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THE MULTIPLICITIES OF THE EQUIVARIANT INDEX OF TWISTED DIRAC OPERATORS

PAUL-ÉMILE PARADAN, MICHÈLE VERGNE

RÉSUMÉ. In this note, we give a geometric expression for the multiplicities of the equivariant index of a Dirac operator twisted by a line bundle.

1. Introduction

This note is an announcement of work whose details will appear later.

Let $M$ be a compact connected manifold. We assume that $M$ is even dimensional and oriented. We consider a spin$^c$ structure on $M$, and denote by $S$ the corresponding irreducible Clifford module. Let $K$ be a compact connected Lie group acting on $M$, and preserving the spin$^c$ structure. We denote by $D : \Gamma(M, S^+) \to \Gamma(M, S^-)$ the corresponding twisted Dirac operator. The equivariant index of $D$, denoted $Q_{K}^{\text{spin}}(M)$, belongs to the Grothendieck group of representations of $K$,

$$Q_{K}^{\text{spin}}(M) = \sum_{\pi \in \hat{K}} m(\pi) \pi.$$

An important example is when $M$ is a compact complex manifold, $K$ a compact group of holomorphic transformations of $M$, and $L$ any holomorphic $K$-equivariant line bundle on $M$ (not necessarily ample). Then the Dolbeault operator twisted by $L$ can be realized as a twisted Dirac operator $D$. In this case $Q_{K}^{\text{spin}}(M) = \sum_{\pi} (-1)^{\dim H^{0,\bar{0}}(M, L)}$.

The aim of this note is to give a geometric description of the multiplicity $m(\pi)$ in the spirit of the Guillemin-Sternberg phenomenon $[Q, R] = 0$ [3, 7, 8, 11, 9].

Consider the determinant line bundle $L = \det(S)$ of the spin$^c$ structure. This is a $K$-equivariant complex line bundle on $M$. The choice of a $K$-invariant hermitian metric and of a $K$-invariant hermitian connection $\nabla$ on $L$ determines an abstract moment map

$$\Phi_{\nabla} : M \to \mathfrak{k}^*$$

by the relation $L(X) - \nabla_X L = \frac{i}{2} \langle \Phi_{\nabla}, X \rangle$, for all $X \in \mathfrak{k}$. We compute $m(\pi)$ in term of the reduced “manifolds” $\Phi_{\nabla}^{-1}(f)/K_f$. This formula extends the result of [10].
However, in this note, we do not assume any hypothesis on the line bundle \( L \), in particular we do not assume that the curvature of the connection \( \nabla \) is a symplectic form. In this pre-symplectic setting, a (partial) answer to this question has been obtained by [6, 4, 5, 1] when \( K \) is a torus. Our method is based on localization techniques as in [9], [10].

2. Admissible coadjoints orbits

We consider a compact connected Lie group \( K \) with Lie algebra \( \mathfrak{k} \). Consider an admissible coadjoint orbit \( O \) (as in [2]), oriented by its symplectic structure. Then \( O \) carries a \( K \)-equivariant bundle of spinors \( S^O \), such that the associated moment map is the injection \( O \) in \( \mathfrak{k}^* \). We denote by \( Q^\text{spin}_K(O) \) the corresponding equivariant index.

Let us describe the admissible coadjoint orbits with their spin\(^c\) index.

Let \( T \) be a Cartan subgroup of \( K \) with Lie algebra \( \mathfrak{t} \). Let \( \Lambda \subset \mathfrak{t}^* \) be the lattice of weights of \( T \) (thus \( e^{i\lambda} \) is a character of \( T \)). Choose a positive system \( \Delta^+ \subset \mathfrak{t}^* \), and let \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \). Let \( \mathfrak{t}^*_\geq 0 \) be the closed Weyl chamber and we denote by \( \mathcal{F} \) the set of the relative interiors of the faces of \( \mathfrak{t}^*_\geq 0 \). Thus \( \mathfrak{t}^*_\geq 0 = \bigsqcup_{\sigma \in \mathcal{F}} \sigma \), and we denote \( \mathfrak{t}^*_> 0 \in \mathcal{F} \) the interior of \( \mathfrak{t}^*_\geq 0 \).

We index the set \( \hat{K} \) of classes of finite dimensional irreducible representations of \( K \) by the set \( (\Lambda + \rho) \cap \mathfrak{t}^*_> 0 \). The irreducible representation \( \pi_\lambda \) corresponding to \( \lambda \in (\Lambda + \rho) \cap \mathfrak{t}^*_> 0 \) is the irreducible representation with infinitesimal character \( \lambda \). Its highest weight is \( \lambda - \rho \).

Let \( \sigma \in \mathcal{F} \). The stabilizer \( K_\xi \) of a point \( \xi \in \sigma \) depends only of \( \sigma \). We denote it by \( K_\sigma \), and by \( \mathfrak{k}_\sigma \) its Lie algebra. We choose on \( \mathfrak{k}_\sigma \) the system of positive roots contained in \( \Delta^+ \), and let \( \rho_\sigma \) be the corresponding \( \rho \).

When \( \mu \in \sigma \), the coadjoint orbit \( K \cdot \mu \) is admissible if and only if \( \mu - \rho + \rho_\sigma \in \Lambda \).

The spin\(^c\) equivariant index of the admissible orbits is described in the following lemma.

**Lemma 2.1.** Let \( K \cdot \mu \) be an admissible orbit : \( \mu \in \sigma \) and \( \mu - \rho + \rho_\sigma \in \Lambda \). If \( \mu + \rho_\sigma \) is regular, then \( \mu + \rho_\sigma \in \rho + \mathfrak{t}^* \). Thus we have

\[
Q^\text{spin}_K(K \cdot \mu) = \begin{cases} 
0 & \text{if } \mu + \rho_\sigma \text{ is singular}, \\
\pi_{\mu + \rho_\sigma} & \text{if } \mu + \rho_\sigma \text{ is regular}.
\end{cases}
\]

In particular, if \( \lambda \in (\Lambda + \rho) \cap \mathfrak{t}^*_\geq 0 \), then \( K \cdot \lambda \) is admissible and \( Q^\text{spin}_K(K \cdot \lambda) = \pi_\lambda \).
Let \( H_k \) be the set of conjugacy classes of the reductive algebras \( t_f, f \in \mathfrak{t}^* \). We denote by \( S_k \) the set of conjugacy classes of the semi-simple parts \([\mathfrak{h}, \mathfrak{h}]\) of the elements \((\mathfrak{h}) \in H_k\). The map \((\mathfrak{h}) \mapsto ([\mathfrak{h}, \mathfrak{h}])\) induces a bijection between \( H_k \) and \( S_k \).

The map \( F \to H_k, \sigma \mapsto (t_{\sigma}) \), is surjective and for \((\mathfrak{h}) \in H_k\) we denote by \( \mathfrak{t}_b^* \subset \mathfrak{t}^* \) the set of elements \( f \in \mathfrak{t}^* \) with infinitesimal stabilizer \( \mathfrak{t}_f \) belonging to the conjugacy class \((\mathfrak{h})\).

We have \( \mathfrak{t}_b^* = K \left( \bigcup_{\sigma \in F(\mathfrak{h})} \sigma \right) \). In particular all coadjoint orbits contained in \( \mathfrak{t}_b^* \) have the same dimension. We say that such a coadjoint orbit is of type \((\mathfrak{h})\). If \((\mathfrak{h}) = (t)\), then \( \mathfrak{t}_b^* \) is the open subset of regular elements.

We denote by \( A(\mathfrak{h}) \) the set of admissible coadjoint orbits of type \((\mathfrak{h})\). This is a discrete subset of orbits in \( \mathfrak{t}_b^* \).

Example 1: Consider the group \( K = SU(3) \) and let \((\mathfrak{h})\) be the conjugacy class such that \( \mathfrak{t}_b^* \) is equal to the set of subregular elements \( f \in \mathfrak{t}^* \). The orbit of \( f \) is of dimension \( \text{dim}(K/T) - 2 \). Let \( \omega_1, \omega_2 \) be the two fundamental weights. Then \( \mathfrak{t}_b^* \cap \mathfrak{t}_r = \sigma_1 \cup \sigma_2 \).

The set \( A(\mathfrak{h}) \) is equal to the collection of orbits \( K \cdot (1 + 2n \omega_i), n \in \mathbb{Z}, i = 1, 2 \). The representation \( Q_{K}^{\text{spin}}(K \cdot (1 + 2n \omega_i)) \) is 0 if \( n = 0 \), otherwise it is the irreducible representation \( \pi_{\rho + (n-1)\omega_i} \). In particular, both representations associated to the admissible orbits \( \frac{3}{2} \omega_1 \) and \( \frac{3}{2} \omega_2 \) are the trivial representation \( \pi_{\rho} \).

3. The theorem

Consider the action of \( K \) in \( M \). Let \((\mathfrak{t}_M)\) be the conjugacy class of the generic infinitesimal stabilizer. On a \( K \)-invariant open and dense subset of \( M \), the conjugacy class of \( \mathfrak{t}_m \) is equal to \((\mathfrak{t}_M)\). Consider the (conjugacy class) \((\mathfrak{t}_M, \mathfrak{t}_M)\).

We start by stating two vanishing lemmas.

Lemma 3.1. If \((\mathfrak{t}_M, \mathfrak{t}_M)\) does not belong to the set \( S_k \), then \( Q_{K}^{\text{spin}}(M) = 0 \) for any \( K \)-invariant spinc structure on \( M \).

If \((\mathfrak{t}_M, \mathfrak{t}_M) = ([\mathfrak{h}, \mathfrak{h}])\) for some \( \mathfrak{h} \in H_k \), any \( K \)-invariant map \( \Phi : M \to \mathfrak{t}^* \) is such that \( \Phi(M) \) is included in the closure of \( \mathfrak{t}_b^* \).

Lemma 3.2. Assume that \((\mathfrak{t}_M, \mathfrak{t}_M) = ([\mathfrak{h}, \mathfrak{h}])\) with \( \mathfrak{h} \in H_k \). Let us consider a spinc structure on \( M \) with determinant bundle \( \mathcal{L} \). If there exists a \( K \)-invariant hermitian connection \( \nabla \) on \( \mathcal{L} \) such that \( \Phi_{\nabla}(M) \cap \mathfrak{t}_b^* = \emptyset \), then \( Q_{K}^{\text{spin}}(M) = 0 \).
Thus from now on, we assume that the action of $K$ on $M$ is such that $([\mathfrak{t}_M, \mathfrak{t}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $\mathfrak{h} \in \mathcal{H}_K$. Let us consider a spin-c structure on $M$ with determinant bundle $L$ and a $K$-invariant hermitian connection with moment map $\Phi_\nabla : M \to \mathfrak{t}^*$. We extend the definition of the index to disconnected even dimensional oriented manifolds by defining $Q_{\text{spin}}^\mathfrak{K}(M)$ to be the sum over the connected components of $M$. If $K$ is the trivial group, $Q_{\text{spin}}^\mathfrak{K}(M) \in \mathbb{Z}$ and is denoted simply by $Q_{\text{spin}}(M)$.

Consider a coadjoint orbit $O = K \cdot f$. The reduced space $M_O$ is defined to be the topological space $\Phi^{-1}_\nabla(O)/K = \Phi^{-1}_\nabla(f)/K_f$. We also denote it by $M_f$. This space might not be connected.

In the next section, we define a $\mathbb{Z}$-valued function $O \mapsto Q_{\text{spin}}^\mathfrak{K}(M_O)$ on the set $A(h)$ of admissible orbits of type $(h)$. We call it the reduced index: 

- if $M_O = \emptyset$, then $Q_{\text{spin}}^\mathfrak{K}(M_O) = 0$,
- when $M_O$ is an orbifold, the reduced index $Q_{\text{spin}}^\mathfrak{K}(M_O)$ is defined as an index of a Dirac operator associated to a natural “reduced” spin-c structure on $M_O$.

Otherwise, it is defined via a limit procedure. Postponing this definition, we have the following theorem.

**Theorem 3.3.** Assume that $([\mathfrak{t}_M, \mathfrak{t}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $(h) \in \mathcal{H}_K$. Then

$$Q_{\mathfrak{K}}^\text{spin}(M) = \sum_{O \in A(h)} Q_{\mathfrak{K}}^\text{spin}(M_O) Q_{\mathfrak{K}}^\text{spin}(O).$$

In the expression above, when $\mathfrak{h}$ is not abelian, $Q_{\mathfrak{K}}^\text{spin}(O)$ can be 0, and several orbits $O \in A(h)$ can give the same representation.

Theorem 3.3 is in the spirit of the $[Q, R] = 0$ theorem. However it has some radically new features. First, as $\Phi_\nabla$ is not the moment map of a Hamiltonian structure, the definition of the reduced space requires more care. For example, the fibers of $\Phi_\nabla$ might not be connected, and the Kirwan set $\Phi_\nabla(M) \cap \mathfrak{t}_s^\gg \geq 0$ is not a convex polytope. Furthermore, this Kirwan set depends of the choice of connection $\nabla$. Second, the map $O \in A(h) \to Q_{\mathfrak{K}}^\text{spin}(O)$ is not injective, when $\mathfrak{h}$ is not abelian. Thus the multiplicities $m_\lambda$ of the representation $\pi_\lambda$ in $Q_{\mathfrak{K}}^\text{spin}(M)$ will be eventually obtained as a sum of reduced indices involving several reduced spaces.

We explicit this last point.

**Theorem 3.4.** Assume that $([\mathfrak{t}_M, \mathfrak{t}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $(h) \in \mathcal{H}_K$. Let $m_\lambda \in \mathbb{Z}$ be the multiplicity of the representation $\pi_\lambda$ in $Q_{\mathfrak{K}}^\text{spin}(M)$. We have

$$m_\lambda = \sum_{\sigma \in \pi(h)} Q_{\mathfrak{K}}^\text{spin}(M_{\lambda - \rho_\sigma}).$$
More explicitly, the sum is taken over the (relative interiors of) faces \( \sigma \) of the Weyl chamber such that

\[
([\xi_M, \xi_M]) = ([\xi_\sigma, \xi_\sigma]), \quad \Phi_\nabla(M) \cap \sigma \neq \emptyset, \quad \lambda \in \{ \sigma + \rho_\sigma \}.
\]

If \( \xi_M \) is abelian, we have simply \( m_\lambda = Q^{\text{spin}}(\Phi^{-1}_\nabla(\lambda)/T) \). In particular, if the group \( K \) is the circle group, and \( \lambda \) is a regular value of the moment map \( \Phi_\nabla \), this result was obtained in [1].

If \( \xi_M \) is not abelian, and the curvature of the connection \( \nabla \) is symplectic, Kirwan convexity theorem implies that the image \( \Phi_\nabla(M) \cap t^* \geq 0 \) is contained in the closure of one single \( \sigma \). Thus there is a unique \( \sigma \) satisfying Conditions (2). In this setting Theorem 3.4 is obtained in [10].

Let us give an example where several \( \sigma \) contribute to the multiplicity of a representation \( \pi_\lambda \). We take the notations of Example 1. We label \( \omega_1, \omega_2 \) so that \( \xi_{\omega_1} \) is the group \( S(U(2) \times U(1)) \) stabilizing the line \( \mathbb{C}e_3 \) in the fundamental representation of \( SU(3) \) in \( \mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \).

Let \( P = \{ 0 \subset L_2 \subset L_3 \subset \mathbb{C}^4 \} \) be the partial flag manifold with \( L_2 \) a subspace of \( \mathbb{C}^4 \) of dimension 2 and \( L_3 \) a subspace of \( \mathbb{C}^4 \) of dimension 3. Denote by \( \mathcal{L}_1, \mathcal{L}_2 \) the equivariant line bundles on \( P \) with fiber at \((L_2, L_3)\) the one-dimensional spaces \( \wedge^2 L_2 \) and \( L_3/L_2 \) respectively. Let \( M \) be the subset of \( P \) where \( L_2 \) is assumed to be a subspace of \( \mathbb{C}^3 \). Thus \( M \) is fibered over \( P_2(\mathbb{C}) \) with fiber \( P_1(\mathbb{C}) \). The group \( SU(3) \) acts naturally on \( M \), and the generic stabilizer of the action is \( SU(2) \). We denote by \( \mathcal{L}_{a,b} \) the line bundle \( \mathcal{L}_1^a \otimes \mathcal{L}_2^b \) restricted to \( M \). This line bundle is equipped with a natural holomorphic and hermitian connection \( \nabla \). Consider the spin-c structure with determinant bundle \( \mathcal{L} = \mathcal{L}_{2a+1, 2b+1} \), where \( a, b \) are positive integers. If \( a \geq b \), the curvature of the line bundle \( \mathcal{L} \) is non degenerate, and we are in the symplectic case. Let us consider \( b > a \). It is easy to see that, in this case, the Kirwan set \( \Phi_\nabla(M) \cap t^* \geq 0 \) is the non convex set \([0, b-a] \omega_1 \cup [0, a+1] \omega_2 \). We compute the character of the representation \( Q^{\text{spin}}_K(M) \) by the Atiyah-Bott fixed point formula, and find

\[
Q^{\text{spin}}_K(M) = \sum_{j=0}^{b-a-2} \pi_{\rho+j \omega_1} \oplus \sum_{j=0}^{a-1} \pi_{\rho+j \omega_2}.
\]
In particular the multiplicity of $\pi_\rho$ (the trivial representation) is equal to 2. We use now Theorem 3.3 and the discussion of Example 1, and obtain (reduced multiplicities are equal to 1)

$$Q^\text{spin}_K(M) = \sum_{j=0}^{b-a-1} Q^\text{spin}_K(K \cdot (1 + 2j/2 \omega_1)) \oplus \sum_{j=0}^a Q^\text{spin}_K(K \cdot (1 + 2j/2 \omega_2)).$$

Using the formulae for $Q^\text{spin}_K(K \cdot (1 + 2n/2 \omega_i))$ given in Example 1, these two formulae (fortunately) coincide. Furthermore we see that both faces $\sigma_1, \sigma_2$ give a non zero contribution to the multiplicity of the trivial representation.

4. Definition of the reduced index

We start by defining the reduced index for the action of an abelian torus $H$ on a connected manifold $Y$. Denote by $\Lambda$ the lattice of weights of $H$. We do not assume $Y$ compact, but we assume that the set of stabilizers $H_m$ of points in $Y$ is finite. Let $\mathfrak{h}_Y$ be the generic infinitesimal stabilizer of the action $H$ on $Y$, and $H_Y$ be the connected subgroup of $H$ with Lie algebra $\mathfrak{h}_Y$. Thus $H_Y$ acts trivially on $Y$. Let us consider a spin$^c$ structure on $Y$ with determinant bundle $L$, and a $H_Y$ invariant connection $\nabla$ on $L$. The image $\Phi_\Delta(Y)$ spans an affine space $I_Y$ parallel to $\mathfrak{h}_Y$. We assume that the fibers of the map $\Phi_\Delta$ are compact. We can easily prove that there exists a finite collection of hyperplanes $W_1, \ldots, W_p$ in $I_Y$ such that the group $H/H_Y$ acts locally freely on $\Phi_\Delta^{-1}(f)$, when $f$ is in $\Phi_\nabla$, but not on any of the hyperplanes $W_i$.

Proposition 4.1. • When $\mu \in I_Y \cap \Lambda$ is a regular value of $\Phi_\nabla : Y \to I_Y$, the reduced space $Y_\mu$ is an oriented orbifold equipped with an induced spin$^c$ structure : we denote $Q^\text{spin}(Y_\mu)$ the corresponding spin$^c$ index.

• For any connected component $C$ of $I_Y \setminus \bigcup_{k=1}^p W^k$, we can associate a periodic polynomial function $q^C : \Lambda \cap I_Y \to \mathbb{Z}$ such that

$$q^C(\mu) = Q^\text{spin}(Y_\mu)$$

for any element $\mu \in \Lambda \cap C$ which is a regular value of $\Phi : Y \to I_Y$.

• If $\mu \in \Lambda$ belongs to the closure of two connected components $C_1$ and $C_2$ of $I_Y \setminus \bigcup_{k=1}^p W^k$, we have

$$q^{C_1}(\mu) = q^{C_2}(\mu).$$

We can now state the definition of the “reduced” index on $\Lambda$ :

• $Q^\text{spin}(Y_\mu) = 0$ if $\mu \notin \Lambda \cap I_Y$,
\* for any \( \mu \in \Lambda \cap I_Y \), we define \( Q^{\text{\text{spin}}}(Y, \mu) \) as being equal to \( q_C(\mu) \) where \( C \) is any connected component containing \( \mu \) in its closure. In fact \( Q^{\text{\text{spin}}}(Y, \mu) \) is computed as an index of a particular spin\(^c\) structure on the orbifold \( \Phi^{-1}_Y(\mu + \epsilon)/H \) for any \( \epsilon \) small and such that \( \mu + \epsilon \) is a regular value of \( \Phi_Y \).

If \( Y \) is not connected, we define the reduced index at a point \( \mu \in \Lambda \) as the sum of reduced indices over all connected components of \( Y \).

More generally, let \( H \) be a compact connected group acting on \( Y \) and such that \([H, H]\) acts trivially on \( Y \). Let \( S_Y \) be an equivariant spin\(^c\) structure on \( Y \) with determinant bundle \( L \). For any \( \mu \in \mathfrak{h}^* \) such that \( \mu([\mathfrak{h}, \mathfrak{h}]) = 0 \), and admissible for \( H \), it is then possible to define \( Q^{\text{\text{spin}}}(Y, \mu) \). Indeed eventually passing to a double cover of the torus \( H/[H, H] \) and translating by the square root of the action of \( H/[H, H] \) on the fiber of \( L \), we are reduced to the preceding case of the action of the torus \( H/[H, H] \) and a \( H/[H, H] \)-equivariant spin\(^c\) structure on \( Y \).

Consider now the action of a connected compact group \( K \) on \( M \). Let \( \sigma \) be a (relative interior) of a face of \( T_{c_0}^* \) which satisfies the following conditions

\[
(\tau, \theta) = (\tau, \theta), \quad \Phi^{-1}_Y(\sigma) \neq \emptyset.
\]

Let us explain how to compute the “reduced” index map \( \mu \rightarrow Q^{\text{\text{spin}}}(M, \mu) \) on the set \( \sigma \cap \{ \Lambda + \rho - \rho_\sigma \} \) that parameterizes the admissible orbits intersecting \( \sigma \). We work with the “slice” \( Y \) defined by \( \sigma \). The set \( U_\sigma := K_\sigma(\cup_{\tau \in Y} \Phi^{-1}_Y(\tau)) \) is an open neighborhood of \( \sigma \) in \( \mathfrak{t}_\sigma^* \) such that the open subset \( KU_\sigma \subset \mathfrak{t}_\sigma^* \) is isomorphic to \( K \times_{K_\sigma} U_\sigma \). We consider the \( K_\sigma \)-invariant subset \( Y = \Phi^{-1}_Y(U_\sigma) \). The following lemma allows us to reduce the problem to the abelian case.

**Lemma 4.2.** \( Y \) is a non-empty submanifold of \( M \) such that \( KY \) is an open subset of \( M \) isomorphic to \( K \times_{K_\sigma} Y \).

- The Clifford module \( S_M \) on \( M \) determines a Clifford module \( S_Y \) on \( Y \) with determinant line bundle \( L_Y = L_M|Y \otimes \mathbb{C}_{-2(\rho - \rho_\sigma)} \). The corresponding moment map is \( \Phi_Y|Y - \rho + \rho_\sigma \).
- The group \([K_\sigma, K_\sigma]\) acts trivially on \( Y \) and on the bundle of spinors \( S_Y \).

We thus consider \( Y \) with action of \( K_\sigma \), and Clifford bundle \( S_Y \). If \( \mu \in \sigma \) is admissible for \( K \), then \( \mu - \rho + \rho_\sigma \in \Lambda \) is admissible for \( K_\sigma \). The reduced space \( M_\mu = \Phi_Y^{-1}(\mu)/K_\sigma \) is equal to the reduced space \( Y_{\mu - \rho + \rho_\sigma} \). As \([K_\sigma, K_\sigma]\) acts trivially on \( (Y, S_Y) \), we are in the abelian case, and we define \( Q^{\text{\text{spin}}}(M, \mu) := Q^{\text{\text{spin}}}(M, \mu - \rho + \rho_\sigma) \).
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Références