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RESOLUTION of the NON-STATIONARY or HARMONIC MAXWELL EQUATIONS by a DISCONTINUOUS FINITE ELEMENT METHOD. APPLICATION to an E.M.I.(electromagnetic impulse) CASE.

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ABSTRACT

An original "flux splitting" method for the solution of conservative hyperbolic systems has been developed earlier for fluid mechanics problems. Now it is applied to the particular linear case of the unstationary Maxwell system in 2D (the method is also valid in 3D) for scattering problems.

A 0-order (piecewise constant) finite element approximation gives birth to a discontinuous Galerkin-type numerical scheme.This totally implicit scheme,associated with a splitting of the operator into its "positive" and "negative" parts, can be explicited.

The same method can also be used for solving the 2D harmonic Maxwell equations (the aim then is R.C.S. computation: RADAR cross section). In this harmonic case, the resulting linear problem is solved through an iterative process which is shown to be convergent.

In both cases, the treatment of a non-reflecting exterior boundary condition is a main point.

An important property of the method is the fact that the mesh can be completely unstructured as well in time as in space.

Some results about the consistency and convergence are presented. Then a numerical application to a truly non-stationary case has been performed: it deals with the effect of an incident electromagnetic impulsion (E.M.I) on a metallic barrier with a slot. The numerical results are interpreted in the form of a series of instant views of the total energy distribution (iso-values), and curves of the time evolution of the electromagnetic field components at some chosen points of the computational domain.

Some views of R.C.S. computations obtained either from the harmonic method in 2D or from an unstationary calculation in the harmonic case are shown as well.

1 - INTRODUCTION

This paper describes a method for solving the unstationary Maxwell equations -written in the 1st order form- in 2D or 3D. The problem studied is the scattering of an incident plane wave by a metallic or multi dielectric composed target of any geometry.

The purpose is to obtain an approximation of the restriction to a bounded domain, of the solution in the whole space, through an explicit resolution on an unstructured mesh. The solution at any space point can be restituted further by an inversion of the Maxwell operator in empty space.

This is achieved thanks to a piecewise constant finite element approximation which can be interpreted as a discontinuous Galerkin-type method.It is associated with a positive/ negative splitting (in the sense of eigenvalues) of the Maxwell operator.

The whole method derives from a general "flux-splitting" method for solving conservative hyperbolic systems.(either linear or non linear) in 2D or 3D. This method has been previously developed especially for solving Euler equations, and is based on entropy convexity properties.

In the harmonic case (the incident plane wave is of sinusoidal type in time) the adaptation of the method leads to a stationary linear problem of a complex variable, which is solved explicitely by a convergent fixed point-type iterative solution process.

From a theoretical point of view, the consistency of the approximation with the exact solution on a bounded domain -associated with an approximated"non-reflecting" boundary condition- has been demonstrated.(the existence and uniqueness of this latter solution being proved according to Lax theorem for Friedrichs-type systems). But furthermore, the consistency "for any compact set" with the solution of the problem in the whole space has also been demonstrated, as well as the convergence of the explicit solution algorithms -for both unstationary and harmonic cases).

The main numerical application is a particular purely non stationary 2D case: the scattering of an incident bi-exponential electromagnetic plane wave (E.M.I.: electromagnetic impulse, simulating a Dirac in time) over a perfectly metallic barrier opened by a slot -it could also be dielectric-.

The ability of the method to work on an unstructured mesh has been exploited here, but only in the spatial aspect: with a mixing of a locally implicit and explicit solutions on elements of different sizes. The next step for the future will be a totally time-space unstructured solution process (using different time steps) -which is coherent and feasible.

Some harmonic results are also furnished, concerning some R.C.S. computations over metallic coated targets. They have been obtained using the harmonic complex version of the method, or through a Fourier transform over one time period of the unstationary solution in the case of an incident harmonic wave.

2 - EXPOSITION OF THE PROBLEM - MAIN THEORETICAL RESULTS

2.1 - Presentation of a Maxwell equations as a Friedrichs system

Let us consider the 3D non stationary Maxwell system, composed of 6 coupled equations. Assuming that the geometry and the initial conditions are invariant in the x_3 direction, then the problem can be split into two independant systems of 3 equations each.

We have chosen to work on the "E-orthogonal" (or transverse electrical) mode which can be written as: $\partial_i A^i \varphi + B \varphi = 0$ using the Einstein convention ,where the variable:

$\varphi = {}^t(E_1, E_2, H_3)$ regroups the non-zero components of the electromagnetic field, and the matrices A^i, B depend only on ε, μ, σ (electric permittivity, magnetic permeability, electric conductivity).

The index 0 corresponds to a time derivation, the indices 1 and 2 correspond to the space

derivatives in the x_1 and x_2 directions.

We are interested in scattering problems, so the total electromagnetic field is split into its incident and scattered parts: $\varphi = \varphi^{inc} + \varphi^s$ and the unknown of the problem becomes the scattered variable: $\varphi^s = {}^t(E_1^s, E_2^s, H_3^s)$. φ^{inc} is given and verifies the Maxwell equations in empty space.

The multidielectric/metallic target Q, which is merged in the air at the initial instant, is characterized by the relative functions: ε_r, μ_r, σ_r, which are assumed to be piecewise constants (they are constant on each dielectric component of the body).

ε_0, μ_0, $\sigma_0 = 0$ are the characteristics of the air, and c_0 is the velocity of light in the air.

Then after the change of variables: $\psi = \sqrt{A^0_{air}}.\varphi$, which means: $\begin{cases} E_i' = \sqrt{\varepsilon_0}.E_i \ , i=1,2 \\ H_3' = \sqrt{\mu_0}.H_3 \end{cases}$

and of the time scale: $t' = c_0.t$, the Maxwell system for the plane mode chosen yields: $\partial_i A_i \psi^s + B \psi^s = F$, where now: A^0, B depend only on ε_r, μ_r, σ_r, A^1, A^2 are unchanged (constant), and the right-hand term is: $F = -B\psi^{inc} + (I - A^0)\partial_t \psi^{inc}$ with the initial condition:

$$\psi^s(x,0^+) = (A^{0-1} - I).\psi^{inc}(x,0)$$

We introduce a bounded open set $\Omega < \mathbf{R}^3$ containing the body Q, then with the choice of a supposedly "non-reflecting" boundary condition on $\Gamma^{ext} = \partial\Omega^{ext} \times]0,T[$, we can describe the whole problem on the temporal cylinder $\Delta = \Omega \times]0,T[$ as a Friedrichs system:

$$(P) \begin{cases} \partial_i A^i \varphi + B\varphi = F \text{ on the interior of } \Delta \\ M(x,t)(\varphi - \varphi^0) = F \text{ on } \partial\Delta = \Gamma \\ \text{or } M(x,t)\,\varphi = g \text{ on } \Gamma \end{cases}$$

where M(x,t) is a 3x3 matrix with C^1 coefficients and φ the normalized scattered variable.

The boundary conditions are divided in 3 types:

* initial condition described above on $\overline{\Omega}$ at $t = 0$
  (there is no final condition at $t = T$, as it is a time-evolution problem).
* non-reflecting boundary condition on Γ^{ext}.
* perfectly reflecting boundary condition on $\partial Q^{met} \times]0,T[$, where ∂Q^{met} is the limit of the metallic part Q^{met} of Q. We suppose that the metal is a perfect conductor ($\sigma = +\infty$) so that the electromagnetic field is zero inside. We can then exclude Q^{met} from the computational domain thanks to an "obstacle" condition.

The operator: $L = \partial_t A^0 + \partial_j A^j + B$ (j=1,2) is a skew-symmetric operator, since:

* A^0 is a symmetric positive definite matrix with <u>constant</u> coefficients with respect to **t**,
* A^i are symmetric with constant coefficients with respect to **t**,
* B is positive with piecewise continuous coefficients, and in this case symmetric.

Let us define some functional spaces as follows:

$H_L(\Delta) = \left\{ f \in (L^2(\Delta))^3 : Lf \in (L^2(\Delta))^3 \right\}$ is a Hilbert space for the scalar product :

$$(f,g) = \int_\Delta f.g \, dxdt + \int_\Delta (Lf, Lg) \, dxdt \quad \text{and:} \quad C^1_M(\overline{\Delta}) = \left\{ f \in (C^1(\overline{\Delta}))^3 : Mf = 0 \right\}$$

We now make use of the Lax-Phillips theorem, which ensures the existence and uniqueness of a solution for Friedrichs systems (skew-symmetric operators): (ref. [12,13,14]).

Let $M = -\frac{1}{2}\left(A^i n^i - N\right)$ where n^i are the components of a unitary outgoing normal vector to $\partial\Delta$.
With the following assumptions:
* $A^i n^i$ is of constant rank in a neighbourhood of $\partial\Delta$.
* N is positive, which means that $(N+N^*) \geq 0$, N^* being the transconjugate of N.
* $\mathrm{Ker}\,(A^i n^i - N) + \mathrm{Ker}\,(A^i n^i + N) = \mathbf{R}^m$, m being the dimension of the unknown "vector" φ (this means that the vector subspace Ker M is maximal for inclusion into the cone: $C = \{\varphi : {}^t\varphi . A^i n^i . \varphi \geq 0\}$)

then the Friedrichs problem $L\varphi = 0$ with a homogeneous boundary condition $M\varphi = 0$ admits a unique solution in the space: $H_L^M(\Delta) = \overline{C_M^1(\overline{\Delta})}^{H_L(\Delta)}$.

The non homogeneous problem with $M\varphi = g$ on $\partial\Delta$, $g \neq 0$ and g smooth enough: $g \in (H^{1/2}(\partial\Delta))^3$ can be reduced to a homogeneous problem in $(H^1(\Delta))^3$, if the solution φ is sufficiently regular (piecewise H^1 on Δ).

2.2 - Description of the exterior boundary condition

The total space flux matrix $A^i n_i = A^1 n_x + A^2 n_y$ can be split into a positive and a negative parts:

$A^i n_i = (A^i n_i)^+ + (A^i n_i)^-$, where $(A^i n_i)^+ = \sup(A^i n_i, 0)$ and $(A^i n_i)^- = \inf(A^i n_i, 0)$.

This decomposition can be obtained via the diagonalization of $A^i n_i$, by separating its eigenvalues with respect to their sign.

This splitting has the following characteristics :
* $(A^i(-n_i))^+ = -(A^i n_i)^-$ and $(A^i(-n_i))^- = -(A^i n_i)^+$
* $(A^i n_i)^+$ ($(A^i n_i)^-$) defines a positive (negative) quadratic form, which leads to a convex/ concave splitting of the flux matrix.
* If we note $|A^i n_i| = (A^i n_i)^+ - (A^i n_i)^-$, then we have on any closed polygon ω:

$$\int_\omega A^i n_i \, d\sigma = 0 \quad \text{and} \quad \int_\omega (A^i n_i)^+ \, d\sigma \,,\; -\int_\omega (A^i n_i)^- \, d\sigma \,,\; \int_{\partial\omega} |A^i n_i| \, d\sigma$$

are positive definite matrices (where at least 3 of the normal vectors with respect to the sides are non colinear).

The choice of the boundary conditions is based upon this splitting -as well as the approximation scheme-.

- Boundary conditions on the metallic surface $\partial\Omega_{met}$

The electromagnetic field components inside the metal (supposed to be a perfect conductor) do not have to be calculated since they are by definition zero. However, the compatibility equations: $\vec{E}^{tot} \wedge \vec{n} = 0$ as we are in 2D, have to be verified on the surface of the metal, $\vec{n}$ being an outgoing normal to $\partial\Omega_{met}$.
An equivalent metallic boundary condition is written as:

$M.\varphi = 0$ or $\boxed{M.\varphi^s = - M.\varphi^{inc}}$

where M fulfills the conditions of the Lax-Phillips theorem, with a matrix N which is antisymmetric in this case: $(N\varphi, \varphi) = 0 \quad \forall\varphi$.

- Exterior "non-reflecting" boundary condition

We chose $M = - (A^i n_i)^-$ (which means that $N = | A^i n_i |$)
which verifies the conditions of the Lax theorem (ensuring the existence and uniqueness of a solution). The exterior boundary condition is then written as :

$M (\varphi - \varphi^{inc}) = 0,$ or $\boxed{- \left(A^i n_i\right)^- \varphi^s = 0}$

This boundary condition can be interpreted as an "impedance" condition (coupling E and H), as it is equivalent to: $(E_T - H_3)^s = 0$, where E_T is the tangential component of E to the boundary.

It has been proved (ref. [4]) that this condition is a 0-order development of the exact non-reflecting boundary condition when the boundary $\partial\Omega_{ext}$ is composed of hyperplanes.

This result still holds when the exterior boundary is a circle and it appears that it can also be true if $\partial\Omega_{ext}$ belongs to a family of curves, parametrized by λ, and diffeomorphic to a circle, with the property: $d(0,M) \rightarrow +\infty$ uniformly when $\lambda \rightarrow +\infty$.

Theoretically, this boundary condition should become more valid with increasing distance from dQ, limit of the target.

2.3 - The variational formulation and approximation scheme for both non-stationary and harmonic problems

2.3.1 - The variational non-stationary formulation

This variational formulation has been developed previously in the general context of non linear hyperbolic systems (ref. [5 to 9]). It is based on the splitting of the flux matrix described formerly:

$A^i n_i = A^0 n_t + A^1 n_x + A^2 n_y$

where $\vec{n} = (n_t, n_x, n_y)$ is a unitary outgoing normal time-space vector to a discontinuity. We use this convention for jumps in 2D:

φ^d interior value $\Big) \rightarrow \vec{n}$ φ^e exterior value $\qquad [\varphi] = \varphi^e - \varphi^d$

By choosing approximation functions which are piecewise constant (see the detailed description below: § 3), the test function ψ being the characteristic function of each finite element, the variational formulation we use for the well-posed Friedrichs system written before is given by :

- $\Delta = \Omega \times]0,T[$ being the time-space domain,
- $\partial\Delta$ its boundary,
- $\mathcal{R}\varphi$ and $\mathcal{D}\varphi$ being respectively the regularity domain, and the set of discontinuity lines for the variable φ,

we must now find φ (piecewise constant on $\overline{\Delta}$) such that:

$$\int_{\Delta\cap\mathcal{D}\varphi} (A^i n_i)^- (\varphi^e-\varphi^d)\, d\sigma + \int_{\Delta\cap\mathcal{R}\varphi} B.\varphi\, dx\, dt + \int_{\delta\Delta} M(\varphi^e-\varphi^d)\, d\sigma = \int_{\Delta\cap\mathcal{R}\varphi} F(\varphi^{inc})\, dx\, dt$$

$\mathcal{D}\varphi$ being localized on the sides of the elements.

The initial condition is imposed strongly (Dirichlet condition) on φ -initialization of the field-. There is no final condition (evolution problem).

Remark:

The curvilinear integral on $\Delta\cap\mathcal{D}\varphi$ is written for both orientations of the normal $\vec{n}$ to the interior sides of the elements.

2.3.2 - The harmonic case

Roughly speaking (see 2.3.3), the harmonic equations are obtained from the unstationary equations by supposing that the incident field is periodic in time and that the unstationary solution φ (x,t) also becomes periodic asymptotically in time.

Then $\varphi(x,t) \approx \frac{1}{2}\left(\psi(x)\, e^{i\omega t} + \overline{\psi}(x)\, e^{-i\omega t}\right)$, where the complex variable $\psi(x)$ is the solution of the so-called harmonic system

$$\begin{cases} i\omega A^0 \psi + \underbrace{\partial_k A^k \psi}_{\text{space only}} + B\psi = F(\varphi^{inc}) \text{ on } \Omega \\ M(\psi - \psi^0) = 0 \text{ on } \partial\Omega \end{cases}$$

with $F(\varphi^{inc}) = i\omega\,((I - A^0) - B)\,.\,{}^t(0, e^{i\omega x}, -e^{i\omega x})$

$\overline{\psi}$ is also the solution of the conjugate system. The variational formulation is a direct result of the one used for the unstationary system, by replacing the matrix B by $(i\,\omega\,A^0+B)$, the time-space domain Δ by Ω, 2D space domain, and the approximated function $\varphi(x,t)$ by $\psi(x)$ piecewise constant on $\overline{\Omega}$ with complex values, the set of discontinuities $\mathcal{D}\psi$ being the set of sides of the elements.

2.3.3 - Consistency results

The following results have been demonstrated (ref. [1,2,3]).

THEOREM 1: Consistency for the non-stationary problem

The non-stationary interior Maxwell problem on a time-space bounded domain Δ, with the exterior boundary condition $(A^i n_i)^- \varphi = f$ (where f is $H^{1/2}$) on $\delta\Delta_{ext}$, is well-posed (in the sense of Lax-Friedrichs theorem).

If the solution is piecewise H^1, it can be proved that the consistency with the problem in the whole space is estimated in $C(\Delta)\sqrt{h}$, where $C(\Delta)$ is a constant depending on the domain Δ, and h is a characteristic dimension of the space mesh ($h \to 0$).

Then several results are valid for the harmonic case:

Definition 1: Partial time Laplace transform

Let $p \in \mathbb{C}$ and $\mathcal{R}e(p) > 0 \quad (p = \varepsilon + i\omega, \varepsilon > 0)$

T is any tempered distribution with parabolic support.

The partial time Laplace Transform of T is the tempered distribution $\mathcal{L}_p T \in \mathcal{S}'(\mathbb{R}^n)$ defined by: $< \mathcal{L}_p T , \varphi > = < T , e^{-pt} \varphi > .$ $\mathcal{L}_p T$ has many properties (ref. [2]).

Definition 2:

ψ is a solution of the harmonic Maxwell problem on a bounded space domain, if it is a solution of :

$$\begin{cases} (0 + i\omega) A^0 \psi + \partial_i A^i \psi = f \\ \text{with exterior boundary conditions of the type: } (A^i n_i)^- \psi = 0 \end{cases}$$ in the following sense :

the solution ψ_ε of the problem :

$$\begin{cases} (\varepsilon + i\omega) A^0 \psi_\varepsilon + \partial_i A^i \psi_\varepsilon = f \\ \text{with the same exterior boundary conditions} \end{cases}$$

admits ψ as a limit in L^2_{loc} when $\varepsilon \to 0$.

Then we obtain :

THEOREM 2:

If the solution φ of the unstationary Maxwell problem is such that $\varphi \in L^2 ((0, +\infty); L^2_{loc})$, then ψ, solution of the harmonic problem as described in definition 2, is such that :

$$\psi = \lim_{\varepsilon \to 0} \varepsilon \, \mathcal{L}_{(\varepsilon+i\omega)}(\varphi) \quad \text{in } L^2_{loc}$$

THEOREM 3:

a) The approximate harmonic Maxwell problem, either in the whole space (on an infinite mesh) or on a bounded space domain (with the appropriate exterior boundary condition), admits a solution ψ_h.

b) If ψ, continuous solution in the sense of definition 2, is smooth enough (piecewise H^1) then there is convergence of the approximate solution :

$\psi_h \to \psi$ in L^2_{loc} when $h \to 0$

Remark :

The convergence is stronger than L^2_{loc} in fact. On any compact set, we can take as a norm :

$$|||\psi|||^2 = |\psi|^2_{L^2} + \int_{\mathcal{D}\psi} |A^i n_i| [\psi] [\overline{\psi}]$$ and we have :

THEOREM 4:

On any compact set, the approximate solution ψ_h of the harmonic Maxwell problem in the whole space (on an infinite mesh) is convergent towards ψ (def. 2) when $h \to 0$, in the sense of $|||\bullet|||$ (which is finer than the L^2 norm) (ψ smooth enough for instance piecewise H^1).

The resolution of the approximate problem is detailed further but we announce yet this result:

THEOREM 5:

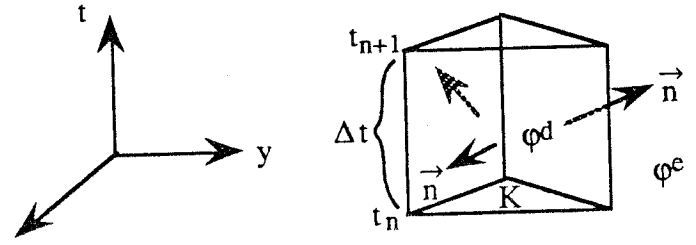
The approximate harmonic Maxwell problem, either in the whole space (infinite mesh) or on a bounded space domain, with the exterior boundary condition $A^i n_i^- \psi = 0$, is solved through a fixed point type method which is convergent.

In the case of the approximate non-stationary problem, we'll see that we have to impose a stability (CFL type) condition to ensure the convergence of the explicit solution process.

3 - APPROXIMATION AND NUMERICAL SCHEME

Let's detail now the discretization of the problem (ref. [1,2,3]). In both cases (non-stationary or harmonic), the test functions as well as the variable function are piecewise constant: they are constant on the finite elements in space, their discontinuities being localized on the sides of the elements. So in the weak formulation discretized on the finite elements, the test functions are equal to the characteristic function on each space element. It is in this way that the approximation may be interpreted as a discontinuous Galerkin type method (it is similar to a finite volume method as well).

3.1 - Non stationary case



Let Ω_h be a triangulation of Ω in space.

$\Delta t = t_{n+1} - t_n$ is the time step

The finite elements we use are triangles or quadrilaterals in space until now (but they could be any convex polyedron), and they are cylindrical in time.

$\omega = K \times]t_n , t_{n+1}[$ is a time-space finite element.

The weak formulation on an interior element is (with $\psi_{/\omega} = 1\omega$ characteristic function of ω, and $\varphi_{/\omega}$ scattered field constant in space and time) :

$$\int_{\delta\omega} \left(A^i n_i\right)^- \left(\varphi^e - \varphi^d\right) d\sigma + \int_{\omega} B\,\varphi\, dx\, dt = \int_{\omega} F\left(\varphi^{inc}\right) dx\, dt$$

with the same convention as previously on the "interior" and "exterior" values (φ^d,φ^e).

The integral on $\delta\omega$ shows the splitting of fluxes into their entering and exiting parts.

If $\delta\omega$ touches a boundary, we must replace the corresponding part of this integral by the appropriate boundary term.

Putting $\varphi^0 = \varphi\,(t_n)$ solution at the previous time step, we obtain the following scheme (with a separation of space and time terms): if $\mathcal{A}_K$ = area (K) and l_p is the length of the p^{th} side of K:

$$\mathcal{A}_K\, A^0\left(\varphi - \varphi^0\right) + \Delta t\,.\,\mathcal{A}_K\, B\,\varphi + \Delta t \sum_{p \in \delta K} l_p \left(A^i n_i\right)^- .\left(\varphi^e - \varphi^d\right) = \int_{\omega} F\left(\varphi^{inc}\right) dx\, dt$$

in space only (i=1,2)

If a side p touches a boundary, the corresponding term in Σ is replaced by a boundary term.

This is a totally implicit scheme and we have proved that it is unconditionally stable.

But we generally solve it by an explicit approach : if we put $\varphi \equiv \varphi^0$ in the "lateral" terms on δK and the term in B we get :

$$\varphi^d = \varphi^0 - \Delta t \,.\, A_0^{-1}.B.\varphi^0 - \frac{\Delta t}{\mathcal{A}_K} \,.\, A_0^{-1} \sum_{p \in \partial K} l_p \left(A^i n_i\right)^- \left(\varphi_0^e - \varphi_0^d\right) + \frac{A_0^{-1}}{\mathcal{A}_K} \int_\omega F\left(\varphi^{inc}\right) dx\, dt$$

This is an explicit scheme, which requires a CFL-type stability condition linking Δt, time step, to Δx, characteristic space dimension.

The stability study (ref. [1]) has been performed through an "entropy" method (in the linear case of Maxwell equations, entropy is the same as energy). Using the global entropy inequality we get a "strong stability" criterion (this means that this criterion ensures stability for any initial conditions φ^0).

It entails that locally on each element : $\Delta t_K \leq \frac{\mathcal{A}_K}{P_K}$ where $\mathcal{A}_K$ is the area, and P_K the perimeter of the element.This is exactly half of the classical estimation in r_K where r_K is the radius of the inscribed circle to the triangle K. So globally : $\Delta t_{\Omega_h} \leq_{K \in \Omega_h} \inf \frac{\mathcal{A}_K}{P_K}$

But, in fact, this strong stability criterion is sure but probably too severe: we don't take in account the properties of the initial condition and the fluxes.A "weak" stability criterion ensuring the stability for a given initial condition would be sufficient.

Practically, we just use the classical criterion mentioned above : $\Delta t \leq \inf_{K \in \Omega_h} r_K$

It enables the use of a greater time step, so it is more efficient in CPU time, and it seems to be sufficient, since no case of instability has ever appeared.

3.2 - Harmonic case

The elements are also triangles or quadrilaterals. The weak formulation, with test functions equal to characteristic functions on the elements, and the variable complex function ψ constant by element with discontinuities localized on the element boundaries, becomes on an interior element K:

$$\int_K \left(i\,\omega\, A^0 + B\right)\psi\, dx + \int_{\partial K} \underbrace{\left(A^k n_k\right)^-}_{\text{in space only}} \cdot \left(\psi^e - \psi^d\right) d\sigma = \int_K F\left(\varphi^{inc}\right) dx$$

with the same convention on interior and exterior values, along with some boundary terms $\int_{\partial K \cap \partial \Omega} M\left(\psi^d - \psi^0\right) d\sigma$ replacing a part of the integral on ∂K, if K touches a limit of the domain.

The interest of this scheme is that it allows an explicit solution process through an iterative fixed-point type algorithm: the field ψ in K at iteration (k) is calculated from the exterior fields at the previous iteration (k-1), with any initialization $\psi^{(0)}$:

$$\begin{cases} k = 0 \qquad \psi^{(0)} = 0 \\ \text{for } k \geq 1 : \displaystyle\int_K (i\,\omega\,A^0 + B)\,\psi^{(k)} + \int_{\delta K} (A^i\,n_i)^-\,(\psi_e^{(k-1)} - \psi_d^{(k)}) = \int_K F(\varphi^{inc})\,dx \end{cases}$$

(if ∂K touches a boundary, the corresponding boundary term is taken at iteration k).

Each global iteration of this fixed point method is solved explicitly and locally on each element by the inversion of a 3x3 matrix which is constant throughout the iterations :

$$\left(\int_K (i\,\omega\,A^0 + B)\,dx \; - \int_{\delta K} (A^i\,n_i)^-\,d\sigma \; \{ + \text{boundary terms} \} \right)$$

The convergence of this method has been proved, as we mentioned before (theorem 4).

3.3 - Advantages of this discretization

Until now we have used either triangles or quadrilaterals in space, but the method allows the use of any set of convex polygons, and moreover the following configuration is allowed:

A node is not necessarily a "vertex" for every element. So, in fact, the method can be totally unstructured in time as well as in space.

(The total space-time destructuration has been developed in 2D for fluid mechanics problems with a self-adaptative mesh refinement using an entropy criterion (ref. [5,10,11])).

By now, for Maxwell equations, we have only taken advantage of the spatial destructuration, in the unstationary case, by mixing the explicit and implicit solution processes, each one working on a different zone of the spatial mesh: these two zones are separated with respect to the sizes of the elements. As the stability criterion imposes a local time-step depending on the characteristic dimension of the element, we choose a "limit" time step: on all the elements of "superior" dimension the explicit scheme is used, and on the elements of "inferior" size we solve the implicit problem, using a fixed point method equivalent to that used in the harmonic case. On an interior "implicit" element K, this process becomes, at the n^{th} time step :

$$\begin{cases} k = 0 \qquad \varphi_n^{(0)} = \varphi\,(t_{n-1}) \quad \text{solution at the previous time step} \\ \text{for } k \geq 1 : \text{(with the previous notations)} \\ \mathcal{A}_k \cdot A^0 \left(\varphi_n^{(k)} - \varphi^{(0)}\right) + \Delta t\; \mathcal{A}_k\, B\, \varphi_n^{(k)} + \Delta t \sum_{p \in \partial K} \underbrace{(A^i\,n_i)^-}_{\text{space only}} \left(\varphi_e^{(k-1)} - \varphi_d^{(k)}\right) l_p = \displaystyle\int_\omega F\left(\varphi^{inc}\right) dx\,dt \end{cases}$$

(plus the eventual boundary terms, taken at iteration (k)).

This process is also convergent, and it can be interpreted as a "natural" block-Gauss Seidel

method. The global iteration is solved locally on each element by inverting a constant matrix:

$$\left\{ \mathcal{A}_k \; A^0 + \Delta t \; \mathcal{A}_k \, B - \Delta t \sum_{p \in \partial K} l_p \, (A^i \, n_i)^- \right\}$$

This implicit/explicit resolution has been used for the numerical case later shown. This approach becomes even more interesting as the mesh becomes finer in space and requires a very small Δt. The choice of the limit Δt, determining the two zones of the elements, needs a compromise between the number of implicit elements -it must stay low, because the implicit resolution is more time consuming than the explicit one- and its own size, since it is the time step used on the whole domain, and it must be large enough for a reasonable CPU time for the unstationary computation.

The principle of this destructuration is very simple, and stability and consistency are ensured for the approximate solution. However it requires a great effort in terms of data organization for the description of the two zones and for the numerical programming.

4 - NUMERICAL EXPERIMENTS

4.1 - Non stationary case

This study has been performed with the support of the C.E.G. (Centre d'Études de Gramat). We have studied the effect of an EMI (electromagnetic impulse), represented by an incident biexponential plane wave simulating a Dirac in time, over a metallic cylindrical barrier opened by a slot. The cylinder is infinite in the x_3 direction and the propagation is along the x_1 direction, in front of the slot, as shown in figure 1.

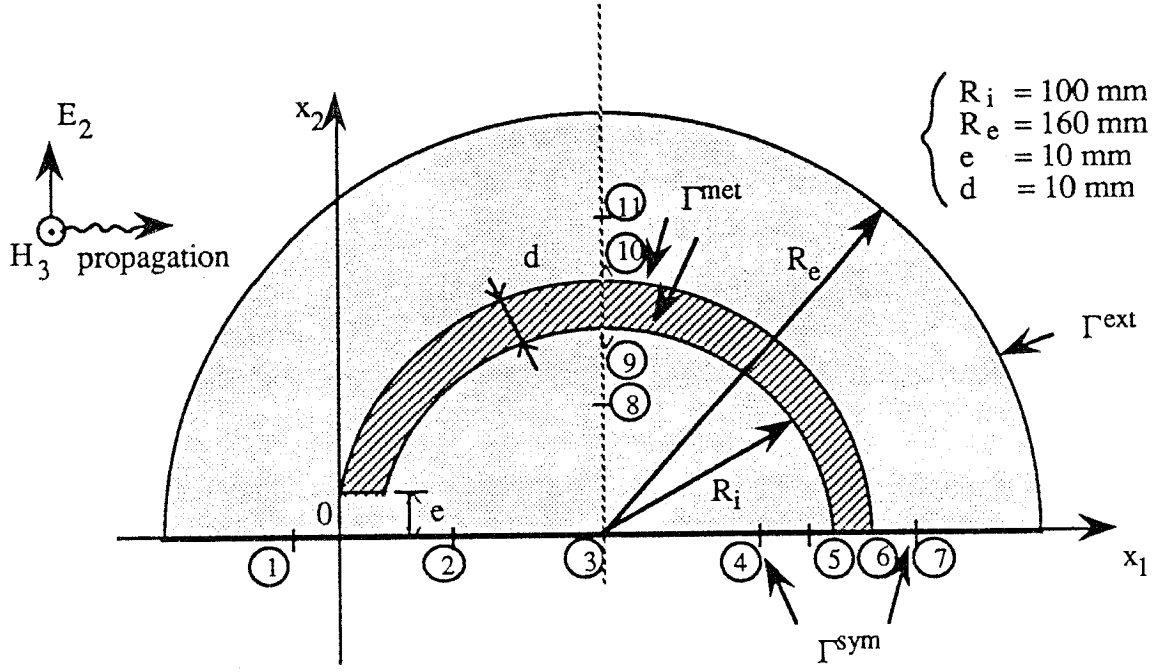


Figure 1
Representation of the half-domain

The width of the slot: e, and the thickness of the metal: d, are two parameters of the geometry (5 mm ≤ 2 e ≤ 20 mm, 1 mm ≤ d ≤ 10 mm). We have chosen a "coarse" case

(allowing a coarser mesh).

The barrier is made of brass: at its limit Γ^{met} we impose the metal boundary condition. (It could also be treated as a dielectric of characteristics ε_r, μ_r, σ_r).

With the previous normalization and change of variables, we work on the transverse electric mode for Maxwell equations with an incident wave written as:

$\varphi^{inc} = {}^t(0, E_2^{inc}, H_3^{inc})$ where $E_2^{inc} = H_3^{inc} = Y(t-x_1)(e^{-\alpha(t-x_1)} - e^{-\beta(t-x_1)})$

and $\begin{cases} \alpha = 4.10^7 / C_0 \quad m^{-1} \\ \beta = 4.76 \times 10^9 / C_0 \quad m^{-1} \end{cases}$ where C_0 is the velocity of light in air.

Y represents the Heaviside function.

In figure 2 (page 16), we can see the shape of this "impulse": its total extinction is obtained at t=30 (with the normalization, t is homogeneous to a length in meters).

As we cannot define a "mean" wavelength, the definition of the characteristic dimension for the mesh in space is based on the velocity of propagation and the time interval to rise to the peak of the impulse: considering that 30 Δt are sufficient to represent this rise, it yields $\Delta x \approx$ 10 mm.

An additional constraint is the presence of the slot which accelerates propagation. So, we have chosen an average $\Delta x \approx 5$ mm with a refined zone of elements around the slot ($\Delta x \approx 2.5$ mm). Globally, the resolution of the mesh is strongly related to the parameters d and e.

* A triangular mesh has been produced by the automatic 2D mesh generator developed at CERT/GAN (figure 3, on page 16). We have introduced the accretive symmetry boundary condition on the x_1-symmetry axis, in order to use only a half mesh, written as:

$M(\varphi - \varphi^{inc}) = 0$ on Γ^{sym} or $\boxed{M.\varphi^s = 0}$

where M verifies the Lax-Phillips theorem and its expression results from symmetry/ antisymmetry properties of $E^1(x,t) / E^2(x,t)$ and $H^3(x,t)$.

* As a comparison, we have also performed the computation on the whole circular domain, with non-symmetric and symmetric meshes, for the same position of the exterior boundary Γ^{ext}: we have obtained exactly the same results for all cases.

The explicit approach is very interesting in this case because it requires very little time: for example, on the CRAY-XMP 416 of the CERT the entire calculation up to extinction (t = 30) needs less than 30' of CPU time for the half mesh:

$\Delta t \approx 4.65 \times 10^{-4}$
number of sides : 3410 → (72 exterior, 117 metal, 55 symmetry on boundaries)
number of elements : 2192
number of nodes : 1219
CPU time / time step ⟨ 0,025"

We have performed the same computation with the mixed locally implicit/explicit approach on the same mesh.

The limit time step is $\Delta t' = 1.5 \times 10^{-3}$ ($\approx 4 \times \Delta t$). Thus we get:

n_{imp} = 492 implicit elements
n_{exp} = 1700 explicit elements

The savings in time is very small in that case: it would need approximately 25' of CRAY-XMP 416 to reach total extinction.

As the number of fixed point iterations on the implicit elements is slowly decreased as the computation goes on, the CPU time per time-step is decreasing slightly also. The numerical results are exactly the same in the two approaches, with the use of the implicit/ explicit process being advantageous only in the cases when the mesh is much finer and the resulting global time step prohibits an explicit resolution (for example: with a very thin metal barrier or narrow slot).

We present some instant views of the iso-total energy density ($W_K(t) = \frac{1}{2}\left(|E|^2_K + |H|^2_K\right)$ in air, constant by element) (fig. 4, 5, 6, 7 on page 17) and curves representing the time evolution of this quantity at the points numbered 1 to 11 (see fig. 1) (fig. 8, 9, 10 on page 16). All of these results were obtained by the mixed implicit/explicit method.

4.2 - Harmonic case

We present here a computational case for RCS (radar cross section) on a metallic NACA 0012 profile, with 2 different meshes:

- one with a circular boundary Γ_{ext} (fig. 11, 13)
- the other with an elliptic boundary Γ_{ext} (fig. 12, 14 on page 18).

The RCS has been computed in two different ways:
- by the harmonic method of resolution for Maxwell equations previously described,
- by the non-stationary resolution, using a periodic incident plane wave (sinusoidal in time):

$\varphi^{inc}_{(x,t)} = \vec{v}\, e^{i\omega\left(t - \vec{k}.\overrightarrow{OM}\right)}$ and with the wave vector $\vec{k} = {}^t(k_1 , k_2)$ then $\vec{V} = {}^t(- k_2 , k_1 , +1)$

In that case, we obtain $\psi(x)$, the complex solution of the harmonic Maxwell equations, through a time Fourier series development, over one time period, of the unstationary solution $\varphi(x,t)$ (we suppose in this case that for $t \geq T_0$, we may consider that $\varphi(x,t)$ is periodic in time, with the same period T as $\varphi^{inc}(x,t)$).

The identification of the 1st Fourier coefficient in the series gives the following expression:

$$\psi(x) = \frac{2}{T}\int_{T_0}^{T_0+T} \varphi(x,t)\,.\,e^{-i\omega t}\,dt$$

The RCS is then calculated by a classical asymptotic formula depending only on the values of ψ (in fact, of E_T and H_3) on a closed curve Σ around the target.So, with the unstationary resolution, we only need storing $\varphi(P,t)$ throughout one time period, for $P \in \Sigma$.

The bistatic RCS curves are identical for both solution processes. But of course the harmonic approach is more economic in terms of CPU time (physicists are interested in monostatic RCS curves, for which an entire computation is necessary for each incident angle:: $\theta \in [0,2\pi]$). In addition to a RCS comparison, we have obtained another interesting result: an evaluation of the "non reflexivity" of the exterior boundary condition $(A^i n_i)^- \varphi = 0$. We can compare from fig. 13 and 14 the iso-total energy density curves at a given instant (t = 7.02) for both meshes. The similarity of the results is remarkable in this case.

Fig. 15 shows a bistatic back scattered RCS curve for the NACA 0012 airfoil lightened by the trailing edge (so $\alpha = 0°$ is the incidence) and then by the leading edge ($\alpha = 180°$). With the harmonic method this computation (for one incident angle) requires around 50" of CPU time on CRAY for about 300 iterations of the fixed point process.

5 - CONCLUSIONS AND FUTURE DEVELOPMENTS

We have obtained interesting results, nevertheless the two methods we have described still have some weak points, theoretical as well numerical.

* The theoretical originality of these methods can be resumed as follows: the orientation of time is defined by the splitting of the operator on an unbounded domain ; this decomposition leads to a well-posed spatial semi-discretized problem (unstationary case). The local type "radiation" conditions at infinity (Silver-Müller,...) are implicitely taken into account, as well as convenient spaces for the resolution of the scattering problem (Lax decomposition, B* Hörmander Space, ... ref. [14,15]).

When the harmonic problem is well-posed, this particular behavior enables a convergent subdomain solution process (avoiding a global matrix inversion).

On any compact set, the "a priori" estimates entail compact injections and monotonies that can be exploited in solution algorithms. Furthermore, an exact non-reflecting boundary condition can be written on any compact set around the target, for the spatial semi-discretized problem or its harmonic equivalent.

* From a numerical point of view, the search for a totally (space and time) unstructured solution process limits us to piecewise constant (by element) approximations, thus the convergence is bounded only by $\sqrt{h}$ (h characteristic mesh dimension).

Moreover, the exact absorbing boundary condition has not been used yet, as at present we can implement it only by the inversion of a boundary linear system (along a strip of 2 elements). But it appears certain from the structure of this system, that we could solve it by a fixed-point type iterative method. This process could become completely chaotic through any interaction with the interior approximate solution (crossed iterations). Until now we haven't worked out this solution process. If this could be performed in a simple way, the lack of precision of the numerical approximation should be balanced by :

- the quality of convergence -monotonous in energy-,
- the destructuration potential (a multigrid aspect is implicitely involved in the scheme. The demonstrations of stability and consistency do not assume a finite number of elements, even on a bounded domain),
- and finally the arbitrary choice of the non reflecting exterior boundary. We can also assure that as the compact set containing the object shrinks, the convergence improves.

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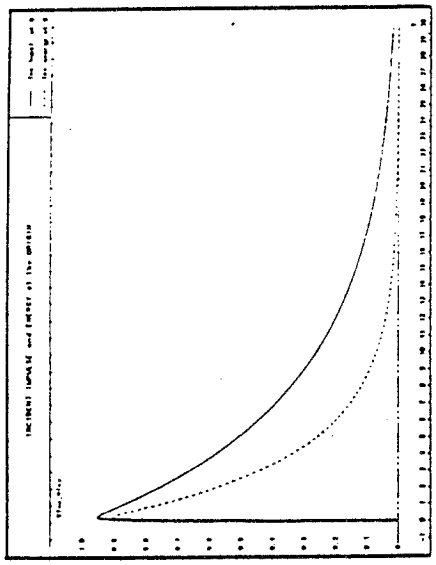


Figure 2: Incident impulse and energy at the origin

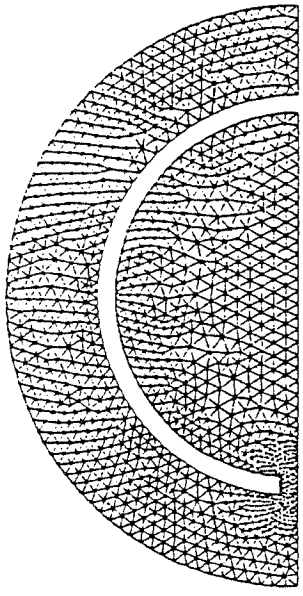
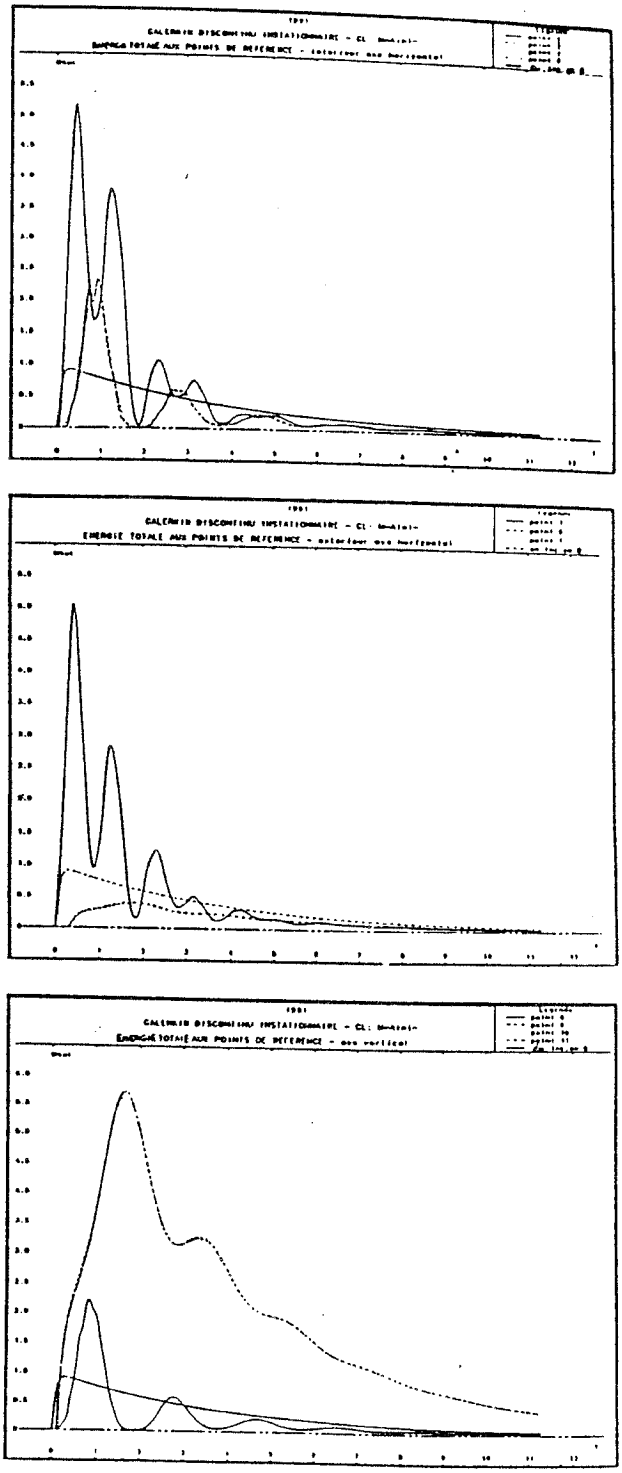
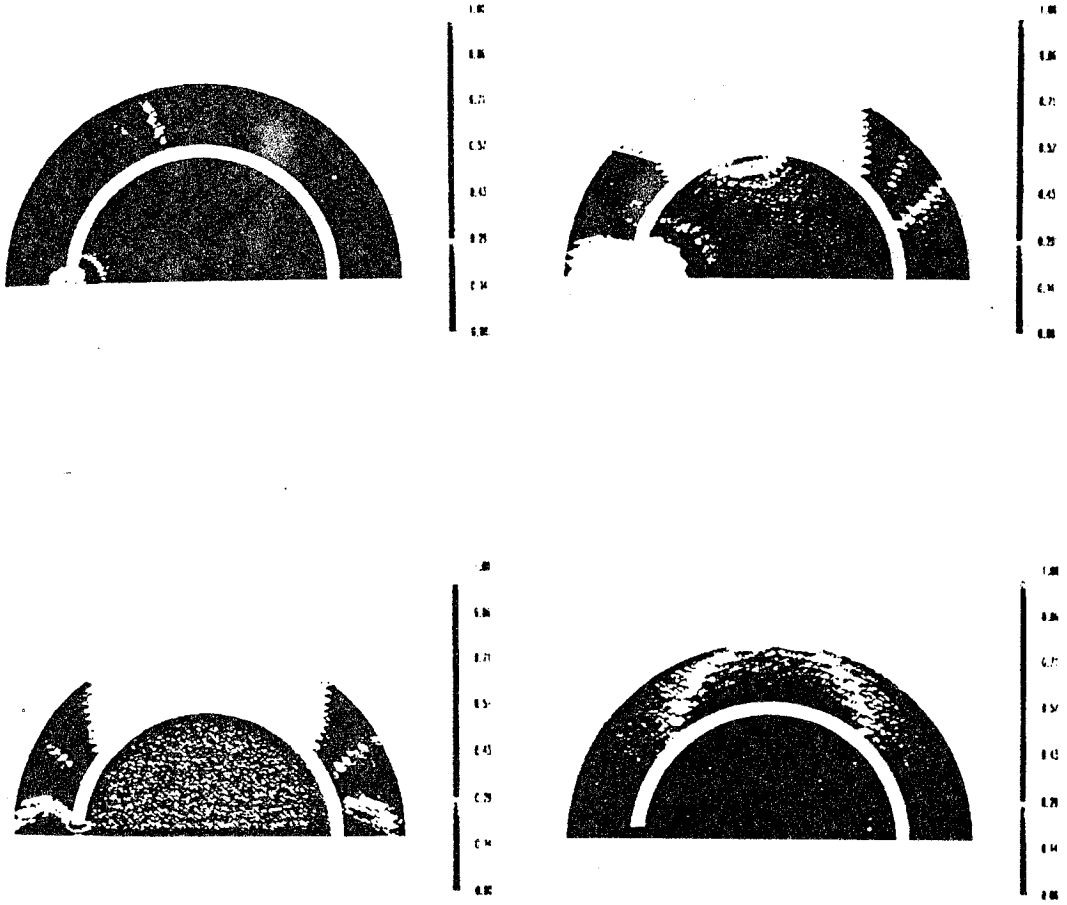


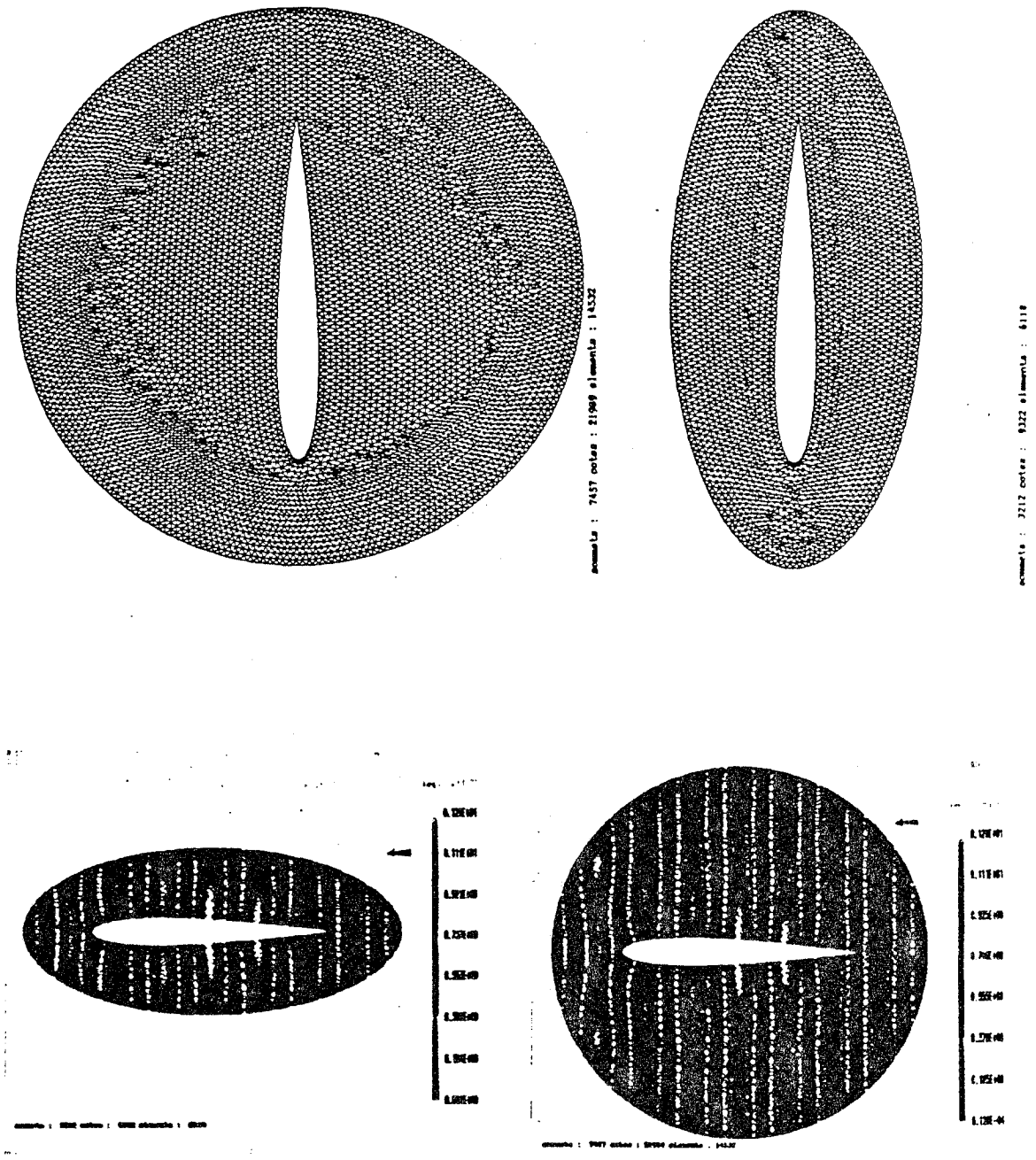
Figure 3: The mesh of the half-domain



Figures 8, 9, 10: Time curves of the total energy density at points : 2,3,4,5 / 1,6,7 / 8,9,10,11



Figures $\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$ Unstationary method : iso-total energy density at
t = 0.109, 0.36, 4.79, 10.82



Figures $\begin{pmatrix} 11 & 12 \\ 13 & 14 \end{pmatrix}$: Meshes around the NACA0012 airfoil.
Iso-total energy density at t=7.02
Case of a periodic incident wave lightening the trailing edge.