



HAL
open science

Erratum to "Multipliers and Morrey spaces"

Pierre Gilles Lemarié-Rieusset

► **To cite this version:**

Pierre Gilles Lemarié-Rieusset. Erratum to "Multipliers and Morrey spaces". *Potential Analysis*, 2014, 41 (4), pp.1359-1362. 10.1007/s11118-014-9407-8 . hal-00974630

HAL Id: hal-00974630

<https://hal.science/hal-00974630>

Submitted on 7 Apr 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Erratum to “Multipliers and Morrey spaces”.

Pierre Gilles Lemarié–Rieusset*

Abstract

We correct the complex interpolation results for Morrey spaces which is false for the first interpolation functor of Calderón, but is exact for the Calderón’s second interpolation functor.

Keywords : Morrey spaces; interpolation.

2010 Mathematics Subject Classification : 42B35

In my paper “Multipliers and Morrey spaces” [3], there is a slight mistake in Theorem 3, concerning the interpolation of Morrey spaces.

Let \mathcal{B} be the collection of all Euclidean balls B on \mathbb{R}^d : $B = B(x_B, r_B) = \{x \in \mathbb{R}^d / |x - x_B| < R_B\}$. For $B \in \mathcal{B}$, we write $|B| = \int_B dx = |B(0, 1)|r_B^d$. Define, for $1 < p \leq q < +\infty$, the space $\dot{M}^{p,q}(\mathbb{R}^n)$ as the space of locally p -integrable functions f such that

$$\|f\|_{\dot{M}^{p,q}} := \sup_{B \in \mathcal{B}} |B|^{1/q-1/p} \left(\int_B |f(x)|^p dx \right)^{1/p} < +\infty.$$

It is easy to check that, when $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$\left(\frac{1}{p}, \frac{1}{q}\right) = (1 - \theta)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$$

for some $\theta \in (0, 1)$, then $\dot{M}^{p_0, q_0} \cap \dot{M}^{p_1, q_1} \subset \dot{M}^{p, q}$ and

$$\|f\|_{\dot{M}^{p,q}} \leq \|f\|_{\dot{M}^{p_0, q_0}}^{1-\theta} \|f\|_{\dot{M}^{p_1, q_1}}^\theta$$

The question I studied was then whether one may find an interpolation functor F of exponent θ such that $F(\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}) = \dot{M}^{p, q}$. If so, one should have the continuous embeddings

$$[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_{\theta, 1} \subset \dot{M}^{p, q} \subset [\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_{\theta, \infty}.$$

*Laboratoire de Mathématiques et Modélisation d’Évry, UMR CNRS 8071, Université d’Évry; e-mail : plemarie@univ-evry.fr

The inclusion $[A_0, A_1]_{\theta,1} \subset F(A_0, A_1)$ is a direct conclusion from the inequality

$$\|f\|_{F(A_0, A_1)} \leq C \|f\|_{A_0}^{1-\theta} \|f\|_{A_1}^{\theta}$$

(obtained by interpolation inequalities for the operator norms of $\lambda \mapsto \lambda f$ from \mathbb{R} to A_0 and from \mathbb{R} to A_1 , hence from \mathbb{R} to $F(A_0, A_1)$). The inclusion $F(A_0, A_1) \subset [A_0, A_1]_{\theta,\infty}$ is proven in [1] under the assumption that $A_0 \cap A_1$ is dense in A_0 and in A_1 . This is not the case for Morrey spaces. However, one may easily adapt the proof, as Morrey spaces are dual spaces (see for instance [6]).

If we assume that $A_0 = B'_0$ and $A_1 = B'_1$ and that $B_0 \cap B_1$ is dense in B_0 and B_1 ; then, for $b \in B_0 \cap B_1$, the linear form $T_b : f \mapsto \langle f|b \rangle$ has a norm less than $\|b\|_{B_0}$ as an operator from A_0 to \mathbb{R} and less than $\|b\|_{B_1}$ as an operator from A_1 to \mathbb{R} , hence as a norm less than $C \|b\|_{B_0}^{1-\theta} \|b\|_{B_1}^{\theta}$; thus, $\tilde{T}_f : b \mapsto \langle f|b \rangle$ is a continuous linear form on $[B_0, B_1]_{\theta,1}$. This gives that $f \in ([B_0, B_1]_{\theta,1})' = [B'_0, B'_1]_{\theta,\infty}$ (since $B_0 \cap B_1$ is dense in B_0 and B_1).

The theorem I proved in [3] is the following one:

Theorem 1

Let $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$\left(\frac{1}{p}, \frac{1}{q}\right) = (1-\theta)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$$

for some $\theta \in (0, 1)$. Then there exists an interpolation functor F of exponent θ such that $F(\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}) = \dot{M}^{p, q}$ if and only if $p_0/q_0 = p_1/q_1$.

The negative result for the case $p_0/q_0 \neq p_1/q_1$ was proven by a generalization of a counterexample by Ruiz and Vega [4] which proves that, in that case, we don't have the embedding of $\dot{M}^{p, q}$ into $[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_{\theta, \infty}$.

The proof for the positive result (on the case $p_0/q_0 = p_1/q_1$) was incorrect. I claimed that in that case we have the complex interpolation $\dot{M}^{p, q} = [\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_{\theta}$. But this is false as pointed to me by Sickel (who has recently characterized the intermediate space $\dot{M}^{p, q} \subset [\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_{\theta}$ in a joint work with Yang and Yuan [5]). Indeed, it is easy to see that, when $p_0/q_0 = p_1/q_1 = p_0/q_0 < 1$ and $p_0 \neq p_1$, $\dot{M}^{p_0, q_0} \cap \dot{M}^{p_1, q_1}$ is not dense in $\dot{M}^{p, q}$, while it is always true that $A_0 \cap A_1$ is dense in $[A_0, A_1]_{\theta}$ (see [1]). Sickel's counterexample is very clear : if $r = \min(p_0, p_1)$ and $s = \max(q_0, q_1)$, we have $\dot{M}^{p, q} \subset \dot{M}^{r, s}$ and $\dot{M}^{p_0, q_0} \cap \dot{M}^{p_1, q_1} \subset \dot{M}^{r, s}$; thus the applications $f \mapsto \rho^{d(1/q-1/r)} 1_{B(0, \rho)} f$ are equicontinuous from $\dot{M}^{p, q}$ to L^r ; for $f \in \dot{M}^{p_0, q_0} \cap \dot{M}^{p_1, q_1}$, we have $\lim_{\rho \rightarrow 0} \rho^{d(1/q-1/r)} \|1_{B(0, \rho)} f\|_r = 0$, while for $f_0 = |x|^{-d/q} \in \dot{M}^{p, q}$, we have $\lim_{\rho \rightarrow 0} \rho^{d(1/q-1/r)} \|1_{B(0, \rho)} f_0\|_r > 0$; thus f_0 does not belong to the closure of $\dot{M}^{p_0, q_0} \cap \dot{M}^{p_1, q_1}$

However, a slight modification of the proof of [3] gives the following theorem :

Theorem 2

Let $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$\left(\frac{1}{p}, \frac{1}{q}\right) = (1 - \theta)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$$

If $p_0/q_0 = p_1/q_1$, then

$$\dot{M}^{p,q} = [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]^\theta.$$

Let us recall that Calderón [2] defined two complex interpolation functors : $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$. We have $[A_0, A_1]_\theta = [A_0, A_1]^\theta$ (with equality when at least one of the two spaces A_0 and A_1 is reflexive).

Proof :

Let us recall the definition of $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$.

Let Ω be the open complex strip $\Omega = \{z \in \mathbb{C} / 0 < \Re z < 1\}$. \mathcal{F} is the space of functions F defined on the closed complex strip $\bar{\Omega}$ such that :

1. F is continuous and bounded from $\bar{\Omega}$ to $A_0 + A_1$
2. F is analytic from Ω to $A_0 + A_1$
3. $t \mapsto F(it)$ is continuous from \mathbb{R} to A_0 , and $\lim_{|t| \rightarrow +\infty} \|F(it)\|_{A_0} = 0$
4. $t \mapsto F(1+it)$ is continuous from \mathbb{R} to A_1 , and $\lim_{|t| \rightarrow +\infty} \|F(1+it)\|_{A_0} = 0$

Then

$$f \in [A_0, A_1]_\theta \Leftrightarrow \exists F \in \mathcal{F}, f = F(\theta)$$

and

$$\|f\|_{[A_0, A_1]_\theta} = \inf_{f=F(\theta)} \max(\sup_{t \in \mathbb{R}} \|F(it)\|_{A_0}, \sup_{t \in \mathbb{R}} \|F(1+it)\|_{A_1}).$$

On the other hand, \mathcal{G} is the space of functions G defined on the closed complex strip $\bar{\Omega}$ such that :

1. $\frac{1}{1+|z|}G$ is continuous and bounded from $\bar{\Omega}$ to $A_0 + A_1$
2. G is analytic from Ω to $A_0 + A_1$
3. $t \mapsto G(it) - G(0)$ is Lipschitz from \mathbb{R} to A_0

4. $t \mapsto G(1 + it) - G(1)$ is Lipschitz from \mathbb{R} to A_1

Then

$$f \in [A_0, A_1]^\theta \Leftrightarrow \exists G \in \mathcal{G}, f = G'(\theta)$$

and

$$\|f\|_{[A_0, A_1]^\theta} = \inf_{f=G'(\theta)} \max\left(\sup_{t_1, t_2 \in \mathbb{R}} \left\| \frac{G(it_2) - G(it_1)}{t_2 - t_1} \right\|_{A_0}, \sup_{t_1, t_2 \in \mathbb{R}} \left\| \frac{G(1 + it_2) - G(1 + it_1)}{t_2 - t_1} \right\|_{A_1}\right).$$

Let us remark that, for continuous functions, (strong) analyticity is equivalent to weak analyticity or even *-weak analyticity when A_0 and A_1 are dual spaces of B_0 and B_1 with $B_0 \cap B_1$ dense in B_0 and B_1 . Indeed, analyticity is equivalent to the fact that, whenever the closed ball $\bar{B}(z_0, r)$ is contained in Ω and $|w - z_0| < r$, then $F(w) = \frac{1}{2i\pi} \int_{|z-z_0|=r} F(z) \frac{dz}{z-w}$. As F is continuous, we have, for $b \in B_0 \cap B_1$,

$$\langle b | \frac{1}{2i\pi} \int_{|z-z_0|=r} F(z) \frac{dz}{z-w} \rangle_{B_0 \cap B_1, A_0 + A_1} = \frac{1}{2i\pi} \int_{|z-z_0|=r} \langle b | F(z) \rangle_{B_0 \cap B_1, A_0 + A_1} \frac{dz}{z-w}$$

The equivalence remains true for *-weakly continuous functions.

However, of course, there is no equivalence between (strong) continuity and *-weak continuity. In the original proof of [3], one made two remarks :

1. Let $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$\left(\frac{1}{p}, \frac{1}{q}\right) = (1 - \theta) \left(\frac{1}{p_0}, \frac{1}{q_0}\right) + \theta \left(\frac{1}{p_1}, \frac{1}{q_1}\right).$$

If F is an interpolation functor of exponent θ that satisfies $F(L^{p_0}, L^{q_1}) = L^p$, then $F(\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}) \subset \dot{M}^{p, q}$. Thus, we have the embeddings of $[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_{\theta, p}$, $[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_\theta$ and $[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]^\theta$ into $\dot{M}^{p, q}$.

2. When moreover $p_0/q_0 = p_1/q_1 = p/q$ we may define for $f \in \dot{M}^{p, q}$ the function $F(z) = \frac{f}{|f|} |f|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}}$. This is a bounded *-weakly continuous function of $z = x + iy$ (for $0 \leq x \leq 1$) with values in $\dot{M}^{p_0, q_0} + \dot{M}^{p_1, q_1}$, holomorphic on the strip $0 < x < 1$, with $\sup_{y \in \mathbb{R}} \|F(iy)\|_{\dot{M}^{p_0, q_0}} < +\infty$, $\sup_{y \in \mathbb{R}} \|F(1 + iy)\|_{\dot{M}^{p_1, q_1}} < +\infty$, and $F(\theta) = f$.

If F was strongly continuous, we would find that $f = F(\theta)$ would belong to $[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]_\theta$. But F is only *-weakly continuous. We may define $G(z) = \int_0^z F(w) dw$. Then we have $G \in \mathcal{G}$, and $G'(\theta) = f$; thus f belongs to $[\dot{M}^{p_0, q_0}, \dot{M}^{p_1, q_1}]^\theta$ \diamond

References

- [1] J. Bergh and J. Löfström. *Interpolation spaces*. Springer-Verlag, 1976.
- [2] A.P. Calderón. Intermediate spaces and interpolation: the complex method. *Studia Math.*, 24:113–190, 1964.
- [3] P.G. Lemarié-Rieusset. Multipliers and Morrey spaces. *Potential Analysis*, 38:741–752, 2013.
- [4] A. Ruiz and L. Vega. Corrigenda to “unique continuation for Schrödinger operators” and a remark on interpolation of Morrey spaces. *Publ. Mat.*, 39:404–411, 1995.
- [5] W. Sickel. Personal communication, march 2014.
- [6] C.T. Zorko. Morrey spaces. *Proc. Amer. Math. Soc.*, 98:586–592, 1986.