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THE MULTIPLICITIES OF THE EQUIVARIANT INDEX OF TWISTED DIRAC OPERATORS

PAUL-EMILE PARADAN, MICHÈLE VERGNE

RÉSUMÉ. In this note, we give a geometric expression for the multiplicities of the equivariant index of a Dirac operator twisted by a line bundle.

1. INTRODUCTION

This note is an announcement of work whose details will appear later.

Let M be a compact connected manifold. We assume that M is even dimensional and oriented. We consider a spin^c structure on M , and denote by \mathcal{S} the corresponding irreducible Clifford module. Let K be a compact connected Lie group acting on M , and preserving the spin^c structure. We denote by $D : \Gamma(M, \mathcal{S}^+) \rightarrow \Gamma(M, \mathcal{S}^-)$ the corresponding twisted Dirac operator. The equivariant index of D , denoted $Q_K^{\text{spin}}(M)$, belongs to the Grothendieck group of representations of K ,

$$Q_K^{\text{spin}}(M) = \sum_{\pi \in \widehat{K}} m(\pi) \pi.$$

An important example is when M is a compact complex manifold, K a compact group of holomorphic transformations of M , and \mathcal{L} any holomorphic K -equivariant line bundle on M (not necessarily ample). Then the Dolbeaut operator twisted by \mathcal{L} can be realized as a twisted Dirac operator D . In this case $Q_K^{\text{spin}}(M) = \sum_q (-1)^q H^{0,q}(M, \mathcal{L})$.

The aim of this note is to give a geometric description of the multiplicity $m(\pi)$ in the spirit of the Guillemin-Sternberg phenomenon $[Q, R] = 0$ [3, 7, 8, 11, 9].

Consider the determinant line bundle $\mathbb{L} = \det(\mathcal{S})$ of the spin^c structure. This is a K -equivariant complex line bundle on M . The choice of a K -invariant hermitian metric and of a K -invariant hermitian connection ∇ on \mathbb{L} determines an abstract moment map

$$\Phi_\nabla : M \rightarrow \mathfrak{k}^*$$

by the relation $\mathcal{L}(X) - \nabla_{X_M} = \frac{i}{2} \langle \Phi_\nabla, X \rangle$, for all $X \in \mathfrak{k}$. We compute $m(\pi)$ in term of the reduced “manifolds” $\Phi_\nabla^{-1}(f)/K_f$. This formula extends the result of [10].

However, in this note, we do not assume any hypothesis on the line bundle \mathbb{L} , in particular we do not assume that the curvature of the connection ∇ is a symplectic form. In this pre-symplectic setting, a (partial) answer to this question has been obtained by [6, 4, 5, 1] when K is a torus. Our method is based on localization techniques as in [9], [10].

2. ADMISSIBLE COADJOINTS ORBITS

We consider a compact connected Lie group K with Lie algebra \mathfrak{k} . Consider an admissible coadjoint orbit \mathcal{O} (as in [2]), oriented by its symplectic structure. Then \mathcal{O} carries a K -equivariant bundle of spinors $\mathcal{S}_{\mathcal{O}}$, such that the associated moment map is the injection \mathcal{O} in \mathfrak{k}^* . We denote by $Q_K^{\text{spin}}(\mathcal{O})$ the corresponding equivariant index.

Let us describe the admissible coadjoint orbits with their spin^c index.

Let T be a Cartan subgroup of K with Lie algebra \mathfrak{t} . Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights of T (thus $e^{i\lambda}$ is a character of T). Choose a positive system $\Delta^+ \subset \mathfrak{t}^*$, and let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let $\mathfrak{t}_{\geq 0}^*$ be the closed Weyl chamber and we denote by \mathcal{F} the set of the relative interiors of the faces of $\mathfrak{t}_{\geq 0}^*$. Thus $\mathfrak{t}_{\geq 0}^* = \coprod_{\sigma \in \mathcal{F}} \sigma$, and we denote $\mathfrak{t}_{> 0}^* \in \mathcal{F}$ the interior of $\mathfrak{t}_{\geq 0}^*$.

We index the set \hat{K} of classes of finite dimensional irreducible representations of K by the set $(\Lambda + \rho) \cap \mathfrak{t}_{> 0}^*$. The irreducible representation π_{λ} corresponding to $\lambda \in (\Lambda + \rho) \cap \mathfrak{t}_{> 0}^*$ is the irreducible representation with infinitesimal character λ . Its highest weight is $\lambda - \rho$.

Let $\sigma \in \mathcal{F}$. The stabilizer K_{ξ} of a point $\xi \in \sigma$ depends only of σ . We denote it by K_{σ} , and by \mathfrak{k}_{σ} its Lie algebra. We choose on \mathfrak{k}_{σ} the system of positive roots contained in Δ^+ , and let ρ_{σ} be the corresponding ρ .

When $\mu \in \sigma$, the coadjoint orbit $K \cdot \mu$ is admissible if and only if $\mu - \rho + \rho_{\sigma} \in \Lambda$. The spin^c equivariant index of the admissible orbits is described in the following lemma.

Lemma 2.1. *Let $K \cdot \mu$ be an admissible orbit : $\mu \in \sigma$ and $\mu - \rho + \rho_{\sigma} \in \Lambda$. If $\mu + \rho_{\sigma}$ is regular, then $\mu + \rho_{\sigma} \in \rho + \bar{\sigma}$. Thus we have*

$$Q_K^{\text{spin}}(K \cdot \mu) = \begin{cases} 0 & \text{if } \mu + \rho_{\sigma} \text{ is singular,} \\ \pi_{\mu + \rho_{\sigma}} & \text{if } \mu + \rho_{\sigma} \text{ is regular.} \end{cases}$$

In particular, if $\lambda \in (\Lambda + \rho) \cap \mathfrak{t}_{> 0}^$, then $K \cdot \lambda$ is admissible and $Q_K^{\text{spin}}(K \cdot \lambda) = \pi_{\lambda}$.*

Let $\mathcal{H}_{\mathfrak{k}}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_f, f \in \mathfrak{k}^*$. We denote by $\mathcal{S}_{\mathfrak{k}}$ the set of conjugacy classes of the semi-simple parts $[\mathfrak{h}, \mathfrak{h}]$ of the elements $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. The map $(\mathfrak{h}) \rightarrow ([\mathfrak{h}, \mathfrak{h}])$ induces a bijection between $\mathcal{H}_{\mathfrak{k}}$ and $\mathcal{S}_{\mathfrak{k}}$.

The map $\mathcal{F} \rightarrow \mathcal{H}_{\mathfrak{k}}, \sigma \mapsto (\mathfrak{k}_{\sigma})$, is surjective and for $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ we denote by

- $\mathcal{F}(\mathfrak{h})$ the set of $\sigma \in \mathcal{F}$ such that $(\mathfrak{k}_{\sigma}) = (\mathfrak{h})$,
- $\mathfrak{k}_{\mathfrak{h}}^* \subset \mathfrak{k}^*$ the set of elements $f \in \mathfrak{k}^*$ with infinitesimal stabilizer \mathfrak{k}_f belonging to the conjugacy class (\mathfrak{h}) .

We have $\mathfrak{k}_{\mathfrak{h}}^* = K(\cup_{\sigma \in \mathcal{F}(\mathfrak{h})} \sigma)$. In particular all coadjoint orbits contained in $\mathfrak{k}_{\mathfrak{h}}^*$ have the same dimension. We say that such a coadjoint orbit is of type (\mathfrak{h}) . If $(\mathfrak{h}) = (\mathfrak{t})$, then $\mathfrak{k}_{\mathfrak{h}}^*$ is the open subset of regular elements.

We denote by $A(\mathfrak{h})$ the set of admissible coadjoint orbits of type (\mathfrak{h}) . This is a discrete subset of orbits in $\mathfrak{k}_{\mathfrak{h}}^*$.

Example 1 : Consider the group $K = SU(3)$ and let (\mathfrak{h}) be the conjugacy class such that $\mathfrak{k}_{\mathfrak{h}}^*$ is equal to the set of subregular element $f \in \mathfrak{k}^*$ (the orbit of f is of dimension $\dim(K/T) - 2$). Let ω_1, ω_2 be the two fundamental weights. Let σ_1, σ_2 be the half lines $\mathbb{R}_{>0}\omega_1, \mathbb{R}_{>0}\omega_2$. Then $\mathfrak{k}_{\mathfrak{h}}^* \cap \mathfrak{k}_{\geq 0}^* = \sigma_1 \cup \sigma_2$. The set $A(\mathfrak{h})$ is equal to the collection of orbits $K \cdot (\frac{1+2n}{2}\omega_i), n \in \mathbb{Z}_{\geq 0}, i = 1, 2$. The representation $Q_K^{\text{spin}}(K \cdot (\frac{1+2n}{2}\omega_i))$ is 0 if $n = 0$, otherwise it is the irreducible representation $\pi_{\rho+(n-1)\omega_i}$. In particular, both representations associated to the admissible orbits $\frac{3}{2}\omega_1$ and $\frac{3}{2}\omega_2$ are the trivial representation π_{ρ} .

3. THE THEOREM

Consider the action of K in M . Let (\mathfrak{k}_M) be the conjugacy class of the generic infinitesimal stabilizer. On a K -invariant open and dense subset of M , the conjugacy class of \mathfrak{k}_m is equal to (\mathfrak{k}_M) . Consider the (conjugacy class) $([\mathfrak{k}_M, \mathfrak{k}_M])$.

We start by stating two vanishing lemmas.

Lemma 3.1. *If $([\mathfrak{k}_M, \mathfrak{k}_M])$ does not belong to the set $\mathcal{S}_{\mathfrak{k}}$, then $Q_K^{\text{spin}}(M) = 0$ for any K -invariant spin^c structure on M .*

If $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$, any K -invariant map $\Phi : M \rightarrow \mathfrak{k}^*$ is such that $\Phi(M)$ is included in the closure of $\mathfrak{k}_{\mathfrak{h}}^*$.

Lemma 3.2. *Assume that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$. Let us consider a spin^c structure on M with determinant bundle \mathbb{L} . If there exists a K -invariant hermitian connection ∇ on \mathbb{L} such that $\Phi_{\nabla}(M) \cap \mathfrak{k}_{\mathfrak{h}}^* = \emptyset$, then $Q_K^{\text{spin}}(M) = 0$.*

Thus from now on, we assume that the action of K on M is such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $\mathfrak{h} \in \mathcal{H}_{\mathfrak{k}}$. Let us consider a spin^c structure on M with determinant bundle \mathbb{L} and a K -invariant hermitian connection with moment map $\Phi_{\nabla} : M \rightarrow \mathfrak{k}^*$.

We extend the definition of the index to disconnected even dimensional oriented manifolds by defining $Q_K^{\text{spin}}(M)$ to be the sum over the connected components of M . If K is the trivial group, $Q_K^{\text{spin}}(M) \in \mathbb{Z}$ and is denoted simply by $Q^{\text{spin}}(M)$.

Consider a coadjoint orbit $\mathcal{O} = K \cdot f$. The reduced space $M_{\mathcal{O}}$ is defined to be the topological space $\Phi_{\nabla}^{-1}(\mathcal{O})/K = \Phi_{\nabla}^{-1}(f)/K_f$. We also denote it by M_f . This space might not be connected.

In the next section, we define a \mathbb{Z} -valued function $\mathcal{O} \mapsto Q^{\text{spin}}(M_{\mathcal{O}})$ on the set $A(\mathfrak{h})$ of admissible orbits of type (\mathfrak{h}) . We call it the reduced index :

- if $M_{\mathcal{O}} = \emptyset$, then $Q^{\text{spin}}(M_{\mathcal{O}}) = 0$,
- when $M_{\mathcal{O}}$ is an orbifold, the reduced index $Q^{\text{spin}}(M_{\mathcal{O}})$ is defined as an index of a Dirac operator associated to a natural “reduced” spin^c structure on $M_{\mathcal{O}}$.

Otherwise, it is defined via a limit procedure. Postponing this definition, we have the following theorem.

Theorem 3.3. *Assume that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. Then*

$$Q_K^{\text{spin}}(M) = \sum_{\mathcal{O} \in A(\mathfrak{h})} Q^{\text{spin}}(M_{\mathcal{O}}) Q_K^{\text{spin}}(\mathcal{O}).$$

In the expression above, when \mathfrak{h} is not abelian, $Q_K^{\text{spin}}(\mathcal{O})$ can be 0, and several orbits $\mathcal{O} \in A(\mathfrak{h})$ can give the same representation.

Theorem 3.3 is in the spirit of the $[Q, R] = 0$ theorem. However it has some radically new features. First, as Φ_{∇} is not the moment map of a Hamiltonian structure, the definition of the reduced space requires more care. For example, the fibers of Φ_{∇} might not be connected, and the Kirwan set $\Phi_{\nabla}(M) \cap \mathfrak{k}_{\geq 0}^*$ is not a convex polytope. Furthermore, this Kirwan set depends of the choice of connection ∇ . Second, the map $\mathcal{O} \in A(\mathfrak{h}) \rightarrow Q_K^{\text{spin}}(\mathcal{O})$ is not injective, when \mathfrak{h} is not abelian. Thus the multiplicities m_{λ} of the representation π_{λ} in $Q_K^{\text{spin}}(M)$ will be eventually obtained as a sum of reduced indices involving several reduced spaces.

We explicit this last point.

Theorem 3.4. *Assume that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. Let $m_{\lambda} \in \mathbb{Z}$ be the multiplicity of the representation π_{λ} in $Q_K^{\text{spin}}(M)$. We have*

$$(1) \quad m_{\lambda} = \sum_{\substack{\sigma \in \mathcal{F}(\mathfrak{h}) \\ \lambda - \rho_{\sigma} \in \sigma}} Q^{\text{spin}}(M_{\lambda - \rho_{\sigma}}).$$

More explicitly, the sum is taken over the (relative interiors of) faces σ of the Weyl chamber such that

$$(2) \quad ([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \quad \Phi_\nabla(M) \cap \sigma \neq \emptyset, \quad \lambda \in \{\sigma + \rho_\sigma\}.$$

If \mathfrak{k}_M is abelian, we have simply $m_\lambda = \mathbb{Q}^{\text{spin}}(\Phi_\nabla^{-1}(\lambda)/T)$. In particular, if the group K is the circle group, and λ is a regular value of the moment map Φ_∇ , this result was obtained in [1].

If \mathfrak{k}_M is not abelian, and the curvature of the connection ∇ is symplectic, Kirwan convexity theorem implies that the image $\Phi_\nabla(M) \cap \mathfrak{k}_{\geq 0}^*$ is contained in the closure of one single σ . Thus there is a unique σ satisfying Conditions (2). In this setting Theorem 3.4 is obtained in [10].

Let us give an example where several σ contribute to the multiplicity of a representation π_λ .

We take the notations of Example 1. We label ω_1, ω_2 so that \mathfrak{k}_{ω_1} is the group $S(U(2) \times U(1))$ stabilizing the line $\mathbb{C}e_3$ in the fundamental representation of $SU(3)$ in $\mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$.

Let $P = \{0 \subset L_2 \subset L_3 \subset \mathbb{C}^4\}$ be the partial flag manifold with L_2 a subspace of \mathbb{C}^4 of dimension 2 and L_3 a subspace of \mathbb{C}^4 of dimension 3. Denote by $\mathcal{L}_1, \mathcal{L}_2$ the equivariant line bundles on P with fiber at (L_2, L_3) the one-dimensional spaces $\wedge^2 L_2$ and L_3/L_2 respectively. Let M be the subset of P where L_2 is assumed to be a subspace of \mathbb{C}^3 . Thus M is fibered over $P_2(\mathbb{C})$ with fiber $P_1(\mathbb{C})$. The group $SU(3)$ acts naturally on M , and the generic stabilizer of the action is $SU(2)$. We denote by $\mathcal{L}_{a,b}$ the line bundle $\mathcal{L}_1^a \otimes \mathcal{L}_2^b$ restricted to M . This line bundle is equipped with a natural holomorphic and hermitian connection ∇ . Consider the spin^c structure with determinant bundle $\mathbb{L} = \mathcal{L}_{2a+1, 2b+1}$, where a, b are positive integers. If $a \geq b$, the curvature of the line bundle \mathbb{L} is non degenerate, and we are in the symplectic case. Let us consider $b > a$. It is easy to see that, in this case, the Kirwan set $\Phi_\nabla(M) \cap \mathfrak{k}_{\geq 0}^*$ is the non convex set $[0, b-a]\omega_1 \cup [0, a+1]\omega_2$. We compute the character of the representation $\mathbb{Q}_K^{\text{spin}}(M)$ by the Atiyah-Bott fixed point formula, and find

$$\mathbb{Q}_K^{\text{spin}}(M) = \sum_{j=0}^{b-a-2} \pi_{\rho+j\omega_1} \oplus \sum_{j=0}^{a-1} \pi_{\rho+j\omega_2}.$$

In particular the multiplicity of π_ρ (the trivial representation) is equal to 2. We use now Theorem 3.3 and the discussion of Example 1, and obtain (reduced multiplicities are equal to 1)

$$Q_K^{\text{spin}}(M) = \sum_{j=0}^{b-a-1} Q_K^{\text{spin}}\left(K \cdot \left(\frac{1+2j}{2}\omega_1\right)\right) \oplus \sum_{j=0}^a Q_K^{\text{spin}}\left(K \cdot \left(\frac{1+2j}{2}\omega_2\right)\right).$$

Using the formulae for $Q_K^{\text{spin}}\left(K \cdot \left(\frac{1+2n}{2}\omega_i\right)\right)$ given in Example 1, these two formulae (fortunately) coincide. Furthermore we see that both faces σ_1, σ_2 give a non zero contribution to the multiplicity of the trivial representation.

4. DEFINITION OF THE REDUCED INDEX

We start by defining the reduced index for the action of an abelian torus H on a connected manifold Y . Denote by Λ the lattice of weights of H . We do not assume Y compact, but we assume that the set of stabilizers H_m of points in Y is finite. Let \mathfrak{h}_Y be the generic infinitesimal stabilizer of the action H on Y , and H_Y be the connected subgroup of H with Lie algebra \mathfrak{h}_Y . Thus H_Y acts trivially on Y . Let us consider a spin^c structure on Y with determinant bundle \mathbb{L} , and a H invariant connection ∇ on \mathbb{L} . The image $\Phi_\Delta(Y)$ spans an affine space I_Y parallel to \mathfrak{h}_Y^\perp . We assume that the fibers of the map Φ_Δ are compact. We can easily prove that there exists a finite collection of hyperplanes W^1, \dots, W^p in I_Y such that the group H/H_Y acts locally freely on $\Phi_\Delta^{-1}(f)$, when f is in $\Phi_\nabla(Y)$, but not on any of the hyperplanes W^i .

Proposition 4.1. • *When $\mu \in I_Y \cap \Lambda$ is a regular value of $\Phi_\nabla : Y \rightarrow I_Y$, the reduced space Y_μ is an oriented orbifold equipped with an induced spin^c structure : we denote $Q^{\text{spin}}(Y_\mu)$ the corresponding spin^c index.*

• *For any connected component \mathcal{C} of $I_Y \setminus \cup_{k=1}^p W^k$, we can associate a periodic polynomial function $q^\mathcal{C} : \Lambda \cap I_Y \rightarrow \mathbb{Z}$ such that*

$$q^\mathcal{C}(\mu) = Q^{\text{spin}}(Y_\mu)$$

for any element $\mu \in \Lambda \cap \mathcal{C}$ which is a regular value of $\Phi : Y \rightarrow I_Y$.

• *If $\mu \in \Lambda$ belongs to the closure of two connected components \mathcal{C}_1 and \mathcal{C}_2 of $I_Y \setminus \cup_{k=1}^p W^k$, we have*

$$q^{\mathcal{C}_1}(\mu) = q^{\mathcal{C}_2}(\mu).$$

We can now state the definition of the “reduced” index on Λ :

• $Q^{\text{spin}}(Y_\mu) = 0$ if $\mu \notin \Lambda \cap I_Y$,

• for any $\mu \in \Lambda \cap I_Y$, we define $Q^{\text{spin}}(Y_\mu)$ as being equal to $q^{\mathcal{C}}(\mu)$ where \mathcal{C} is any connected component containing μ in its closure. In fact $Q^{\text{spin}}(Y_\mu)$ is computed as an index of a particular spin^c structure on the orbifold $\Phi_{\nabla}^{-1}(\mu + \epsilon)/H$ for any ϵ small and such that $\mu + \epsilon$ is a regular value of Φ_{∇} .

If Y is not connected, we define the reduced index at a point $\mu \in \Lambda$ as the sum of reduced indices over all connected components of Y .

More generally, let H be a compact connected group acting on Y and such that $[H, H]$ acts trivially on Y . Let \mathcal{S}_Y be an equivariant spin^c structure on Y with determinant bundle \mathbb{L} . For any $\mu \in \mathfrak{h}^*$ such that $\mu([\mathfrak{h}, \mathfrak{h}]) = 0$, and admissible for H , it is then possible to define $Q^{\text{spin}}(Y_\mu)$. Indeed eventually passing to a double cover of the torus $H/[H, H]$ and translating by the square root of the action of $H/[H, H]$ on the fiber of \mathbb{L} , we are reduced to the preceding case of the action of the torus $H/[H, H]$, and a $H/[H, H]$ -equivariant spin^c structure on Y .

Consider now the action of a connected compact group K on M . Let σ be a (relative interior) of a face of $\mathfrak{t}_{\geq 0}^*$ which satisfies the following conditions

$$(3) \quad ([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \quad \Phi_{\nabla}^{-1}(\sigma) \neq \emptyset.$$

Let us explain how to compute the “reduced” index map $\mu \rightarrow Q^{\text{spin}}(M_\mu)$ on the set $\sigma \cap \{\Lambda + \rho - \rho_\sigma\}$ that parameterizes the admissible orbits intersecting σ . We work with the “slice” Y defined by σ . The set $U_\sigma := K_\sigma(\cup_{\sigma \subset \overline{\tau}} \tau)$ is an open neighborhood of σ in \mathfrak{k}_σ^* such that the open subset $KU_\sigma \subset \mathfrak{k}^*$ is isomorphic to $K \times_{K_\sigma} U_\sigma$. We consider the K_σ -invariant subset $Y = \Phi_{\nabla}^{-1}(U_\sigma)$. The following lemma allows us to reduce the problem to the abelian case.

Lemma 4.2. • *Y is a non-empty submanifold of M such that KY is an open subset of M isomorphic to $K \times_{K_\sigma} Y$.*

• *The Clifford module \mathcal{S}_M on M determines a Clifford module \mathcal{S}_Y on Y with determinant line bundle $\mathbb{L}_Y = \mathbb{L}_M|_Y \otimes \mathbb{C}_{-2(\rho - \rho_\sigma)}$. The corresponding moment map is $\Phi_{\nabla}|_Y - \rho + \rho_\sigma$.*

• *The group $[K_\sigma, K_\sigma]$ acts trivially on Y and on the bundle of spinors \mathcal{S}_Y .*

We thus consider Y with action of K_σ , and Clifford bundle \mathcal{S}_Y . If $\mu \in \sigma$ is admissible for K , then $\mu - \rho + \rho_\sigma \in \Lambda$ is admissible for K_σ . The reduced space $M_\mu = \Phi_{\nabla}^{-1}(\mu)/K_\sigma$ is equal to the reduced space $Y_{\mu - \rho + \rho_\sigma}$. As $[K_\sigma, K_\sigma]$ acts trivially on (Y, \mathcal{S}_Y) , we are in the abelian case, and we define $Q^{\text{spin}}(M_\mu) := Q^{\text{spin}}(Y_{\mu - \rho + \rho_\sigma})$.

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