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CONTINUOUS FIELDS OF PROPERLY INFINITE C*-ALGEBRAS

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Abstract. Any unital separable continuous C(X)-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space X has finite topological dimension. We show that this is still true when the compact space X has infinite topological dimension.

1. Introduction

One of the basic C*-algebras studied in the classification programme launched by G. Elliott ([Ell94]) of nuclear C*-algebras through K-theoretical invariants is the Cuntz C*-algebra O∞ generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous C(X)-algebra A the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra D is a trivial C(X)-algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra D tensorially absorbs the Jiang-Su algebra Z ([Win09]). Hence, this C*-algebra D is K1-injective ([Rør04]) and the C(X)-algebra A satisfies $A \cong D \otimes C(X)$ ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product $\prod_{n=0}^{\infty} S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type $2^\infty$ ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dăd09, §3] for all pair $(\Gamma_0, \Gamma_1)$ of discrete countable torsion groups a unital separable continuous C(X)-algebra A such that:

- the base space X is the compact Hilbert cube $X = \mathcal{X}$ of infinite dimension,
- all the fibres $A_x (x \in \mathcal{X})$ are isomorphic to the strongly self-absorbing Cuntz C*-algebra $O_2$ generated by two isometries $s_1, s_2$ satisfying $1_{O_2} = s_1s_1^* + s_2s_2^*$,
- $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $Y_0 \subset [0, 1]$ is the canonical Cantor set.

These K-theoretical conditions imply that the C(X)-algebra A is not a trivial one. But these arguments does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra $O_\infty$ ([Cun77]), in so far as $K_0(O_\infty) = \mathbb{Z}$ is a torsion free group.

We still do not know whether all unital continuous C(X)-algebra with fibres $O_\infty$ are trivial. But we show that any unital separable continuous C(X)-algebra with fibres $O_\infty$ is at least properly infinite (see [Blan13, §6]).

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2. Notations

We present in this section the main notations which are used in this article. We denote by \( \mathbb{N} = \{0, 1, 2, \ldots\} \) the set of positive integers and we denote by \([S]\) the closed linear span of any subset \( S \) in a Banach space.

**Definition 2.1.** ([Dix69], [Kas88], [Blan97]) Let \( X \) be a compact Hausdorff space and let \( C(X) \) be the \( C^* \)-algebra of continuous function on \( X \).

- A unital \( C(X) \)-algebra is a unital \( C^* \)-algebra \( A \) endowed with a unital morphism of \( C^* \)-algebra from \( C(X) \) to the centre of \( A \).
- For all closed subset \( F \subset X \) and all element \( a \in A \), one denotes by \( a|_{F} \) the image of \( a \) in the quotient \( A|_{F} := A/C_0(X \setminus F) \cdot A \). If \( x \in X \) is a point in \( X \), one calls fibre at \( x \) the quotient \( A|_{\{x\}} := A|_{\{x\}} \) and one write \( a_x \) for \( a|_{\{x\}} \).
- The \( C(X) \)-algebra \( A \) is said to be continuous if the upper semicontinuous map \( x \in X \mapsto \|a_x\| \in \mathbb{R}_+ \) is continuous for all \( a \in A \).

**Remarks 2.2.** a) ([Cun81], [BRR08]) For all integer \( n \geq 2 \), the \( C^* \)-algebra \( T_n := T(C_n) \) is the universal unital \( C^* \)-algebra generated by \( n \) isometries \( s_1, \ldots, s_n \) satisfying the relation
\[
s_1s_1^* + \ldots + s_ns_n^* \leq 1 . \quad (2.1)\]
b) A unital \( C^* \)-algebra \( A \) is properly infinite if and only if one the following equivalent conditions holds ([Cun77], [Rør03, Proposition 2.1]):

- the \( C^* \)-algebra \( A \) contains two isometries with mutually orthogonal range projections, i.e. \( A \) unitally contains a copy of \( T_2 \),
- the \( C^* \)-algebra \( A \) contains a unital copy of the simple Cuntz \( C^* \)-algebra \( O_\infty \) generated by infinitely many isometries with pairwise orthogonal ranges.

3. Global proper infiniteness

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous \( C(X) \)-algebras with properly infinite fibres.

**Proposition 3.1.** Let \( X \) be a second countable perfect compact Hausdorff space, i.e. without any isolated point, and let \( A \) be a separable unital continuous \( C(X) \)-algebra with properly infinite fibres.

1) There exist:

- a finite integer \( n \geq 1 \),
- a covering \( X = \bigcup_{k=1}^{n} F_k \) by the interiors of closed balls \( F_1, \ldots, F_n \),
- unital embeddings of \( C^* \)-algebra \( \sigma_k : O_\infty \hookrightarrow A|_{F_k} \) (\( 1 \leq k \leq n \)).

2) The tensor product \( M_p(\mathbb{C}) \otimes A \) is properly infinite for all large enough integers \( p \).

**Proof.** 1) For all point \( x \in X \), the semiprojectivity of the \( C^* \)-subalgebra \( O_\infty \hookrightarrow A_x \) ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood \( F \subset X \) of the point \( x \) and a unital embedding \( O_\infty \otimes C(F) \hookrightarrow A|_F \) of \( C(F) \)-algebra. The compactness of the topological space \( X \) enables to conclude.
Remark 3.2. If $X$ is an ordinary second countable compact Hausdorff space and $A$ is a separable unital continuous $C(X)$-algebra, then $\tilde{X} := X \times [0, 1]$ is a perfect compact space, $\tilde{A} := A \otimes C([0, 1])$ is a unital continuous $C(\tilde{X})$-algebra and any unital morphism $\mathcal{O}_\infty \to \tilde{A}$ induces a unital morphism $\mathcal{O}_\infty \to A$ by composition with the projection map $\tilde{A} \to A$ coming from the injection $x \in X \mapsto (x, 0) \in \tilde{X}$.

The proper infiniteness of the tensor product $M_p(C) \otimes A$ does not always imply that the $C^*$-algebra $A$ is properly infinite ([HR98]). Indeed, there exists a unital $C^*$-algebra $A$ which is not properly infinite, but such that the tensor product $M_2(C) \otimes A$ is properly infinite ([Ror03, Proposition 4.5]). We nevertheless have the following corollary.

**Corollary 3.3.** Let $j_0, j_1$ denote the two canonical unital embeddings of the Cuntz extension $T_2$ in the full unital free product $T_2 \ast C T_2$ and let $\bar{u} \in \mathcal{U}(T_2 \ast C T_2)$ be a $K_1$-trivial unitary satisfying $j_1(s_1) = \bar{u} \cdot j_0(s_1)$ ([BRR08, Lemma 2.4]).

Then the following conditions are equivalent:

(a) The full unital free product $T_2 \ast C T_2$ is $K_1$-injective.
(b) The unitary $\bar{u}$ belongs to the connected component $\mathcal{U}_0(T_2 \ast C T_2)$ of $1_{T_2 \ast C T_2}$.
(c) Every separable unital continuous $C(X)$-algebra $A$ with properly infinite fibres is a properly infinite $C^*$-algebra.

**Proof.** (a)$\Rightarrow$(b) A unital $C^*$-algebra $A$ is called $K_1$-injective if and only if all $K_1$-trivial unitaries $v \in \mathcal{U}(A)$ are homotopic to the unit $1_A$ in $\mathcal{U}(A)$ (see e.g. [Roh09]). Thus, (b) is a special case of (a) since $K_1(T_2 \ast C T_2) = \{1\}$ (see e.g. [Blan10, Lemma 4.4]).

(b)$\Rightarrow$(c) Let $A$ be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = \tilde{F}_1 \cup \ldots \cup \tilde{F}_n$ such that there exist unital embeddings $\sigma_k : T_2 \to A|_{F_k}$ ($1 \leq k \leq n$). Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A|_{G_k}$ such that the two projections $w_k w_k^*$ and $1_{G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A|_{G_k}$:

- If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.
- If $k \in \{1, \ldots, n-1\}$ and the isometry $w_k \in A|_{G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital $C^*$-algebra $\pi_k : T_2 \ast C T_2 \to A|_{G_k \cap F_{k+1}}$ satisfying:

\[
\begin{align*}
\pi_k(j_0(s_1)) &= w_k|_{G_k \cap F_{k+1}}, \\
\pi_k(j_1(s_1)) &= \pi_{k+1}(s_1)|_{G_k \cap F_{k+1}} = \pi_k(\bar{u}) \cdot w_k|_{G_k \cap F_{k+1}}.
\end{align*}
\]

If the unitary $\bar{u}$ belongs to $\mathcal{U}_0(T_2 \ast C T_2)$, then $\pi_k(\bar{u})$ is homotopic to $1_{A|_{G_k \cap F_{k+1}}} = \pi_k(1_{T_2 \ast C T_2})$ in $\mathcal{U}(A|_{G_k \cap F_{k+1}})$, so that $\pi_k(\bar{u})$ admits a unitary lifting $z_{k+1}$ in $\mathcal{U}_0(A|_{F_{k+1}})$ (see e.g. [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A|_{G_{k+1}}$ satisfying the two constraints:

\[
w_{k+1}|_{G_k} = w_k \quad \text{ and } \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1)
\]
verifies that the two projections $w_{k+1}w_{k+1}^*$ and $1_{A_{G_{k+1}}} - w_{k+1}w_{k+1}^*$ are properly infinite and full in $A_{G_{k+1}}$.

The proper infiniteness of the projection $w_nw_n^* \in A_{G_n} = A$ implies that the unit $1_A = w_n^*w_n = w_n^*w_n^*w_n$ is also a properly infinite projection in $A$, i.e. the C*-algebra $A$ is properly infinite.

(c)⇒(a) The C*-algebra $D := \{ f \in C([0,1], T_2 \ast C T_2) : f(0) \in j_0(T_2) \text{ and } f(1) \in j_1(T_2) \}$ is a unital continuous $C([0,1])$-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra $D$ is properly infinite, a statement which is equivalent to the $K_1$-injectivity of $T_2 \ast C T_2$ ([Blan10, Proposition 4.2]).

\[ \square \]

**Proposition 3.4.** Let $a \in T_2$ be the sum $a := 1 - s_1s_1^* + s_1$.

1) The operator $a$ is a unitary in $T_2$.

2) The unitary $u := j_1(a) \cdot j_0(a)^* \in T_2 \ast C T_2$ belongs to the connected component $U_0(T_2 \ast C T_2)$. It also satisfies $j_1(s_1) \cdot j_0(s_1) = u \cdot j_0(s_1s_1^*)$.

3) The C*-algebra $T_2 \ast C T_2$ is $K_1$-injective.

**Proof.**

1) Let $H$ be the Hilbert space $H := \ell^2(\mathbb{N})$ with canonical orthonormal basis $\{ e_k : k \in \mathbb{N} \}$ and identify $T_2$ with the C*-algebra generated by the two isometries $s_1, s_2$ given by $s_1 \cdot e_k := e_{2k+i}$ for $i = 1, 2$ and $k \in \mathbb{N}$.

$a$ is an isometry since $a^*a = ((1 - s_1^*s_1^*) + s_1^*)((1 - s_1^*s_1^*) + s_1) = (1 - s_1^*s_1^*) + s_1s_1^* = 1.

For all vector $\xi \in H$, we also have
\[
\begin{align*}
   aa^*\xi &= 0 \\
   \iff a^*\xi &= s_1^*\xi + (1 - s_1s_1^*)^*\xi = 0 \\
   \Rightarrow (1 - s_1s_1^*)^*\xi &= 0 \text{ and } s_1^*\xi = 0 \\
   \Rightarrow \xi &= s_1s_1^*\xi \text{ and } \xi = 0 \\
   \Rightarrow \xi &= 0.
\end{align*}
\]

Thus, the coisometry $a^*$ is injective and the operator $a$ is a unitary.

2) The unitary $a$ is homotopic to 1 in $U(T_2)$ (see e.g. [Blan10, Proposition 3.2]). Hence, the product $u = j_1(a) \cdot j_0(s_1^*)$ is homotopic to 1 in $U(T_2 \ast C T_2)$.

Besides, $u \cdot j_0(s_1s_1^*) = j_1(a) \cdot j_0(s_1^*) = j_1(s_1) \cdot j_0(s_1)^*$.

3) This derives for Corollary 3.3. \[ \square \]

**4. The Pimsner-Toeplitz algebra of a Hilbert $C(X)$-module**

We look in this section at the special case of unital continuous $C(X)$-algebras with fibres $O_\infty$ corresponding to the Pimsner-Toeplitz $C(X)$-algebras of Hilbert $C(X)$-modules with infinite dimension fibres.

**Definition 4.1.** ([Pim95]) Let $X$ be a compact Hausdorff space and $E$ a full Hilbert $C(X)$-module $E$, i.e. without any zero fibre.

a) The full Fock Hilbert $C(X)$-module $F(E)$ of $E$ is the direct sum of Hilbert $C(X)$-module
\[
F(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_c C(X))^m}.
\]
coaction is the closed linear span 

$$\alpha$$ morphism of 

$\ast$ the Hopf C

$\ast$ the unit circle and with coproduct 

$\cdot$ Let 

$\cdot$ The Pimsner-Toeplitz C(X)-algebra $\mathcal{T}(E)$ of $E$ is the unital subalgebra of the C(X)-algebra $\mathcal{L}_e C(X)(\mathcal{F}(E))$ of adjointable C(X)-linear operator acting on $\mathcal{F}(E)$ generated by the creation operators $L_i$ algebra

$\ast$ 1) There exist a covering $X = \bigcup_{i=1}^m F_i$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ sections $\zeta_1, \ldots, \zeta_m$ in $E$ such that $\mathcal{T}(E) = C^* < \mathcal{T}(E)^\alpha, \ell(\zeta_1), \ldots, \ell(\zeta_m) >$ and $\|\zeta_k\|_x = 1$ for all $k \in \{1, \ldots, m\}$ and $x \in F_k$.

$\ast$ 2) The Pimsner-Toeplitz C(X)-algebra $\mathcal{T}(E)$ is properly infinite.

Proof. 1) For all point $x \in X$, there exists a section $\zeta \in E$ satisfying $\|\zeta_x\| = 1$, whence an isomorphism of C*-algebra $\mathcal{T}(E)_x \cong \mathcal{T}(E_x) = C^* < \mathcal{T}(E_x)^\alpha, \ell(\zeta_x) >$. The semi-projectivity of the C*-algebra $\mathcal{O}_x \cong \mathcal{T}(E)_x$ and the compactness of the space $X$ imply that there exist a finite covering $X = \bigcup_{i=1}^m F_i$ by the interiors of closed subsets $F_1, \ldots, F_m$ and $m$ contractions $\zeta_1, \ldots, \zeta_m$ in $E$ such that $\|\zeta_k\|_x = 1$ for all index $k \in \{1, \ldots, m\}$ and all point $x \in F_k$, so that $\ell(E)|_{F_k} = \ell(E) \cdot \ell(\zeta_k)^* \cdot \ell(\zeta_k)|_{F_k}$ and $\mathcal{T}(E)|_{F_k} = C^* < \mathcal{T}(E)^\alpha, \ell(\zeta_k) > |_{F_k}$.

2) The equivalence $(a) \Leftrightarrow (c)$ in Corollary 3.3 implies that $\mathcal{T}(E)$ is properly infinite. □

Question 4.3. Is there a unitary $v \in \mathcal{U}(\mathcal{T}_2* \mathcal{T}_2)$ such that $j_1(s_1s_1^*) = v \cdot j_0(s_1s_1^*) \cdot v^*$?
References


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