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CONTINUOUS FIELDS OF PROPERLY INFINITE C*-ALGEBRAS

ETIENNE BLANCHARD

Abstract. Any unital separable continuous C(X)-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space X has finite topological dimension. We show that this is still true when the compact space X has infinite topological dimension.

1. Introduction

One of the basic C*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C*-algebras through K-theoretical invariants is the Cuntz C*-algebra O∞ generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous C(X)-algebra A whose fibres are isomorphic to the same strongly self-absorbing C*-algebra D is a trivial C(X)-algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra D tensorially absorbs the Jiang-Su algebra Z ([Win09]). Hence, this C*-algebra D is K1-injective ([Rør04]) and the C(X)-algebra A satisfies A ∼= D ⊗ C(X) ([DW08].) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product Π_{n=0}^∞ S2 such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2∞ ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dăd09, §3] for all pair (Γ₀, Γ₁) of discrete countable torsion groups a unital separable continuous C(X)-algebra A such that:

- the base space X is the compact Hilbert cube X = X of infinite dimension,
- all the fibres Aₓ (x ∈ X) are isomorphic to the strongly self-absorbing Cuntz C*-algebra O₂ generated by two isometries s₁, s₂ satisfying 1O₂ = s₁s₁* + s₂s₂*,
- K₁(A) ∼= C(Y₀, Γ₁) for i = 0, 1, where Y₀ ⊂ [0, 1] is the canonical Cantor set.

These K-theoretical conditions imply that the C(X)-algebra A is not a trivial one. But these arguments does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra O∞ ([Cun77]), in so far as K₀(O∞) = Z is a torsion free group.

We still do not know whether all unital continuous C(X)-algebra with fibres O∞ are trivial. But we show that any unital separable continuous C(X)-algebra with fibres O∞ is at least properly infinite (see [Blan13, §6]).

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2. Notations

We present in this section the main notations which are used in this article. We denote by \( \mathbb{N} = \{0, 1, 2, \ldots\} \) the set of positive integers and we denote by \([S]\) the closed linear span of any subset \( S \) in a Banach space.

**Definition 2.1.** ([Dix69], [Kas88], [Blan97]) Let \( X \) be a compact Hausdorff space and let \( C(X) \) be the \( C^* \)-algebra of continuous function on \( X \).

- A unital \( C(X) \)-algebra is a unital \( C^* \)-algebra \( A \) endowed with a unital morphism of \( C^* \)-algebra from \( C(X) \) to the centre of \( A \).
- For all closed subset \( F \subset X \) and all element \( a \in A \), one denotes by \( a|_F \) the image of \( a \) in the quotient \( A|_F := A/C_0(X \setminus F) \cdot A \). If \( x \in X \) is a point in \( X \), one calls fibre at \( x \) the quotient \( A_x := A|_{\{x\}} \) and one write \( a|x \) for \( a|_{\{x\}} \).
- The \( C(X) \)-algebra \( A \) is said to be continuous if the upper semicontinuous map \( x \in X \mapsto \|a_x\| \in \mathbb{R}_+ \) is continuous for all \( a \in A \).

**Remarks 2.2.**

a) ([Cun81], [BRR08]) For all integer \( n \geq 2 \), the \( C^* \)-algebra \( T_n := T(C^n) \) is the universal unital \( C^* \)-algebra generated by \( n \) isometries \( s_1, \ldots, s_n \) satisfying the relation
\[
s_1s_1^* + \ldots + s_ns_n^* \leq 1.
\] (2.1)
b) A unital \( C^* \)-algebra \( A \) is properly infinite if and only if one the following equivalent conditions holds ([Cun77], [Rør03, Proposition 2.1]):

- the \( C^* \)-algebra \( A \) contains two isometries with mutually orthogonal range projections, i.e. \( A \) unitally contains a copy of \( T_2 \),
- the \( C^* \)-algebra \( A \) contains a unital copy of the simple Cuntz \( C^* \)-algebra \( O_\infty \) generated by infinitely many isometries with pairwise orthogonal ranges.

3. Global proper infiniteness

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous \( C(X) \)-algebras with properly infinite fibres.

**Proposition 3.1.** Let \( X \) be a second countable perfect compact Hausdorff space, i.e. without any isolated point, and let \( A \) be a separable unital continuous \( C(X) \)-algebra with properly infinite fibres.

1) There exist:

(a) a finite integer \( n \geq 1 \),
(b) a covering \( X = F_1 \cup \ldots \cup F_n \) by the interiors of closed balls \( F_1, \ldots, F_n \),
(c) unital embeddings of \( C^* \)-algebra \( \sigma_k : O_\infty \hookrightarrow A|_{F_k} \) (\( 1 \leq k \leq n \)).

2) The tensor product \( M_p(\mathbb{C}) \otimes A \) is properly infinite for all large enough integers \( p \).

**Proof.** 1) For all point \( x \in X \), the semiprojectivity of the \( C^* \)-subalgebra \( O_\infty \hookrightarrow A_x \) ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood \( F \subset X \) of the point \( x \) and a unital embedding \( O_\infty \otimes C(F) \hookrightarrow A|_F \) of \( C(F) \)-algebra. The compactness of the topological space \( X \) enables to conclude.
2) Proposition \cite[Proposition 2.7]{BRR08} entails that $M_{2^n-1}(A)$ is properly infinite and \cite[Proposition 2.1]{Ror97} implies that $M_p(A)$ for all integer $p \geq 2^n - 1$. \qed

Remark 3.2. If $X$ is an ordinary second countable compact Hausdorff space and $A$ is a separable unital continuous $C(X)$-algebra, then $\widetilde{X} := X \times [0, 1]$ is a perfect compact space, $\widetilde{A} := A \otimes C([0, 1])$ is a unital continuous $C(\widetilde{X})$-algebra and any unital morphism $\mathcal{O}_\infty \rightarrow \widetilde{A}$ induces a unital morphism $\mathcal{O}_\infty \rightarrow A$ by composition with the projection map $A \rightarrow \widetilde{A}$ coming from the injection $x \in X \mapsto (x, 0) \in \widetilde{X}$.

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the $C^*$-algebra $A$ is properly infinite (\cite{HR98}). Indeed, there exists a unital $C^*$-algebra $A$ which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is properly infinite (\cite[Proposition 4.5]{Ror03}). We nevertheless have the following corollary.

Corollary 3.3. Let $j_0, j_1$ denote the two canonical unital embeddings of the Cuntz extension $T_2$ in the full unital free product $T_2 \ast_{\mathbb{C}} T_2$ and let $\tilde{u} \in \mathcal{U}(T_2 \ast_{\mathbb{C}} T_2)$ be a $K_1$-trivial unitary satisfying $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$ (\cite[Lemma 2.4]{BRR08}).

Then the following conditions are equivalent:

(a) The full unital free product $T_2 \ast_{\mathbb{C}} T_2$ is $K_1$-injective.
(b) The unitary $\tilde{u}$ belongs to the connected component $\mathcal{U}_0(T_2 \ast_{\mathbb{C}} T_2)$ of $1_{T_2 \ast_{\mathbb{C}} T_2}$.
(c) Every separable unital continuous $C(X)$-algebra $A$ with properly infinite fibres is a properly infinite $C^*$-algebra.

Proof. (a)⇒(b) A unital $C^*$-algebra $A$ is called $K_1$-injective if and only if all $K_1$-trivial unitaries $v \in \mathcal{U}(A)$ are homotopic to the unit $1_A$ in $\mathcal{U}(A)$ (see e.g. \cite{Roh09}). Thus, (b) is a special case of (a) since $K_1(T_2 \ast_{\mathbb{C}} T_2) = \{1\}$ (see e.g. \cite[Lemma 4.4]{Blan10}).

(b)⇒(c) Let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = F_1 \cup \ldots \cup F_n$ such that there exist unital embeddings $\sigma_k : T_2 \rightarrow A|_{F_k}$ ($1 \leq k \leq n$). Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A|_{G_k}$ such that the two projections $w_k w_k^*$ and $1_{|_{G_k}} - w_k w_k^*$ are properly infinite and full in the restriction $A|_{G_k}$:

If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

If $k \in \{1, \ldots, n-1\}$ and the isometry $w_k \in A|_{G_k}$ is already constructed, then Lemma 2.4 of \cite{BRR08} implies that there exist an homomorphism of unital $C^*$-algebra $\pi_k : T_2 \ast_{\mathbb{C}} T_2 \rightarrow A|_{G_k \cap F_{k+1}}$ satisfying:

\begin{align*}
\pi_k(j_0(s_1)) &= w_k|_{G_k \cap F_{k+1}} \\
\pi_k(j_1(s_1)) &= \sigma_{k+1}(s_1)|_{G_k \cap F_{k+1}} = \pi_k(\tilde{u}) \cdot w_k|_{G_k \cap F_{k+1}}.
\end{align*}

If the unitary $\tilde{u}$ belongs to $\mathcal{U}_0(T_2 \ast_{\mathbb{C}} T_2)$, then $\pi_k(\tilde{u})$ is homotopic to $1_{A|_{G_k \cap F_{k+1}}} = \pi_k(1_{T_2 \ast_{\mathbb{C}} T_2})$ in $\mathcal{U}(A|_{G_k \cap F_{k+1}})$, so that $\pi_k(\tilde{u})$ admits a unitary lifting $z_{k+1}$ in $\mathcal{U}_0(A|_{F_{k+1}})$ (see e.g. \cite[Lemma 2.1.7]{LLR00}). The only isometry $w_{k+1} \in A|_{G_{k+1}}$ satisfying the two constraints:

\begin{align*}
w_{k+1}|_{G_k} &= w_k \\
w_{k+1}|_{F_{k+1}} &= (z_{k+1})^* \cdot \sigma_{k+1}(s_1)
\end{align*}
verifies that the two projections \( w_{k+1}w_{k+1}^* \) and \( 1_{A|_{G_{k+1}}} - w_{k+1}w_{k+1}^* \) are properly infinite and full in \( A|_{G_{k+1}} \).

The proper infiniteness of the projection \( w_nw_n^* \in A|_{G_n} = A \) implies that the unit \( 1_A = w_nw_n^*w_nw_n \) is also a properly infinite projection in \( A \), i.e. the C*-algebra \( A \) is properly infinite.

(c)⇒(a) The C*-algebra \( D := \{ f \in C([0,1], \mathcal{D}_2 \mathcal{C} \mathcal{D}_2) : f(0) \in \mathcal{D}_0(\mathcal{D}_2) \text{ and } f(1) \in \mathcal{D}_1(\mathcal{D}_2) \} \) is a unital continuous \( C([0,1]) \)-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra \( D \) is properly infinite, a statement which is equivalent to the K₁-injectivity of \( \mathcal{T}_2 \mathcal{C} \mathcal{T}_2 \) ([Blan10, Proposition 4.2]). \( \square \)

**Proposition 3.4.** Let \( a \in \mathcal{T}_2 \) be the sum \( a := 1 - s_1s_1^* + s_1 \).

1) The operator \( a \) is a unitary in \( \mathcal{T}_2 \).
2) The unitary \( u := j_1(a) \cdot j_0(a)^* \in \mathcal{T}_2 \mathcal{C} \mathcal{T}_2 \) belongs to the connected component \( \mathcal{U}_0(\mathcal{T}_2 \mathcal{C} \mathcal{T}_2) \). It also satisfies \( j_1(s_1) \cdot j_0(s_1) = u \cdot j_0(s_1s_1^*) \).
3) The C*-algebra \( \mathcal{T}_2 \mathcal{C} \mathcal{T}_2 \) is K₁-injective.

**Proof.** 1) Let \( H \) be the Hilbert space \( H := \ell^2(\mathbb{N}) \) with canonical orthonormal basis \( \{ e_k ; k \in \mathbb{N} \} \) and identify \( \mathcal{T}_2 \) with the C*-algebra generated by the two isometries \( s_1, s_2 \) given by \( s_i \cdot e_k := e_{2k+i} \) for \( i = 1, 2 \) and \( k \in \mathbb{N} \).

\( a \) is an isometry since
\[
aa^* = (1 - s_1^*s_1)s_1^* + s_1^*(1 - s_1^*s_1) + s_1 = (1 - s_1^*s_1) + s_1s_1s_1^* = 1.
\]

For all vector \( \xi \in H \), we also have
\[
aa^*\xi = 0 \iff a^*\xi = s_1^*\xi + (1 - s_1s_1^*)^*\xi = 0
\]
\[\Rightarrow (1 - s_1s_1^*)^*\xi = 0 \text{ and } s_1^*\xi = 0
\]
\[\Rightarrow \xi = s_1s_1^*\xi \text{ and } s_1^*\xi = 0
\]
\[\Rightarrow \xi = 0.
\]
Thus, the coisometry \( a^* \) is injective and the operator \( a \) is a unitary.

2) The unitary \( a \) is homotopic to 1 in \( \mathcal{U}(\mathcal{T}_2) \) (see e.g. [Blan10, Proposition 3.2]). Hence, the product \( u = j_1(a) \cdot j_0(s_1^*) \) is homotopic to 1 in \( \mathcal{U}(\mathcal{T}_2 \mathcal{C} \mathcal{T}_2) \).

Besides, \( u \cdot j_0(s_1s_1^*) = j_1(a) \cdot j_0(s_1^*) = j_1(s_1) \cdot j_0(s_1)^* \).

3) This derives for Corollary 3.3. \( \square \)

4. The Pimsner-Toeplitz algebra of a Hilbert C(X)-module

We look in this section at the special case of unital continuous C(X)-algebras with fibres \( \mathcal{O}_\infty \) corresponding to the Pimsner-Toeplitz C(X)-algebras of Hilbert C(X)-modules with infinite dimension fibres.

**Definition 4.1.** ([Pim95]) Let \( X \) be a compact Hausdorff space and \( E \) a full Hilbert C(X)-module \( E \), i.e. without any zero fibre.

a) The full Fock Hilbert C(X)-module \( \mathcal{F}(E) \) of \( E \) is the direct sum of Hilbert C(X)-module
\[
\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes \mathcal{C}(X))^m}, \quad (4.1)
\]
where \( E^\otimes_{C(X)} \) is the closed linear span of morphisms of \( C \). Let \( \ell \) be a covering of \( \mathcal{F}(E) \) where \( E \). The Pimsner-Toeplitz \( T(E) \) of \( T(E) \) is the unital subalgebra of the \( C(X) \)-algebra \( T(E) \) of adjointable \( C(X) \)-linear operator acting on \( \mathcal{F}(E) \) generated by the creation operators \( \ell(\zeta) (\zeta \in E) \), where:

\[
\begin{align*}
- \quad & \ell(\zeta) (f \cdot \hat{1}_{C(X)}) := f \cdot \zeta = \zeta \cdot f \quad \text{for} \quad f \in C(X) \quad \text{and} \\
- \quad & \ell(\zeta_1 \otimes \ldots \otimes \zeta_k) := \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k \quad \text{for} \quad \zeta_1, \ldots, \zeta_k \in E \quad \text{if} \quad k \geq 1.
\end{align*}
\]

(4.2)

c) Let \( (C^*(\mathbb{Z}), \Delta) \) be the compact quantum group generated by a unitary \( u \) with spectrum the unit circle and with coproduct \( \Delta(u) = u \otimes u \). Then, there is a unique coaction \( \alpha \) of the Hopf \( C^* \)-algebra \( (C^*(\mathbb{Z}), \Delta) \) on the Pimsner-Toeplitz \( C(X) \)-algebra \( T(E) \) such that \( \alpha(\ell(\zeta)) = \ell(\zeta) \otimes u \) for all \( \zeta \in E \), i.e.

\[
\alpha : \quad \begin{align*}
\mathcal{T}(E) & \rightarrow \mathcal{T}(E) \otimes C^*(\mathbb{Z}) = C(\mathbb{T}, \mathcal{T}(E)) \\
\ell(\zeta) & \mapsto \ell(\zeta) \otimes u = (z \mapsto \ell(z \zeta))
\end{align*}
\]

(4.3)

The fixed point \( C(X) \)-subalgebra \( \mathcal{T}(E)^\alpha = \{ a \in \mathcal{T}(E) ; \alpha(a) = a \otimes 1 \} \) under this coaction is the closed linear span

\[
\mathcal{T}(E)^\alpha = \left[ C(X).1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^* \right].
\]

(4.4)

Besides, the projection \( P \in \mathcal{L}(\mathcal{F}(E)) \) onto the submodule \( E \) induces a quotient morphism of \( C(X) \)-algebra \( a \in \mathcal{T}(E)^\alpha \mapsto \mathcal{Q}(a) := P \cdot a \cdot P \in \mathcal{K}(E) + C(X) \cdot 1 \subset \mathcal{L}(E) \).

**Proposition 4.2.** Let \( X \) be a second countable compact Hausdorff perfect space and let \( E \) be a separable Hilbert \( C(X) \)-module with infinite dimensional fibres.

1) There exist a covering \( X = \overset{\circ}{F}_1 \cup \ldots \cup \overset{\circ}{F}_m \) by the interiors of closed subsets \( F_1, \ldots, F_m \) and \( m \) sections \( \zeta_1, \ldots, \zeta_m \) in \( E \) such that \( \mathcal{T}(E) = C^* < \mathcal{T}(E)^\alpha, \ell(\zeta_1), \ldots, \ell(\zeta_m) > \) and \( \| (\zeta_k)_x \| = 1 \) for all \( k \in \{1, \ldots, m\} \) and \( x \in F_k \).

2) The Pimsner-Toeplitz \( C(X) \)-algebra \( T(E) \) is properly infinite.

**Proof.** 1) For all point \( x \in X \), there exists a section \( \zeta \in E \) satisfying \( \| \zeta_x \| = 1 \), whence an isomorphism of \( C^* \)-algebra \( \mathcal{T}(E)_x \cong \mathcal{T}(E_x) = C^* < \mathcal{T}(E^\alpha)_x, \ell(\zeta_1), \ldots, \ell(\zeta_m) > \). The semiprojectivity of the \( C^* \)-algebra \( \mathcal{O}_\infty \cong \mathcal{T}(E)_x \) and the compactness of the space \( X \) imply that there exist a finite covering \( X = \overset{\circ}{F}_1 \cup \ldots \cup \overset{\circ}{F}_m \) by the interiors of closed subsets \( F_1, \ldots, F_m \) and \( m \) contractions \( \zeta_1, \ldots, \zeta_m \) in \( E \) such that \( \| (\zeta_k)_x \| = 1 \) for all index \( k \in \{1, \ldots, m\} \) and all point \( x \in F_k \), so that \( \ell(E)_{F_k} = \ell(E) \cdot \ell(\zeta_k) \cdot \ell(\zeta_k)_{F_k} \) and \( \mathcal{T}(E)_{F_k} = C^* < \mathcal{T}(E)^\alpha, \ell(\zeta_k) >_{|F_k} \).

2) The equivalence \( (a) \Leftrightarrow (c) \) in Corollary 3.3 implies that \( T(E) \) is properly infinite. \( \Box \)

**Question 4.3.** Is there a unitary \( v \in \mathcal{U}(\mathcal{T}_2 \ast_C \mathcal{T}_2) \) such that \( j_1(s_1 s_1^*) = v \cdot j_0(s_1 s_1^*) \cdot v^* \)?
References


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