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# CONTINUOUS FIELDS OF PROPERLY INFINITE C\*-ALGEBRAS

ETIENNE BLANCHARD

ABSTRACT. The only separable unital continuous  $C([0, 1])$ -algebra with fibres isomorphic to the Cuntz algebra  $\mathcal{O}_\infty$  is the trivial continuous field  $\mathcal{O}_\infty \otimes C([0, 1])$ . But there exist non properly infinite separable unital continuous  $C([0, 1])$ -algebras with properly infinite fibres.

## 1. INTRODUCTION

One of the basic C\*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C\*-algebras through K-theoretical invariants is the Cuntz C\*-algebra  $\mathcal{O}_\infty$  generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C\*-algebra is pretty rigid in so far as it is a strongly self-absorbing C\*-algebra ([TW07]): Any separable unital continuous  $C(X)$ -algebra  $A$  the fibres of which are isomorphic to the same strongly self-absorbing C\*-algebra  $D$  is a trivial  $C(X)$ -algebra provided the compact Hausdorff base space  $X$  has finite topological dimension. (Indeed, the strongly self-absorbing C\*-algebra  $D$  tensorially absorbs the Jiang-Su algebra  $\mathcal{Z}$  ([Win09]). Hence, this C\*-algebra  $D$  is  $K_1$ -injective ([Rør04]) and the  $C(X)$ -algebra  $A$  satisfies  $A \cong D \otimes C(X)$  ([DW08].) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C\*-bundle over the infinite dimensional compact product  $\prod_{n=0}^\infty S^2$  such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type  $2^\infty$  ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dăd09, §3] for all pair  $(\Gamma_0, \Gamma_1)$  of discrete countable torsion groups a unital separable continuous  $C(X)$ -algebra  $A$  such that:

- the base space  $X$  is the compact Hilf cube  $X = \mathfrak{X}$  of infinite dimension,
- all the fibres  $A_x$  ( $x \in \mathfrak{X}$ ) are isomorphic to the strongly self-absorbing Cuntz C\*-algebra  $\mathcal{O}_2$  generated by two isometries  $s_1, s_2$  satisfying  $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$ ,
- $K_i(A) \cong C(Y_0, \Gamma_i)$  for  $i = 0, 1$ , where  $Y_0 \subset [0, 1]$  is the canonical Cantor set.

These K-theoretical conditions imply that the  $C(\mathfrak{X})$ -algebra  $A$  is not a trivial one. These arguments does not anymore work when the strongly self-absorbing algebra  $D$  is the Cuntz algebra  $\mathcal{O}_\infty$  ([Cun77]), in so far as  $K_0(\mathcal{O}_\infty) = \mathbb{Z}$  is a torsion free group.

We show directly in this article that the Pimsner-Toeplitz algebra ([Pim95]) of the nontrivial Dixmier-Douady Hilbert  $C(X)$ -module  $E_{DD}$  ([DD63]) is a nontrivial unital  $C(X)$ -algebra with fibres  $\mathcal{O}_\infty$ . This notably implies that there exists a properly infinite C\*-algebra  $A$  which is not  $K_1$ -injective, *i.e.* the mapping  $\mathcal{U}(A)/\mathcal{U}_0(A) \rightarrow K_1(A)$  is not

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injective, and there exist separable unital continuous  $C([0, 1])$ -algebras with properly infinite fibres which are not properly infinite  $C^*$ -algebras.

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## 2. NOTATIONS

We present in this section the main notations which are used in this article. We denote by  $\mathbb{N} = \{0, 1, 2, \dots\}$  the set of positive integers and we denote by  $[S]$  the closed linear span of any subset  $S$  in a Banach space.

**Definition 2.1.** ([Dix69], [Kas88], [Blan97]) *Let  $X$  be a compact Hausdorff space and let  $C(X)$  be the  $C^*$ -algebra of continuous function on  $X$ .*

- *A unital  $C(X)$ -algebra is a unital  $C^*$ -algebra  $A$  endowed with a unital morphism of  $C^*$ -algebra from  $C(X)$  to the centre of  $A$ .*
- *For all closed subset  $F \subset X$  and all element  $a \in A$ , one denotes by  $a|_F$  the image of  $a$  in the quotient  $A|_F := A/C_0(X \setminus F) \cdot A$ . If  $x \in X$  is a point in  $X$ , one calls fibre at  $x$  the quotient  $A_x := A|_{\{x\}}$  and one write  $a_x$  for  $a|_{\{x\}}$ .*
- *The  $C(X)$ -algebra  $A$  is said to be continuous if the upper semicontinuous map  $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$  is continuous for all  $a \in A$ .*

*Remarks 2.2.* a) ([Cun81], [BRR08]) For all integer  $n \geq 2$ , the  $C^*$ -algebra  $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$  is the universal unital  $C^*$ -algebra generated by  $n$  isometries  $s_1, \dots, s_n$  satisfying the relation

$$s_1 s_1^* + \dots + s_n s_n^* \leq 1. \quad (2.1)$$

b) A unital  $C^*$ -algebra  $A$  is *properly infinite* if and only if one the following equivalent conditions holds ([Cun77], [Rør03, Proposition 2.1]):

- the  $C^*$ -algebra  $A$  contains two isometries with mutually orthogonal range projections, *i.e.*  $A$  unittally contains a copy of  $\mathcal{T}_2$ ,
- the  $C^*$ -algebra  $A$  contains a unital copy of the simple Cuntz  $C^*$ -algebra  $\mathcal{O}_\infty$  generated by infinitely many isometries with pairwise orthogonal ranges.

## 3. GLOBAL PROPER INFINITENESS

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous  $C(X)$ -algebras with properly infinite fibres.

**Proposition 3.1.** *Let  $X$  be a second countable perfect compact Hausdorff space and let  $A$  be a separable unital continuous  $C(X)$ -algebra with properly infinite fibres.*

1) *There exist:*

- (a) *a finite integer  $n \geq 1$ ,*
- (b) *a covering  $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  by the interiors of closed balls  $F_1, \dots, F_n$ ,*
- (c) *unital embeddings of  $C^*$ -algebra  $\sigma_k : \mathcal{O}_\infty \hookrightarrow A|_{F_k}$  ( $1 \leq k \leq n$ ).*

2) *The tensor product  $M_p(\mathbb{C}) \otimes A$  is properly infinite for all large enough integers  $p$ .*

*Proof.* 1) For all point  $x \in X$ , the semiprojectivity of the  $C^*$ -subalgebra  $\mathcal{O}_\infty \hookrightarrow A_x$  ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood  $F \subset X$  of the point  $x$  and a unital embedding  $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A|_F$  of  $C(F)$ -algebra. The compactness of the topological space  $X$  enables to conclude.

2) Proposition [BRR08, Proposition 2.7] entails that  $M_{2^{n-1}}(A)$  is properly infinite and [Rør97, Proposition 2.1] implies that  $M_p(A)$  for all integer  $p \geq 2^{n-1}$ .  $\square$

*Remark 3.2.* If  $X$  is an ordinary second countable compact Hausdorff space and  $A$  is a separable unital continuous  $C(X)$ -algebra, then  $\tilde{X} := X \times [0, 1]$  is a perfect space,  $\tilde{A} := A \otimes C([0, 1])$  is a unital continuous  $C(\tilde{X})$ -algebra and any morphism  $\mathcal{O}_\infty \rightarrow \tilde{A}$  induces a morphism  $\mathcal{O}_\infty \rightarrow A$  by composition with the projection map  $\tilde{A} \rightarrow A$  coming from the injection  $x \in X \mapsto (x, 0) \in \tilde{X}$ .

The proper infiniteness of the tensor product  $M_p(\mathbb{C}) \otimes A$  does not always imply that the  $C^*$ -algebra  $A$  is properly infinite ([HR98]). Indeed, there exists a unital  $C^*$ -algebra  $A$  which is not properly infinite, but such that the tensor product  $M_2(\mathbb{C}) \otimes A$  is properly infinite ([Rør03, Proposition 4.5]). We nevertheless have the following corollary.

**Corollary 3.3.** *Let  $j_0, j_1$  denote the two canonical unital embeddings of the  $C^*$ -algebra  $\mathcal{T}_2$  in the full unital free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  and let  $\tilde{u} \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  be a  $K_1$ -trivial unitary satisfying  $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$  ([BRR08, Lemma 2.4]).*

*Then the following conditions are equivalent:*

- (a) *The full unital free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  is  $K_1$ -injective.*
- (b) *The unitary  $\tilde{u}$  belongs to the connected component  $\mathcal{U}_0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  of  $1_{\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2}$ .*
- (c) *Every separable unital continuous  $C(X)$ -algebra  $A$  with properly infinite fibres is a properly infinite  $C^*$ -algebra.*

*Proof.* (a) $\Rightarrow$ (b) A unital  $C^*$ -algebra  $A$  is called  $K_1$ -injective if and only if every unitary  $v \in \mathcal{U}(A)$  is homotopic to the unit  $1_A$  in  $\mathcal{U}(A)$  (see e.g. [Roh09]). Thus, (b) is a special case of (a).

(b) $\Rightarrow$ (c) Let  $A$  be a separable unital continuous  $C(X)$ -algebra with properly infinite fibres. Take a finite covering such that there exist unital embeddings  $\sigma_k : \mathcal{T}_2 \rightarrow A|_{F_k}$  ( $1 \leq k \leq n$ ). Set  $G_k := F_1 \cup \dots \cup F_k \subset X$  for all  $1 \leq k \leq n$  and let us construct by induction isometries  $w_k \in A|_{G_k}$  such that the two projections  $w_k w_k^*$  and  $1|_{G_k} - w_k w_k^*$  are properly infinite and full in the restriction  $A|_{G_k}$ :

– If  $k = 1$ , the isometry  $w_1 := \sigma_1(s_1)$  has the requested properties.

– If  $k \in \{1, \dots, n-1\}$  and the isometry  $w_k \in A|_{G_k}$  is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital  $C^*$ -algebra  $\pi_k : \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2 \rightarrow A|_{G_k \cap F_{k+1}}$  and a  $K_1$ -trivial unitary  $u_{k+1} \in \mathcal{U}(A|_{G_k \cap F_{k+1}})$  satisfying:

$$\begin{aligned} & - \pi_k(j_0(s_1)) = w_k|_{G_k \cap F_{k+1}} \quad \text{and} \\ & - \pi_k(j_1(s_1)) = \sigma_{k+1}(s_1)|_{G_k \cap F_{k+1}} = u_{k+1} \cdot w_k|_{G_k \cap F_{k+1}}. \end{aligned} \tag{3.1}$$

If the unitary  $\tilde{u}$  belongs to  $\mathcal{U}_0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ , then the unitary  $u_{k+1} = \pi_k(\tilde{u})$  is homotopic to  $1_{A|_{G_k \cap F_{k+1}}} = \pi_k(1_{\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2})$  in  $\mathcal{U}(A|_{G_k \cap F_{k+1}})$ , so that  $u_{k+1}$  admits a unitary lifting  $z_{k+1}$

in  $\mathcal{U}_0(A|_{F_{k+1}})$  (see *e.g.* [LLR00, Lemma 2.1.7]). The only isometry  $w_{k+1} \in A|_{G_{k+1}}$  satisfying the two constraints:

$$w_{k+1}|_{G_k} = w_k \quad \text{and} \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1) \quad (3.2)$$

verifies that the two projections  $w_{k+1}w_{k+1}^*$  and  $1|_{G_{k+1}} - w_{k+1}w_{k+1}^*$  are properly infinite and full in  $A|_{G_{k+1}}$ .

The proper infiniteness of the projection  $w_n w_n^* \in A|_{G_n} = A$  implies that the unit  $1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n$  is also a properly infinite projection in  $A$ , *i.e.* the  $C^*$ -algebra  $A$  is properly infinite.

(c) $\Rightarrow$ (a) The  $C^*$ -algebra  $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_{\mathcal{C}} \mathcal{T}_2); f(0) \in \mathcal{J}_0(\mathcal{T}_2) \text{ and } f(1) \in \mathcal{J}_1(\mathcal{T}_2)\}$  is a unital continuous  $C([0, 1])$ -algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the  $C^*$ -algebra  $\mathcal{D}$  is properly infinite, a statement which is equivalent to the  $K_1$ -injectivity of  $\mathcal{T}_2 *_{\mathcal{C}} \mathcal{T}_2$  ([Blan10, Proposition 4.2]).  $\square$

#### 4. THE PIMSNER-TOEPLITZ ALGEBRA OF A HILBERT $C(X)$ -MODULE

We look in this section at the special case of unital continuous  $C(X)$ -algebras with fibres  $\mathcal{O}_\infty$  corresponding to the Pimsner-Toeplitz  $C(X)$ -algebras of Hilbert  $C(X)$ -modules with infinite dimension fibres.

**Definition 4.1.** ([Pim95]) *Let  $X$  be a compact Hausdorff space and  $E$  a full Hilbert  $C(X)$ -module  $E$ , *i.e.* without any zero fibre.*

a) *The full Fock Hilbert  $C(X)$ -module  $\mathcal{F}(E)$  of  $E$  is the direct sum of Hilbert  $C(X)$ -module*

$$\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)} m)}, \quad (4.1)$$

where  $E^{(\otimes_{C(X)} m)} := \begin{cases} C(X) & \text{if } m = 0, \\ E \otimes_{C(X)} \dots \otimes_{C(X)} E \text{ (} m \text{ terms)} & \text{if } m \geq 1. \end{cases}$

b) *The Pimsner-Toeplitz  $C(X)$ -algebra  $\mathcal{T}(E)$  of  $E$  is the unital subalgebra of the  $C(X)$ -algebra  $\mathcal{L}_{C(X)}(\mathcal{F}(E))$  of adjointable  $C(X)$ -linear operator acting on  $\mathcal{F}(E)$  generated by the creation operators  $\ell(\zeta)$  ( $\zeta \in E$ ), where:*

$$\begin{aligned} - \ell(\zeta)(f \cdot \hat{1}_{C(X)}) &:= f \cdot \zeta = \zeta \cdot f & \text{for } f \in C(X) & \text{and} \\ - \ell(\zeta)(\zeta_1 \otimes \dots \otimes \zeta_k) &:= \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_k & \text{for } \zeta_1, \dots, \zeta_k \in E & \text{if } k \geq 1. \end{aligned} \quad (4.2)$$

c) *There is a unique coaction  $\alpha$  of the group  $C^*$ -algebra  $(C^*(\mathbb{Z}), \Delta)$  on the Pimsner-Toeplitz  $C(X)$ -algebra  $\mathcal{T}(E)$  such that  $\alpha(\ell(\zeta)) = \ell(\zeta) \otimes \mathbf{u}$  for all  $\zeta \in E$ , *i.e.**

$$\begin{aligned} \alpha : \mathcal{T}(E) &\rightarrow \mathcal{T}(E) \otimes C^*(\mathbb{Z}) = C(\mathbb{T}, \mathcal{T}(E)) \\ \ell(\zeta) &\mapsto \ell(\zeta) \otimes \mathbf{u} = (z \mapsto \ell(z\zeta)) \end{aligned} \quad (4.3)$$

The fixed point  $C(X)$ -subalgebra  $\mathcal{T}(E)^\alpha = \{a \in \mathcal{T}(E); \alpha(a) = a \otimes 1\}$  under this coaction is the closed linear span

$$\mathcal{T}(E)^\alpha = [C(X).1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^*]. \quad (4.4)$$

Besides, the projection  $P \in \mathcal{L}(\mathcal{F}(E))$  onto the submodule  $E$  induces a quotient morphism of  $C(X)$ -algebra  $a \in \mathcal{T}(E)^\alpha \mapsto \bar{\mathbf{q}}(a) := P \cdot a \cdot P \in \mathcal{K}(E) + C(x) \cdot 1 \subset \mathcal{L}(E)$ .

**Proposition 4.2.** *Let  $X$  be a second countable compact Hausdorff perfect space, i.e. without any isolated point, and let  $E$  be a separable Hilbert  $C(X)$ -module with infinite dimensional fibres.*

1) *There exist a covering  $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_m$  by the interiors of closed subsets  $F_1, \dots, F_m$  and  $m$  sections  $\zeta_1, \dots, \zeta_m$  in  $E$  such that  $\mathcal{T}(E) = C^* \langle \mathcal{T}(E)^\alpha, \ell(\zeta_1), \dots, \ell(\zeta_m) \rangle$  and  $\|(\zeta_k)_x\| = 1$  for all  $k \in \{1, \dots, m\}$  and all  $x \in F_k$ .*

2) *If the full unital free product  $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$  is  $K_1$ -injective, then there exists a section  $\xi \in E$  such that  $\|\xi_x\| = 1$  for all  $x \in X$ .*

*Proof.* 1) For all point  $x \in X$ , there exists a section  $\zeta \in E$  such that  $\|\zeta_x\| = 1$ , whence an isomorphism of  $C^*$ -algebra  $\mathcal{T}(E)_x \cong \mathcal{T}(E_x) = C^* \langle \mathcal{T}(E_x)^\alpha, \ell(\zeta_x) \rangle$ . The semiprojectivity of the  $C^*$ -algebra  $\mathcal{O}_\infty \cong \mathcal{T}(E)_x$  and the compactness of the space  $X$  then imply that there exist a finite covering  $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_m$  by the interiors of closed subsets  $F_1, \dots, F_m$  and  $m$  contractions  $\zeta_1, \dots, \zeta_m$  in  $E$  such that  $\|(\zeta_k)_x\| = 1$  for all index  $k \in \{1, \dots, m\}$  and all point  $x \in F_k$ .

2) Set  $G_k := F_1 \cup \dots \cup F_k$  for all  $k \in \{1, \dots, m\}$  (as in Corollary 3.3) and let us construct inductively sections  $\xi(k) \in E|_{G_k}$  such that  $\|\xi(k)_x\| = 1$  for all  $x \in G_k$ .

– If  $k = 1$ , the section  $\xi(1) := (\zeta_1)|_{F_1}$  has the requested properties.

– If  $k \in \{1, \dots, m-1\}$  and a convenient section  $\xi(k)$  in  $E|_{G_k}$  has already been constructed, then there exists a unital  $*$ -homomorphism  $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \rightarrow \mathcal{T}(E)|_{G_k \cap F_{k+1}}$  such that  $\pi_k(j_0(s_1)) = \ell(\xi(k))|_{G_k \cap F_{k+1}}$  and  $\pi_k(j_1(s_1)) = \ell(\zeta_{k+1})|_{G_k \cap F_{k+1}}$ . The partial isometry  $\ell(\zeta_{k+1})|_{G_k \cap F_{k+1}} \cdot \ell(\xi(k))^*|_{G_k \cap F_{k+1}}$  belongs to the subalgebra  $\mathcal{T}(E)^\alpha|_{G_k \cap F_{k+1}}$  and it extends in  $\mathcal{T}(E)|_{G_k \cap F_{k+1}}$  to a  $K_1$ -trivial unitary  $u_k$  by [BRR08, Lemma 2.4], i.e.

$$\ell(\zeta_{k+1})|_{G_k \cap F_{k+1}} \cdot \ell(\xi(k))^*|_{G_k \cap F_{k+1}} = \ell(\zeta_{k+1})\ell(\zeta_{k+1})^*|_{G_k \cap F_{k+1}} \cdot u_k = u_k \cdot \ell(\xi(k))\ell(\xi(k))^*|_{G_k \cap F_{k+1}},$$

since  $1 - \ell(\xi(k))\ell(\xi(k))^*|_{G_k \cap F_{k+1}} \geq \pi_k(j_0(s_2 s_2^*))$  and  $1 - \ell(\zeta_{k+1})\ell(\zeta_{k+1})^*|_{G_k \cap F_{k+1}} \geq \pi_k(j_1(s_2 s_2^*))$  are two properly infinite and full projections in  $\mathcal{T}(E)|_{G_k \cap F_{k+1}}$ . If  $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$  is  $K_1$ -injective, this unitary  $u_k$  belongs to  $\mathcal{U}_0(\mathcal{T}(E)|_{G_k \cap F_{k+1}})$  and it admits a unitary lifting  $z_{k+1} \in \mathcal{U}_0(\mathcal{T}(E))$ . The only section  $\xi(k+1) \in E|_{G_{k+1}}$  such that  $\xi(k+1)|_{G_k} = \xi(k)$  and  $\xi(k+1)|_{F_{k+1}} = \overline{\mathbf{q}}(z_{k+1})^* \cdot \xi_{k+1}|_{F_{k+1}}$  satisfies the desired conditions.  $\square$

**Theorem 4.3.** *The full unital free product  $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$  is properly infinite, but not  $K_1$ -injective.*

*Proof.* Let  $\mathfrak{X}$  be the complex Hilbert cube  $\mathfrak{X} := \{z \in \mathbb{C}; |z| \leq 1\}^{\mathbb{N}}$ . It is a compact space when equipped with the distance  $d(x, y) = \sum_p 2^{-p-2} |x_p - y_p|$ . Let also  $E_{DD}$  be the non-trivial separable Hilbert  $C(\mathfrak{X})$ -module contracted by J. Dixmier and A. Douady ([DD63]): It has infinite dimensional fibres and every section  $\zeta \in E_{DD}$  satisfies  $\zeta_x = 0$  for at least one  $x \in \mathfrak{X}$ . Thus proposition 4.2 implies that  $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$  is not  $K_1$ -injective.  $\square$

The  $C([0, 1])$ -algebra  $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2); f(0) \in j_0(\mathcal{T}_2) \text{ and } f(1) \in j_1(\mathcal{T}_2)\}$  introduced in Corollary 3.3 is an example of the following corollary:

**Corollary 4.4.** *There exists a separable unital continuous  $C([0, 1])$ -algebra with properly infinite fibres which is not properly infinite.*

*Question 4.5.* The Pimsner-Toeplitz algebra  $\mathcal{T}(E_{DD})$  is locally purely infinite ([BK04b, Definition 1.3]) since all its simple quotients are isomorphic to the Cuntz algebra  $\mathcal{O}_\infty$  ([BK04b, Proposition 5.1]). But is  $\mathcal{T}(E_{DD})$  properly infinite?

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