cdot N_1$ in Δ^∞ and let $\Delta^\infty_{1,1}$ be a subgroup of the image of $\widetilde{\Lambda}\cdot N_{1,1}$.

Lemma 2.10. *Let $m \geq 1$ and assume $[L:\widetilde{\Lambda}]$ is finite but L is infinite.*

1. *The image of the map $H_m(\Delta^\infty_{1,1},\mathbb{Z}) \to H_m(\Delta^\infty_1,\mathbb{Z})$ is killed by multiplication by $[\widetilde{\Lambda}:\Lambda]$.*
2. *If $\widetilde{\Lambda}/\Lambda$ is cyclic and m is even then the image of the map $H_m(\Delta^\infty_{1,1},\mathbb{Z}) \to H_m(\Delta^\infty_1,\mathbb{Z})$ is trivial.*

Proof. It will suffice to prove those results for $\Delta^\infty_{1,1}$ equal to the image of $\widetilde{\Lambda}\cdot N_{1,1}$ in Δ^∞_1. Note that we have compatible isomorphisms $\Delta^\infty_1 \simeq \widetilde{\Lambda}/\Lambda \times N_1$ and $\Delta^\infty_{1,1} \simeq \widetilde{\Lambda}/\Lambda \times N_{1,1}$. If $N_1' \subset N_1$ consists of the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)$ with c in L^* then by [10, Theorem 1.9] the natural map $N_1 \to N_1'$ induces an isomorphism on homology. The composition $\Delta^\infty_1 \simeq \widetilde{\Lambda}/\Lambda \times N_1 \to \widetilde{\Lambda}/L \times N_1'$ then induces an isomorphism on homology. Since this composition maps $\Delta^\infty_{1,1}$ to $\widetilde{\Lambda}/L \times \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$, the result is clear. $\square$

Let $p_1,\ldots,p_s$ with $s \geq 1$ be distinct points in $\mathbb{P}^1_L$. Fix elements $A_1,\ldots,A_s$ in Δ with $p_j = A_j[1,0]$. Then $A_{j_1}^{-1}A_{j_2}$ is in Δ^∞_1 if and only if $j_1 = j_2$.

We shall consider elements in $P_3(L)$ satisfying the following. (In Proposition 4.5 we shall prove that it is fulfilled by the elements obtained from our algorithm in Section 4.)

Assumption 2.11. *Suppose α in $P_3(L)$ is such that the following hold.*

(1) *It is a sum of* (g_0, g_1, g_2, g_3) *with each* g_i *in a coset* $A_{j_i}\Delta_1^\infty$.
(2) *In* $P_2(L)_\Delta$, *we can write* $\mathrm{d}\alpha = \sum_{j=1}^s A_j\beta_j$ *for* β_j *in the kernel of*

$$\mathrm{d} : \mathbb{Z}[\Delta_{1,1}^\infty \times \Delta_{1,1}^\infty \times \Delta_{1,1}^\infty]_{\Delta_{1,1}^\infty} \to \mathbb{Z}[\Delta_{1,1}^\infty \times \Delta_{1,1}^\infty]_{\Delta_{1,1}^\infty}.$$

Note that each (g_0, g_1, g_2, g_3) in the first condition is in $P_3(L)$ for $p = (1, 0)$: $A_{j_1}\gamma_1[1, 0] = A_{j_2}\gamma_2[1, 0]$ with the γ_i in Δ_1^∞ implies $j_1 = j_2$, and $\gamma_1(1, 0) = \gamma_2(1, 0)$ already in $\mathcal{L}$ because of the definition of Δ_1^∞.

We now prove that under mild hypotheses we can associate an element in $\ker(\delta_{2,L})$ to any α satisfying Assumption 2.11.

Proposition 2.12. *Assume* $[\widetilde{\Lambda} : \Lambda]$ *is finite, and let* $t \geq 1$ *be an integer such that the image of* $H_2(\Delta_{1,1}^\infty, \mathbb{Z})$ *in* $H_2(\Delta_1^\infty, \mathbb{Z})$ *is annihilated by multiplication by* t. *To* α *satisfying Assumption 2.11 we can then associate a well-defined element in* $\ker(\delta_{2,L})$ *by computing the map in* (2.8) *with* $p = [1, 0]$ *on those* (g_0, g_1, g_2, g_3) *in* $t\alpha$ *for which the* $g_0, \ldots, g_3$ *lie in four distinct cosets* $A_j\Delta_1^\infty$.

Proof. Note that a t exists by Lemma 2.10 if L is infinite, and it obviously exists if L is finite. We can write $t\beta_j = \mathrm{d}\tilde{\beta}_j$ in $\mathbb{Z}[\Delta_1^\infty \times \Delta_1^\infty \times \Delta_1^\infty]_{\Delta_1^\infty}$ for some $\tilde{\beta}_j$ in $\mathbb{Z}[\Delta_1^\infty \times \Delta_1^\infty \times \Delta_1^\infty \times \Delta_1^\infty]$, which is unique up to adding an element in the kernel of $\mathbb{Z}[\Delta_1^\infty \times \Delta_1^\infty \times \Delta_1^\infty \times \Delta_1^\infty] \to \mathbb{Z}[\Delta_1^\infty \times \Delta_1^\infty \times \Delta_1^\infty]_{\Delta_1^\infty}$. From the definitions we clearly have that the standard resolution of Δ_1^∞ is a subcomplex of $P_\bullet(L)$, so that

$$\tilde{\alpha} = t\alpha - \sum_{j=1}^s A_j\tilde{\beta}_j$$

gives a well-defined class in $H_3(P_\bullet(L)_\Delta)$ modulo the image of $H_3(\Delta_1^\infty, \mathbb{Z})$. Note that under the composition (2.8) of the homomorphisms obtained from (2.7) and (2.2), the $\tilde{\beta}_j$ do not contribute to the image (and hence this image is independent of the choice of the $\tilde{\beta}_j$) because $\gamma'[1, 0] = [1, 0]$ for all γ' in Δ_1^∞, so they only result in combinations of degenerate generators in $C_3(\mathcal{L}^2)$, which are killed by $f_{3,L}$. Similarly, the only terms $(A_{j_1}\gamma_1, A_{j_2}\gamma_2, A_{j_3}\gamma_3, A_{j_4}\gamma_4)$ (with all γ_i in Δ_1^∞) in α that contribute are those where all A_{j_i} are distinct. ☐

Remark 2.13. If $\widetilde{\Lambda}$ is finite, and $a = \mathrm{lcm}(2, |\widetilde{\Lambda}|)$, then according to Remark 2.6, multiplication by a gives us a map $\ker(\delta_{2,L})/\text{torsion} \to B(L)/\text{torsion}$. If L is infinite then the latter is isomorphic with $K_3(L)_{\text{ind}}/\text{torsion}$. So for finite $\widetilde{\Lambda}$ and infinite L, Proposition 2.12 gives a way of constructing an element in $K_3(L)_{\text{ind}}/\text{torsion}$ out of α. We shall use this in Section 4 in order to show that for an imaginary quadratic number field k, we can obtain elements of infinite order in $K_3(k)/\text{torsion} \simeq \mathbb{Z}$ from suitable triangulations of hyperbolic 3-space.

3. The Borel-Serre boundary

Let k be an imaginary quadratic number field, and let $\mathcal{O}$ be its ring of algebraic integers. We work with the Borel–Serre bordification $\widehat{\mathcal{H}}$ of hyperbolic 3-space $\mathcal{H}$, which yields a compact orbit space ${}_{\Gamma}\backslash\widehat{\mathcal{H}}$ [7, appendix]. Recall from [7, page 512] the identification, for a torsion-free subgroup Γ of finite index in $SL_2(\mathcal{O})$, between the integral homology of the cusp stabiliser Γ_i and the integral homology of the torus $\mathbf{T}_i$ joined at the ith cusp by the Borel–Serre compactification. Namely,

$$\oplus_i \mathrm{H}_n(\Gamma_i;\, \mathbb{Z}) \cong \oplus_i \mathrm{H}_n(\mathbf{T}_i;\, \mathbb{Z}) \cong \mathrm{H}_n(\partial({}_{\Gamma}\backslash\mathcal{H});\, \mathbb{Z}),$$

where the direct sum runs over the orbits of cusps modulo Γ, and the identification passes over to the boundary of the quotient of hyperbolic 3–space $\mathcal{H}$ by Γ. Consider the map α induced on homology when attaching the boundary in the Borel–Serre compactification of ${}_{\Gamma}\backslash\mathcal{H}$.

It turns out [5] that the above relative homology group $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ is precisely the kernel of the map α_2 induced on second degree homology by the attaching map α. So the answer given in [5] to a question of Serre provides an isomorphism from $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ to the one-generator group $\langle \cup_i \mathbf{T}_i \rangle \cong \mathbb{Z}$.

This is the case because we can locate the induced maps α_n in the first of the long exact sequences (1.1). Taking into account that the cohomological dimension of Γ is 2, this long exact sequence concentrates in

$$\begin{tikzcd}[column sep=small]
 & \ldots \arrow[r, "\alpha_3 = 0"] & \mathrm{H}_3(\Gamma) = 0 \arrow[dll, out=-10, in=170] \\
\langle \cup_i \mathbf{T}_i \rangle \arrow[r] & \bigoplus_i \langle \mathbf{T}_i \rangle \arrow[r, "\alpha_2"] & \bigoplus_i \langle \mathbf{T}_i \rangle / _{\langle \cup_i \mathbf{T}_i \rangle} \oplus \mathrm{H}_2^{cusp} \arrow[dll, out=-10, in=170] \\
\mathrm{H}_2^{cusp} \bigoplus_i \langle x_i \rangle \arrow[r] & \bigoplus_i \langle x_i, y_i \rangle \arrow[r, "\alpha_1"] & \bigoplus_i \langle y_i \rangle \oplus \mathrm{H}_1^{cusp} \arrow[dll, out=-10, in=170] \\
\mathrm{H}_1^{cusp} \oplus \left\{ \substack{\text{vanishing} \\ \text{ideal}} \right\} \arrow[r] & \bigoplus_i \mathbb{Z} \arrow[r, "\alpha_0", "\text{augmentation}"'] & \mathbb{Z} \to 0
\end{tikzcd}$$

where x_i and y_i denote the cycles generating $\mathrm{H}_1(\mathbf{T}_i)$, and the maps without labels are the obvious restriction maps making the sequence exact. Note that the cuspidal homology satisfies equality of ranks between H_2^{cusp} and H_1^{cusp}, because the naive Euler-Poincaré characteristic vanishes.

The isomorphism from $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ to the one-generator group $\langle \cup_i \mathbf{T}_i \rangle \cong \mathbb{Z}$ tells us that a representative for the generator of $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ is given by the Borel–Serre bordification $\widehat{F}$ of an ideal fundamental polyhedron F for the action of Γ on $\mathcal{H}$, because the boundary of ${}_{\Gamma}\backslash\widehat{F}$ is $\cup_i \mathbf{T}_i$. In $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$, the polyhedron $\widehat{F}$ is given as a chain of matrix quadruples: we triangulate $\widehat{F}$ into tetrahedra, fix a basepoint, and then replace each tetrahedron by the quadruple of matrices that map the basepoint to its four corners. The vertices of the tetrahedra are actually on one Δ–orbit, because we have chosen F to be an ideal polyhedron (all of its vertices are cusps). If $\mathcal{O}$ is a principal ideal domain, then these vertices are even on one Γ–orbit, because there is just one $SL_2(\mathcal{O})$–orbit of cusps in this case.

4. Explicit determination of the homology classes

In this section, we outline an algorithm to compute a chain of matrix quadruples satisfying Assumption 2.11, for an imaginary quadratic number field k with ring of algebraic integers $\mathcal{O}$.

We call a polyhedron in hyperbolic 3–space $\mathcal{H}$ an ideal polyhedron, if all of its vertices are cusps. Our algorithm inputs an ideal polyhedron F tessellated by finitely many fundamental domains for the action of $SL_2(\mathcal{O})$ on $\mathcal{H}$, such that the facets of F occur in pairs with an $SL_2(\mathcal{O})$-identification between them. Yasaki [12] has conceived and implemented an algorithm which, for all $\mathcal{O}$, produces ideal polyhedra satifying this condition. He has obtained explicit polyhedra for all discriminants of absolute value smaller than 100, and all imaginary quadratic fields of class number 1 and 2. So we know that our input object exists and is practically available for our computations. We equip the Borel–Serre boundary of hyperbolic 3-space $\mathcal{H}$ with a coordinate system by taking the cusp in $\mathbb{C} \cup \{\infty\}$ as the first coordinate, and the position on the attached real plane, expressed as a number in $\mathbb{C}$, as the second coordinate.

Before we proceed we need some preliminaries. Recall that $SL_2(\mathbb{C})$ acts on the boundary of hyperbolic 3-space by fractional linear transformations, via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The $SL_2(\mathcal{O})$-orbits of $\mathbb{P}^1_k$ for this action correspond to the ideal class group of k, with the principal ideals corresponding to the orbit of ∞.

Lemma 4.1 (Flöge)**.** *Let $\frac{\alpha}{\beta}$ in $\mathbb{P}^1_k$ be any cusp with (α, β) not a principal ideal. Then its stabiliser in $PSL_2(\mathcal{O})$ is isomorphic to the Abelian group (with respect to natural addition)*

$$\mathcal{A} := \left\{ c \in \mathcal{O} \mid c\frac{\alpha}{\beta} \in \mathcal{O} \text{ and } c\frac{\alpha^2}{\beta^2} \in \mathcal{O} \right\}$$

by the map

$$c \mapsto \pm \begin{pmatrix} 1 - c\frac{\alpha}{\beta} & c(\frac{\alpha}{\beta})^2 \\ -c & 1 + c\frac{\alpha}{\beta} \end{pmatrix}.$$

Proof. Given in [2, 9.3]. Note that Flöge used the action formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az-b}{cz-d}$. ☐

Corollary 4.2. *Let $\frac{\alpha}{\beta}$ in $\mathbb{P}^1_k$ be any cusp with (α, β) not a principal ideal, and let $M_{\alpha,\beta} = \begin{pmatrix} \alpha & -\frac{1}{\beta} \\ \beta & 0 \end{pmatrix}$. Then $M^{-1}_{\alpha,\beta}$ conjugates the stabiliser of $\frac{\alpha}{\beta}$ in $SL_2(\mathcal{O})$ to the group $\pm \begin{pmatrix} 1 & \frac{\mathcal{O}}{\alpha^2} \cap \frac{\mathcal{O}}{\beta^2} \\ 0 & 1 \end{pmatrix}$.*

Proof. We perform the conjugation using the explicit shape of the cusp stabiliser given in Flöge's lemma above, finding

$$\begin{pmatrix} 0 & \frac{1}{\beta} \\ -\beta & \alpha \end{pmatrix} \left(\pm \begin{pmatrix} 1 - c\frac{\alpha}{\beta} & c(\frac{\alpha}{\beta})^2 \\ -c & 1 + c\frac{\alpha}{\beta} \end{pmatrix} \right) \begin{pmatrix} \alpha & -\frac{1}{\beta} \\ \beta & 0 \end{pmatrix} = \pm \begin{pmatrix} 1 & \frac{c}{\beta^2} \\ 0 & 1 \end{pmatrix}.$$

Then by Lemma 4.1, the cusp stabiliser gets conjugated to the group $\pm \begin{pmatrix} 1 & \frac{1}{\beta^2}\mathcal{A} \\ 0 & 1 \end{pmatrix}$, with $\mathcal{A} := \left\{ c \in \mathcal{O} \mid c\frac{\alpha}{\beta} \in \mathcal{O} \text{ and } c\frac{\alpha^2}{\beta^2} \in \mathcal{O} \right\}$. We have $\frac{\mathcal{A}}{\beta^2} = \frac{\mathcal{O}}{\alpha^2} \cap \frac{\mathcal{O}}{\alpha\beta} \cap \frac{\mathcal{O}}{\beta^2}$, which equals $\frac{\mathcal{O}}{\alpha^2} \cap \frac{\mathcal{O}}{\beta^2}$ by considering the factorisations of the fractional ideals for $\mathcal{O}$. ☐

We now once and for all fix representatives (α, β) of the ideal class group of k, and matrices $M_{\alpha,\beta}$ as above. (For the principal ideal we take $(1, 0)$ and $M_{1,0} = \mathrm{id}_2$ the 2×2-identity matrix.) Let $C_1, \ldots, C_h$ (with h the class number of k) be the matrices thus obtained. Then every cusp in $\mathbb{P}^1_k$ is of the form $gC_i \cdot \infty$ for some g in $SL_2(\mathcal{O})$, with i uniquely determined by the $SL_2(\mathcal{O})$-orbit of the cusp.

Note that $g_1 C_{i_1} \cdot \infty = g_2 C_{i_2} \cdot \infty$, with the g_i in $SL_2(\mathcal{O})$, implies that $i_1 = i_2$, and with $i = i_1 = i_2$ we have that $C_i^{-1} g_2^{-1} g_1 C_i$ is in

$$\left\{ \begin{pmatrix} u & \mu \\ 0 & u^{-1} \end{pmatrix} \text{ with } u \text{ in } \mathcal{O}^* \text{ and } \mu \text{ in } \mathcal{I} \right\} \tag{4.3}$$

for some fractional ideal $\mathcal{I}$ of k. Namely, for the orbit of ∞ this is clear because the corresponding C_i is the 2×2-identity matrix; and for a cusp $C_i \cdot \infty$ corresponding to a non-principal ideal this follows from Corollary 4.2, using that $\mathcal{O}^* = \{\pm 1\}$, because the only two imaginary quadratic fields with $\mathcal{O}^* \neq \{\pm 1\}$ are $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-1})$, which have class number 1.

Take F as at the beginning of this section, i.e., it is an ideal polyhedron tessellated by finitely many fundamental domains for the action of $SL_2(\mathcal{O})$ on $\mathcal{H}$, such that the facets of F occur in pairs with an $SL_2(\mathcal{O})$-identification between them. Let $p_1, \dots, p_s$ be the cusps in the boundary of F. Fix elements $A_1, \dots, A_s$ in $GL_2(k)$ of the form gC_i with g in $SL_2(\mathcal{O})$ such that $p_j = A_j \cdot \infty$. On the plane attached to the cusp p_j, we induce the origin and axes for the second coordinate by transport with the matrix A_j of the origin and axes on the plane at the cusp ∞. Denote the group $\{\pm \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right) \mid \mu \in \mathcal{O}\}$ by $\mathcal{U}$. We then have the following algorithm.

Algorithm 4.4. *Input:* An ideal polyhedron F tessellated by finitely many fundamental domains for the action of $SL_2(\mathcal{O})$ on $\mathcal{H}$, such that the facets of F occur in $SL_2(\mathcal{O})$–conjugate pairs of opposite orientation, and such that the basepoint $b := (\infty, 0)$ lies in the boundary of F.

First step: Triangulate the Borel–Serre bordification $\widehat{F}$ of F.

- Triangulate the surface of $\widehat{F}$. *This is always possible since $\widehat{F}$ is a polyhedron and hence its surface is homeomorphic to a 2–sphere.* As $\widehat{F}$ is an ideal polyhedron, we can choose all of the vertices of the triangulation to lie again at the cusps; and we do so.
- Omit the triangles that touch the basepoint b, in order to obtain a set of base triangles.
- For each base triangle, record the tetrahedron that it spans together with b. *This collection of tetrahedra is the triangulation that we want.*

Second and final step: Express $\widehat{F}$ as a chain of matrix quadruples, such that each of the ideal tetrahedra is specified by the quadruple of matrices that map the basepoint b to its four corners. We obtain these matrices by concatenating

- a $\mathcal{U}$–move in the plane attached by the Borel–Serre construction at the cusp ∞ with
- the fixed matrix A_i moving the cusp ∞ to the cusp where the i-th corner of the tetrahedron is located.

We observe that the matrices chosen here are unique up to multiplication by $\pm \mathrm{id}_2$. Moreover, if we have $g_i \cdot b = h g_j \cdot b$ for moves g_i and g_j, and h in $SL_2(\mathcal{O})$, then $g_i^{-1} h g_j$ is in the group in (4.3) and stabilises b, hence is of the form $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ with u in $\mathcal{O}^*$. Taking, in the notation of Section 2, $\Lambda = \widetilde{\Lambda} = \mathcal{O}^*$, and $\Delta = GL_2(k)/\widetilde{\Lambda} \cdot \mathrm{id}_2$, we have well-defined quadruples (g_0, g_1, g_2, g_3) of classes in Δ, and the geometric identification of points under the action of $SL_2(\mathcal{O})$ now gives rise to identification of the classes in Δ of the corresponding chosen matrices. Finally, we notice that from (4.3) it is also clear that the quadruples are all in $P_3(k)$.

We now show that the resulting element α in $P_3(k)$ satisfies Assumption 2.11.

Proposition 4.5. *The formal sum α of quadruples of matrix classes produced by Algorithm 4.4 has the following properties.*

(1) *It is a sum of (g_0, g_1, g_2, g_3) with each g_i in some coset $A_{j_i}\mathcal{U}$.*
(2) *In $P_2(k)_\Delta$, we can write $\mathrm{d}\alpha = \sum_{j=1}^s A_j \beta_j$ for β_j in the kernel of*

$$\mathrm{d} : \mathbb{Z}[\mathcal{U} \times \mathcal{U} \times \mathcal{U}]_{\mathcal{U}} \to \mathbb{Z}[\mathcal{U} \times \mathcal{U}]_{\mathcal{U}}.$$

Proof. (1) The second step of Algorithm 4.4 places the matrices g_i into some coset $A_{j_i}\mathcal{U}$.

(2) By Poincaré's theorem on fundamental polyhedra [4], the properties of F give rise to a finite index subgroup Γ of $SL_2(\mathcal{O})$, such that F is a fundamental domain for Γ, strict in its interior, and such that inside hyperbolic space, the polyhedron F has boundary zero modulo the action of Γ. Denoting the stabiliser in Γ of the cusp ∞ by Γ_∞, this allows us to write $\mathrm{d}\alpha = \sum_{j=1}^{s} A_j\beta_j$ in $\mathbb{Z}[\Delta \times \Delta \times \Delta]_\Gamma$ with β_j in $\mathbb{Z}[\Gamma_\infty \times \Gamma_\infty \times \Gamma_\infty]_{\Gamma_\infty}$, where Γ_∞ is clearly contained in $\mathcal{U}$. As F is an ideal polyhedron, the orbit space F/Γ is a hyperbolic manifold, and hence the image of Γ in $PSL_2(\mathcal{O})$ is torsion-free. By the construction of the Borel-Serre compactification [7], the boundary of $\widehat{F}$ modulo Γ consists of one 2–torus at each cusp. Tori are closed surfaces that admit no boundary, so for all j we obtain $\mathrm{d}(\beta_j) = 0$.

□

Note that it is possible to obtain an explicit description of the group Γ involved in the above proof [6], but that this is outside of the aims of the present paper.

We shall now show that the classes obtained in Algorithm 4.4 give rise to non-zero classes in $K_3(\mathcal{O})/\text{torsion} = K_3(k)/\text{torsion} \simeq \mathbb{Z}$ under the method discussed in Remark 2.13.

We observe that $K_3^M(k)$ is 2-torsion for any number field k by [11, Theorem 8] (and in fact trivial if k is totally imaginary). Therefore $K_3(k)_{\text{ind}}/\text{torsion} = K_3(k)/\text{torsion}$. As stated in loc. cit., the rank of $K_3(k)$ is given by the number of infinite places, so that for an imaginary quadratic field k we have $K_3(k)_{\text{ind}}/\text{torsion} = K_3(k)/\text{torsion} \simeq \mathbb{Z}$.

Theorem 4.6. *Let k be an imaginary quadratic number field with ring of algebraic integers $\mathcal{O}$, and let α in $\mathbb{Z}[\Delta\times\Delta\times\Delta\times\Delta]_\Delta$ be a class obtained from Algorithm 4.4. Then Proposition 2.12 applies to α with $\Lambda = \widetilde{\Lambda} = \mathcal{O}^*$. Multiplying the resulting element in $\ker(\delta_{2,k})$ by $a = \mathrm{lcm}(2, |\widetilde{\Lambda}|)$ as in Remark 2.6, we obtain a non-zero element in $K_3(k)/\text{torsion}$.*

Proof. That α satisfies the conditions in Assumption 2.11 is stated in Proposition 4.5. According to Lemma 2.10(2), Proposition 2.12 applies to α with $t = 1$.

Clearly Remark 2.6 applies, so it only remains to show that the resulting element in $B(k)/\text{torsion} \subseteq \mathfrak{p}(k)/\text{torsion}$ is non-trivial. But the dilogarithm $D : \mathbb{C}^\flat \to \mathbb{R}$ applied to the generators of $\mathbb{Z}[\mathbb{C}^\flat]$ satisfies all the relations imposed when creating $\mathfrak{p}(k)$ or $\overline{B}_2(k)$. Moreover, if an ideal tetrahedron in $\mathcal{H}$ has vertices a_0, a_1, a_2 and a_3 in $\mathbb{P}^1_\mathbb{C}$, then its hyperbolic volume equals $cr_3(a_0, a_1, a_2, a_3)$. (For a discussion of those facts see, e.g., [13, §2].)

But according to Proposition 2.12, the only (g_0, g_1, g_2, g_3) in α that contribute to the image in $\mathfrak{p}(L)$ are those for which the four cusps $Q_i = g_i \cdot [1, 0]$ in $\mathbb{P}^1_\mathbb{C}$ are distinct. Applying the dilogarithm to the resulting generator $[cr_3(Q_0, Q_1, Q_2, Q_3)]$ in $\mathfrak{p}(L)$ then gives the volume of the corresponding ideal tetrahedron in $\mathcal{H}$ in the triangulation of $\mathcal{H}$ obtained by collapsing the boundary in the Borel-Serre bordification to single points at the cusps. Under this process all other (g_0, g_1, g_2, g_3) collapse to degenerate tetrahedra, so in total we obtain the volume of the triangulation, which is non-zero. □

5. Examples

In this section, we consider some explicit examples for the machinery developed in Sections 2 and 4, by applying our algorithm to two cases where the ring of integers is Euclidean. For $\mathcal{O}$ Euclidean, Cremona [1] has given a coordinate description of an ideal polyhedron in hyperbolic 3-space fulfilling our algorithm's input condition, and we use the coordinates printed there.

5.1. **The Eisenstein integers case.**

We consider the case of the Eisenstein integers, namely the ring $\mathcal{O}_{-3} := \mathcal{O}_k$ for $k = \mathbb{Q}(\sqrt{-3})$. Let $\rho := \frac{1+\sqrt{-3}}{2}$.

The ideal polyhedron F specified for $\mathcal{O}_{-3}$ in [1] admits as symmetry group the group of order 12 generated by $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \rho - 1 \\ \rho & 0 \end{pmatrix}$. Therefore the group Γ discussed in the proof of Proposition 4.5 is a normal subgroup of index 12 in $SL_2(\mathcal{O}_{-3})$, such that the quotient of $SL_2(\mathcal{O}_{-3})$ by Γ is that symmetry group.

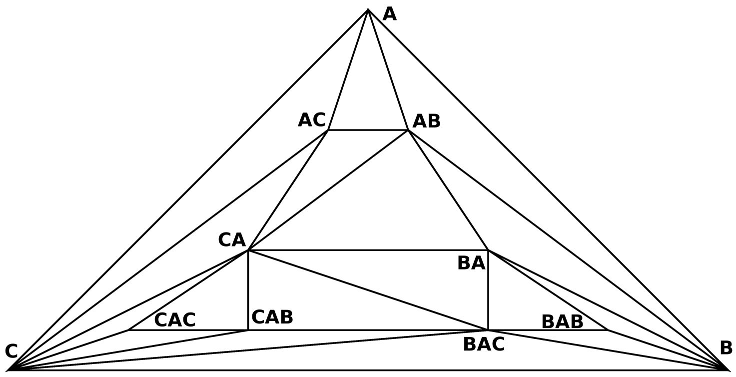

FIGURE 5.1. Base triangles for $\mathcal{O}_{-3}$. Each vertex is specified by the matrix which moves the point $(\infty, 0)$ into it.

Then the base triangles of the triangulation of the Borel–Serre compactification $\widehat{F}$ of F for the action of Γ are given in Figure 5.1. Recall that the ideal tetrahedra of our triangulation are spanned as the convex hulls of a base triangle together with the basepoint. The set of cusps in the boundary of F is $\{0, 1, \rho, \infty\}$. We will make use of the following matrices in Γ, which map some (but not necessarily all) of the elements of this set of cusps back into this set.

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrices B and C are in the cusp stabiliser Γ_∞; concatenating them *after* the matrix A gives the moves from the cusp at ∞ to the cusps at the corners of F; and we generate them with the three basic moves $A \cdot \infty = 0$, $\ B \cdot 0 = \rho$ and $C \cdot 0 = 1$. Combining with words in B and C concatenated *before* the matrix A, allows us to express all of the vertices of the base triangles as images of the basepoint $b = (\infty, 0)$. The coordinates of the vertices of the base triangles in Figure 5.1 are then the following:

$$\begin{array}{llll} (\infty,\rho) = B\cdot b, & (\infty,1) = C\cdot b, & (0,0) = A\cdot b, & (0,1) = AC\cdot b, \\ (0,\rho) = AB\cdot b, & (1,0) = CA\cdot b, & (1,1) = CAC\cdot b, & (1,\rho) = CAB\cdot b, \\ (\rho,0) = BA\cdot b, & (\rho,1) = BAC\cdot b, & (\rho,\rho) = BAB\cdot b. & \end{array}$$

The tetrahedra of the triangulation, specified by these base triangles, of $\widehat{F}$ then give rise to the following chain of matrix quadruples α:

$$\begin{gathered} (1, CAC, C, CAB) + (1, CAB, C, BAC) + (1, BAC, C, B) + (1, BAC, B, BAB) \\ +(1, BA, BAC, BAB) + (1, BA, BAB, B) + (1, AB, BA, B) + (1, A, AB, B) \\ +(1, A, AC, AB) + (1, A, C, AC) + (1, AC, C, CA) + (1, CA, C, CAC) + (1, CA, CAC, CAB) \\ +(1, CA, CAB, BAC) + (1, CA, BAC, BA) + (1, CA, BA, AB) + (1, CA, AB, AC). \end{gathered}$$

Here the quadruple $(1, CA, BA, AB)$ specifies the only tetrahedron of non-zero volume. Letting those matrices act on $[1, 0]$ and taking the cross-ratio of the resulting four points yields $-1/(\rho-1) = \rho$, so that the element we obtain in $\ker(\delta_{2,k})$ is $[\rho]$. Note that in this triangulation we use only Γ, not the full group $SL_2(\mathcal{O})$, and we can strengthen the discussion after Algorithm 4.4 and Proposition 4.5. Namely, because the image of Γ in $PSL_2(\mathcal{O})$ is torsion-free by the proof of Proposition 4.5, we can use $\Lambda = \widetilde{\Lambda} = \{\pm 1\}$ and $\Delta = GL_2(k)/\{\pm \mathrm{id}_2\}$ in Theorem 4.6. Then $a = 2$, and the non-trivial element we obtain in $B(k)/\text{torsion} \simeq K_3(k)/\text{torsion}$ is $2[\rho]$.

5.2. **The case of discriminant -7.** Let $\omega := \frac{1+\sqrt{-7}}{2}$ in $k = \mathbb{Q}(\sqrt{-7})$. The ideal polyhedron F specified for $\mathcal{O}_{-7}$ in [1] admits as symmetry group the group of order 6 generated by

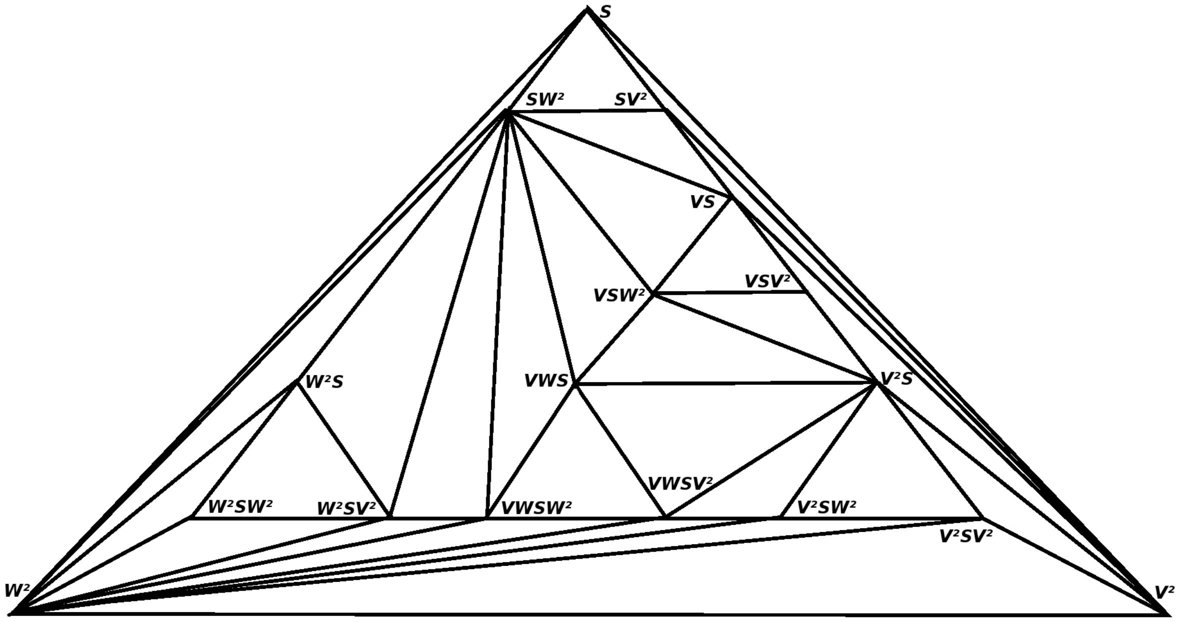

FIGURE 5.2. Base triangles for $\mathcal{O}_{-7}$. Each vertex is specified by the matrix which moves the point $(\infty, 0)$ into it.

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & \omega \\ \omega - 1 & 1 \end{pmatrix}.$$

Therefore the group Γ discussed in the proof of Proposition 4.5 is a normal subgroup of index 6 in $SL_2(\mathcal{O}_{-7})$, such that the quotient of $SL_2(\mathcal{O}_{-7})$ by Γ is that symmetry group.

The set of cusps in the boundary of our ideal polyhedron F is $\{0, 1, \omega, \frac{\omega}{2}, \frac{\omega+1}{2}, \infty\}$. We will make use of the following matrices, which map some (but not necessarily all) of the elements of this set of cusps back into this set.

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad V := \begin{pmatrix} 1 & \frac{\omega}{2} \\ 0 & 1 \end{pmatrix}, \quad W := \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

The matrices V and W are not in $SL_2(\mathcal{O})$, so we do not obtain an element of $\mathrm{H}_3(\Gamma; \Gamma_1, \ldots, \Gamma_s)$ along the way as was the case with the element α for the Eisensteinian integers. But this is no problem, because for Theorem 4.6, we only need the matrices to be in $\Delta^{\infty}_{1,1}$. Concatenating words in V and W *after* the matrix S gives the moves from the cusp at ∞ to the cusps at the corners of F; and we generate them with the three basic moves $S \cdot \infty = 0$, $\;V \cdot 0 = \frac{\omega}{2}$ and $W \cdot 0 = \frac{1}{2}$. The matrices V^2 and W^2 are in the cusp stabiliser Γ_∞; combining with words in V^2 and W^2 concatenated *before* the matrix S, allows us to express all of the vertices of the base triangles as images of the basepoint $b = (\infty, 0)$. The coordinates of the vertices of the base triangles in Figure 5.2 are then the following:

$$\begin{array}{llll}
(\infty, \omega) = V^2 \cdot b, & (\infty, 1) = W^2 \cdot b, & (0,0) = S \cdot b, & (0,1) = SW^2 \cdot b, \\
(0, \omega) = SV^2 \cdot b, & (1,0) = W^2 S \cdot b, & (1,1) = W^2 SW^2 \cdot b, & (1, \omega) = W^2 SV^2 \cdot b, \\
(\omega, 0) = V^2 S \cdot b, & (\omega, 1) = V^2 SW^2 \cdot b, & (\omega, \omega) = V^2 SV^2 \cdot b. & \\
(\frac{\omega+1}{2}, \omega) = VWSV^2 \cdot b, & (\frac{\omega+1}{2}, 0) = VWS \cdot b & (\frac{\omega+1}{2}, 1) = VWSW^2 \cdot b & \\
(\frac{\omega}{2}, 0) = VS \cdot b & (\frac{\omega}{2}, 1) = VSW^2 \cdot b & (\frac{\omega}{2}, \omega) = VSV^2 \cdot b &
\end{array}$$

The tetrahedra of the triangulation, specified by these base triangles, then give the following α:

$$(1, SW^2, VWS, VSW^2) + (1, SW^2, W^2SV^2, VWSW^2) + (1, VSW^2, VWS, V^2S)$$

$$
\begin{gathered}
+(1,S,SW^2,SV^2)+(1,S,W^2,SW^2)+(1,SW^2,W^2,W^2S)+(1,W^2S,W^2,W^2SW^2) \\
+(1,W^2SV^2,W^2SW^2,W^2)+(1,VWSW^2,W^2SV^2,W^2)+(1,VWSV^2,VWSW^2,W^2) \\
+(1,V^2SW^2,VWSV^2,W^2)+(1,V^2SV^2,V^2SW^2,W^2)+(1,V^2SV^2,W^2,V^2) \\
+(1,V^2SV^2,V^2,V^2S)+(1,SV^2,V^2,S)+(1,V^2S,V^2,VSV^2)+(1,VSV^2,V^2,VS) \\
+(1,VS,V^2,SV^2)+(1,SV^2,SW^2,VS)+(1,SW^2,VSW^2,VS) \\
+(1,SW^2,VWSW^2,VWS)+(1,SW^2,W^2S,W^2SV^2)+(1,W^2S,W^2SW^2,W^2SV^2) \\
+(1,VS,VSW^2,VSV^2)+(1,VSV^2,VSW^2,V^2S)+(1,VWS,VWSV^2,V^2S) \\
+(1,VWSV^2,V^2SW^2,V^2S)+(1,V^2SW^2,V^2SV^2,V^2S)+(1,VWS,VWSW^2,VWSV^2).
\end{gathered}
$$

Here the three quadruples in the first row specify the only tetrahedra of non-zero volume. For each of those quadruples, letting the matrices act on $[1,0]$ and taking the cross-ratio of the resulting four points yields $\omega/(\omega+1)=(2+\omega)/4$, $(\omega+1)/2$ and ω so that the element we obtain in $\ker(\delta_{2,k})$ is $[\omega/(\omega+1)]+[-(1-\omega)/\omega]+[\omega]$. (Note that $2=\omega(1-\omega)$ and $1+\omega=-(1-\omega)^2$.) Because k contains only two roots of unity, Theorem 4.6 gives that $2[\omega/(\omega+1)]+2[-(1-\omega)/\omega]+2[\omega]$ is a non-trivial element in $B(k)/\text{torsion} \simeq K_3(k)/\text{torsion}$.

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