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On some historical aspects of Riemann zeta function, 3

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Abstract

In this third note, we deepen, from either an historical and historiographical standpoint, the arguments treated in the previous preprint hal.archives-ouvertes.fr/hal-00907136 - version 1.

Roughly, the history of entire function theory starts with the theorems of factorization of a certain class of complex functions, later called *entire transcendental functions* by Weierstrass (see (Loria 1950, Chapter XLIV, Section 741) and (Klein 1979, Chapter VI)), which made their explicit appearance around the mid-1800s, within the wider realm of complex function theory which had its paroxysmal moment just in the 19th century. But, if one wished to identify, with a more precision, that chapter of complex function theory which was the crucible of such a theory, then the history would lead to elliptic function theory and related factorization theorems for doubly periodic elliptic functions, these latter being meant as a generalization of trigonometric functions. Following (Stillwell 2002, Chapter 12, Section 12.6), the early idea which was as at the basis of the origin of elliptic functions as obtained by inversion of elliptic integrals, is due to Gauss, Abel and Jacobi. Gauss has such an idea in the late 1790s but didn't publish it; Abel had the idea in 1823 and published it in 1827, independently of Gauss. Jacobi's independence instead is not quite so clear. He seems to have been approaching the idea of inversion in 1827, but he was only spurred by the appearance of Abel's paper. At any rate, his ideas subsequently developed at an explosive rate, up until he published the first and major book on elliptic functions, the celebrated *Fundamenta Nova Theoriæ Functionum Ellipticarum* in 1829. Following (Enriques 1982, Book III, Chapter I, Section 6), on the legacy left, amongst others, by J.L. Lagrange, N.H. Abel, C.G.J. Jacobi, A.L. Cauchy and L. Euler, Riemann and Weierstrass quickly became the outstanding figures of the 19th century mathematics. Agreeing with Poincaré in his 1908 *Science et méthode*, Riemann was an extremely brilliant intuitive mathematician, whereas Weierstrass was primarily a logician,

both personifying, therefore, those two complementary and opposite typical aspects characterizing the mathematical work. Beyond what had been made by Cauchy, they created the main body of the new complex function theory in the period from about 1850 to 1880 (see (Klein 1979, Chapter VI)). Both received a strong impulse from Jacobi work. The first elements of the theory of functions according to Weierstrass date back to a period which roughly goes on from 1842-43 to 1854; in the meanwhile, Riemann published, in the early 1850s, his first works on the foundations of complex analysis, followed by the celebrated works on Abelian functions (which are elliptic functions so named by Jacobi) of the years 1856-57, which dismayed Weierstrass himself, influencing his next research program. This last point should be taken with a certain consideration. Instead, following (Klein 1979, Chapter VI), in the period from 1830s to the early 1840s, Weierstrass began to self-taughtly study Jacobi's *Fundamenta nova theoriæ functionum ellipticarum*, hence attended Christoph Gudermann (1798-1852) lectures on elliptic functions. He wrote his first paper in 1841 on modular functions, followed by some other papers wrote between 1842 and 1849 on general function theory and differential equations. His first relevant papers were written in the years 1854-56 on hyperelliptic or Abelian functions, which engaged him very much. Afterwards, in the wake of his previous work on analytic, elliptic and Abelian functions, Weierstrass was led to consider the so-called *natural boundaries* (that is to say, curves or points - later called *essential singularities* - in which the function is not regularly defined) of an analytic function to which Riemann put little attention. The first and rigorous treatment of these questions was given by Weierstrass in his masterful 1876 paper entitled *Zur Theorie der eindeutigen anatytschen Funktionen*, where many new results were achieved, amongst which is the well-known *Casorati-Weierstrass theorem* (as we today know it) and the product factorization theorem. Klein (1979, Chapter VI) states that the content of this seminal paper surely dated back to an earlier period, and was chiefly motivated by his research interests in elliptic functions. As pointed out in (Hancock 1910, Introduction), nevertheless, it is quite difficult to discern the right contribution to the elliptic function theory due to Weierstrass from other previous mathematicians, because of the objective fact that Weierstrass started to publish his lessons and researches only after the mid-1860s.

Weierstrass' theory of entire functions and their product decompositions, according to Klein, has found its most brilliant application in the (Weierstrass) theory of elliptic functions, to be precise, in the construction of the basic Weierstrassian σ -function $\sigma(u)$; perhaps - Klein says - Weierstrass' theory of entire functions even originated from his theory of elliptic functions (see also (Bottazzini & Gray 2013, Chapter 6, Section 6.6.3)). Nevertheless, already Gauss and Abel were gone very close to this σ -function and its properties. Again Klein says that he wishes to conclude his discussion of Weierstrass' theory

of complex functions, adding only some remarks on the history, referring to R. Fricke distinguished review on elliptic functions for more information (see (Burkhardt et al. 1899-1927, Zweiter Teil, B.3, pp. 177-348)). If we now ask - again Klein says - from where Weierstrass got the impulse to represent his functions by infinite products, we find his principal forerunner in G. Eisenstein (1823-1852), who was also a friend of Riemann with whom often talked about mathematical questions. In this paper (see (Eisenstein 1847)), Eisenstein did not attain the fully symmetric normal form, because he still lacked the exponential factors attached to the individual prime factors that will be then introduced by Weierstrass for inducing the product to converge in an absolutely manner. As he himself declared, Weierstrass got this idea from Gauss, who had proceeded in a similar way with his product expansion of the gamma function in 1812 (see the paper on the hypergeometric series in Weierstrass' *Mathematische Werke*, Band ...). It therefore turns out clear that elliptic and hyperelliptic function theory exerted a notable role in preparing the *humus* in which grew up the Weierstrass work on factorization theorem, and not only this: in general, it exerted a great influence on Riemann and Weierstrass work (see (Bottazzini & Gray 2013, Chapter 4, Section 4.5)). Following (Burkhardt et al. 1899-1927, Zweiter Teil, B.3, Nr. 15-17, 25, 45, 55), amongst others, Abel¹, Euler, Jacobi, Cayley and Gauss (see (Bottazzini & Gray 2013, Chapter 1, Section 1.5.1.1; Chapter 4, Section 4.2.3.1-2)) had already provided product expansions of certain elliptic functions, but it was Eisenstein (see (Eisenstein 1847)), with his infinite product expansion *ansatz*, the closest forerunner of the Weierstrass work on his σ function, in turn based on the previous work made by Jacobi and Gauss. Furthermore, already Cauchy², since 1843, gave some useful formulas involving infinite products and infinite series which maybe could have played a certain role in the 1859 Riemann paper in deducing some properties of that functional equation related to his ξ function (see later), having seen too the simple fact that Riemann himself known very well Cauchy's work. Therefore, in conclusion, Weierstrass himself acknowledges, in different

¹In (Greenhill 1892, Chapter IX, Section 258), the author states that the well-known expressions for the circular and hyperbolic functions in the form of finite and infinite products, have their analogues for the elliptic functions as laid down by Abel in some his researches on elliptic functions published in the celebrated *Crelle's Journal*, Issues 2 and 3, years 1827 and 1828. Following (Hancock 1910, Chapter V, Article 89), Abel showed, in the 1820s, that elliptic functions, considered as the inverse of the elliptic integrals, could be expressed as the quotient of infinite products, then systematically reconsidered in a deeper manner by Jacobi.

²In this regards see (Bellacchi 1894, Chapter X), where an interesting discussion of the Jacobi series is made, amongst other things highlighting that already Abel, on the wake of what was done by Johann Bernoulli, had introduced infinite product expansions of certain elliptic functions that later Jacobi, in turn, converted into infinite series by means of trigonometric arguments, so giving rise to new elliptic functions (see (Greenhill 1892, Chapter IX, Section 258)).

places, his debit both to Gauss and Cauchy, in achieving his celebrated results on entire function factorization theorem.

In 1858, Riemann wrote his unique paper on number theory, which marked a revolution in mathematics. According to Laugwitz (1999, Introduction, Sections 4.1 and 4.2), real and complex analysis has always influenced Riemann work: algebraic geometry appears, in his works, as a part of complex analysis; he treats number theory with methods of complex function theory; he subsumes physical applications into partial differential equations; he replaces the usual axiomatic conception of geometry by his novel (Riemannian) geometry, which is a part of real analysis of several variables; and he develops the topology of manifolds as a new discipline derived from analysis. Riemann known the elements of algebraic analysis according to J-L. Lagrange and L. Euler, through the lessons of his teacher, M.A. Stern (1807-1894), who was one of the last schoolmasters of the subject. Riemann handled the gamma function in a secure and self-confident way and has dealt with differential equations and recursions in the Euler's manner. The Stern lessons were of very fundamental importance to achieve many Riemann's results, even if the celebrated 1748 Euler's *Introductio in Analysin Infinitorum* was one of the most influential textbooks of the time. Nevertheless, Klein (1979, Chapter VI) states that Riemann began already to study elliptic and Abelian functions since the late 1840s, because this subject, in the meantime, has become of a certain vogue in Germany. In the 1855-56 winter term, following the Dirichlet's research lines, Riemann lectured on functions of a complex quantity, in particular elliptic and Abelian functions, while in the 1856-57 winter term he lectured on the same subject, but now with special regard to hypergeometric series and related transcendentals. These lectures, from which he drew publications on Abelian and hypergeometric functions, were partially repeated in the following semesters. Klein (1979, Chapter VI) points out that the years 1857-62 marked the high-point of Riemann's creativity. Moreover, Klein states that before to characterize the specific Riemannian function theory work, he wishes to put forward a remark that may cause some surprise: Riemann did much important work in the theory of functions that does not fit into the framework of his typical theory. Klein refers to the notable 1859 paper on the number of primes less than a given magnitude, where it is introduced «*the Riemann zeta-function $\zeta(\sigma + it)$ given by an analytic expression, namely an infinite product. This product is converted into a definite integral, which can then be evaluated by shifting the path of integration. The whole procedure is function theory à la Cauchy*». Therefore, according to what Felix Klein states, the mathematical background that was at the basis of the Riemannian analytic treatment of his ζ function, essentially lies on the Cauchy's theory of complex functions. This is also confirmed by (Bottazzini & Gray 2013, Chapter 5, Section 5.1), coherently with what has just been said above in regards to the importance played

by Cauchy's work on the Riemann's one.

According to (Laugwitz 1999, Chapter 1), notwithstanding the era of ferment that concerned the 19th century mathematics, an autonomous and systematic account of the foundations of complex analysis is findable, for the first time, in the Riemann's works and lecture notes through the winter term 1855-56 to the winter term 1861-62, the latter having been published by the physicist Carl Ernst Abbe (1840-1905) in the summer term 1861 (see (Ullrich 2003) and references therein) when he was a student of W. Weber and Riemann in Göttingen. The only systematic and congruous historical attempt to organically recognize the various Riemann's lessons has been pursued by E. Neuenschwander in (Neuenschwander 1996). In any case, Riemann was fully imbedded into the real and complex analysis scenery of the first middle of the 19th century, which seen involved the outstanding figures of Cauchy, Weierstrass and Riemann himself, whose researches were intertwined amongst them more times. According to (Laugwitz 1999, Chapter 1, Section 1.1.5), just in connection with the drawing up of his paper on the same subject, Riemann was aware of the Weierstrass' papers on Abelian functions wrote between 1853 and 1856-57, for which it is evident that a certain influence of the latter on the Riemann's one - at least, as concerns such a period - there was, even if Weierstrass will publish these his works only later. Again following (Neuenschwander 1996), one of the key themes of the last 1861 *sommersemester* Riemann lectures on analytic functions, was the determination of a complex function from its singularities. Thereafter, he clarifies that this problem regards only single-valued functions defined on $\mathbb{C} \cup \{0\}$ whose unique singularities are poles (the names *pole* and *essential singularity* are respectively due to C.A. Briot and J.C. Bouquet and to Weierstrass). In turn, the resolution of this problem requires the previous knowledge of the zeros of the function which has to be determined. At first, Riemann considered the case of a function having a finite number of zeros and poles, then he went over the the next question, namely to determine a function with infinitely many zeros whose unique point of accumulation is ∞ (which, inter alia, concerns too the Riemann zeta function theory). But, again following (Laugwitz 1999, Chapter 1, Section 1.1.6), in doing so Riemann went over very close to the next Weierstrass' work on the infinite product representation of an entire function, using special cases to explain the general procedure. Detlef Laugwitz points out that Riemann has pursued this latter task in such a way that, by following his directions, one could immediately give a proof of the known Weierstrass' product theorem, even if Riemann ultimately failed in reaching the general case; nevertheless, for what follows, this last claim has a certain importance from our historical standpoint.

Indeed, in his renowned paper on the distribution of prime numbers³, Rie-

³This paper was presented by Riemann, after his nomination as full professor in July 1859, to the Berlin Academy for his consequent election as a corresponding member of this

mann stated the following function

$$\xi(t) \doteq \left(\frac{1}{2} \Gamma \left(\frac{s}{2} \right) s(s-1) \pi^{-s/2} \zeta(s) \right)_{s=1/2+it} \quad (1)$$

later called *Riemann ξ -function*. It is an entire function. Riemann conjectured that $(\xi(t) = 0) \Rightarrow (\Im(t) = 0)$, that is to say, the famous *Riemann hypothesis* (RH), as it will be called later. Whereupon, he stated that

«This function $\xi(t)$ is finite for all finite values of t , and allows itself to be developed in powers of tt as a very rapidly converging series. Since, for a value of s whose real part is greater than 1, $\log \zeta(s) = -\sum \log(1-p^{-s})$ remains finite, and since the same holds for the logarithms of the other factors of $\xi(t)$, it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $i/2$ and $-i/2$. The number of roots of $\xi(t) = 0$, whose real parts lie between 0 and T is approximately $(T/2) \log(T/2)\pi - T/2\pi$; because the integral $\int d \log \xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $i/2$ and $-i/2$ and whose real parts lie between 0 and T , is (up to a fraction of the order of magnitude of the quantity $1/T$) equal to $(T \log(T/2)\pi - T/2\pi)i$; this integral however is equal to the number of roots of $\xi(t) = 0$ lying within this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation. If one denotes by α all the roots of the equation $\xi(t) = 0$, one can express $\log \xi(t)$ as

$$\sum \log \left(1 - \frac{tt}{\alpha\alpha} \right) + \log \xi(0) \quad (2)$$

for, since the density of the roots of the quantity t grows with t only as $\log t/2\pi$, it follows that this expression converges and becomes for an infinite t only infinite as $t \log \xi(t)$; thus it differs from $\log \xi(t)$ by a function of tt , that for a finite t remains continuous and finite and, when divided by tt , becomes infinitely small for infinite t . This difference is consequently a constant, whose value can be determined through setting $t = 0$. With the assistance of these methods, the number of prime numbers that are smaller than x can now be determined».

latter. To be precise, following (Bottazzini 2003), just due to this election, Riemann and Dedekind visited Berlin, where they met E.E. Kummer, L. Kronecker and K. Weierstrass. According to (Dedekind 1876), very likely, it was just from this meeting that sprung out of the celebrated 1859 Riemann number theory paper that was, then, sent to Weierstrass himself, to be published in the November issue of the *Monatsberichte der Berliner Akademie*.

So, in his celebrated 1859 paper, Riemann himself had already reached an infinite product factorization of $\xi(t)$, namely the (2), which can be equivalently written as follows

$$\log \xi(t) = \sum_{\alpha} \log \left(1 - \frac{t}{\alpha}\right) + \log \xi(0) = \log \xi(0) \prod_{\alpha} \left(1 - \frac{t}{\alpha}\right) \quad (3)$$

from which it follows that $\xi(t) = \xi(0) \prod_{\alpha} (1 - t/\alpha)$. Thus, questions related to entire function factorizations had already been foreshadowed in this Riemann work. Therefore, we now wish to outline the main points concerning the very early history of entire function factorization theorems, having taken the 1859 Riemann paper as an occasional starting point of this historical question, in which, inter alia, a particular entire function factorization - i.e. the (3) - has been used. In short, this 1859 Riemann paper has been a valuable *καίρως* to begin to undertake one of the many study's branch which may depart from this milestone of the history of mathematics, to be precise that branch concerning the entire function theory which runs parallel to certain aspects of the theory of Riemann zeta function, with interesting meeting points with physics. One of the very few references which allude to these Riemann paper aspects is the article by W.F. Osgood in (Burkhardt et al. 1899-1927, Zweiter Teil, B.1.III, pp. 79-80), where, discussing of the gender of an entire function, an infinite product expansion of the function $\sin \pi s/\pi s$ is considered; to be precise, since Johann Bernoulli to Euler, the following form⁴ had already been deduced (see (Bellacchi 1894, Chapter XI))

$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$

which has gender 0. Following (Stopple 2003, Chapter 6, Section 6.1), Euler's idea is to write the function $\sin \pi x/\pi x$ as a product over its zeros, analogous

⁴As $n \rightarrow \infty$ and $t \neq 0$, Weierstrass proved to be $\sin \pi t/\pi t = \prod_{n=-\infty}^{\infty} (1 - t/n)e^{t/n}$ - see (Bellacchi 1894, Chapter XI). But, according to (Bellacchi 1894, Chapter XI) and (Hancock 1910, Chapter I, Arts. 13, 14), Cauchy was the first to have treated (in the *Exercices de Mathématiques, IV*) the subject of decomposition into prime factors of circular functions and related convergence questions, from a more general standpoint. Although Cauchy did not complete the theory, he however recognized that, if a is a root of an integral (or entire) transcendental function $f(s)$, then it is necessary, in many cases, to join to the product of the infinite number of factors such as $(1 - s/a)$, a certain exponential factor $e^{P(s)}$, where $P(s)$ is a power series in positive powers of s . Weierstrass gave then a complete treatment of this subject. On the other hand, besides what has already been said above, also in (Greenhill 1892, Chapter IX, Section 258)) it is pointed out that, since Abel's work, the infinite product expansions of trigonometric functions have been formal models from which to draw inspiration, by analogy, for further generalizations or extensions.

to factoring a polynomial in terms of its roots. For example, if a quadratic polynomial $f(x) = ax^2 + bx + c$ has roots α, β different from 0, then we can write $f(x) = c(1 - x/\alpha)(1 - x/\beta)$. On the other hand, $\sin \pi x = 0$ when $x = 0, \pm 1, \pm 2, \dots$ and since $\sin \pi x/x = 1 - \pi^2 x^2/6 + O(x^4)$, $\sin \pi x/\pi x \xrightarrow{x \rightarrow 0} 1$ and $\sin \pi x/\pi x = 0$ when $x = \pm 1, \pm 2, \dots$, Euler guessed that $\sin \pi x/\pi x$ had a factorization as a product

$$\begin{aligned} \frac{\sin \pi x}{\pi x} &= \left(1 - \frac{x}{1}\right) \left(1 + \frac{x}{1}\right) \left(1 - \frac{x}{2}\right) \left(1 + \frac{x}{2}\right) \dots = \\ &= \left(1 - \frac{x^2}{1}\right) \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{16}\right) \dots \end{aligned}$$

which will lead later to a valid proof of this factorization. Then, even in the context of the history of entire function factorization theorems, W.F. Osgood points out that already Riemann, just in his famous 1859 paper, had considered an entire function, the $\xi(s)$, as a function of s^2 with gender 0, but without giving any rigorous prove of this fact, which will be done later by J. Hadamard. In the next sections, when we will deepen the works of Hadamard and Pólya on the entire function theory related to Riemann zeta function, we also will try to clarify, as far as possible, these latter aspects of the 1859 Riemann paper which mainly constitute one of the central cores of the present work.

Following (Stoppa 2003, Chapter 10, Section 10.1), it was Riemann to realize that a product formula for $\xi(s)$ would have had a great significance for the study of prime numbers. The first rigorous proof of this product formula was due to Hadamard but, as himself remember, it took almost three decades before he reached to a satisfactory proof of it. Likewise, also H.M. Edwards (1974, Chapter 1, Sections 1.8-1.19) affirms that the parts concerned with (2) are the most difficult portion of the 1859 Riemann's paper (see also (Bottazzini & Gray 2013, Chapter 5, Section 5.10)). Their goal is essentially to prove that $\xi(s)$ can be expressed as an infinite product, stating that

«[...] any polynomial $p(t)$ can be expanded as a finite product $p(t) = p(0) \prod_{\rho} (1 - t/\rho)$ where ρ ranges over the roots of the equation $p(t) = 0$ [except that the product formula for $p(t)$ is slightly different if $p(0) = 0$]; hence the product formula (2) states that $\xi(t)$ is like a polynomial of infinite degree. Similarly, Euler thought of $\sin x$ as a "polynomial of infinite degree"⁵ when he conjectured, and finally proved, the formula $\sin x = \pi x \prod_{n \in \mathbb{N}} (1 - (x/n)^2)$. On other hand, [...] $\xi(t)$ is like a polynomial of infinite degree, of which a finite number of its terms gives a very good approximation in any finite part of the

⁵Following (Bottazzini & Gray 2013, Chapter 8, Section 8.5.1), amongst the functions that behave very like a polynomial, there is the Riemann ξ function. In this regards, see also what will be said in the next section 5 about Lee-Yang theorems and, in general, the theory concerning the location of the zeros of polynomials.

plane. [...] Hadamard (in 1893) proved necessary and sufficient conditions for the validity of the product formula $\xi(t) = \xi(0) \prod_{\rho} (1 - t/\rho)$ but the steps of the argument by which Riemann went from the one to the other are obscure, to say the very least».

The last sentence of this Edwards' quotation is historically quite interesting and would deserve further attention and investigation. Furthermore, H.M. Edwards states too that

«[...] a recurrent theme in Riemann's work is the global characterization of analytic functions by their singularities. See, for example, the Inauguraldissertation, especially Article 20 of Riemann's *Werke* (pp. 37-39) or Part 3 of the introduction to the Riemann article "Theorie der Abel'schen Functionen", which is entitled "Determination of a function of a complex variable by boundary values and singularities". See also Riemann's introduction to Paper XI of the his collected works, where he writes about " [...] our method, which is based on the determination of functions by means of their singularities (*Umtetigkeiten und Unendlichwerden*) [...]". Finally, see the textbook of Ahlfors (1979), namely the section 4.5 of Chapter 8, entitled "Riemann's Point of View"»,

according to which Riemann was therefore a strong proponent of the idea that an analytic function can be defined by its singularities and general properties, just as well as or perhaps better than through an explicit expression, in this regards showing, with Riemann, that the solutions of a hypergeometric differential equation can be characterized by properties of this type. In short, all this strongly suggests us the need for a deeper re-analysis of Riemann *œuvre* concerning these latter arguments, as well as a historical seek for the mathematical background which was at the origins of his celebrated 1859 number theory paper. From what has just been said, it turns out clear that a look at the history of entire function theory, within the general and wider complex function theory framework, is needed to clarify some of the historical aspects of this influential seminal paper which, as Riemann himself recognized, presented some obscure points. In this regards, also Gabriele Torelli (1901, Chapter VIII, Sections 60-64) claimed this last aspect, pointing out, in particular, on the Riemann's ansatz according to which the entire function $\xi(t)$ is equal, via (3), to the Weierstrass' infinite product of primary factors without any exponential factor. As is well-known, this basic question will be brilliantly solved by J. Hadamard in his famous 1892 paper that, inter alia, will mark a crucial moment in the history of entire function theory (see (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2) and next sections).

An historical account of entire factorization theorems from Weierstrass on-

ward. To begin, we wish to preliminarily follow the basic textbook on complex analysis of Giulio Vivanti (1859-1949), an Italian mathematician whose main research field was into complex analysis, becoming an expert of the entire function theory. He wrote some notable treatises on entire, modular and polyhedral analytic functions: a first edition of a prominent treatise on analytic functions appeared in 1901, under the title *Teoria delle funzioni analitiche*, published by Ulrico Hoepli in Milan, where the first elements of the theory of analytic functions, worked out in the late 19th-century quarter, are masterfully exposed into three main parts, giving a certain load to the Weierstrass' approach respect to the Cauchy's and Riemann's ones. The importance of this work immediately arose, so that a German edition was carried out, in collaboration with A. Gutzmer, and published in 1906 by B.G. Teubner in Leipzig, under the title *Theorie der eindeutigen analytischen funktionen. Umarbeitung unter mitwirkung des verfassers deutsch herausgegeben von A. Gutzmer*, which had to be considered as a kind of second enlarged and revised edition of the 1901 first Italian edition according to what Vivanti himself said in the preface to the 1928 second Italian edition, entitled *Elementi della teoria delle funzioni analitiche e delle trascendenti intere*, again published by Ulrico Hoepli in Milan, and written following the above German edition in which many new and further arguments and results were added, also as regards entire functions. Almost all the Vivanti's treatises are characterized by the presence of a detailed and complete bibliographical account of the related literature, this showing the great historical attention that he always put in drawing up his works. Therefore, he also was a valid historian of mathematics besides to be an able researcher (see (Janovitz & Mercanti 2008, Chapter 1) and references therein), so that his works are precious sources for historical studies, in our case as concerns entire functions. The above mentioned Vivanti's textbook on complex analysis has been one of the most influential Italian treatises on the subject. It has also had wide international fame thanks to its German edition.

Roughly speaking, the transcendental entire functions may be formally considered as a generalization, in the complex field, of polynomial functions. Following (Vivanti 1928, Sections 134-135), (Markuševič 1988, Chapter VII) and (Pierpont 1914, Chapter VIII, Sections 127 and 140), the great analogy subsisting between these two last function classes suggested the search for an equal formal analogy between the corresponding main properties. To be precise, the main properties of polynomials concerned either with the existence of zeros (Gauss' theorem) and the linear factor decomposition of a polynomial, so that it was quite obvious trying to see whether these could be, in a certain way, extended to entire functions. As regards the Gauss' theorem, it was immediately realized that it couldn't subsist because of the simple counterexample given by the fundamental elementary transcendental function e^x which does not have any zero in the whole of complex plane. On the other hand, just

this last function will provide the basis for building up the most general entire function which is never zero, which has the general form $e^{G(x)}$, where $G(x)$ is an arbitrary entire function, and is said to be an exponential factor. Then, the next problem consisted in finding those entire functions having zeros and hence how it is possible to build up them from their zero set. In this regards, it is well-known that, if $P(z)$ is an arbitrary non-zero polynomial with zeros $z_1, \dots, z_n \in \mathbb{C} \setminus 0$, having $z = 0$ as a zero with multiplicity λ (supposing $\lambda = 0$ if $P(0) = 0$), then we have the following well-known finite product factorization⁶

$$P(z) = Cz^\lambda \prod_{j=1}^n \left(1 - \frac{z}{z_j}\right) \quad (4)$$

where $C \in \mathbb{C} \setminus 0$ is a constant, so that a polynomial, except a constant factor, may be determined by its zeros. For transcendental entire functions, this last property is much more articulated respect to the polynomial case: indeed, whilst the indeterminacy for polynomials is given by a constant C , for transcendental entire functions it is larger and related to the presence of an exponential factor which is need to be added to warrant the convergence of infinite product development. A great part of history of the approach and resolution of this last problem is the history of entire function factorization. Nevertheless, we also wish to report what says Giacomo Bellacchi (1894, Chapter XI, Section 98) about this last problem. To be precise, he states that

«Se $a_1, a_2, a_3, \dots, a_n, \dots$ simboleggino le radici semplici di una funzione olomorfa $f(z)$, ed il quoziente $f(z) : \prod (z - a_n)$ non si annulli per alcuna di esse, la sua derivata logaritmica $\psi'(z) = f'(z)/f(z) - \sum (1/(z - a_n))$ è olomorfa in tutto il piano; moltiplicando i due membri per dz ed integrando, Cauchy giunse

⁶It is noteworthy the historical fact pointed out by Giuseppe Bagnera (1927, Chapter III, Section 12, Number 73), in agreement to what has been likewise said above, according to which already Cauchy himself had considered first forms of infinite product developments, after the Euler's work. Also Bagnera then, in this his work, quotes Betti's work on elliptic functions and related factorization theorems. Instead, it is quite strange that the Italian mathematician Giacomo Bellacchi (1838-1924) does not cite Betti, in his notable historical work on elliptic functions (Bellacchi 1894) in regards to entire function factorization theorems which are treated in the last chapter of this his work; this is also even more strange because Chapter XI of his book is centered around the 1851 Riemann dissertation on complex function theory, without quoting the already existed Italian translation just due to Betti. Furthermore, Bellacchi studied at the *Scuola Normale Superiore* of Pisa in the 1860s, for which it is impossible that he had not known Betti (see (Maroni 1924)). On the other hand, also (Loria 1950, Chapter XLIV, Section 741) refers that Weierstrass found inspiration for his factorization theorem, a result of uncommon importance according to Gino Loria, generalizing a previous Cauchy's formula: indeed, both Cauchy and Gauss are quoted at p. 120 of the 1879 French translation of the original 1876 Weierstrass paper. This, to further confirmation of what has been said above.

alla formula $f(z) = Ce^{\psi(z)} \prod(1 - z/a_n)$, dove C è una costante»

[«If $a_1, a_2, a_3, \dots, a_n, \dots$ represent the simple roots of a holomorphic function $f(z)$, and the ratio $f'(z) : \prod(z - a_n)$ is not zero for each root, then its logarithmic derivative $\psi'(z) = f'(z)/f(z) - \sum(1/(z - a_n))$ is holomorphic in the whole of plane; multiplying both sides by dz and integrating, Cauchy reached the formula $f(z) = Ce^{\psi(z)} \prod(1 - z/a_n)$, where C is a constant»],

so that it seems, according to Bellacchi, that already Cauchy had described the utility of exponentials as convergence-producing factors, in a series of his papers published in the Tome XVII of the *Comptes Rendus de l'Académie des Sciences (France)*; this supposition is also confirmed by Hancock (1910, Chapter I, Art. 14). Nevertheless, following (Vivanti 1928, Sections 135-141), the rise of the first explicit formulation of the entire function factorization theorem was given by Weierstrass in 1876 (see (Weierstrass 1876)) and was mainly motivated by the purpose to give a solution to the latter formal problem, concerning the convergence of the infinite product development of a transcendental entire function $f(z)$ having an infinite number of zeros, namely $z = 0$, with multiplicity λ , and z_1, \dots, z_n, \dots such that $0 < |z_j| \leq |z_{j+1}|, z_j \neq z_{j+1} \quad j = 1, 2, \dots$, trying to extend the case related to a finite number of zeros z_1, \dots, z_n , in which such a factorization is given by

$$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^n \left(1 - \frac{z}{z_j}\right), \quad (5)$$

to the case of infinite zeros, reasoning, by analogy, as follows. The set of infinite zeros z_j is a countable set having only one accumulation point, that at infinite. Therefore, for every infinite increasing natural number sequence $\{\rho_i\}_{i \in \mathbb{N}}$, it will be always possible to arrange the zeros z_j according to their modulus in such a manner to have the following non-decreasing sequence $|z_1| \leq |z_2| \leq \dots$ with $\lim_{n \rightarrow \infty} |z_n| = \infty$. In such a case, if one wants, by analogy, to extend (5) as follows

$$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right), \quad (6)$$

then it will not be possible to fully avoid divergence's problems inherent to the related infinite product. The first hint towards a possible overcoming of these difficulties, was suggested to Weierstrass (see (Weierstrass 1856a)) by looking at the form of the inverse of the Euler integral of the second kind - that is to say, the *gamma function* - and given by

$$\frac{1}{\Gamma(z)} = z \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) \left(\frac{j}{1+j}\right)^z = z \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) e^{-z \log \frac{j+1}{j}}, \quad (7)$$

from which he described the possible utility of the exponential factors there involved to, as the saying goes, force the convergence of the infinite product of the last equality; these his ideas concretized only in 1876 with the explicit formulation of his celebrated theorem on the entire function factorization.

As we have said above, Weierstrass (1856a) attributes, however, the infinite product expansion (7) to Gauss, but some next historical studies attribute to Euler this formula, that he gave in the famous 1748 *Introductio in Analysin Infinitorum*. Indeed, as has been said above, from the 1879 French translation of the original 1876 Weierstrass paper, it turns out that both Cauchy and Gauss are quoted (at page 120), before to introduce the primary factors. Nevertheless, P. Ullrich (1989, Section 3.5) says that the real motivation to these Weierstrass' results about entire function factorization were mainly due to attempts to characterize the factorization of quotients of meromorphic functions on the basis of their zero sets, rather than to solve the above problem related to the factorization of a polynomial in dependence on its zeros. Furthermore, Ullrich (1989, Section 3.5) observes too that other mathematicians dealt with questions concerning entire function factorization methods, amongst whom are just Enrico Betti and Bernard Riemann, the latter, in his important 1861 *sommersemester* lectures on analytic functions, arguing, as has already been said, upon the construction of particular complex functions with simple zeros, even if, all things considered, he didn't give, according to Ullrich (1989, Section 3.5), nothing more what Euler done about gamma function through 1729 to his celebrated 1748 treatise on infinitesimal analysis⁷. Instead, as we have seen above, D. Laugwitz (1999, Chapter 1, Section 1.1.6) states that Riemann's work on meromorphic functions was ahead of the Weierstrass' one, having been carried out with originality and simplicity. To this point, for our purposes, it would be of a certain importance to deepen the possible relationships between Riemann and Weierstrass, besides to what has been said above: for instance, in this regards, Laugwitz (1999, Chapter 1, Section 1.1.5) says that Riemann was aware of the Weierstrass' works until 1856-57, in connection with the composition of his paper on Abelian functions, in agreement with what has been said in the previous sections. Again according to (Laugwitz 1999, Chapter 1, Section 1.1.6), one of the key themes of Riemann's work on complex function theory was the determination of a function from its singularities which, in turn, implies the approach of another problem, the one concerning the determination of a function from its zeros. In this regards, Riemann limited himself to consider the question to determine a function with infinitely many zeros whose only point of accumulation is ∞ . What he is after is the product representation later named after Weierstrass. Riemann uses a special case to explain the

⁷Following (Lunts 1950), (Markušević 1988, Chapter VII) and references therein, also Lobačevskij, since 1830s, made some notable studies on gamma function which preempted times.

general procedure. He does it in such a way that by following his direction one could immediately give a proof of the Weierstrass product theorem. Therefore, it would be hoped a deeper study of these 1857-61 Riemann's lectures on complex function theory to historically clarify this last question which is inside the wider historical framework concerning the work of Riemann in complex function theory.

Furthermore, to this point, there seems not irrelevant to further highlight, although in a very sketchily manner, some of the main aspects of the history of gamma function. To this end, we follow the as many notable work of Reinhold Remmert (1998) which, besides to mainly be an important textbook on some advanced complex analysis topics, it is also a very valuable historical source on the subject, which seems to remember the style of the above mentioned Vivanti's textbook whose German edition, on the other hand, has always been a constant reference point in drawing up the Remmert's textbook itself. Following (Remmert 1998, Chapter 2), the early origins of gamma function should be searched into the attempts to extend the function $n!$ to real arguments. Euler was the first to approach this problem since 1729, giving a first expression of this function, in a letter to Goldbach (see (Whittaker & Watson 1927, Chapter XII, Section 12.1)), providing a first infinite product expression of this new function, but only for real values. Gauss, who did not know Euler work, also taking into account Newton's work on interpolation (see (Schering 1881, Sections XI and XII)), in 1811 considered as well complex values, denoting such a new function with Π , while it was Legendre, in 1814, to introduce a unified notation both for Euler and Gauss functions, denoting these latter with $\Gamma(z)$ and speaking, since then, of *gamma function*. Afterwards, in 1854, Weierstrass began to consider an Euler infinite product expansion of the function $1/\Gamma(z)$, that he denoted with $Fc(z)$ and is given by $1/\Gamma(z) = ze^{\gamma z} \prod_{j=1}^{\infty} (1 + z/j)e^{-z/j}$, where γ is the well-known *Euler-Mascheroni constant*⁸, from which he maybe recognized, for the first time, the importance of the use of exponential factors as infinite product convergence-producing elements. However, according to (Whittaker & Watson 1927, Chapter XII, Section 12.1), the formula (7) had already been obtained by F.W. Newman (1848) starting from Euler's expression of gamma function given by (7).

Following (Vivanti 1928, Section 135-141), (Remmert 1998, Chapter 3) and, above all, (Bottazzini & Gray 2013, Section 6.7), Weierstrass extended the product (5) in such a manner to try to avoid divergence problems with the *ad hoc* introduction, into the product expansion, of certain forcing convergence factors. This attempt was successfully attended, since 1874, as a solution to a particular question - the one which may be roughly summarized as the attempt to build up an entire transcendental function with prescribed zeros - which arose within the general Weierstrass' intent to solve the wider

⁸See (Pepe 2012) for a contextual brief history of the Euler-Mascheroni constant.

problem to find a representation for a single-valued function as a quotient of two convergent power series. To be precise, he reached, amongst other things, the following main result

«Given a countable set of non-zero complex points z_1, z_2, \dots , such that $0 < |z_1| \leq |z_2| \leq \dots$ with $\lim_{n \rightarrow \infty} |z_n| = \infty$, then it is possible to find, in infinite manners, a non-decreasing sequence of natural numbers p_1, p_2, \dots such that the series $\sum_{j=1}^{\infty} |z/z_j|^{p_j+1}$ be convergent for every finite value of z , in such a manner that the most general entire function which is zero, with their own multiplicity, in the points z_1, z_2, \dots , and has a zero of order λ in the origin, is given by⁹

$$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) E_j(z) \quad (8)$$

where $E_j(z) = (1-z)(\sum_{i=1}^j z^i/i)$ for $j \geq 1$ and $E_0(z) = 1-z$, $g(z)$ being an arbitrary entire function, and the infinite product is absolutely convergent for each finite value of $|z|$ ».

The factors $E_j(z)$ will be later called *Weierstrass' factors*, whilst the numbers p_j will be called *convergence exponents*. The sequence $E_j(z)_{j \in \mathbb{N}_0}$ plays a very fundamental role in the Weierstrass' theorem: from the equation

$$1 - z = \exp(\log(1 - z)) = \exp\left(-\sum_{i \geq 1} z^i/i\right), \quad (9)$$

Weierstrass obtained the formula $E_j(z) = \exp(-\sum_{i > j} z^i/j)$ in proving convergence properties which, on the other hand, would have been easier obtained by means of the following estimates

$$|E_j(z) - 1| \leq |z|^{j+1}, \forall j \in \mathbb{N}_0, \forall z \in \mathbb{C}, |z| \leq 1 \quad (10)$$

that have been proved only later. Amongst the first ones to have made this, seems there having been L. Fejér (see (Hille 1959, Section 8.7)), but the argument appears as early as 1903 in a paper of Luciano Orlando¹⁰ (1903) which starts from Weierstrass' theorem as treated by Borel's monograph on entire functions. As has already been said above, Weierstrass was led to develop his

⁹Historically, in relation to (8), the function $f(z)$ was usually denoted, d'après Weierstrass, by $G(z)$, whilst $z^\lambda \prod_{j=1}^{\infty} (1 - z/z_j) E_j(z)$ was named *canonical (or primitive) function* - see (Sansone 1972, Chapter IV, Section 3), where there are too many interesting historical notes.

¹⁰Luciano Orlando (1887-1915) was an Italian mathematician prematurely died in the First World War - see the very brief obituary (Marcolongo 1918) as well as (Rouse Ball 1937, Appendix II, pp. 430-431). His supervisors were G. Bagnera and R. Marcolongo who led him to make researches in algebraic integral equation theory and mathematical physics.

theory by the chief objective to establish the general expression for all analytic functions meromorphic in \mathbb{C} except in finitely many points, reaching the scope after a series of previous futile attempts only in 1876, with notable results, spelt out in (Weierstrass 1876), concerning the class of transcendental entire functions. But what was new and sensational in the Weierstrass' construction was just the introduction and the application of the so-called *convergence-producing factors* (or *primary factors* or *Weierstrass' factors*) which strangely have no influence on the behavior and distribution of the zeros.

Towards the theory of entire functions. In his necrology to Weierstrass, Poincaré (1899, Section 6) said that Weierstrass' major contribution to the development of function theory was just the discovery of primary factors. Also Hermite was, in a certain sense, astonished and intrigued from the introduction of this new Weierstrass' notion of prime factor, which he considered of capital importance in analysis and making later notable studies in this direction; he also suggested to Émile Picard to do a French translation of the original 1876 Weierstrass' work, so opening a French research trend on this area. En passant, we also point out the fact that, from the notion of prime factor and from the convergence of the infinite product $\prod_{j \in \mathbb{N}} E_j(z/a_j)$, representing an entire transcendental function vanishing, in a prescribed way, in each a_j , Hilbert drew inspiration to formulate his valuable algebraic notion of prime ideal¹¹. Following (Pincherle 1922, Chapter IX, Section 137), (Vivanti 1928, Section 136), (Burckel 1979, Chapter XI), (Remmert 1998, Chapters 3 and 6), (Ullrich 1989, Section 3.5) and (Bottazzini & Gray 2013, Sections 5.11.5 and 6.7), since the late 1850s, Enrico Betti had already reached notable results, about convergence properties of infinite products of the type (6), very near to the Weierstrass' ones related to the resolution of a fundamental problem of entire function theory, the so-called *Weierstrass' problem*¹² (see (Pincherle 1922, Chapter IX, Section 137)). Betti exposed these outcomes in his celebrated 1859-60 Pisa lectures on advanced analysis entitled *La teorica delle funzioni ellittiche* (see (Betti 1903-1913, Tomo I, XXII)), published in the *Tomes III* and

¹¹Usually, the notion of prime ideal of the commutative algebra, with related operations, would want to be stemmed from the factorization of natural numbers.

¹²Following (Forsyth 1918, Chapter V, Section 50) and (Bottazzini & Gray 2013, Section 4.2.3.2), in relation to the infinite product expression of an entire transcendental function prior to 1876 Weierstrass' paper, attention should be also paid to a previous 1845 work of A. Cayley on doubly periodic functions. Furthermore, following (Tannery & Molk 1893, Section 85), into some previous 1847 works of G. Eisenstein on elliptic functions, some notable problems having to do with the construction of analytic functions with prescribed zeros as a quotient of entire functions with the involvement of certain transcendental entire functions of exponential type (similar to the Weierstrass problem as historically related to meromorphic functions), had already been considered. See also certain function's quotients stemmed from the developments of certain determinants given in (Gordan 1874). In any case, all these historical considerations confirm, once again, that the prolegomena of entire function factorization theorems should be searched in the general history of elliptic functions.

IV of the *Annali di matematica pura ed applicata*, Series I, after having published, in the Tome II of these *Annali*, an Italian translation of the celebrated 1851 Riemann's inaugural dissertation on complex function theory, which can be considered as an introduction to his next lectures on elliptic functions. Indeed, in these latter, Betti before all places an Introduction on the general principles on complex functions, essentially based on these 1851 Riemann lectures. From the point 3. onward of this Introduction, Betti starts to deal with entire functions, their finite and infinite zeros (there called *roots*), as well as on possible quotients between them. In particular, taking into account what is said in (Briot & Bouquet 1859), he considered infinite products of the type $\prod_{\rho}(1 - z/\rho)$, where ρ are the zeros of an entire function, with the introduction of a factor of the type e^w , where w is an arbitrary entire function, to make convergent this infinite product. Furthermore, Betti dealt with this type of infinite products starting to consider infinite product representations of the following particular function $es(z) = z \prod_{m=1}^{\infty} (m/(m+1))^z (1 + z/m)$, which satisfies some functional equations and verifies the relation $\Gamma(z) = 1/es(z)$. Therefore, as Weierstrass too will do later, Betti started from the consideration of the infinite product expansion of the inverse of the gamma function for studying the factorization of entire functions. Therefore, Betti guessed the utility of the convergence factors having exponential form, looking at the infinite product expansion of Gamma function, similarly to what Weierstrass will do. Afterwards, Betti proved some theorems which can be considered particular cases of the next Weierstrass' results, concluding affirming that

«Da questi teoremi si deduce che le funzioni intere potranno decomporci in un numero infinito di fattori di primo grado ed esponenziali, e qui comparisce una prima divisione delle funzioni intere. Quelle che hanno gl'indici delle radici in linea retta, e quelle che le hanno disposte comunque nel piano; le prime, che sono espresse da un prodotto semplicemente infinito, le chiameremo di prima classe, le seconde, che sono espresse da un prodotto doppiamente infinito, le diremo di seconda classe. Le funzioni di prima classe si dividono anch'esse in due specie, la prima, che comprende quelle che hanno gl'indici delle radici disposti simmetricamente rispetto a un punto, e che possono esprimersi per un prodotto infinito di fattori di primo grado, le altre, che hanno gl'indici delle radici disposti comunque sopra la retta, le quali si decomporranno in fattori di primo grado ed esponenziali. Ogni funzione intera di prima classe della prima specie potrà decomporci nel prodotto di più funzioni intere della stessa classe di seconda specie, e data una funzione della seconda specie se ne potrà sempre trovare un'altra che moltiplicata per la medesima dia per prodotto una funzione della prima specie. Le funzioni di seconda classe si dividono anch'esse in due specie; la prima comprenderà quelle che hanno gl'indici delle radici disposti egualmente nei quattro angoli di due assi ortogonali, in modo che facendo una rotazione intorno all'origine di un quarto di circolo, gl'indici

di tutte le radici vengano a sovrapporsi, le quali funzioni possono esprimersi per un prodotto doppiamente infinito di fattori di primo grado; la seconda comprenderà quelle che hanno gl'indici disposti comunque, e si decompongono in un prodotto doppiamente infinito di fattori di primo grado e di fattori esponenziali. Data una funzione della seconda specie se ne potrà sempre trovare un'altra che moltiplicata per quella dia una funzione della prima specie».

[«From these theorems, we deduce that entire functions might be decomposed into an infinite number of first degree factors and exponential factors, so that here there is a first classification of entire functions according to that their root's indexes lie along a line or are arbitrarily placed in the plane; the former are said to be of first class and are expressed by a simply infinite product, while the latter are said to be of second class and are expressed by a doubly infinite product. The functions of the first class are, in turn, classified into two kinds: the first one comprises those functions having the root's indexes symmetrically placed respect to a point and that can be expressed by an infinite product of first degree factors; the second one comprises those functions having root's indexes arbitrarily placed along a line and that can be expressed by an infinite product both of first degree factors and of exponential factors. Each entire function of first class and of first kind might be decomposed into the product of other entire functions of the same class and of the second kind; furthermore, given a function of the second kind, it is always possible to find another function that multiplied by the former, the product gives rise to another function of the first kind. Likewise, the functions of the second class are divided into two kinds: the first one comprises those functions having the root's indexes equally placed into the four angles of the two orthogonal cartesian axes in such a manner that all these are overlapped through a $\pi/2$ radian rotation around the origin, and are decomposable into a doubly infinite product of first degree factors; the second one includes those functions having the root's indexes arbitrarily placed and that are decomposable into a doubly infinite product of first degree factors and exponential factors. Furthermore, given a function of the second kind, it is always possible to find another function that multiplied by the former, the product gives rise to a function of first kind»].

Then, Betti carries on treating entire functions in the first part of his lessons on elliptic functions, followed by a second part devoted to quotients of functions, mentioning either the paper (Weierstrass 1856a) and the paper (Weierstrass 1856b). Therefore, Betti's work on entire function factorization, made in the period 1860-63, was very forerunner of the Weierstrass' one: this is confirmed either by (Rouse Ball 1937, Appendix II, pp. 376-384)) and by (Federigo Enriques 1982, Book III, Chapter I, Section 6), in which it is pointed out that the fundamental Weierstrass' theorem on the factorization of entire transcendental functions from their zeros, had already been discovered by Betti, highlighting

however as the Pisa's mathematician, with singular personal disinterestedness, wanted not claim it as due to him. Indeed, following Francesco Cecioni's comments about some works of Ulisse Dini (see (Dini 1953-59, Volume II)), it turns out that Betti's work could easily reach, only with very slight modifications, the same generality and abstraction of the Weierstrass' one, as Dini explicitly proved in (Dini 1881); furthermore, Dini proved too that Betti's work could be able to give a particular case, given in the years 1876-77, of the general Gösta Mittag-Leffler theorem - see (Mittag-Leffler 1884), (Vivanti 1928, Section 145), (Loria 1950, Chapter XLIV, Section 752) and (Bottazzini & Gray 2013, Section 6.7.6) - independently by what Weierstrass himself was doing in the same period, in regards to these latter arguments. Cecioni says that this Dini's work had already been worked out since 1880, whilst the Weierstrass' theorem was published in 1876 - see (Weierstrass 1876). Thus, much before, namely in 1860, Betti had proved, as we have already said, a particular but important case of this theorem, albeit he didn't go beyond, because the results achieved by him were enough to his pragmatic scopes concerning Abelian and elliptic functions¹³, and, as also Pincherle (1922, Chapter IX, Section 135) has claimed, the Weierstrass' method was essentially the same of the Betti's one with slight modifications. In the years 1876-77, also G. Mittag-Leffler proved a particular case of a more general theorem that he will give later, to be precise in 1884, after a long series of previous works in which he gradually, through particular cases, reached the general form of this his theorem as nowadays we know it. In the meanwhile, Weierstrass reconsidered Mittag-Leffler's works, since the early 1880s, in relation to what himself have done on the same subject. Also F. Casorati (1880-82) had some interesting ideas similar to the Mittag-Leffler's ones, giving further contributions to the subject (see (Loria 1950, Chapter XLIV, Section 750)). Almost at the same time, amongst others, Ernst Schering (1881), Charles Hermite (1881), Émile Picard (1881), Felice Casorati (1882), Ulisse Dini (1881), Paolo Gazzaniga¹⁴ (1882) and Claude

¹³In this regards, also Salvatore Pincherle (1899, Chapter IX, Section 175) reports that Betti solved the Weierstrass' problem in a quite general case.

¹⁴Some historical sources refer of Paolo Cazzaniga, whereas others refer of Paolo Gazzaniga, but, very likely, they are the same person, that is to say, Paolo Gazzaniga (1853-1930), an Italian mathematician graduated from Pavia University in 1878 under the supervision of Felice Casorati. In the years 1878-1883, he was interim assistant professor at Pavia, then he spent a period of study in Germany under the Weierstrass and Kronecker supervision. Afterwards, from 1888, he became professor at the high school Tito Livio in Padua, teaching too in the local University. He was also one of the most influential teachers of Tullio Levi-Civita during his high school studied. Gazzaniga's researches mainly concerned with applied algebra and number theory. Furthermore, Paolo Gazzaniga has to be distinguished from Tito Camillo Cazzaniga (1872-1900), a prematurely died Italian mathematician, graduated from Pavia University in 1896, whose researches concerned with matrix theory and analytic functions according to the research trend of Ernesto Pascal (1865-1940) during his teaching in Pavia. Both Tito Cazzaniga (see (Rouse Ball 1937, Appendix II, pp. 412-413))

Guichard (1884), achieved notable results about the general problem to build up a complex function with prescribed singularities, although related to a generality degree less than that of the Mittag-Leffler results. Thus, the history of the Mittag-Leffler theorem makes too its awesome appearance within the general history of meromorphic functions, a part of which may be retraced in the same Mittag-Leffler 1884 paper in which, amongst other things, also the 1881 work of Ulisse Dini is quoted. However, both Schering (1881, Section XVI) and Casorati (1880-82, p. 269, footnote (**)), in discussing the above mentioned Mittag-Leffler results, quote Betti's work on Weierstrass' theorem; in particular, the former speaks of Betti's convergence factors and the latter states that

«Anche il sig. Dini, nella sua Nota sopra citata, dimostra questo teorema, riducendo lo studio del prodotto infinito a quello della serie dei logaritmi dei fattori; riduzione di cui s'era già valso felicemente, per il caso di distribuzione degli zeri a distanze non mai minori di una quantità fissa d , il sig. Betti nella Introduzione della sua Monografia delle funzioni ellittiche (Annali di Matematica, Tomo III, Roma, 1860), dove precede assai più oltre di Gauss nella via che mena al teorema del sig. Weierstrass».

[«Also Mr. Dini, in his Note of above, proves this theorem, reducing the study of the infinite product to the study of the series of the logarithms of the factors; reduction, this, that had already been used by Mr. Betti in the Introduction to his monograph on elliptic functions (Annali di Matematica, Tome III, Rome, 1860) for the case of a distribution of zeros having reciprocal distances not less than a fixed quantity d ; in doing so, he much foregoes Gauss in a fashion which leads to the theorem of Mr. Weierstrass».]

Therefore, from the Mittag-Leffler's works onward, together to all those works made by other mathematicians amongst whom are Dini, Schering, Casorati, Hermite, Picard, Cazzaniga, Guichard and Weierstrass himself, it starts the theory of entire transcendental functions whose early historical lines have been traced in the previous sections. In any case, with Mittag-Leffler, we have the most general theorems for the construction, by infinite products, of a meromorphic function with prescribed singularities. To be precise, following (Gonchar et al. 1997, Part I, Introduction) and (Vivanti 1901, Section 215), the works by Weierstrass, Mittag-Leffler and Picard, dating back to the 1870s, marked the beginning of the systematic studies of the theory of entire and meromorphic functions. The Weierstrass and Mittag-Leffler theorems gave a general description of the structure of entire and meromorphic functions, while the representation of entire functions as an infinite product *à la* Weierstrass, served as basis for studying properties of entire and meromorphic functions. The Picard the-

and Paolo Gazzaniga are quoted in (Vivanti 1901) but not in (Vivanti 1928).

orem opened the theory of value distribution of meromorphic functions, while the J.L.W.V. Jensen works, at the late 1890s, were of a great importance for further developments of the theory of entire and meromorphic functions which started, in the same period, to gradually become a separate and autonomous mathematical discipline after the pioneering works of Laguerre, Hadamard, Poincaré, Lindelöf and Borel, up until the Rolf Nevanlinna work of the early 1900s. All that will be in-depth studied in the next section, where we shall deal with the main lines of the history of entire and meromorphic functions whose theory basically starts just from the entire function factorization theorems.

En passant, we also note that the Weierstrass' entire function factorization theorem has had further remarkable applications in many other pure and applied mathematical contexts. In this place, we wish to point out another possible interesting historical connection. To be precise, following (Markušević 1967, Volume II, Chapters 8 and 9), (Burckel 1979, Chapter VII) and (Remmert 1998, Chapter 4), a very similar problem to that considered by Weierstrass was the one considered in (Markušević 1967, Volume II, Chapter 8, Theorem 8.5) where, roughly, a bounded analytic function with prescribed zeros is constructed by means of certain infinite products introduced by Wilhelm Blaschke (1915) in relation to questions related to the Giuseppe Vitali convergence theorem for sequences of holomorphic functions, and defined upon those complex numbers assigned as given zeros of that function that has to be determined. We shall return later on such aspects concerning *Blaschke products*, which form a special class of Weierstrass' products, when we will go on in deepening the history of entire function theory; due to this, to our historical purposes, we would want to try to analyze whether the previous Weierstrass' work on entire function factorization theorems have played a certain role in the Blaschke's work.

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