Continuous fields of properly infinite C*-algebras

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Abstract. Any unital separable continuous $C(X)$-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space X has finite topological dimension. We show that this is still true when the compact space X has infinite topological dimension.

1. Introduction

One of the basic C*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C*-algebras through K-theoretical invariants is the Cuntz C*-algebra $\mathcal{O}_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous $C(X)$-algebra A the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra D is a trivial $C(X)$-algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra D tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$ ([Win09]). Hence, this C*-algebra D is K_1-injective ([Rør04]) and the $C(X)$-algebra A satisfies $A \cong D \otimes C(X)$ ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product $\Pi_{n=0}^{\infty} S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2^∞ ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dăd09, §3] for all pair (Γ_0, Γ_1) of discrete countable torsion groups a unital separable continuous $C(X)$-algebra A such that:

– the base space X is the compact Hilbert cube $X = \mathfrak{X}$ of infinite dimension,
– all the fibres A_x ($x \in \mathfrak{X}$) are isomorphic to the strongly self-absorbing Cuntz C*-algebra $\mathcal{O}_2$ generated by two isometries s_1, s_2 satisfying $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$,
– $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $Y_0 \subset [0,1]$ is the canonical Cantor set.

These K-theoretical conditions imply that the $C(X)$-algebra A is not a trivial one. But these arguments does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra $\mathcal{O}_\infty$ ([Cun77]), in so far as $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ is a torsion free group.

We still do not know whether all unital continuous $C(X)$-algebra with fibres $\mathcal{O}_\infty$ are trivial. But we show that any unital separable continuous $C(X)$-algebra with fibres $\mathcal{O}_\infty$ is at least properly infinite (see [Blan13, §6]).

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2. Notations

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of positive integers and we denote by $[S]$ the closed linear span of any subset S in a Banach space.

Definition 2.1. *(*[Dix69], [Kas88], [Blan97]*) Let X be a compact Hausdorff space and let $C(X)$ be the C*-algebra of continuous function on X.*

- *A unital $C(X)$-algebra is a unital C*-algebra A endowed with a unital morphism of C*-algebra from $C(X)$ to the centre of A.*
- *For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a_{|F}$ the image of a in the quotient $A_{|F} := A/C_0(X \setminus F) \cdot A$. If $x \in X$ is a point in X, one calls fibre at x the quotient $A_x := A_{|\{x\}}$ and one write a_x for $a_{|\{x\}}$.*
- *The $C(X)$-algebra A is said to be continuous if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$ is continuous for all $a \in A$.*

Remarks 2.2. a) ([Cun81], [BRR08]) For all integer $n \geq 2$, the C*-algebra $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$ is the universal unital C*-algebra generated by n isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1 s_1^* + \ldots + s_n s_n^* \leq 1 \, . \tag{2.1}$$

b) A unital C*-algebra A is *properly infinite* if and only if one the following equivalent conditions holds ([Cun77], [Rør03, Proposition 2.1]):

- the C*-algebra A contains two isometries with mutually orthogonal range projections, *i.e.* A unitally contains a copy of $\mathcal{T}_2$,
- the C*-algebra A contains a unital copy of the simple Cuntz C*-algebra $\mathcal{O}_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges.

3. Global proper infiniteness

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous $C(X)$-algebras with properly infinite fibres.

Proposition 3.1. *Let X be a second countable perfect compact Hausdorff space, i.e. without any isolated point, and let A be a separable unital continuous $C(X)$-algebra with properly infinite fibres.*

1) There exist:

(a) *a finite integer $n \geq 1$,*
(b) *a covering $X = \overset{o}{F_1} \cup \ldots \cup \overset{o}{F_n}$ by the interiors of closed balls $F_1, \ldots, F_n$,*
(c) *unital embeddings of C*-algebra $\sigma_k : \mathcal{O}_\infty \hookrightarrow A_{|F_k}$ $(1 \leq k \leq n)$.*

2) The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integers p.

Proof. 1) For all point $x \in X$, the semiprojectivity of the C*-subalgebra $\mathcal{O}_\infty \hookrightarrow A_x$ ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood $F \subset X$ of the point x and a unital embedding $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A_{|F}$ of $C(F)$-algebra. The compactness of the topological space X enables to conclude.

2) Proposition [BRR08, Proposition 2.7] entails that $M_{2^{n-1}}(A)$ is properly infinite and [Rør97, Proposition 2.1] implies that $M_p(A)$ for all integer $p \geq 2^{n-1}$. $\square$

Remark 3.2. If X is an ordinary second countable compact Hausdorff space and A is a separable unital continuous $C(X)$-algebra, then $\widetilde{X} := X \times [0,1]$ is a perfect compact space, $\widetilde{A} := A \otimes C([0,1])$ is a unital continuous $C(\widetilde{X})$-algebra and any unital morphism $\mathcal{O}_\infty \to \widetilde{A}$ induces a unital morphism $\mathcal{O}_\infty \to A$ by composition with the projection map $\widetilde{A} \to A$ coming from the injection $x \in X \mapsto (x,0) \in \widetilde{X}$.

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the C*-algebra A is properly infinite ([HR98]). Indeed, there exists a unital C*-algebra A which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is properly infinite ([Rør03, Proposition 4.5]). We nevertheless have the following corollary.

Corollary 3.3. *Let $\jmath_0, \jmath_1$ denote the two canonical unital embeddings of the Cuntz extension $\mathcal{T}_2$ in the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ and let $\tilde{u} \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ be a K_1-trivial unitary satisfying $\jmath_1(s_1) = \tilde{u} \cdot \jmath_0(s_1)$ (*[BRR08, Lemma 2.4]*).*

Then the following conditions are equivalent:

(a) *The full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1-injective.*
(b) *The unitary $\tilde{u}$ belongs to the connected component $\mathcal{U}_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ of $1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2}$.*
(c) *Every separable unital continuous $C(X)$-algebra A with properly infinite fibres is a properly infinite C*-algebra.*

Proof. (a)⇒(b) A unital C*-algebra A is called K$_1$-injective if and only if all K_1-trivial unitaries $v \in \mathcal{U}(A)$ are homotopic to the unit 1_A in $\mathcal{U}(A)$ (see *e.g.* [Roh09]). Thus, (b) is a special case of (a) since $K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = \{1\}$ (see *e.g.* [Blan10, Lemma 4.4]).

(b)⇒(c) Let A be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = \overset{o}{F_1} \cup \ldots \cup \overset{o}{F_n}$ such that there exist unital embeddings $\sigma_k : \mathcal{T}_2 \to A_{|F_k}$ $(1 \leq k \leq n)$. Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A_{|G_k}$ such that the two projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A_{|G_k}$:

– If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

– If $k \in \{1, \ldots, n-1\}$ and the isometry $w_k \in A_{|G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital C*-algebra $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to A_{|G_k \cap F_{k+1}}$ satisfying:

$$\begin{array}{ll} - & \pi_k(\jmath_0(s_1)) = w_{k|G_k \cap F_{k+1}}\,, \\ - & \pi_k(\jmath_1(s_1)) = \sigma_{k+1}(s_1)_{|G_k \cap F_{k+1}} = \pi_k(\tilde{u}) \cdot w_{k|G_k \cap F_{k+1}}\,. \end{array} \tag{3.1}$$

If the unitary $\tilde{u}$ belongs to $\mathcal{U}_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$, then $\pi_k(\tilde{u})$ is homotopic to $1_{A_{|G_k \cap F_{k+1}}} = \pi_k(1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2})$ in $\mathcal{U}(A_{|G_k \cap F_{k+1}})$, so that $\pi_k(\tilde{u})$ admits a unitary lifting z_{k+1} in $\mathcal{U}_0(A_{|F_{k+1}})$ (see *e.g.* [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A_{|G_{k+1}}$ satisfying the two constraints:

$$w_{k+1|G_k} = w_k \qquad \text{and} \qquad w_{k+1|F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1) \tag{3.2}$$

verifies that the two projections $w_{k+1}w_{k+1}^*$ and $1_{|G_{k+1}} - w_{k+1}w_{k+1}^*$ are properly infinite and full in $A_{|G_{k+1}}$.

The proper infiniteness of the projection $w_n w_n^* \in A_{|G_n} = A$ implies that the unit $1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n$ is also a properly infinite projection in A, *i.e.* the C*-algebra A is properly infinite.

(c)⇒(a) The C*-algebra $\mathcal{D} := \{ f \in C([0,1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)\,;\, f(0) \in \jmath_0(\mathcal{T}_2) \text{ and } f(1) \in \jmath_1(\mathcal{T}_2) \}$ is a unital continuous $C([0,1])$-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra $\mathcal{D}$ is properly infinite, a statement which is equivalent to the K$_1$-injectivity of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([Blan10, Proposition 4.2]). □

Proposition 3.4. *Let $a \in \mathcal{T}_2$ be the sum $a := 1 - s_1 s_1^* + s_1$.*

1) The operator a is a unitary in $\mathcal{T}_2$.

2) The unitary $u := \jmath_1(a) \cdot \jmath_0(a)^* \in \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ belongs to the connected component $\mathcal{U}_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$. It also satisfies $\jmath_1(s_1) \cdot \jmath_0(s_1^*) = u \cdot \jmath_0(s_1 s_1^*)$.

3) The C-algebra $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1-injective.*

Proof. 1) Let H be the Hilbert space $H := \ell^2(\mathbb{N})$ with canonical orthonormal basis $\{ e_k\,;\, k \in \mathbb{N}\}$ and identify $\mathcal{T}_2$ with the C*-algebra generated by the two isometries s_1, s_2 given by $s_i \cdot e_k := e_{2k+i}$ for $i = 1, 2$ and $k \in \mathbb{N}$.

a is an isometry since $a^*a = ((1 - s_1^* s_1) + s_1)^*((1 - s_1^* s_1) + s_1) = ((1 - s_1^* s_1) + s_1^* s_1 = 1$. For all vector $\xi \in H$, we also have

$$\begin{aligned} aa^*\xi = 0 \;&\Leftrightarrow a^*\xi = s_1^*\xi + (1 - s_1 s_1^*)^*\xi = 0 \\ &\Rightarrow (1 - s_1 s_1^*)^*\xi = 0 \text{ and } s_1^*\xi = 0 \\ &\Rightarrow \xi = s_1 s_1^*\xi \text{ and } {}^*_1\xi = 0 \\ &\Rightarrow \xi = 0\,. \end{aligned}$$

Thus, the coisometry a^* is injective and the operator a is a unitary.

2) The unitary a is homotopic to 1 in $\mathcal{U}(\mathcal{T}_2)$ (see *e.g.* [Blan10, Proposition 3.2]). Hence, the product $u = \jmath_1(a) \cdot \jmath_0(s_1^*)$ is homotopic to 1 in $\mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

Besides, $u \cdot \jmath_0(s_1 s_1^*) = \jmath_1(a) \cdot \jmath_0(s_1^*) = \jmath_1(s_1) \cdot \jmath_0(s_1)^*$.

3) This derives for Corollary 3.3. □

4. The Pimsner-Toeplitz algebra of a Hilbert $C(X)$-module

We look in this section at the special case of unital continuous $C(X)$-algebras with fibres $\mathcal{O}_\infty$ corresponding to the Pimsner-Toeplitz $C(X)$-algebras of Hilbert $C(X)$-modules with infinite dimension fibres.

Definition 4.1. *(*[Pim95]*) Let X be a compact Hausdorff space and E a full Hilbert $C(X)$-module E, i.e. without any zero fibre.*
a) The full Fock Hilbert $C(X)$-module $\mathcal{F}(E)$ of E is the direct sum of Hilbert $C(X)$-module

$$\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(\mathfrak{X})})\, m}\,, \tag{4.1}$$

where $E^{(\otimes_{C(\mathfrak{X})})\, m} := \begin{cases} C(X) & \text{if } m = 0\,, \\ E \otimes_{C(X)} \ldots \otimes_{C(X)} E \ \ (m \,\text{terms}) & \text{if } m \geq 1\,. \end{cases}$

b) The Pimsner-Toeplitz $C(X)$*-algebra* $\mathcal{T}(E)$ *of* E *is the unital subalgebra of the* $C(X)$*-algebra* $\mathscr{L}_{C(\mathfrak{X})}(\mathcal{F}(E)\,)$ *of adjointable* $C(X)$*-linear operator acting on* $\mathcal{F}(E)$ *generated by the creation operators* $\ell(\zeta)$ $(\zeta \in E)$*, where:*

$$\begin{array}{llll} - \ \ \ell(\zeta)\,(f \cdot \hat{1}_{C(X)}) & := f \cdot \zeta = \zeta \cdot f & \textit{for} \ \ f \in C(X) & \text{and} \\ - \ \ \ell(\zeta)\,(\zeta_1 \otimes \ldots \otimes \zeta_k) & := \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k & \textit{for} \ \ \zeta_1, \ldots, \zeta_k \in E & \text{if } k \geq 1\,. \end{array} \tag{4.2}$$

c) Let $(C^*(\mathbb{Z}), \Delta)$ *be the compact quantum group generated by a unitary* $\mathbf{u}$ *with spectrum the unit circle and with coproduct* $\Delta(\mathbf{u}) = \mathbf{u} \otimes \mathbf{u}$*. Then, there is a unique coaction* α *of the Hopf C*-algebra* $(C^*(\mathbb{Z}), \Delta)$ *on the Pimsner-Toeplitz* $C(X)$*-algebra* $\mathcal{T}(E)$ *such that* $\alpha\big(\ell(\zeta)\big) = \ell(\zeta) \otimes \mathbf{u}$ *for all* $\zeta \in E$*, i.e.*

$$\begin{array}{lcccc} \alpha: \ \mathcal{T}(E) & \to & \mathcal{T}(E) \otimes C^*(\mathbb{Z}) & = & C(\mathbb{T}, \mathcal{T}(E)) \\ \quad\ \ \ell(\zeta) & \mapsto & \ell(\zeta) \otimes \mathbf{u} & = & (z \mapsto \ell(z\zeta)\,) \end{array} \tag{4.3}$$

The fixed point $C(X)$-subalgebra $\mathcal{T}(E)^\alpha = \{a \in \mathcal{T}(E)\,;\, \alpha(a) = a \otimes 1\}$ under this coaction is the closed linear span

$$\mathcal{T}(E)^\alpha = \Big[C(X).1 + \sum_{k \geq 1} \ell(E)^k \cdot \big(\ell(E)^k\big)^*\Big]\,. \tag{4.4}$$

Besides, the projection $P \in \mathcal{L}(\,\mathcal{F}(E))$ onto the submodule E induces a quotient morphism of $C(X)$-algebra $a \in \mathcal{T}(E)^\alpha \mapsto \overline{\mathbf{q}}(a) := P \cdot a \cdot P \in \mathcal{K}(E) + C(X) \cdot 1 \subset \mathcal{L}(E)$.

Proposition 4.2. *Let* X *be a second countable compact Hausdorff perfect space and let* E *be a separable Hilbert* $C(X)$*-module with infinite dimensional fibres.*

1) There exist a covering $X = \overset{o}{F_1} \cup \ldots \cup \overset{o}{F_m}$ *by the interiors of closed subsets* $F_1, \ldots, F_m$ *and* m *sections* $\zeta_1, \ldots, \zeta_m$ *in* E *such that* $\mathcal{T}(E) = C^* < \mathcal{T}(E)^\alpha, \ell(\zeta_1), \ldots, \ell(\zeta_m) >$ *and* $\|(\zeta_k)_x\| = 1$ *for all* $k \in \{1, \ldots, m\}$ *and* $x \in F_k$ *.*

2) The Pimsner-Toeplitz $C(X)$*-algebra* $\mathcal{T}(E)$ *is properly infinite.*

Proof. 1) For all point $x \in X$, there exists a section $\zeta \in E$ satisfying $\|\zeta_x\| = 1$, whence an isomorphism of C*-algebra $\mathcal{T}(E)_x \cong \mathcal{T}(E_x) = C^* < \mathcal{T}(E_x)^\alpha, \ell(\zeta_x) >$. The semiprojectivity of the C*-algebra $\mathcal{O}_\infty \cong \mathcal{T}(E)_x$ and the compactness of the space X imply that there exist a finite covering $X = \overset{o}{F_1} \cup \ldots \cup \overset{o}{F_m}$ by the interiors of closed subsets $F_1, \ldots, F_m$ and m contractions $\zeta_1, \ldots, \zeta_m$ in E such that $\|(\zeta_k)_x\| = 1$ for all index $k \in \{1, \ldots, m\}$ and all point $x \in F_k$, so that $\ell(E)_{|F_k} = \ell(E) \cdot \ell(\zeta_k)^* \cdot \ell(\zeta_k)_{|F_k}$ and $\mathcal{T}(E)_{|F_k} = C^* < \mathcal{T}(E)^\alpha, \ell(\zeta_k) >_{|F_k}$.

2) The equivalence (a)⇔(c) in Corollary 3.3 implies that $\mathcal{T}(E)$ is properly infinite. □

Question 4.3. Is there a unitary $v \in \mathcal{U}((\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)^\alpha)$ such that $\jmath_1(s_1 s_1^*) = v \cdot \jmath_0(s_1 s_1^*) \cdot v^*$?

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