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CONTINUOUS FIELDS OF PROPERLY INFINITE C*-ALGEBRAS

ETIENNE BLANCHARD

Abstract. Any unital separable continuous C(\(X\))-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space \(X\) has finite topological dimension. We show that this is still true when the compact space \(X\) has infinite topological dimension.

1. Introduction

One of the basic C*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C*-algebras through K-theoretical invariants is the Cuntz C*-algebra \(O_\infty\) generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous C(\(X\))-algebra \(A\) the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra \(D\) is a trivial C(\(X\))-algebra provided the compact Hausdorff base space \(X\) has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra \(D\) tensorially absorbs the Jiang-Su algebra \(Z\) ([Win09]). Hence, this C*-algebra \(D\) is \(K_1\)-injective ([Rør04]) and the C(\(X\))-algebra \(A\) satisfies \(A \cong D \otimes C(X)\) ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product \(\prod_{n=0}^{\infty} S^2\) such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type \(2^\infty\) ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dăd09, §3] for all pair \((\Gamma_0, \Gamma_1)\) of discrete countable torsion groups a unital separable continuous C(\(X\))-algebra \(A\) such that:

- the base space \(X\) is the compact Hilbert cube \(X = \mathcal{X}\) of infinite dimension,
- all the fibres \(A_x\) \((x \in \mathcal{X})\) are isomorphic to the strongly self-absorbing Cuntz C*-algebra \(O_2\) generated by two isometries \(s_1, s_2\) satisfying \(1_{O_2} = s_1s_1^* + s_2s_2^*\),
- \(K_i(A) \cong C(Y_0, \Gamma_i)\) for \(i = 0, 1\), where \(Y_0 \subset [0, 1]\) is the canonical Cantor set.

These K-theoretical conditions imply that the C(\(X\))-algebra \(A\) is not a trivial one. But these arguments does not anymore work when the strongly self-absorbing algebra \(D\) is the Cuntz algebra \(O_\infty\) ([Cun77]), in so far as \(K_0(O_\infty) = \mathbb{Z}\) is a torsion free group.

We still do not know whether all unital continuous C(\(X\))-algebra with fibres \(O_\infty\) are trivial. But we show that any unital separable continuous C(\(X\))-algebra with fibres \(O_\infty\) is at least properly infinite (see [Blan13, §6]).

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2. Notations

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of positive integers and we denote by $[S]$ the closed linear span of any subset $S$ in a Banach space.

Definition 2.1. ([Dix69], [Kas88], [Blan97]) Let $X$ be a compact Hausdorff space and let $C(X)$ be the $C^*$-algebra of continuous function on $X$.

- A unital $C(X)$-algebra is a unital $C^*$-algebra $A$ endowed with a unital morphism of $C^*$-algebra from $C(X)$ to the centre of $A$.
- For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a|_F$ the image of $a$ in the quotient $A|_F := A/C_0(X \setminus F) \cdot A$. If $x \in X$ is a point in $X$, one calls fibre at $x$ the quotient $A_x := A_{\{x\}}$ and one write $a_x$ for $a|_{\{x\}}$.
- The $C(X)$-algebra $A$ is said to be continuous if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}$ is continuous for all $a \in A$.

Remarks 2.2. a) ([Cun81], [BRR08]) For all integer $n \geq 2$, the $C^*$-algebra $T_n := T(C^n)$ is the universal unital $C^*$-algebra generated by $n$ isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1s_1^* + \ldots + s_ns_n^* \leq 1. \quad (2.1)$$

b) A unital $C^*$-algebra $A$ is properly infinite if and only if one the following equivalent conditions holds ([Cun77], [Rør03, Proposition 2.1]):

- the $C^*$-algebra $A$ contains two isometries with mutually orthogonal range projections, i.e. $A$ unitally contains a copy of $T_2$,
- the $C^*$-algebra $A$ contains a unital copy of the simple Cuntz $C^*$-algebra $O_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges.

3. Global proper infiniteness

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous $C(X)$-algebras with properly infinite fibres.

Proposition 3.1. Let $X$ be a second countable perfect compact Hausdorff space, i.e. without any isolated point, and let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres.

1) There exist:

(a) a finite integer $n \geq 1$,
(b) a covering $X = \overset{n}{\underset{k=1}{\cup}} \overset{\circ}{F_k}$ by the interiors of closed balls $F_1, \ldots, F_n$,
(c) unital embeddings of $C^*$-algebra $\sigma_k : O_\infty \hookrightarrow A|_{F_k}$ ($1 \leq k \leq n$).

2) The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integers $p$.

Proof. 1) For all point $x \in X$, the semiprojectivity of the $C^*$-subalgebra $O_\infty \hookrightarrow A_x$ ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood $F \subset X$ of the point $x$ and a unital embedding $O_\infty \otimes C(F) \hookrightarrow A|_F$ of $C(F)$-algebra. The compactness of the topological space $X$ enables to conclude.
2) Proposition [BRR08, Proposition 2.7] entails that $M_{2^n-1}(A)$ is properly infinite and [Rør97, Proposition 2.1] implies that $M_p(A)$ for all integer $p \geq 2^{n-1}$. \hfill $\square$

**Remark 3.2.** If $X$ is an ordinary second countable compact Hausdorff space and $A$ is a separable unital continuous $C(X)$-algebra, then $\hat{X} := X \times [0,1]$ is a perfect compact space, $\hat{A} := A \otimes C([0,1])$ is a unital continuous $C(\hat{X})$-algebra and any unital morphism $\mathcal{O}_\infty \to \hat{A}$ induces a unital morphism $\mathcal{O}_\infty \to A$ by composition with the projection map $A \to A$ coming from the injection $x \in X \mapsto (x,0) \in \hat{X}$.

The proper infiniteness of the tensor product $M_p(C) \otimes A$ does not always imply that the $C^*$-algebra $A$ is properly infinite ([HR98]). Indeed, there exists a unital $C^*$-algebra $A$ which is not properly infinite, but such that the tensor product $M_2(C) \otimes A$ is properly infinite ([Rør03, Proposition 4.5]). We nevertheless have the following corollary.

**Corollary 3.3.** Let $j_0,j_1$ denote the two canonical unital embeddings of the Cuntz extension $T_2$ in the full unital free product $T_2 \ast_\mathbb{C} T_2$ and let $\tilde{u} \in \mathcal{U}(T_2 \ast_\mathbb{C} T_2)$ be a $K_1$-trivial unitary satisfying $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$ ([BRR08, Lemma 2.4]).

Then the following conditions are equivalent:

(a) The full unital free product $T_2 \ast_\mathbb{C} T_2$ is $K_1$-injective.
(b) The unitary $\tilde{u}$ belongs to the connected component $\mathcal{U}_0(T_2 \ast_\mathbb{C} T_2)$ of $1_{T_2 \ast_\mathbb{C} T_2}$.
(c) Every separable unital continuous $C(X)$-algebra $A$ with properly infinite fibres is a properly infinite $C^*$-algebra.

**Proof.** (a)$\Rightarrow$(b) A unital $C^*$-algebra $A$ is called $K_1$-injective if and only if all $K_1$-trivial unitaries $v \in \mathcal{U}(A)$ are homotopic to the unit $1_A$ in $\mathcal{U}(A)$ (see e.g. [Roh09]). Thus, (b) is a special case of (a) since $K_1(T_2 \ast_\mathbb{C} T_2) = \{1\}$ (see e.g. [Blan10, Lemma 4.4]).

(b)$\Rightarrow$(c) Let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = \tilde{F}_1 \cup \ldots \cup \tilde{F}_n$ such that there exist unital embeddings $\sigma_k : T_2 \to A_{|F_k}$ ($1 \leq k \leq n$). Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A_{|G_k}$ such that the two projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A_{|G_k}$:

- If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.
- If $k \in \{1,\ldots,n-1\}$ and the isometry $w_k \in A_{|G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital $C^*$-algebra $\pi_k : T_2 \ast_\mathbb{C} T_2 \to A_{|G_k \cap F_{k+1}}$ satisfying:

$$\begin{align*}
&\pi_k(j_0(s_1)) = w_k|G_k \cap F_{k+1}, \\
&\pi_k(j_1(s_1)) = \sigma_{k+1}(s_1)|G_k \cap F_{k+1} = \pi_k(\tilde{u}) \cdot w_k|G_k \cap F_{k+1}.
\end{align*}$$  \hfill (3.1)

If the unitary $\tilde{u}$ belongs to $\mathcal{U}_0(T_2 \ast_\mathbb{C} T_2)$, then $\pi_k(\tilde{u})$ is homotopic to $1_{A_{|G_k \cap F_{k+1}}} = \pi_k(1_{T_2 \ast_\mathbb{C} T_2})$ in $\mathcal{U}(A_{|G_k \cap F_{k+1}})$, so that $\pi_k(\tilde{u})$ admits a unitary lifting $z_{k+1}$ in $\mathcal{U}_0(A_{|F_{k+1}})$ (see e.g. [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A_{|F_{k+1}}$ satisfying the two constraints:

$$w_{k+1}|F_k = w_k \quad \text{and} \quad w_{k+1}|F_{k+1} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1)$$  \hfill (3.2)
verifies that the two projections \( w_{k+1}w_{k+1}^* \) and \( 1_{\mathcal{G}_{k+1}} - w_{k+1}w_{k+1}^* \) are properly infinite and full in \( A_{\mathcal{G}_{k+1}} \).

The proper infiniteness of the projection \( w_n^*w_n \in A_{\mathcal{G}_n} = A \) implies that the unit \( 1_A = w_n^*w_n = w_n^*w_n \cdot w_n \) is also a properly infinite projection in \( A \), i.e. the C*-algebra \( A \) is properly infinite.

(c)⇒(a) The C*-algebra \( \mathcal{D} := \{ f \in C([0,1], \mathcal{T}_2 \ast_C \mathcal{T}_2) : f(0) \in j_0(\mathcal{T}_2) \text{ and } f(1) \in j_1(\mathcal{T}_2) \} \) is a unital continuous \( C([0,1]) \)-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra \( \mathcal{D} \) is properly infinite, a statement which is equivalent to the \( K_1 \)-injectivity of \( \mathcal{T}_2 \ast_C \mathcal{T}_2 \) ([Blan10, Proposition 4.2]). \( \square \)

**Proposition 3.4.** Let \( a \in \mathcal{T}_2 \) be the sum \( a := 1 - s_1s_1^* + s_1 \).

1) The operator \( a \) is a unitary in \( \mathcal{T}_2 \).

2) The unitary \( u := j_1(a) \cdot j_0(a) \in \mathcal{T}_2 \ast_C \mathcal{T}_2 \) belongs to the connected component \( U_0(\mathcal{T}_2 \ast_C \mathcal{T}_2) \). It also satisfies \( j_1(s_1) \cdot j_0(s_1) = u \cdot j_0(s_1s_1^*) \).

3) The C*-algebra \( \mathcal{T}_2 \ast_C \mathcal{T}_2 \) is \( K_1 \)-injective.

**Proof.** 1) Let \( H \) be the Hilbert space \( H := \ell^2(\mathbb{N}) \) with canonical orthonormal basis \( \{ e_k ; k \in \mathbb{N} \} \) and identify \( \mathcal{T}_2 \) with the C*-algebra generated by the two isometries \( s_1, s_2 \) given by \( s_i \cdot e_k := e_{2k+i} \) for \( i = 1, 2 \) and \( k \in \mathbb{N} \).

\( a \) is an isometry since \( a^*a = ((1 - s_1^*s_1) + s_1)^*((1 - s_1^*s_1) + s_1) = ((1 - s_1^*s_1) + s_1^*s_1) = 1 \).

For all vector \( \xi \in H \), we also have

\[
\begin{align*}
aa^*\xi &= 0 \\
\overset{\text{since } a \text{ is an isometry}}{\Rightarrow} a^*\xi &= s_1^*\xi + (1 - s_1s_1^*)^*\xi = 0 \\
\Rightarrow (1 - s_1s_1^*)^*\xi &= 0 \quad \text{and} \quad s_1^*\xi = 0 \\
\Rightarrow \xi &= s_1s_1^*\xi \quad \text{and} \quad \xi = 0 \\
\Rightarrow \xi &= 0.
\end{align*}
\]

Thus, the coisometry \( a^* \) is injective and the operator \( a \) is a unitary.

2) The unitary \( a \) is homotopic to \( 1 \) in \( \mathcal{U}(\mathcal{T}_2) \) (see e.g. [Blan10, Proposition 3.2]). Hence, the product \( u = j_1(a) \cdot j_0(s_1^*) \) is homotopic to \( 1 \) in \( \mathcal{U}(\mathcal{T}_2 \ast_C \mathcal{T}_2) \).

Besides, \( u \cdot j_0(s_1s_1^*) = j_1(a) \cdot j_0(s_1^*) = j_1(s_1) \cdot j_0(s_1)^* \).

3) This derives for Corollary 3.3. \( \square \)

### 4. The Pimsner-Toeplitz Algebra of a Hilbert C(X)-Module

We look in this section at the special case of unital continuous \( C(X) \)-algebras with fibres \( \mathcal{O}_m \) corresponding to the Pimsner-Toeplitz \( C(X) \)-algebras of Hilbert \( C(X) \)-modules with infinite dimension fibres.

**Definition 4.1.** ([Pim95]) Let \( X \) be a compact Hausdorff space and \( E \) a full Hilbert \( C(X) \)-module \( E \), i.e. without any zero fibre.

a) The full Fock Hilbert \( C(X) \)-module \( \mathcal{F}(E) \) of \( E \) is the direct sum of Hilbert \( C(X) \)-module

\[
\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\mathcal{O}_{C(X)})m}.
\] (4.1)
where $E^{(\otimes_{C(X)})^m} := \begin{cases} C(X) & \text{if } m = 0, \\ E \otimes C(X) \ldots \otimes C(X) E & \text{(m terms) if } m \geq 1. \end{cases}$

b) The Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ of $E$ is the unital subalgebra of the $C(X)$-algebra $\mathcal{L}(C(X)(\mathcal{F}(E)))$ of adjointable $C(X)$-linear operator acting on $\mathcal{F}(E)$ generated by the creation operators $\ell(\zeta) (\zeta \in E)$, where:

\begin{align*}
- \quad \ell(\zeta) (f \cdot \hat{1}_{C(X)}) &:= f \cdot \zeta = \zeta \cdot f \quad \text{for } f \in C(X) \quad \text{and} \\
- \quad \ell(\zeta) (\zeta_1 \otimes \ldots \otimes \zeta_k) &:= \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k \quad \text{for } \zeta_1, \ldots, \zeta_k \in E \quad \text{if } k \geq 1. \quad (4.2)
\end{align*}

c) Let $(C^*(\mathbb{Z}), \Delta)$ be the compact quantum group generated by a unitary $u$ with spectrum the unit circle and with coproduct $\Delta(u) = u \otimes u$. Then, there is a unique coaction $\alpha$ of the Hopf $C^*$-algebra $(C^*(\mathbb{Z}), \Delta)$ on the Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ such that $\alpha(\ell(\zeta)) = \ell(\zeta) \otimes u$ for all $\zeta \in E$, i.e.

\begin{align*}
\alpha : \quad \mathcal{T}(E) &\rightarrow \mathcal{T}(E) \otimes C^*(\mathbb{Z}) = C(T, \mathcal{T}(E)) \\
\ell(\zeta) &\mapsto \ell(\zeta) \otimes u = (\zeta \mapsto \ell(\zeta u)) \quad (4.3)
\end{align*}

The fixed point $C(X)$-subalgebra $\mathcal{T}(E)^\alpha = \{a \in \mathcal{T}(E); \alpha(a) = a \otimes 1\}$ under this coaction is the closed linear span

$$
\mathcal{T}(E)^\alpha = \left[ C(X) \cdot 1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^* \right]. \quad (4.4)
$$

Besides, the projection $P \in \mathcal{L}(\mathcal{F}(E))$ onto the submodule $E$ induces a quotient morphism of $C(X)$-algebra $a \in \mathcal{T}(E)^\alpha \mapsto \mathfrak{q}(a) := P \cdot a \cdot P \in K(E) + C(X) \cdot 1 \subset \mathcal{L}(E)$.

**Proposition 4.2.** Let $X$ be a second countable compact Hausdorff perfect space and let $E$ be a separable Hilbert $C(X)$-module with infinite dimensional fibres.

1) There exist a covering $X = F_1 \cup \ldots \cup F_m$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ sections $\zeta_1, \ldots, \zeta_m$ in $E$ such that $T(E) = C^* < T(E)^\alpha, \ell(\zeta_1), \ldots, \ell(\zeta_m) >$ and $\|(\zeta_k)_x\| = 1$ for all $k \in \{1, \ldots, m\}$ and $x \in F_k$.

2) The Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ is properly infinite.

**Proof.** 1) For all point $x \in X$, there exists a section $\zeta \in E$ satisfying $\|\zeta_x\| = 1$, whence an isomorphism of $C^*$-algebra $\mathcal{T}(E)_x \cong T(E_x) = C^* < T(E_x)^\alpha, \ell(\zeta_x) >$. The semi-projectivity of the $C^*$-algebra $O_\infty \cong T(E)_x$ and the compactness of the space $X$ imply that there exist a finite covering $X = F_1 \cup \ldots \cup F_m$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ contractions $\zeta_1, \ldots, \zeta_m$ in $E$ such that $\|\zeta_k\| = 1$ for all index $k \in \{1, \ldots, m\}$ and all point $x \in F_k$, so that $\ell(E)|F_k = \ell(E) \cdot \ell(\zeta_k)^* \cdot \ell(\zeta_k)|F_k$ and $T(E)|F_k = C^* < T(E)^\alpha, \ell(\zeta_k) >|F_k$.

2) The equivalence $(a) \Leftrightarrow (c)$ in Corollary 3.3 implies that $\mathcal{T}(E)$ is properly infinite. \hfill \Box

**Question 4.3.** Is there a unitary $v \in \mathcal{U}(\mathcal{F}_2 \ast \mathcal{F}_2)^\alpha$ such that $j_1(s_1 s_1^*) = v \cdot j_0(s_1 s_1^*) \cdot v^*$?
References


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