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► **To cite this version:**

Paul-Emile Paradan, Michèle Vergne. The multiplicities of the equivariant index of twisted Dirac operators. *Comptes Rendus. Mathématique*, 2014, 352 (9), 10.1016/j.crma.2014.05.001 . hal-00975130

**HAL Id: hal-00975130**

**<https://hal.science/hal-00975130>**

Submitted on 8 Apr 2014

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# THE MULTIPLICITIES OF THE EQUIVARIANT INDEX OF TWISTED DIRAC OPERATORS

PAUL-EMILE PARADAN, MICHÈLE VERGNE

RÉSUMÉ. In this note, we give a geometric expression for the multiplicities of the equivariant index of a Dirac operator twisted by a line bundle.

## 1. INTRODUCTION

This note is an announcement of work whose details will appear later.

Let  $M$  be a compact connected manifold. We assume that  $M$  is even dimensional and oriented. We consider a  $\text{spin}^c$  structure on  $M$ , and denote by  $\mathcal{S}$  the corresponding irreducible Clifford module. Let  $K$  be a compact connected Lie group acting on  $M$ , and preserving the  $\text{spin}^c$  structure. We denote by  $D : \Gamma(M, \mathcal{S}^+) \rightarrow \Gamma(M, \mathcal{S}^-)$  the corresponding twisted Dirac operator. The equivariant index of  $D$ , denoted  $Q_K^{\text{spin}}(M)$ , belongs to the Grothendieck group of representations of  $K$ ,

$$Q_K^{\text{spin}}(M) = \sum_{\pi \in \widehat{K}} m(\pi) \pi.$$

An important example is when  $M$  is a compact complex manifold,  $K$  a compact group of holomorphic transformations of  $M$ , and  $\mathcal{L}$  any holomorphic  $K$ -equivariant line bundle on  $M$  (not necessarily ample). Then the Dolbeaut operator twisted by  $\mathcal{L}$  can be realized as a twisted Dirac operator  $D$ . In this case  $Q_K^{\text{spin}}(M) = \sum_q (-1)^q H^{0,q}(M, \mathcal{L})$ .

The aim of this note is to give a geometric description of the multiplicity  $m(\pi)$  in the spirit of the Guillemin-Sternberg phenomenon  $[Q, R] = 0$  [3, 7, 8, 11, 9].

Consider the determinant line bundle  $\mathbb{L} = \det(\mathcal{S})$  of the  $\text{spin}^c$  structure. This is a  $K$ -equivariant complex line bundle on  $M$ . The choice of a  $K$ -invariant hermitian metric and of a  $K$ -invariant hermitian connection  $\nabla$  on  $\mathbb{L}$  determines an abstract moment map

$$\Phi_\nabla : M \rightarrow \mathfrak{k}^*$$

by the relation  $\mathcal{L}(X) - \nabla_{X_M} = \frac{i}{2} \langle \Phi_\nabla, X \rangle$ , for all  $X \in \mathfrak{k}$ . We compute  $m(\pi)$  in term of the reduced “manifolds”  $\Phi_\nabla^{-1}(f)/K_f$ . This formula extends the result of [10].

However, in this note, we do not assume any hypothesis on the line bundle  $\mathbb{L}$ , in particular we do not assume that the curvature of the connection  $\nabla$  is a symplectic form. In this pre-symplectic setting, a (partial) answer to this question has been obtained by [6, 4, 5, 1] when  $K$  is a torus. Our method is based on localization techniques as in [9], [10].

## 2. ADMISSIBLE COADJOINTS ORBITS

We consider a compact connected Lie group  $K$  with Lie algebra  $\mathfrak{k}$ . Consider an admissible coadjoint orbit  $\mathcal{O}$  (as in [2]), oriented by its symplectic structure. Then  $\mathcal{O}$  carries a  $K$ -equivariant bundle of spinors  $\mathcal{S}_{\mathcal{O}}$ , such that the associated moment map is the injection  $\mathcal{O}$  in  $\mathfrak{k}^*$ . We denote by  $Q_K^{\text{spin}}(\mathcal{O})$  the corresponding equivariant index.

Let us describe the admissible coadjoint orbits with their  $\text{spin}^c$  index.

Let  $T$  be a Cartan subgroup of  $K$  with Lie algebra  $\mathfrak{t}$ . Let  $\Lambda \subset \mathfrak{t}^*$  be the lattice of weights of  $T$  (thus  $e^{i\lambda}$  is a character of  $T$ ). Choose a positive system  $\Delta^+ \subset \mathfrak{t}^*$ , and let  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ . Let  $\mathfrak{t}_{\geq 0}^*$  be the closed Weyl chamber and we denote by  $\mathcal{F}$  the set of the relative interiors of the faces of  $\mathfrak{t}_{\geq 0}^*$ . Thus  $\mathfrak{t}_{\geq 0}^* = \coprod_{\sigma \in \mathcal{F}} \sigma$ , and we denote  $\mathfrak{t}_{> 0}^* \in \mathcal{F}$  the interior of  $\mathfrak{t}_{\geq 0}^*$ .

We index the set  $\hat{K}$  of classes of finite dimensional irreducible representations of  $K$  by the set  $(\Lambda + \rho) \cap \mathfrak{t}_{> 0}^*$ . The irreducible representation  $\pi_{\lambda}$  corresponding to  $\lambda \in (\Lambda + \rho) \cap \mathfrak{t}_{> 0}^*$  is the irreducible representation with infinitesimal character  $\lambda$ . Its highest weight is  $\lambda - \rho$ .

Let  $\sigma \in \mathcal{F}$ . The stabilizer  $K_{\xi}$  of a point  $\xi \in \sigma$  depends only of  $\sigma$ . We denote it by  $K_{\sigma}$ , and by  $\mathfrak{k}_{\sigma}$  its Lie algebra. We choose on  $\mathfrak{k}_{\sigma}$  the system of positive roots contained in  $\Delta^+$ , and let  $\rho_{\sigma}$  be the corresponding  $\rho$ .

When  $\mu \in \sigma$ , the coadjoint orbit  $K \cdot \mu$  is admissible if and only if  $\mu - \rho + \rho_{\sigma} \in \Lambda$ . The  $\text{spin}^c$  equivariant index of the admissible orbits is described in the following lemma.

**Lemma 2.1.** *Let  $K \cdot \mu$  be an admissible orbit :  $\mu \in \sigma$  and  $\mu - \rho + \rho_{\sigma} \in \Lambda$ . If  $\mu + \rho_{\sigma}$  is regular, then  $\mu + \rho_{\sigma} \in \rho + \bar{\sigma}$ . Thus we have*

$$Q_K^{\text{spin}}(K \cdot \mu) = \begin{cases} 0 & \text{if } \mu + \rho_{\sigma} \text{ is singular,} \\ \pi_{\mu + \rho_{\sigma}} & \text{if } \mu + \rho_{\sigma} \text{ is regular.} \end{cases}$$

*In particular, if  $\lambda \in (\Lambda + \rho) \cap \mathfrak{t}_{> 0}^*$ , then  $K \cdot \lambda$  is admissible and  $Q_K^{\text{spin}}(K \cdot \lambda) = \pi_{\lambda}$ .*

Let  $\mathcal{H}_{\mathfrak{k}}$  be the set of conjugacy classes of the reductive algebras  $\mathfrak{k}_f, f \in \mathfrak{k}^*$ . We denote by  $\mathcal{S}_{\mathfrak{k}}$  the set of conjugacy classes of the semi-simple parts  $[\mathfrak{h}, \mathfrak{h}]$  of the elements  $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ . The map  $(\mathfrak{h}) \rightarrow ([\mathfrak{h}, \mathfrak{h}])$  induces a bijection between  $\mathcal{H}_{\mathfrak{k}}$  and  $\mathcal{S}_{\mathfrak{k}}$ .

The map  $\mathcal{F} \rightarrow \mathcal{H}_{\mathfrak{k}}, \sigma \mapsto (\mathfrak{k}_{\sigma})$ , is surjective and for  $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$  we denote by

- $\mathcal{F}(\mathfrak{h})$  the set of  $\sigma \in \mathcal{F}$  such that  $(\mathfrak{k}_{\sigma}) = (\mathfrak{h})$ ,
- $\mathfrak{k}_{\mathfrak{h}}^* \subset \mathfrak{k}^*$  the set of elements  $f \in \mathfrak{k}^*$  with infinitesimal stabilizer  $\mathfrak{k}_f$  belonging to the conjugacy class  $(\mathfrak{h})$ .

We have  $\mathfrak{k}_{\mathfrak{h}}^* = K(\cup_{\sigma \in \mathcal{F}(\mathfrak{h})} \sigma)$ . In particular all coadjoint orbits contained in  $\mathfrak{k}_{\mathfrak{h}}^*$  have the same dimension. We say that such a coadjoint orbit is of type  $(\mathfrak{h})$ . If  $(\mathfrak{h}) = (\mathfrak{t})$ , then  $\mathfrak{k}_{\mathfrak{h}}^*$  is the open subset of regular elements.

We denote by  $A(\mathfrak{h})$  the set of admissible coadjoint orbits of type  $(\mathfrak{h})$ . This is a discrete subset of orbits in  $\mathfrak{k}_{\mathfrak{h}}^*$ .

**Example 1 :** Consider the group  $K = SU(3)$  and let  $(\mathfrak{h})$  be the conjugacy class such that  $\mathfrak{k}_{\mathfrak{h}}^*$  is equal to the set of subregular element  $f \in \mathfrak{k}^*$  (the orbit of  $f$  is of dimension  $\dim(K/T) - 2$ ). Let  $\omega_1, \omega_2$  be the two fundamental weights. Let  $\sigma_1, \sigma_2$  be the half lines  $\mathbb{R}_{>0}\omega_1, \mathbb{R}_{>0}\omega_2$ . Then  $\mathfrak{k}_{\mathfrak{h}}^* \cap \mathfrak{k}_{\geq 0}^* = \sigma_1 \cup \sigma_2$ . The set  $A(\mathfrak{h})$  is equal to the collection of orbits  $K \cdot (\frac{1+2n}{2}\omega_i), n \in \mathbb{Z}_{\geq 0}, i = 1, 2$ . The representation  $Q_K^{\text{spin}}(K \cdot (\frac{1+2n}{2}\omega_i))$  is 0 if  $n = 0$ , otherwise it is the irreducible representation  $\pi_{\rho+(n-1)\omega_i}$ . In particular, both representations associated to the admissible orbits  $\frac{3}{2}\omega_1$  and  $\frac{3}{2}\omega_2$  are the trivial representation  $\pi_{\rho}$ .

### 3. THE THEOREM

Consider the action of  $K$  in  $M$ . Let  $(\mathfrak{k}_M)$  be the conjugacy class of the generic infinitesimal stabilizer. On a  $K$ -invariant open and dense subset of  $M$ , the conjugacy class of  $\mathfrak{k}_m$  is equal to  $(\mathfrak{k}_M)$ . Consider the (conjugacy class)  $([\mathfrak{k}_M, \mathfrak{k}_M])$ .

We start by stating two vanishing lemmas.

**Lemma 3.1.** *If  $([\mathfrak{k}_M, \mathfrak{k}_M])$  does not belong to the set  $\mathcal{S}_{\mathfrak{k}}$ , then  $Q_K^{\text{spin}}(M) = 0$  for any  $K$ -invariant  $\text{spin}^c$  structure on  $M$ .*

If  $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$  for some  $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$ , any  $K$ -invariant map  $\Phi : M \rightarrow \mathfrak{k}^*$  is such that  $\Phi(M)$  is included in the closure of  $\mathfrak{k}_{\mathfrak{h}}^*$ .

**Lemma 3.2.** *Assume that  $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$  with  $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$ . Let us consider a  $\text{spin}^c$  structure on  $M$  with determinant bundle  $\mathbb{L}$ . If there exists a  $K$ -invariant hermitian connection  $\nabla$  on  $\mathbb{L}$  such that  $\Phi_{\nabla}(M) \cap \mathfrak{k}_{\mathfrak{h}}^* = \emptyset$ , then  $Q_K^{\text{spin}}(M) = 0$ .*

Thus from now on, we assume that the action of  $K$  on  $M$  is such that  $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$  for some  $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$ . Let us consider a  $\text{spin}^c$  structure on  $M$  with determinant bundle  $\mathbb{L}$  and a  $K$ -invariant hermitian connection with moment map  $\Phi_{\nabla} : M \rightarrow \mathfrak{k}^*$ .

We extend the definition of the index to disconnected even dimensional oriented manifolds by defining  $Q_K^{\text{spin}}(M)$  to be the sum over the connected components of  $M$ . If  $K$  is the trivial group,  $Q_K^{\text{spin}}(M) \in \mathbb{Z}$  and is denoted simply by  $Q^{\text{spin}}(M)$ .

Consider a coadjoint orbit  $\mathcal{O} = K \cdot f$ . The reduced space  $M_{\mathcal{O}}$  is defined to be the topological space  $\Phi_{\nabla}^{-1}(\mathcal{O})/K = \Phi_{\nabla}^{-1}(f)/K_f$ . We also denote it by  $M_f$ . This space might not be connected.

In the next section, we define a  $\mathbb{Z}$ -valued function  $\mathcal{O} \mapsto Q^{\text{spin}}(M_{\mathcal{O}})$  on the set  $A(\mathfrak{h})$  of admissible orbits of type  $(\mathfrak{h})$ . We call it the reduced index :

- if  $M_{\mathcal{O}} = \emptyset$ , then  $Q^{\text{spin}}(M_{\mathcal{O}}) = 0$ ,
- when  $M_{\mathcal{O}}$  is an orbifold, the reduced index  $Q^{\text{spin}}(M_{\mathcal{O}})$  is defined as an index of a Dirac operator associated to a natural “reduced”  $\text{spin}^c$  structure on  $M_{\mathcal{O}}$ .

Otherwise, it is defined via a limit procedure. Postponing this definition, we have the following theorem.

**Theorem 3.3.** *Assume that  $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$  with  $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ . Then*

$$Q_K^{\text{spin}}(M) = \sum_{\mathcal{O} \in A(\mathfrak{h})} Q^{\text{spin}}(M_{\mathcal{O}}) Q_K^{\text{spin}}(\mathcal{O}).$$

In the expression above, when  $\mathfrak{h}$  is not abelian,  $Q_K^{\text{spin}}(\mathcal{O})$  can be 0, and several orbits  $\mathcal{O} \in A(\mathfrak{h})$  can give the same representation.

Theorem 3.3 is in the spirit of the  $[Q, R] = 0$  theorem. However it has some radically new features. First, as  $\Phi_{\nabla}$  is not the moment map of a Hamiltonian structure, the definition of the reduced space requires more care. For example, the fibers of  $\Phi_{\nabla}$  might not be connected, and the Kirwan set  $\Phi_{\nabla}(M) \cap \mathfrak{k}_{\geq 0}^*$  is not a convex polytope. Furthermore, this Kirwan set depends of the choice of connection  $\nabla$ . Second, the map  $\mathcal{O} \in A(\mathfrak{h}) \rightarrow Q_K^{\text{spin}}(\mathcal{O})$  is not injective, when  $\mathfrak{h}$  is not abelian. Thus the multiplicities  $m_{\lambda}$  of the representation  $\pi_{\lambda}$  in  $Q_K^{\text{spin}}(M)$  will be eventually obtained as a sum of reduced indices involving several reduced spaces.

We explicit this last point.

**Theorem 3.4.** *Assume that  $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$  with  $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ . Let  $m_{\lambda} \in \mathbb{Z}$  be the multiplicity of the representation  $\pi_{\lambda}$  in  $Q_K^{\text{spin}}(M)$ . We have*

$$(1) \quad m_{\lambda} = \sum_{\substack{\sigma \in \mathcal{F}(\mathfrak{h}) \\ \lambda - \rho_{\sigma} \in \sigma}} Q^{\text{spin}}(M_{\lambda - \rho_{\sigma}}).$$

More explicitly, the sum is taken over the (relative interiors of) faces  $\sigma$  of the Weyl chamber such that

$$(2) \quad ([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \quad \Phi_\nabla(M) \cap \sigma \neq \emptyset, \quad \lambda \in \{\sigma + \rho_\sigma\}.$$

If  $\mathfrak{k}_M$  is abelian, we have simply  $m_\lambda = \mathbb{Q}^{\text{spin}}(\Phi_\nabla^{-1}(\lambda)/T)$ . In particular, if the group  $K$  is the circle group, and  $\lambda$  is a regular value of the moment map  $\Phi_\nabla$ , this result was obtained in [1].

If  $\mathfrak{k}_M$  is not abelian, and the curvature of the connection  $\nabla$  is symplectic, Kirwan convexity theorem implies that the image  $\Phi_\nabla(M) \cap \mathfrak{k}_{\geq 0}^*$  is contained in the closure of one single  $\sigma$ . Thus there is a unique  $\sigma$  satisfying Conditions (2). In this setting Theorem 3.4 is obtained in [10].

Let us give an example where several  $\sigma$  contribute to the multiplicity of a representation  $\pi_\lambda$ .

We take the notations of Example 1. We label  $\omega_1, \omega_2$  so that  $\mathfrak{k}_{\omega_1}$  is the group  $S(U(2) \times U(1))$  stabilizing the line  $\mathbb{C}e_3$  in the fundamental representation of  $SU(3)$  in  $\mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ .

Let  $P = \{0 \subset L_2 \subset L_3 \subset \mathbb{C}^4\}$  be the partial flag manifold with  $L_2$  a subspace of  $\mathbb{C}^4$  of dimension 2 and  $L_3$  a subspace of  $\mathbb{C}^4$  of dimension 3. Denote by  $\mathcal{L}_1, \mathcal{L}_2$  the equivariant line bundles on  $P$  with fiber at  $(L_2, L_3)$  the one-dimensional spaces  $\wedge^2 L_2$  and  $L_3/L_2$  respectively. Let  $M$  be the subset of  $P$  where  $L_2$  is assumed to be a subspace of  $\mathbb{C}^3$ . Thus  $M$  is fibered over  $P_2(\mathbb{C})$  with fiber  $P_1(\mathbb{C})$ . The group  $SU(3)$  acts naturally on  $M$ , and the generic stabilizer of the action is  $SU(2)$ . We denote by  $\mathcal{L}_{a,b}$  the line bundle  $\mathcal{L}_1^a \otimes \mathcal{L}_2^b$  restricted to  $M$ . This line bundle is equipped with a natural holomorphic and hermitian connection  $\nabla$ . Consider the  $\text{spin}^c$  structure with determinant bundle  $\mathbb{L} = \mathcal{L}_{2a+1, 2b+1}$ , where  $a, b$  are positive integers. If  $a \geq b$ , the curvature of the line bundle  $\mathbb{L}$  is non degenerate, and we are in the symplectic case. Let us consider  $b > a$ . It is easy to see that, in this case, the Kirwan set  $\Phi_\nabla(M) \cap \mathfrak{k}_{\geq 0}^*$  is the non convex set  $[0, b-a]\omega_1 \cup [0, a+1]\omega_2$ . We compute the character of the representation  $\mathbb{Q}_K^{\text{spin}}(M)$  by the Atiyah-Bott fixed point formula, and find

$$\mathbb{Q}_K^{\text{spin}}(M) = \sum_{j=0}^{b-a-2} \pi_{\rho+j\omega_1} \oplus \sum_{j=0}^{a-1} \pi_{\rho+j\omega_2}.$$

In particular the multiplicity of  $\pi_\rho$  (the trivial representation) is equal to 2. We use now Theorem 3.3 and the discussion of Example 1, and obtain (reduced multiplicities are equal to 1)

$$Q_K^{\text{spin}}(M) = \sum_{j=0}^{b-a-1} Q_K^{\text{spin}}\left(K \cdot \left(\frac{1+2j}{2}\omega_1\right)\right) \oplus \sum_{j=0}^a Q_K^{\text{spin}}\left(K \cdot \left(\frac{1+2j}{2}\omega_2\right)\right).$$

Using the formulae for  $Q_K^{\text{spin}}\left(K \cdot \left(\frac{1+2n}{2}\omega_i\right)\right)$  given in Example 1, these two formulae (fortunately) coincide. Furthermore we see that both faces  $\sigma_1, \sigma_2$  give a non zero contribution to the multiplicity of the trivial representation.

#### 4. DEFINITION OF THE REDUCED INDEX

We start by defining the reduced index for the action of an abelian torus  $H$  on a connected manifold  $Y$ . Denote by  $\Lambda$  the lattice of weights of  $H$ . We do not assume  $Y$  compact, but we assume that the set of stabilizers  $H_m$  of points in  $Y$  is finite. Let  $\mathfrak{h}_Y$  be the generic infinitesimal stabilizer of the action  $H$  on  $Y$ , and  $H_Y$  be the connected subgroup of  $H$  with Lie algebra  $\mathfrak{h}_Y$ . Thus  $H_Y$  acts trivially on  $Y$ . Let us consider a  $\text{spin}^c$  structure on  $Y$  with determinant bundle  $\mathbb{L}$ , and a  $H$  invariant connection  $\nabla$  on  $\mathbb{L}$ . The image  $\Phi_\Delta(Y)$  spans an affine space  $I_Y$  parallel to  $\mathfrak{h}_Y^\perp$ . We assume that the fibers of the map  $\Phi_\Delta$  are compact. We can easily prove that there exists a finite collection of hyperplanes  $W^1, \dots, W^p$  in  $I_Y$  such that the group  $H/H_Y$  acts locally freely on  $\Phi_\Delta^{-1}(f)$ , when  $f$  is in  $\Phi_\nabla(Y)$ , but not on any of the hyperplanes  $W^i$ .

**Proposition 4.1.** • *When  $\mu \in I_Y \cap \Lambda$  is a regular value of  $\Phi_\nabla : Y \rightarrow I_Y$ , the reduced space  $Y_\mu$  is an oriented orbifold equipped with an induced  $\text{spin}^c$  structure : we denote  $Q^{\text{spin}}(Y_\mu)$  the corresponding  $\text{spin}^c$  index.*

• *For any connected component  $\mathcal{C}$  of  $I_Y \setminus \cup_{k=1}^p W^k$ , we can associate a periodic polynomial function  $q^\mathcal{C} : \Lambda \cap I_Y \rightarrow \mathbb{Z}$  such that*

$$q^\mathcal{C}(\mu) = Q^{\text{spin}}(Y_\mu)$$

for any element  $\mu \in \Lambda \cap \mathcal{C}$  which is a regular value of  $\Phi : Y \rightarrow I_Y$ .

• *If  $\mu \in \Lambda$  belongs to the closure of two connected components  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of  $I_Y \setminus \cup_{k=1}^p W^k$ , we have*

$$q^{\mathcal{C}_1}(\mu) = q^{\mathcal{C}_2}(\mu).$$

We can now state the definition of the “reduced” index on  $\Lambda$  :

•  $Q^{\text{spin}}(Y_\mu) = 0$  if  $\mu \notin \Lambda \cap I_Y$ ,

• for any  $\mu \in \Lambda \cap I_Y$ , we define  $Q^{\text{spin}}(Y_\mu)$  as being equal to  $q^{\mathcal{C}}(\mu)$  where  $\mathcal{C}$  is any connected component containing  $\mu$  in its closure. In fact  $Q^{\text{spin}}(Y_\mu)$  is computed as an index of a particular  $\text{spin}^c$  structure on the orbifold  $\Phi_{\nabla}^{-1}(\mu + \epsilon)/H$  for any  $\epsilon$  small and such that  $\mu + \epsilon$  is a regular value of  $\Phi_{\nabla}$ .

If  $Y$  is not connected, we define the reduced index at a point  $\mu \in \Lambda$  as the sum of reduced indices over all connected components of  $Y$ .

More generally, let  $H$  be a compact connected group acting on  $Y$  and such that  $[H, H]$  acts trivially on  $Y$ . Let  $\mathcal{S}_Y$  be an equivariant  $\text{spin}^c$  structure on  $Y$  with determinant bundle  $\mathbb{L}$ . For any  $\mu \in \mathfrak{h}^*$  such that  $\mu([\mathfrak{h}, \mathfrak{h}]) = 0$ , and admissible for  $H$ , it is then possible to define  $Q^{\text{spin}}(Y_\mu)$ . Indeed eventually passing to a double cover of the torus  $H/[H, H]$  and translating by the square root of the action of  $H/[H, H]$  on the fiber of  $\mathbb{L}$ , we are reduced to the preceding case of the action of the torus  $H/[H, H]$ , and a  $H/[H, H]$ -equivariant  $\text{spin}^c$  structure on  $Y$ .

Consider now the action of a connected compact group  $K$  on  $M$ . Let  $\sigma$  be a (relative interior) of a face of  $\mathfrak{t}_{\geq 0}^*$  which satisfies the following conditions

$$(3) \quad ([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \quad \Phi_{\nabla}^{-1}(\sigma) \neq \emptyset.$$

Let us explain how to compute the “reduced” index map  $\mu \rightarrow Q^{\text{spin}}(M_\mu)$  on the set  $\sigma \cap \{\Lambda + \rho - \rho_\sigma\}$  that parameterizes the admissible orbits intersecting  $\sigma$ . We work with the “slice”  $Y$  defined by  $\sigma$ . The set  $U_\sigma := K_\sigma(\cup_{\sigma \subset \tau} \tau)$  is an open neighborhood of  $\sigma$  in  $\mathfrak{k}_\sigma^*$  such that the open subset  $KU_\sigma \subset \mathfrak{k}^*$  is isomorphic to  $K \times_{K_\sigma} U_\sigma$ . We consider the  $K_\sigma$ -invariant subset  $Y = \Phi_{\nabla}^{-1}(U_\sigma)$ . The following lemma allows us to reduce the problem to the abelian case.

**Lemma 4.2.** •  *$Y$  is a non-empty submanifold of  $M$  such that  $KY$  is an open subset of  $M$  isomorphic to  $K \times_{K_\sigma} Y$ .*

• *The Clifford module  $\mathcal{S}_M$  on  $M$  determines a Clifford module  $\mathcal{S}_Y$  on  $Y$  with determinant line bundle  $\mathbb{L}_Y = \mathbb{L}_M|_Y \otimes \mathbb{C}_{-2(\rho - \rho_\sigma)}$ . The corresponding moment map is  $\Phi_{\nabla}|_Y - \rho + \rho_\sigma$ .*

• *The group  $[K_\sigma, K_\sigma]$  acts trivially on  $Y$  and on the bundle of spinors  $\mathcal{S}_Y$ .*

We thus consider  $Y$  with action of  $K_\sigma$ , and Clifford bundle  $\mathcal{S}_Y$ . If  $\mu \in \sigma$  is admissible for  $K$ , then  $\mu - \rho + \rho_\sigma \in \Lambda$  is admissible for  $K_\sigma$ . The reduced space  $M_\mu = \Phi_{\nabla}^{-1}(\mu)/K_\sigma$  is equal to the reduced space  $Y_{\mu - \rho + \rho_\sigma}$ . As  $[K_\sigma, K_\sigma]$  acts trivially on  $(Y, \mathcal{S}_Y)$ , we are in the abelian case, and we define  $Q^{\text{spin}}(M_\mu) := Q^{\text{spin}}(Y_{\mu - \rho + \rho_\sigma})$ .



## ACKNOWLEDGMENTS

We wish to thank the Research in Pairs program at Mathematisches Forschungsinstitut Oberwolfach (January 2014), which gave us the opportunity to work on these questions.

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