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LOCAL AND GLOBAL PROPER INFINITENESS FOR CONTINUOUS $C(X)$ -ALGEBRAS

ETIENNE BLANCHARD

ABSTRACT. Any non-zero quotient of a properly infinite C^* -algebra is properly infinite. We show that in the other direction, there exist continuous $C([0, 1])$ -algebras with properly infinite fibres, which are not properly infinite.

1. INTRODUCTION

The classification programme of nuclear C^* -algebras through K -theoretical invariants launched by G. Elliott ([Ell94]) led A. Toms and W. Winter to introduce the strong self-absorption property for simple unital C^* -algebras ([TW07]). This notion is pretty rigid: Any separable unital continuous $C(X)$ -algebra A the fibres of which are isomorphic to the same strongly self-absorbing C^* -algebra D is a trivial $C(X)$ -algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C^* -algebra D tensorially absorbs the Jiang-Su algebra \mathcal{Z} ([Win09]). Hence, this C^* -algebra D is K_1 -injective ([Ror04]) and the $C(X)$ -algebra A satisfies $A \cong D \otimes C(X)$ ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C^* -bundle over the infinite dimensional compact product $\prod_{n=0}^{\infty} S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2^{∞} ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dad09, §3] for all pair (Γ_0, Γ_1) of discrete countable torsion groups a unital separable continuous $C(X)$ -algebra A such that:

- the base space X is the compact Hilbert cube $X = \mathfrak{X}$ of infinite dimension,
- all the fibres A_x ($x \in \mathfrak{X}$) are isomorphic to the strongly self-absorbing Cuntz C^* -algebra \mathcal{O}_2 generated by two isometries s_1, s_2 satisfying $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$,
- $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $Y_0 \subset [0, 1]$ is the canonical Cantor set.

These K -theoretical conditions imply that the $C(\mathfrak{X})$ -algebra A is not a trivial one. But this argument does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra \mathcal{O}_{∞} ([Cun77]), in so far as $K_0(\mathcal{O}_{\infty}) = \mathbb{Z}$ is a torsion free group.

In this note, we tackle this trivialization problem for unital continuous $C(X)$ -algebras with fibres \mathcal{O}_{∞} in a different way: All unital continuous $C(X)$ -algebras with properly infinite fibres are properly infinite C^* -algebras (a problem already raised in [BRR08]) if and only if the full unital free product $\mathbf{A} := \mathcal{T}_2 *_C \mathcal{T}_2$ of two distinct copies of the unital extension \mathcal{T}_2 of the Cuntz algebras \mathcal{O}_2 by the compact operators ([Cun77]) is a K_1 -injective C^* -algebra, *i.e.* the canonical map from $\mathcal{U}(\mathbf{A})/\mathcal{U}^0(\mathbf{A})$ to $K_1(\mathbf{A})$ is injective

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(Corollary 4.2). We show in Proposition 4.4 that the C^* -algebra \mathbf{A} is not K_1 -injective.

As regards the possible K_1 -injective quotients of this C^* -algebra \mathbf{A} , the reduced unital free product $(\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_2, \psi_2)$ embeds in the reduced unital free product $(\mathfrak{B}, \psi) := (\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$ for some states π_2 and ψ_3 with faithful GNS representations ([BD01]) on the Cuntz extensions \mathcal{T}_2 and \mathcal{T}_3 . Works by M. Rieffel ([Rief83], [Rief87]), K. Dykema, U. Haagerup, M. Rørdam ([DHR97], [Roh09]) enable us to prove that this C^* -algebra \mathfrak{B} is simple and K_1 -injective (Proposition 5.2).

I especially thank E. Kirchberg for a few inspiring remarks.

2. NOTATIONS

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of positive integers and we denote by $[S]$ the closed linear span of any subset S in a Banach space.

Definition 2.1. ([Dix69], [Kas88], [Blan97]) *Let X be a compact Hausdorff space and let $C(X)$ be the C^* -algebra of continuous function on X .*

- *A unital $C(X)$ -algebra is a unital C^* -algebra A endowed with a unital morphism of C^* -algebra from $C(X)$ to the centre of A .*
- *For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a|_F$ the image of a in the quotient $A|_F := A/C_0(X \setminus F) \cdot A$. If $x \in X$ is a point in X , one calls fibre at x the quotient $A_x := A|_{\{x\}}$ and one write a_x for $a|_{\{x\}}$.*
- *The $C(X)$ -algebra A is said to be continuous if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$ is continuous for all $a \in A$.*

Definition 2.2. ([Pim95]) *Let X be a compact Hausdorff space.*

a) *The full Fock Hilbert $C(X)$ -module $\mathcal{F}(E)$ of a Hilbert $C(X)$ -module E is the direct sum of Hilbert $C(X)$ -module*

$$\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)})^m}, \quad (2.1)$$

where $E^{(\otimes_{C(X)})^m} := \begin{cases} C(X) & \text{if } m = 0, \\ E \otimes_{C(X)} \dots \otimes_{C(X)} E & (m \text{ terms}) \text{ if } m \geq 1. \end{cases}$

b) *The Pimsner-Toeplitz $C(X)$ -algebra $\mathcal{T}(E)$ of a full Hilbert $C(X)$ -module E , i.e. with non-zero fibres, is the unital subalgebra of the $C(X)$ -algebra $\mathcal{L}_{C(X)}(\mathcal{F}(E))$ of adjointable $C(X)$ -linear operator acting on $\mathcal{F}(E)$ generated by the creation operators $\ell(\zeta)$ ($\zeta \in E$), where:*

$$\begin{aligned} - \ell(\zeta)(f \cdot \hat{1}_{C(X)}) &:= f \cdot \zeta = \zeta \cdot f & \text{for } f \in C(X) & \text{and} \\ - \ell(\zeta)(\zeta_1 \otimes \dots \otimes \zeta_k) &:= \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_k & \text{for } \zeta_1, \dots, \zeta_k \in E & \text{if } k \geq 1. \end{aligned} \quad (2.2)$$

Remarks 2.3. a) ([Cun81], [BRR08]) For all integer $n \geq 2$, the C^* -algebra $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$ is the universal unital C^* -algebra generated by n isometries s_1, \dots, s_n satisfying the relation

$$s_1 s_1^* + \dots + s_n s_n^* \leq 1. \quad (2.3)$$

b) A unital C^* -algebra A is *properly infinite* if and only if one the following equivalent conditions holds ([Cun77], [Ror03, Proposition 2.1]):

- A contains two isometries with mutually orthogonal range projections,
- A contains a unital copy of the simple Cuntz C^* -algebra \mathcal{O}_∞ generated by infinitely many isometries with pairwise orthogonal ranges.

3. LOCAL PROPER INFINITENESS

Let $(C^*(\mathbb{Z}), \Delta)$ be the unital Hopf C^* -algebra ([Wor95]) generated by a unitary \mathbf{u} with spectrum $\mathbb{T} := \{z \in \mathbb{C}; z^*z = 1\}$ (often written S^1) and with coproduct $\Delta(\mathbf{u}) = \mathbf{u} \otimes \mathbf{u}$.

If X is a compact Hausdorff space and E is a separable Hilbert $C(X)$ -module with non-zero fibres, there is a unique coaction α_E of $(C^*(\mathbb{Z}), \Delta)$ on the Pimsner-Toeplitz $C(X)$ -algebra $\mathcal{T}(E)$ such that $\alpha_E(\ell(\zeta)) = \ell(\zeta) \otimes \mathbf{u}$ for all $\zeta \in E$, *i.e.*

$$\begin{aligned} \alpha_E : \mathcal{T}(E) &\rightarrow \mathcal{T}(E) \otimes C^*(\mathbb{Z}) &= C(\mathbb{T}, \mathcal{T}(E)) \\ \ell(\zeta) &\mapsto \ell(\zeta) \otimes \mathbf{u} &= (z \mapsto \ell(z\zeta)) \end{aligned} \quad (3.1)$$

The fixed point $C(X)$ -subalgebra $\mathcal{T}(E)^{\alpha_E} = \{a \in \mathcal{T}(E); \alpha_E(a) = a \otimes 1\}$ under this coaction is the closed linear span

$$\mathcal{T}(E)^{\alpha_E} = \left[C(X).1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^* \right]. \quad (3.2)$$

Besides, the following local proper infiniteness property holds.

Proposition 3.1. ([Blac04]) *Let X be a compact Hausdorff space and let E be a separable Hilbert $C(X)$ -module the fibres of which have dimension greater than 2.*

If x is a point in X , then there exists a closed neighbourhood $F \subset X$ of x such that the restriction $\mathcal{T}(E)|_F$ of the Pimsner-Toeplitz $C(X)$ -algebra $\mathcal{T}(E)$ is properly infinite.

Proof. Let ζ_1 and ζ_2 be two norm 1 sections in E satisfying $[\langle \zeta_i, \zeta_j \rangle(x)] = 1_{M_2(\mathbb{C})}$, *i.e.* $\{(\zeta_1)_x, (\zeta_2)_x\}$ is an orthonormal system in the Hilbert space E_x . By continuity, the matrix $[\langle \zeta_i, \zeta_j \rangle(y)] \in M_2(\mathbb{C})$ is invertible for all y in a neighbourhood of x and one can even suppose that $[\langle \zeta_i, \zeta_j \rangle(y)] = 1_{M_2(\mathbb{C})}$ for all points y in a closed neighbourhood $F \subset X$ of the point x , up to replacing the section $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ by $c \cdot \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ for some continuous function $c \in C(X; M_2(\mathbb{C}))$ satisfying $c(x) = 1_{M_2(\mathbb{C})}$. This orthonormalization procedure means that

$$\ell(\zeta_i)^* \ell(\zeta_j)|_F = \langle \zeta_i, \zeta_j \rangle|_F = \delta_{i=j} \cdot 1_F \quad \text{for all } i, j = 1 \text{ or } 2.$$

Thus, the restricted unit $1_F = \ell(\zeta_1)^* \ell(\zeta_1)|_F = \ell(\zeta_2)^* \ell(\zeta_2)|_F$ is greater than the sum $\ell(\zeta_1) \ell(\zeta_1)^*|_F + \ell(\zeta_2) \ell(\zeta_2)^*|_F$ and is therefore a properly infinite projection in the Pimsner-Toeplitz algebra $\mathcal{T}(E)|_F = C^* \langle \mathcal{T}(E)^{\alpha_E}|_F, \ell(\zeta_1)|_F \rangle$. \square

Remark 3.2. One can also prove Proposition 3.1 thanks to the semiprojectivity of the Cuntz C^* -algebra \mathcal{O}_∞ ([Blac04, Theorem 3.2]).

4. GLOBAL PROPER INFINITENESS

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following global version of Proposition 3.1 on proper infiniteness for continuous $C(X)$ -algebras with properly infinite fibres.

Proposition 4.1. *Let X be a second countable perfect compact Hausdorff space and let A be a separable unital continuous $C(X)$ -algebra with properly infinite fibres.*

1) *There exist:*

- (a) *a finite integer $n \geq 1$,*
- (b) *a covering $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$ by the interiors of closed balls F_1, \dots, F_n ,*
- (c) *unital embeddings of C^* -algebra $\sigma_k : \mathcal{O}_\infty \hookrightarrow A|_{F_k}$ ($1 \leq k \leq n$).*

2) *The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integers p .*

3) *For all integers i, j in $\{1, \dots, n\}$, there is a unitary $u_{i,j} \in \mathcal{U}(A|_{F_i \cap F_j})$ such that*

$$\sigma_i(s_m)|_{F_i \cap F_j} = u_{i,j} \cdot \sigma_j(s_m)|_{F_i \cap F_j} \quad \text{for all } m \in \mathbb{N}.$$

Proof. 1) For all point $x \in X$, the semiprojectivity of the C^* -subalgebra $\mathcal{O}_\infty \hookrightarrow A_x$ ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood $F \subset X$ of the point x and a unital embedding $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A|_F$ of $C(F)$ -algebra. The compactness of the topological space X enables to conclude.

2) derives from Proposition 3.1, [BRR08, Proposition 2.7] and [Ror97, Proposition 2.1].

3) Set $u_{i,j} := \sum_{m \in \mathbb{N}} \sigma_i(s_m)|_{F_i \cap F_j} \cdot \sigma_j(s_m)^*|_{F_i \cap F_j}$ ([Blan13, Proposition 6.3]). \square

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the C^* -algebra A is properly infinite. Indeed, there exists a unital C^* -algebra A which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is properly infinite ([Ror03, Proposition 4.5]). We nevertheless have the following corollary.

Corollary 4.2. *Let j_0, j_1 denote the two canonical unital embeddings of the C^* -algebra \mathcal{T}_2 in the full unital free product $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ and let $\tilde{u} \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ be a K_1 -trivial unitary satisfying $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$ ([BRR08, Lemma 2.4]).*

Then the following conditions are equivalent:

- (a) *The full unital free product $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ is K_1 -injective.*
- (b) *The unitary \tilde{u} belongs to the connected component $\mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ of $1_{\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2}$.*
- (c) *Every separable unital continuous $C(X)$ -algebra A with properly infinite fibres is a properly infinite C^* -algebra.*

Proof. (a) \Rightarrow (b) A unital C^* -algebra A is called K_1 -injective if and only if every unitary $v \in \mathcal{U}(A)$ is homotopic to the unit 1_A in $\mathcal{U}(A)$. Thus, (b) is a special case of (a).

(b) \Rightarrow (c) Let A be a separable unital continuous $C(X)$ -algebra with properly infinite fibres. Take a finite covering such that there exist unital embeddings $\sigma_k : \mathcal{T}_2 \rightarrow A|_{F_k}$ ($1 \leq k \leq n$). Set $G_k := F_1 \cup \dots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by

induction isometries $w_k \in A_{|G_k}$ such that the two projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A_{|G_k}$:

- If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.
- If $k \in \{1, \dots, n-1\}$ and the isometry $w_k \in A_{|G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital C^* -algebra $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \rightarrow A_{|G_k \cap F_{k+1}}$ and a K_1 -trivial unitary $u_{k+1} \in \mathcal{U}(A_{|G_k \cap F_{k+1}})$ satisfying:

$$\begin{aligned} - \quad & \pi_k(j_0(s_1)) = w_k|_{G_k \cap F_{k+1}} \quad \text{and} \\ - \quad & \pi_k(j_1(s_1)) = \sigma_{k+1}(s_1)|_{G_k \cap F_{k+1}} = u_{k+1} \cdot w_k|_{G_k \cap F_{k+1}}. \end{aligned} \quad (4.1)$$

If the unitary \tilde{u} belongs to $\mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$, then the unitary $u_{k+1} = \pi_k(\tilde{u})$ is homotopic to $1_{A_{|G_k \cap F_{k+1}}} = \pi_k(1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2})$ in $\mathcal{U}(A_{|G_k \cap F_{k+1}})$, so that u_{k+1} admits a unitary lifting z_{k+1} in $\mathcal{U}^0(A_{|F_{k+1}})$ (see *e.g.* [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A_{|G_{k+1}}$ satisfying the two constraints:

$$w_{k+1}|_{G_k} = w_k \quad \text{and} \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1) \quad (4.2)$$

verifies that the two projections $w_{k+1} w_{k+1}^*$ and $1_{|G_{k+1}} - w_{k+1} w_{k+1}^*$ are properly infinite and full in $A_{|G_{k+1}}$.

The proper infiniteness of the projection $w_n w_n^* \in A_{|G_n} = A$ implies that the unit $1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n$ is also a properly infinite projection in A , *i.e.* the C^* -algebra A is properly infinite.

(c) \Rightarrow (a) The C^* -algebra $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2); f(0) \in j_0(\mathcal{T}_2) \text{ and } f(1) \in j_1(\mathcal{T}_2)\}$ is a unital continuous $C([0, 1])$ -algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C^* -algebra \mathcal{D} is properly infinite, a statement which is equivalent to the K_1 -injectivity of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([Blan10, Proposition 4.2]). \square

There is an extra grading property for Pimsner-Toeplitz algebras in case the C^* -algebra $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1 -injective.

Corollary 4.3. *Let X be a second countable compact Hausdorff space and let E be a separable Hilbert $C(X)$ -module the fibres of which are all Hilbert spaces of dimension greater than 2.*

*If the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1 -injective, then there exists an isometry w in the Pimsner-Toeplitz algebra $\mathcal{T}(E)$ such that*

- $\mathcal{T}(E) = C^* \langle \mathcal{T}(E)^{\alpha_E}, w \rangle$,
- the projections $w w^*$ and $1 - w w^*$ are properly infinite and full in $\mathcal{T}(E)$,
- $\alpha_E(w) = w \otimes \mathbf{u}$ (an equality often described as "w has degree 1").

Proof. Let $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$ by a finite covering of the compact Hausdorff space X by the interiors of closed subsets F_1, \dots, F_n and let $\zeta_1, \zeta_2, \dots, \zeta_{2n}$ be $2n$ norm 1 sections in E such that $\langle \zeta_{2k-i}, \zeta_{2k-j} \rangle(y) = \delta_{i=j}$ for all indices $k \in \{1, \dots, n\}$, $0 \leq i, j \leq 1$ and all points y in F_k .

Set $G_k := F_1 \cup \dots \cup F_k \subset X$ for all $k \in \{1, \dots, n\}$ and let us construct by induction n isometries $w_k \in \mathcal{T}(E)|_{G_k}$ ($1 \leq k \leq n$) such that:

- (a) the projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in $\mathcal{T}(E)|_{G_k}$,

- (b) $\mathcal{T}(E)|_{G_k} = C^* \langle \mathcal{T}(E)^{\alpha_E}|_{G_k}, w_k \rangle$,
- (c) $\alpha_{E_{\mathcal{T}(E)|_{G_k}}}(w_k) = w_k \otimes \mathbf{u}$.

– If $k = 1$, the isometry $w_1 = \ell(\zeta_1)|_{F_1}$ has the three requested properties.

– If $k \in \{1, \dots, n-1\}$ and the isometry $w_k \in \mathcal{T}(E)|_{G_k}$ is already constructed, then there exist by (4.2) a unitary $z_{k+1} \in \mathcal{U}^0(\mathcal{T}(E)|_{F_{k+1}})$ and an isometry $w_{k+1} \in \mathcal{T}(E)|_{G_{k+1}}$ satisfying

$$w_{k+1}|_{G_k} = w_k \quad \text{and} \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \ell(\zeta_{2k-1})|_{F_{k+1}}, \quad (4.3)$$

provided the C*-algebra $\mathcal{T}_2 *_C \mathcal{T}_2$ is K_1 -injective.

This isometry w_{k+1} then has the two first desired properties by the previous corollary. Besides, $\alpha_{E|_{G_{k+1}}}(w_{k+1}) = w_{k+1} \otimes \mathbf{u}$ since $\alpha_{E|_{G_k}}(w_k) = w_k \otimes \mathbf{u}$, $\alpha_{E|_{F_{k+1}}}(\ell(\zeta_{2k-1})|_{F_{k+1}}) = \ell(\zeta_{2k-1})|_{F_{k+1}} \otimes \mathbf{u}$ and $\alpha_{E|_{F_{k+1}}}(z_{k+1}) = z_{k+1} \otimes 1$. \square

Proposition 4.4. *The full unital free product $\mathbf{A} = \mathcal{T}_2 *_C \mathcal{T}_2$ is not K_1 -injective.*

Proof. Let $\mathfrak{X} := [0, 1]^\infty$ be the compact Hilbert cube of infinite topological dimension, let \mathcal{E} be the non-trivial Hilbert $C(\mathfrak{X})$ -module \mathcal{E} constructed by J. Dixmier and A. Douady ([DD63], [BK04a, Proposition 3.6]) and let $\mathcal{T}(\mathcal{E})$ be its Pimsner-Toeplitz $C(\mathfrak{X})$ -algebra. There exist by Proposition 3.1 finitely many sections ζ_1, \dots, ζ_n in \mathcal{E} such that

$$\mathcal{T}(\mathcal{E}) = C^* \langle \mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}, \ell(\zeta_1), \dots, \ell(\zeta_n) \rangle. \quad (4.4)$$

If the C*-algebra $\mathcal{T}_2 *_C \mathcal{T}_2$ is K_1 -injective, there exists by Corollary 4.3 a degree 1 isometry $w \in \mathcal{T}(\mathcal{E})$ such that

$$\mathcal{T}(\mathcal{E}) = C^* \langle \mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}, w \rangle. \quad (4.5)$$

We then have $w = b_1 \cdot \ell(\zeta_1) + \dots + b_n \cdot \ell(\zeta_n)$ for some elements b_1, \dots, b_n in $\mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}$. And the equality $1_{C(X)} = \langle \hat{1}_{C(X)}, w^* w \cdot \hat{1}_{C(X)} \rangle$ implies that the section $w \cdot \hat{1}_{C(X)} = b_1 \cdot \zeta_1 + \dots + b_n \cdot \zeta_n \in \mathcal{E} \subset \mathcal{F}(\mathcal{E})$ verifies $\|(w \cdot \hat{1}_{C(X)})_x\| = 1$ for all $x \in \mathfrak{X}$. But this is absurd since any section ζ in the Dixmier-Douady Hilbert $C(\mathfrak{X})$ -module \mathcal{E} satisfies $\zeta_y = 0$ for at least one point $y \in \mathfrak{X}$ (see [DD63], or [BK04a, Proposition 3.6]). \square

Remarks 4.5. a) Proposition 4.4 answers questions 2.8, 2.9 and 2.13 in [BRR08]: If j_0 and j_1 denote the two canonical embeddings of \mathcal{O}_∞ in the full unital free product $\mathcal{O}_\infty *_C \mathcal{O}_\infty$, the continuous $C([0, 1])$ -algebra $\mathcal{A} := \{f \in C([0, 1], \mathcal{O}_\infty *_C \mathcal{O}_\infty); f(0) \in j_0(\mathcal{O}_\infty) \text{ and } f(1) \in j_1(\mathcal{O}_\infty)\}$ is not properly infinite whereas the tensor product $M_2(\mathcal{A})$ is properly infinite ([BRR08, Proposition 4.4]).

b) If \mathfrak{A} denotes the universal UHF-algebra with $K_1(\mathfrak{A}) = \mathbb{Q}$, there is a non-trivial unital continuous $C(\mathfrak{X})$ -algebra D with $D_x \cong \mathcal{O}_\infty \otimes \mathfrak{A}$ for all $x \in \mathfrak{X}$ ([Dad09]). But is there a non-trivial unital continuous $C(\mathfrak{X})$ -algebra with fibres isomorphic to \mathcal{O}_∞ ?

Note that $K_0(\mathcal{O}_\infty \otimes \mathfrak{A}) = \mathbb{Q} \neq \mathbb{Z} = K_0(\mathcal{O}_\infty)$.

c) An automorphism of $C(X)$ -algebra on the Pimsner-Toeplitz $C(X)$ -algebra $\mathcal{T}(E)$ is not always grading preserving, even if the base space X is reduced to a point, as noticed by E. Kirchberg ([KR00]): If E is the Hilbert space $E = \ell^2(\mathbb{N})$, the purely infinite simple unital nuclear C*-algebra $\mathcal{T}(E) \cong \mathcal{O}_\infty$ has real rank zero. Accordingly, there exists a unitary $u \in \mathcal{U}(\mathcal{T}(E))$ such that $u \cdot \mathcal{T}(E)^{\alpha_E} \cdot u^* \not\subset \mathcal{T}(E)^{\alpha_E}$.

d) The projection $1 - w^k(w^k)^* = 1 - ww^* + \dots + w^{k-1}(1 - ww^*)(w^{k-1})^*$ is the sum of k pairwise orthogonal properly infinite and full projections in $\mathcal{T}(E)$. Hence, $1 - w^k(w^k)^*$ is also a full and properly infinite projection .

5. A K_1 -INJECTIVE SUPQUOTIENT

The universal Cuntz extension \mathcal{T}_2 generated by two isometries with orthogonal ranges unittally embeds in the Cuntz extension \mathcal{T}_3 ([Cun81, Lemma 3.1]). As a consequence, the full unital free product $\mathcal{T}_2 *_C \mathcal{T}_2$ unittally embeds in the full unital free product $\mathcal{T}_2 *_C \mathcal{T}_3$ ([ADEL04, Proposition 2.4]). We show in this section that some *reduced* quotient of $\mathcal{T}_2 *_C \mathcal{T}_3$ is a simple K_1 -injective C^* -algebra which contains the algebraic unital free product $\mathcal{T}_2 \otimes_C \mathcal{T}_3$.

Let $\ell^2(\mathbb{N})$ be the Hilbert space of all the sequences $\xi = (\xi_k) \in \mathbb{C}^{\mathbb{N}}$ which satisfy $\|\xi\|_2^2 = \sum_k |\xi_k|^2 < \infty$ and take an orthonormal basis $\{e_0, e_1, e_2, \dots\}$ in $\ell^2(\mathbb{N})$. Define for all integer $n \geq 2$:

- the n isometries $s_{n,1}, \dots, s_{n,n}$ in $\mathbb{B}(\ell^2(\mathbb{N}))$ satisfying $s_{n,i} \cdot e_k = e_{nk+i}$ ($k \in \mathbb{N}$),
- the Cuntz extension $\mathcal{T}_n := C^* \langle s_{n,1}, \dots, s_{n,n} \rangle \subset \mathbb{B}(\ell^2(\mathbb{N}))$ ([Cun77]),
- the state $\psi_n(a) = \frac{1}{n} \sum_{i=1}^n \langle e_i, a \cdot e_i \rangle$ on the C^* -algebra $\mathcal{T}_n \subset \mathbb{B}(\ell^2(\mathbb{N}))$,
- the unitary $u_n := s_{n,1}s_{n,2}^* + \dots + s_{n,n}s_{n,1}^* + p \in \mathcal{T}_n$, where p is the rank 1 projection $p = 1 - \sum_i s_{n,i}s_{n,i}^* = \theta_{e_0, e_0}$ in $\mathbb{B}(\ell^2(\mathbb{N}))$.

Lemma 5.1. 1) $\psi_n(a) = \psi_n(u_n^* a u_n)$ for any operator $a \in \mathcal{T}_n$ and $\psi_n((u_n)^j) = 0$ for any integer $j \in \{1, \dots, n-1\}$,

2) The state ψ_n on \mathcal{T}_n has a faithful GNS representation.

3) The restriction of the state ψ_n to the C^* -subalgebra $C^*(u_n) \subset \mathcal{T}_n$ is faithful.

Proof. 1) The unitary u_n satisfies $(u_n)^n = 1$ and $(u_n)^j \cdot e_1 = e_{n-j+1}$ for all $1 \leq j \leq n$. Thus, there exists an isomorphism of C^* -algebra $\alpha_n : C^*(u_n) \rightarrow C^*(\mathbb{Z}/n\mathbb{Z})$ and

$$\begin{aligned} \psi_n(a) &= \frac{1}{n} \sum_{i=1}^n \langle (u_n)^i \cdot e_1, a (u_n)^i \cdot e_1 \rangle \\ &= \frac{1}{n} \sum_{i=1}^n \langle (u_n)^{i+1} \cdot e_1, a (u_n)^{i+1} \cdot e_1 \rangle = \psi_n(u_n^* a u_n). \end{aligned}$$

We also have $\psi_n((u_n)^j) = \langle e_1, (u_n)^j \cdot e_1 \rangle = \langle e_1, e_{n-j+1} \rangle = 0$ for all $j \in \{1, \dots, n-1\}$.

2) The C^* -algebra \mathcal{T}_n is contained in $\mathbb{B}(\ell^2(\mathbb{N}))$ and the Hilbert space $\ell^2(\mathbb{N})$ is the closure of $\mathcal{T}_n \cdot e_0$. Thus, if $a \in \mathcal{T}_n \setminus \{0\}$ is non-zero, there exists an element $b \in \mathcal{T}_n$ such that

$$\begin{aligned} 0 < \|ab \cdot e_0\|^2 &= \|ab s_{n,1}^* \cdot e_1\|^2 = \langle e_1, s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^* e_1 \rangle \\ &= \sum_{i=1}^n \langle e_i, s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^* e_i \rangle \\ &= n \cdot \psi_n(s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^*). \end{aligned}$$

3) The composition $\psi_n \circ \alpha_n^{-1}$ equals the only Haar state h_n on the finite dimensional Hopf C^* -algebra $(C^*(\mathbb{Z}/n\mathbb{Z}), \Delta)$, which is faithful on $C^*(\mathbb{Z}/n\mathbb{Z})$. \square

As a consequence, the following holds:

Proposition 5.2. *Let $(\mathfrak{B}, \psi) := (\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$ be the reduced unital free product defined by D. Voiculescu ([Voi85]).*

- 1) *The C^* -algebra \mathfrak{B} contains the algebraic unital free product generated by \mathcal{T}_2 and \mathcal{T}_3*
- 2) *The C^* -algebra \mathfrak{B} is simple and has stable rank 1.*
- 3) *The C^* -algebra \mathfrak{B} is K_1 -injective.*

Proof. 1) This embedding result is essentially contained in Proposition 4.2 of [DS01]. But we include here the main steps of that proof for completeness. Define:

- the tensor product C^* -algebra $B := \mathcal{T}_2 \otimes \mathcal{T}_3$,
- the state $\phi := \psi_2 \otimes \psi_3$ on the C^* -algebra B ,
- the full countably generated Hilbert B -bimodule $E := L^2(B, \phi) \otimes B$,
- the full Fock Hilbert B -bimodule $\mathcal{F}(E) = B \oplus E \oplus (E \otimes E) \oplus \dots$,
- the Pimsner-Toeplitz C^* -algebra $\mathcal{T}(E)$ generated in $\mathcal{L}(\mathcal{F}(E))$ by the *creation* operators $\ell(\xi)$, $\xi \in E$ ([Pim95]).
- the conditional expectation $\mathfrak{E} : \mathcal{T}(E) \rightarrow B$ given by compression with the orthogonal projection from $\mathcal{F}_B(E)$ onto the first summand $B \subset \mathcal{F}(E)$ ([DS01]).

The C^* -algebra B unitaly embeds in $\mathcal{T}(E) \subset \mathcal{L}(\mathcal{F}(E))$ by

$$b \cdot \ell(\Lambda_\phi b_1 \otimes b_2) \cdot b' = \ell(\Lambda_\phi b b_1 \otimes b_2 b').$$

Take a unitary v in the C^* -subalgebra $C^*\langle \ell(\Lambda_\phi 1 \otimes 1) \rangle \subset \mathcal{T}(E)$ such that $\mathfrak{E}(v^j) = 0$ for all non-zero integer $j \in \mathbb{Z}$ and let $\pi_2 : \mathcal{T}_2 \rightarrow \mathcal{T}(E)$, $\pi_3 : \mathcal{T}_3 \rightarrow \mathcal{T}(E)$ be the two unital $*$ -morphisms $\pi_2(a) = v^2 \cdot (a \otimes 1_{\mathcal{T}_3}) \cdot v^{-2}$ and $\pi_3(a) = v^3 \cdot (1_{\mathcal{T}_2} \otimes a) \cdot v^{-3}$. Then, the algebraic amalgamated free product $\mathcal{T}_2 \otimes_C \mathcal{T}_3$ studied by B. Blackadar ([Blac78]) and G. Pedersen ([Ped94]) is (isomorphic to) the unital $*$ -algebra generated by $\pi_2(\mathcal{T}_2)$ and $\pi_3(\mathcal{T}_3)$ in $\mathcal{T}(E)$, whereas Voiculescu's reduced unital free product $(\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$ ([Voi85]) is (isomorphic to) its closure (see *e.g.* [Blan09, Theorem 4.1]).

2) The previous Lemma 5.1 and [BD01, Theorem 1.3] imply that there is a unital embedding $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_C (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \hookrightarrow (\mathfrak{B}, \psi) = (\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$, if h_2 and h_3 denote the Haar states on the compact groups $(C^*(\mathbb{Z}/2\mathbb{Z}), \Delta_2)$ and $(C^*(\mathbb{Z}/3\mathbb{Z}), \Delta_3)$. Hence, the reduced free product \mathfrak{B} is simple ([Av82, Proposition 3.1+its corollary]) and this unital C^* -algebra has stable rank 1 ([DHR97, Theorem 3.8]), *i.e.* the open subset of invertible elements is dense in \mathfrak{B} .

3) With the notations of [Rief83], the set $Lg_1(\mathfrak{B})$ of invertible elements is dense in the stable rank 1 C^* -algebra \mathfrak{B} , so that this C^* -algebra \mathfrak{B} is K_1 -injective by [Rief83, Proposition 1.6] and [Rief87, Theorem 2.10]. (A self-contained proof is also available in Theorem 3.2.11 of [Roh09].) \square

Remark 5.3. For all strictly positive integer n , the groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are normal subgroups in the cyclic groups $\mathbb{Z}/2n\mathbb{Z}$ and $\mathbb{Z}/3n\mathbb{Z}$. Thus, there is a unital embedding $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_C (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \subset (C^*(\mathbb{Z}/2n_1 \cdot \mathbb{Z}), h_{2n_1}) *_C (C^*(\mathbb{Z}/3n_2 \cdot \mathbb{Z}), h_{3n_2})$

([BD01]) and the reduced unital free product $(\mathcal{T}_{2n_1}, \psi_{2n_1}) *_\mathbb{C} (\mathcal{T}_{3n_2}, \psi_{3n_2})$ is both simple and K_1 -injective for all strictly positive integers n_1, n_2 .

Question 5.4. *Is the reduced unital free product $(\mathcal{T}_2, \psi_2) *_\mathbb{C} (\mathcal{T}_2, \psi_2)$ also K_1 -injective?*

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