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Littlewood-Paley functionals on graphs

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Abstract

Let Γ be a graph equipped with a Markov operator P . We introduce discrete fractional Littlewood-Paley square functionals and prove their L^p -boundedness under various geometric assumptions on the graph Γ .

Contents

1	Introduction	2
1.1	Presentation of the discrete framework	3
1.1.1	General setting	3
1.1.2	Geometric assumptions and estimates for the Markov operator	4
1.2	Main results	5
2	Preliminary results	6
2.1	Estimates on the kernels	6
2.2	Results on the Hardy-Littlewood maximal function	7
2.3	L^p boundedness for Calderón-Zygmund operators	8
3	Littlewood-Paley functionals	9
3.1	$L^2(\Gamma)$ -boundedness of g_β^2	9
3.2	$L^p(\Gamma)$ -boundedness of g_β , $2 < p < +\infty$	10
3.3	L^p -boundedness of \tilde{g}_β , $2 \leq p < p_0$	13
3.4	L^p -boundedness of g_β and \tilde{g}_β , $1 < p \leq 2$	16
3.5	Reverse L^p inequalities for g_β and \tilde{g}_β	17
4	L^p-boundedness of \tilde{g}_0, $1 < p < 2$	18
4.1	Proof of Theorem 4.2	18
4.2	Proof of Theorem 1.14	19
4.3	Proof of Corollary 4.3	21
A	Further estimates for Markov chains	22
A.1	Time regularity estimates	22
A.2	Gaffney-type inequalities	24
B	Estimates for the Taylor coefficients of $(1 - z)^{-\beta}$	27
C	Reverse Hölder estimates for sequences	28

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Throughout the paper, we use the following notations. If E is a nonempty set and A and B are some quantities depending on $x \in E$, the notation $A(x) \lesssim B(x)$ means that there exists C such that $A(x) \leq C B(x)$ for all $x \in E$, while $A(x) \simeq B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$. If E and F are Banach spaces and $T : E \rightarrow F$ is a bounded linear operator, $\|T\|_{E \rightarrow F}$ stands for the operator norm of T . When $E = L^p$ and $F = L^q$ for $1 \leq p, q \leq +\infty$, $\|T\|_{L^p \rightarrow L^q}$ will also be denoted by $\|T\|_{p,q}$.

1 Introduction

This paper is devoted to the L^p -boundedness of Littlewood-Paley type square functionals on graphs. The prototype of these functionals is the g -function in the Euclidean space, defined in the following way. If f is, say, in $\mathcal{D}(\mathbb{R}^n)$ and $u(x, t)$ denotes “the” harmonic extension of f , that is $u(x, t) = P_t * f(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$, where P_t stands for the Poisson kernel, define

$$g_1 f(x) := \left(\int_0^{+\infty} \left(\left| \frac{\partial u}{\partial t}(x, t) \right|^2 + \sum_{1 \leq i \leq n} \left| \frac{\partial u}{\partial x_i}(x, t) \right|^2 \right) \frac{dt}{t} \right)^{1/2}.$$

It is a well-known fact ([20, Chapter 4, Theorem 1]) that, for all $p \in (1, +\infty)$,

$$\|g_1 f\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.1)$$

This result was extended in various directions, and we only recall some of them. In the Euclidean framework, the harmonic extension can be replaced by e^{-tL} , where L is a second order uniformly elliptic operator in divergence form. In this case, the range of p in (1.1) is related to the L^p boundedness of e^{-tL} or $t\nabla e^{-tL}$ (see [2, Chapter 7]).

If, in the functional g , one is only interested in the “horizontal” part, *i.e.* the derivative with respect to t , then the L^p boundedness of the corresponding Littlewood-Paley functional holds in the much more general context of measured spaces endowed with appropriate Markov semigroups ([21, Corollaries 1 and 2]). Notice also that similar results can be proved when the derivative $\frac{\partial}{\partial t}$ is replaced by a “fractional” derivative ([7]).

Littlewood-Paley functionals were also considered in the context of complete Riemannian manifolds. Let M be a complete Riemannian manifold, ∇ be the Riemannian gradient and Δ the Laplace-Beltrami operator. Consider the “vertical” functionals

$$Gf(x) := \left(\int_0^{+\infty} \left| t \nabla e^{-t\sqrt{\Delta}} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}$$

and

$$Hf(x) := \left(\int_0^{+\infty} \left| \sqrt{t} \nabla e^{-t\Delta} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Several L^p -boundedness results for G and H are known. Let us recall here that, when $1 < p \leq 2$, G and H are $L^p(M)$ -bounded when M is an *arbitrary* complete Riemannian manifold ([9, Theorem 1.2]), while the $L^p(M)$ -boundedness of G and H for $p > 2$ holds under much stronger assumptions, expressed in terms of the domination of the gradient of the semigroup by the semigroup applied to the gradient ([8, Proposition 3.1]).

Littlewood-Paley functionals on graphs were also considered. In [15], if Δ is a Laplace operator on a graph Γ , a “vertical” Littlewood-Paley functional, involving the (continuous-time) semigroup generated by Δ , is proved to be $L^p(\Gamma)$ -bounded for all $1 < p \leq 2$ under very weak assumptions on Γ . In [5], “discrete time” Littlewood-Paley functionals are proved to be $L^p(\Gamma)$ -bounded under geometric assumptions on Γ (about the volume growth of balls, or L^2 Poincaré inequalities), while similar results are obtained for weighted L^p -norms in [4]. Note also that the L^p -boundedness of discrete time Littlewood-Paley functionals in abstract settings was recently established in [1].

The present paper is devoted to the proof of the L^p -boundedness on graphs of some discrete time fractional Littlewood-Paley horizontal or vertical functionals. Before stating our results, let us present the graphs under consideration.

1.1 Presentation of the discrete framework

1.1.1 General setting

Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geq 0$ a symmetric weight on $\Gamma \times \Gamma$. The couple (Γ, μ) induces a (weighted unoriented) graph structure if we define the set of edges by

$$E = \{(x, y) \in \Gamma \times \Gamma, \mu_{xy} > 0\}.$$

We call then x and y neighbors (or $x \sim y$) if $(x, y) \in E$.

We will assume that the graph is connected and locally uniformly finite. A graph is connected if for all $x, y \in \Gamma$, there exists a path $x = x_0, x_1, \dots, x_N = y$ such that for all $1 \leq i \leq N$, $x_{i-1} \sim x_i$ (the length of such path is then N). A graph is said to be locally uniformly finite if there exists $M_0 \in \mathbb{N}$ such that for all $x \in \Gamma$, $\#\{y \in \Gamma, y \sim x\} \leq M_0$ (i.e. the number of neighbors of a vertex is uniformly bounded).

The graph is endowed with its natural metric d , which is the shortest length of a path joining two points.

For all $x \in \Gamma$ and all $r > 0$, the ball of center x and radius r is defined as $B(x, r) = \{y \in \Gamma, d(x, y) < r\}$.

In the opposite way, the radius of a ball B is the only integer r such that $B = B(x_B, r)$ (with x_B the center of B). Therefore, for all balls $B = B(x, r)$ and all $\lambda > 0$, we set $\lambda B := B(x, \lambda r)$ and define $C_j(B) = 2^{j+1}B \setminus 2^j B$ for all $j \geq 2$ and $C_1(B) = 4B$.

We define the weight $m(x)$ of a vertex $x \in \Gamma$ by $m(x) = \sum_{x \sim y} \mu_{xy}$. More generally, the volume of a subset $E \subset \Gamma$ is defined as $m(E) := \sum_{x \in E} m(x)$. We use the notation $V(x, r)$ for the volume of the ball $B(x, r)$, and in the same way, $V(B)$ represents the volume of a ball B .

We define now the $L^p(\Gamma)$ spaces. For all $1 \leq p < +\infty$, we say that a function f on Γ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p m(x) \right)^{\frac{1}{p}} < +\infty,$$

while $L^\infty(\Gamma)$ is the set of functions satisfying

$$\|f\|_\infty := \sup_{x \in \Gamma} |f(x)| < +\infty.$$

Let us define for all $x, y \in \Gamma$ the discrete-time reversible Markov kernel p associated to the measure m by $p(x, y) = \frac{\mu_{xy}}{m(x)m(y)}$. The discrete kernel $p_l(x, y)$ is then defined recursively for all $l \geq 0$ by

$$\begin{cases} p_0(x, y) = \frac{\delta(x, y)}{m(y)} \\ p_{l+1}(x, y) = \sum_{z \in \Gamma} p(x, z) p_l(z, y) m(z). \end{cases} \quad (1.2)$$

Remark 1.1. Note that this definition of p_l differs from the one of p_l in [18], [5] or [13], because of the $m(y)$ factor. However, p_l coincides with K_l in [14]. Remark that in the case of the Cayley graphs of finitely generated discrete groups, where $m(x) = 1$ for all x , the definitions coincide.

Notice that for all $l \geq 1$, we have

$$\|p_l(x, \cdot)\|_{L^1(\Gamma)} = \sum_{y \in \Gamma} p_l(x, y) m(y) = \sum_{d(x, y) \leq l} p_l(x, y) m(y) = 1 \quad \forall x \in \Gamma, \quad (1.3)$$

and that the kernel is symmetric:

$$p_l(x, y) = p_l(y, x) \quad \forall x, y \in \Gamma. \quad (1.4)$$

For all functions f on Γ , we define P as the operator with kernel p , i.e.

$$Pf(x) = \sum_{y \in \Gamma} p(x, y) f(y) m(y) \quad \forall x \in \Gamma. \quad (1.5)$$

It is easily checked that P^l is the operator with kernel p_l .

Remark 1.2. Even if the definition of p_l is different from [18] or [5], P^l is the same operator in both cases.

Since $p(x, y) \geq 0$ and (1.3) holds, one has, for all $p \in [1, +\infty]$,

$$\|P\|_{p \rightarrow p} \leq 1. \quad (1.6)$$

Remark 1.3. Let $1 < p < +\infty$. Since, for all $l \geq 0$, $\|P^l\|_{p \rightarrow p} \leq 1$, the operators $(I - P)^\beta$ and $(I + P)^\beta$ are L^p -bounded for all $\beta > 0$ (see [12], p. 423).

We define a nonnegative Laplacian on Γ by $\Delta = I - P$. One has then

$$\begin{aligned} \langle (I - P)f, f \rangle_{L^2(\Gamma)} &= \sum_{x, y \in \Gamma} p(x, y)(f(x) - f(y))f(x)m(x)m(y) \\ &= \frac{1}{2} \sum_{x, y \in \Gamma} p(x, y)|f(x) - f(y)|^2 m(x)m(y), \end{aligned} \quad (1.7)$$

where we use (1.3) for the first equality and (1.4) for the second one. The last calculus proves that the following operator

$$\nabla f(x) = \left(\frac{1}{2} \sum_{y \in \Gamma} p(x, y)|f(y) - f(x)|^2 m(y) \right)^{\frac{1}{2}},$$

called “length of the gradient” (and the definition of which is taken from [10]), satisfies

$$\langle (I - P)f, f \rangle_{L^2(\Gamma)} = \|\nabla f\|_{L^2(\Gamma)}^2. \quad (1.8)$$

1.1.2 Geometric assumptions and estimates for the Markov operator

Under suitable geometric assumptions on Γ , the iterates of P satisfy various $L^p - L^q$ estimates, which we now review.

Our first assumption is:

Definition 1.4. A graph (Γ, μ) satisfies (LB) if there exists $\epsilon > 0$ such that

$$\mu_{xx} \geq \epsilon m(x) \quad \forall x \in \Gamma. \quad (LB)$$

Remark 1.5. Let us state a stronger assumption than (LB): there exists $\epsilon > 0$ such that for all $x \in \Gamma$, $x \sim x$ and

$$\mu_{xy} \geq \epsilon m(x) \quad \forall x \sim y. \quad (LB_2)$$

Even if (LB₂) plays a crucial role in some parabolic regularity estimates on graphs ([13]), it will play no role in our results.

The second assumption is the following one:

Definition 1.6 (Doubling property). The weighted graph (Γ, μ) satisfies the doubling property if there exists $C > 0$ such that

$$V(x, 2r) \leq CV(x, r) \quad \forall x \in \Gamma, \forall r > 0. \quad (DV)$$

Recall that, under the assumption (DV), there exists $d > 0$ such that

$$V(\theta x, r) \lesssim \theta^d V(x, r) \quad \forall r > 0, x \in \Gamma, \theta \geq 1. \quad (1.9)$$

In the sequel, a local version of (DV) will also be needed:

Definition 1.7. Say that (Γ, μ) satisfies (LDV) if there exists $c > 0$ such that

$$V(x, 2) \leq cm(x) \quad \forall x \in \Gamma. \quad (LDV)$$

Let us also state the Poincaré inequalities needed in the sequel.

Definition 1.8 (Poincaré inequality on balls). *Let $s \in [1, +\infty)$. The weighted graph (Γ, μ) satisfies the Poincaré inequality (P_s) if there exists $C > 0$ such that, for all $x \in \Gamma$, all $r > 0$ and all functions on Γ*

$$\frac{1}{V(x, r)} \sum_{y \in B(x, r)} |f(y) - f_B|^s m(y) \leq C \frac{r^s}{V(x, 2r)} \sum_{y \in B(x, 2r)} |\nabla f(y)|^s m(y), \quad (P_s)$$

where

$$f_B = \frac{1}{V(B)} \sum_{x \in B} f(x) m(x). \quad (1.10)$$

Remark 1.9. *It is a known fact that (P_{s_1}) implies (P_{s_2}) if $s_1 \leq s_2$ (cf [17]).*

Let us now introduce some estimates on p_l , which will be needed in the statement of our results.

Definition 1.10 (On diagonal upper estimate of p_l). *We say that (Γ, μ) satisfies (DUE) if there exists $C > 0$ such that, for all $x \in \Gamma$ and all $l \in \mathbb{N}^*$,*

$$p_l(x, x) \leq \frac{C}{V(x, \sqrt{l})}. \quad (DUE)$$

Definition 1.11. *Let $p \in [1, +\infty]$. Say that a weighted graph (Γ, μ) verifies (GG_p) if*

$$\|\nabla P^l f\|_{L^p} \leq \frac{C_p}{\sqrt{l}} \|f\|_{L^p} \quad \forall l \in \mathbb{N}^*, \forall f \in L^p(\Gamma). \quad (GG_p)$$

Remark 1.12. *Note that the assumption (GG_∞) holds when Γ is the Cayley graph of a finitely generated discrete group (as well as assumption (P_1) , see [17]). Indeed, in this case,*

$$\nabla_x p_l(x, y) \lesssim \left(\frac{1}{lV(x, \sqrt{l})V(y, \sqrt{l})} \right)^{\frac{1}{2}} \exp\left(-c \frac{d^2(x, y)}{l}\right).$$

1.2 Main results

For all $\beta > 0$, all functions f on Γ and all $x \in \Gamma$, define

$$g_\beta f(x) = \left(\sum_{l \geq 1} l^{2\beta-1} |(I - P)^\beta P^{l-1} f(x)|^2 \right)^{\frac{1}{2}}.$$

For all $\beta > -\frac{1}{2}$, all functions f on Γ and all $x \in \Gamma$, define

$$\tilde{g}_\beta f(x) = \left(\sum_{l \geq 1} l^{2\beta} |\nabla (I - P)^\beta P^{l-1} f(x)|^2 \right)^{\frac{1}{2}}.$$

Here is our main result:

Theorem 1.13. *Let (Γ, μ) be a weighted graph satisfying (DV) , (LB) and (DUE) . Then*

1. g_β is of weak type $(1, 1)$, which means that there exists $C > 0$ such that, for all $\lambda > 0$,

$$m(\{x \in \Gamma; g_\beta f(x) > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(\Gamma)},$$

and of strong type (p, p) for all $1 < p < +\infty$, i.e. there exists a constant $C > 0$ such that

$$\|g_\beta f\|_{L^p} \leq C \|f\|_{L^p} \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma).$$

2. \tilde{g}_β is of weak type $(1, 1)$, and of strong type (p, p) for all $1 < p \leq 2$. Moreover, if (Γ, μ) satisfies (P_2) and (GG_q) for some $q > 2$, then \tilde{g}_β is of strong type (p, p) for $p \in (2, q)$.

3. For all $1 < p < +\infty$,

$$\|f\|_{L^p} \leq C \|g_\beta f\|_{L^p} \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma),$$

for all $2 \leq p < +\infty$

$$\|f\|_{L^p} \leq C \|\tilde{g}_\beta f\|_{L^p} \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma)$$

and if (P_2) and (GG_q) are true for some $q > 2$, then for all $q' < p < 2$ (with $\frac{1}{q'} + \frac{1}{q} = 1$),

$$\|f\|_{L^p} \leq C \|\tilde{g}_\beta f\|_{L^p} \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma).$$

Our second result deals with the L^p -boundedness of \tilde{g}_0 , under very weak assumptions on Γ :

Theorem 1.14. *Let (Γ, μ) be a graph satisfying (LB) and (LDV) . Then \tilde{g}_0 is L^p -bounded for all $p \in (1, 2]$.*

Remark 1.15. *The range $\beta > -\frac{1}{2}$ for the L^p -boundedness of \tilde{g}_β is related to the presence of ∇ in \tilde{g}_β .*

Remark 1.16. 1. *The L^p -boundedness of g_1 was proved in [5, Theorem 1.16]. Theorem 1.13 extends this fact to a fractional version of g_1 . Moreover, we prove a similar estimate for the vertical Littlewood-Paley functional \tilde{g}_β and also establish converse inequalities.*

2. *The L^p -boundedness of g_β can be deduced from arguments in [1]. Indeed, since g_1 is of strong type (p, p) for all $p \in (1, +\infty)$ by [5, Theorem 1.16], [1, Theorem 3.1] yields that P is an R -Ritt operator, and the fact that g_β is of strong type (p, p) for all $p \in (1, +\infty)$ follows from [1, Theorem 3.3]. However, these arguments do not yield the fact that g_β is of weak type $(1, 1)$. Moreover, they do not provide any information about \tilde{g}_β .*

Section 2 is devoted to the introduction of the tools used in the sequel. In particular, we state various off-diagonal estimates of the Markov kernel, which are proven in the Appendix A. Theorems 1.13 and 1.14 are respectively proven in Section 3 and 4.

Acknowledgements: the author would like to thank C. Le Merdy for pointing out reference [1] to him.

2 Preliminary results

2.1 Estimates on the kernels

In this paragraph, we gather various estimates on p_l which will be instrumental in our proofs. The conjunction of (LB) , (DV) and (DUE) provide us with further estimates on p_l . First, one has ([11, Theorem 5.2, Theorem 6.1]):

Proposition 2.1. *Let (Γ, μ) be a weighted graph satisfying (DV) and (LB) . Then, assumption (DUE) is equivalent to the off-diagonal upper estimate:*

$$p_l(x, y) \leq C \left(\frac{1}{V(x, \sqrt{l})V(y, \sqrt{l})} \right)^{\frac{1}{2}} \exp \left(-c \frac{d^2(x, y)}{l} \right) \quad \forall x, y \in \Gamma, \forall l \in \mathbb{N}^*. \quad (UE)$$

Remark 2.2. *An immediate consequence of (DV) is that, for all $x, y \in \Gamma$ and $l \in \mathbb{N}^*$,*

$$p_{l-1}(x, y) \leq C \left(\frac{1}{V(x, \sqrt{l})V(y, \sqrt{l})} \right)^{\frac{1}{2}} \exp \left(-c \frac{d^2(x, y)}{l} \right).$$

Remark 2.3. *Assume that Γ is a graph satisfying (DV) . It is easily checked that assumption (UE) is equivalent to*

$$p_l(x, y) \leq \frac{C}{V(y, \sqrt{l})} \exp \left(-c \frac{d^2(x, y)}{l} \right) \quad (2.1)$$

or

$$p_l(x, y) \leq \frac{C}{V(x, \sqrt{l})} \exp \left(-c \frac{d^2(x, y)}{l} \right). \quad (2.2)$$

We will now state some “time regularity” estimates for higher order differences of p_l (first proved for first order differences by Christ ([6]) but an easier proof was given by Dungey in [14]).

Theorem 2.4. *Let (Γ, μ) be a weighted graph. Assume that Γ satisfies (DV), (LB) and (DUE). We define $D(r)$ as the following operator which acts on sequences*

$$(D(r)u)_l = u_l - u_{l+r}.$$

Then, for all $j \geq 0$ there exist two constants $C_j, c_j > 0$ such that, for all $l \geq 1$ and all $x, y \in \Gamma$,

$$|(D(1)^j p)_l(x, y)| \leq \frac{C_j}{l^j V(x, \sqrt{l})} \exp\left(-c_j \frac{d^2(x, y)}{l}\right). \quad (TD - UE)$$

Theorem 2.4 (actually a slightly more general version) will be established in Section A.1 in the appendix. From the previous estimates, we derive the following result, the proof of which will be given in Section A.2 in the appendix.

Theorem 2.5. *Let (Γ, μ) be a weighted graph satisfying (DV), (LB) and (DUE). The following Gaffney type inequalities hold: for all $j \in \mathbb{N}$, there exist $c, C > 0$ such that for all sets $E, F \subset \Gamma$, all $x_0 \in \Gamma$, all $l \in \mathbb{N}^*$ satisfying one of the following conditions*

$$(i) \sup \{d(x_0, y), y \in F\} \leq 3d(E, F),$$

$$(ii) \sup \{d(x_0, y), y \in F\} \leq \sqrt{l},$$

$$(iii) \sup \{d(x_0, x), x \in E\} \leq 3d(E, F),$$

$$(iv) \sup \{d(x_0, x), x \in E\} \leq \sqrt{l},$$

and all functions f supported in F , we have, for all $j \in \mathbb{N}$,

$$\|(I - P)^j P^l f\|_{L^2(E)} \leq \frac{C}{l^j} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F)} \quad (GT_2)$$

and

$$\begin{aligned} \|\nabla(I - P)^j P^l f\|_{L^2(E)} &\leq \frac{C}{l^{j+\frac{1}{2}}} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F)} \\ \|\nabla(I - P)^j P^l f\|_{L^2(E)} &\leq \frac{C}{l^{j+\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^2(F)}. \end{aligned} \quad (GGT_2)$$

Remark 2.6. *The theorem above will be used for*

$$(E, F) \in \{(B, C_j(B)), B \text{ ball}, j \geq 2\} \cup \{(C_j(B), B), B \text{ ball}, j \geq 2\}.$$

2.2 Results on the Hardy-Littlewood maximal function

Definition 2.7. *Denote by \mathcal{M} the Hardy-Littlewood maximal operator*

$$\mathcal{M}f(x) = \sup \frac{1}{V(B)} \sum_{y \in B} |f(y)| m(y)$$

where the supremum is taken over the balls B of Γ containing x .

In the same way, for $s \geq 1$, \mathcal{M}_s will denote

$$\mathcal{M}_s f = (\mathcal{M}|f|^s)^{\frac{1}{s}}.$$

The following observation will turn to be useful: under the assumption (UE), for all $k \geq 1$, all functions f on Γ and all $x_0, x \in \Gamma$ with $d(x, x_0) \leq \sqrt{k}$,

$$|P^k f(x)| \leq \mathcal{M}f(x_0). \quad (2.3)$$

Indeed,

$$\begin{aligned}
|P^k f(x)| &= \left| \sum_{y \in \Gamma} p_k(x, y) f(y) m(y) \right| \\
&\lesssim \sum_{y \in \Gamma} \frac{1}{V(x, \sqrt{k})} \exp\left(-c \frac{d(x, y)^2}{k}\right) |f(y)| m(y) \\
&\lesssim \frac{1}{V(x, \sqrt{k})} \sum_{d(x, y) < \sqrt{k}} |f(y)| m(y) + \sum_{j \geq 0} \frac{e^{-c2^{2j}}}{V(x, \sqrt{k})} \sum_{2^j \sqrt{k} \leq d(x, y) < 2^{j+1} \sqrt{k}} |f(y)| m(y) \\
&\lesssim \frac{1}{V(x, \sqrt{k})} \sum_{d(x, y) < \sqrt{k}} |f(y)| m(y) \\
&\quad \sum_{j \geq 0} \frac{2^{(j+1)d} e^{-c2^{2j}}}{V(x, 2^{j+1} \sqrt{k})} \sum_{2^j \sqrt{k} \leq d(x, y) < 2^{j+1} \sqrt{k}} |f(y)| m(y) \\
&\leq \left(1 + \sum_{j \geq 0} 2^{(j+1)d} e^{-c2^{2j}} \right) \mathcal{M}f(x_0) \\
&\lesssim \mathcal{M}f(x_0),
\end{aligned}$$

where we use for the fifth line the doubling property and the fact that $d^2(x, x_0) \leq k$.

Proposition 2.8. *Let (Γ, μ) be a weighted graph satisfying (DV). If $(q, q_0, \beta) \in (1, +\infty]^2 \times [0, 1)$ satisfy $\frac{1}{q} = \frac{1}{q_0} - \beta$, then \mathcal{M}^β is bounded from $L^{q_0}(\Gamma)$ to $L^q(\Gamma)$.*

We also recall the Fefferman-Stein inequality.

Theorem 2.9. *Let (Γ, μ) be a weighted graph satisfying (DV) and $s \geq 1$. Then, if $p, q \in (s, +\infty)$, there exists $C_{p,q} > 0$ such that for all sequences $(f_n)_{n \in \mathbb{N}}$ of measurable functions defined on Γ ,*

$$\left\| \left[\sum_{n=0}^{+\infty} (\mathcal{M}_s f_n)^q \right]^{\frac{1}{q}} \right\|_p \leq C_{p,q} \left\| \left[\sum_{n=0}^{+\infty} |f_n|^q \right]^{\frac{1}{q}} \right\|_p.$$

This result is proven in \mathbb{R}^d in [16] and the proof easily extends to spaces of homogeneous type.

2.3 L^p boundedness for Calderón-Zygmund operators

We will make use of the following theorems about Calderón-Zygmund operators “without kernels”, which can be found in [5], Theorem 1.14 and Theorem 1.17. See also [2], Theorem 1.1 and 1.2.

Before stating these results, recall (see Theorem 1.13) that a sublinear operator T is of weak type (p, p) ($1 \leq p < +\infty$) if there exists $C > 0$ such that, for all $\lambda > 0$ and all $f \in L^p(\Gamma)$,

$$m(\{x \in \Gamma; |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda^p} \|f\|_{L^p(\Gamma)}^p.$$

Furthermore, T is said to be of strong type (p, p) if there exists $C > 0$ such that, for all $f \in L^p(\Gamma)$,

$$\|Tf\|_{L^p(\Gamma)} \leq C \|f\|_{L^p(\Gamma)}.$$

Theorem 2.10. *Let $p_0 \in (2, +\infty]$. Assume that Γ satisfies the doubling property (DV) and let T be a sublinear operator of strong type $(2, 2)$ defined on Γ . For all balls B , let A_B be a linear operator acting on $L^2(\Gamma)$. Assume that there exists a constant $C > 0$ such that, for all $f \in L^2(\Gamma)$, all $x \in \Gamma$ and all balls $B \ni x$,*

$$\frac{1}{V(B)^{\frac{1}{2}}} \|T(I - A_B)f\|_{L^2(B)} \leq C \mathcal{M}_2 f(x) \tag{2.4}$$

and

$$\frac{1}{V(B)^{\frac{1}{p_0}}} \|T A_B f\|_{L^{p_0}(B)} \leq C \mathcal{M}_2 |Tf|(x). \tag{2.5}$$

Then, for all $p \in (2, p_0)$, T is of strong type (p, p) .

Theorem 2.11. *Let $p_0 \in [1, 2)$. Assume that Γ satisfies the doubling property (DV) and let T be a sublinear operator of strong type $(2, 2)$. For all balls B , let A_B be a linear operator acting on $L^2(\Gamma)$. Assume that, for all $j \geq 1$, there exists $\varphi(j) > 0$ such that, for all $B \subset \Gamma$ and all functions supported in B and all $j \geq 2$,*

$$\frac{1}{V(2^{j+1}B)^{\frac{1}{2}}} \|T(I - A_B)f\|_{L^2(C_j(B))} \leq \varphi(j) \frac{1}{V(B)^{\frac{1}{p_0}}} \|f\|_{L^{p_0}} \quad (2.6)$$

and for all $j \geq 1$

$$\frac{1}{V(2^{j+1}B)^{\frac{1}{p_0}}} \|A_B f\|_{L^2(C_j(B))} \leq \varphi(j) \frac{1}{V(B)^{\frac{1}{p_0}}} \|f\|_{L^{p_0}}. \quad (2.7)$$

If $\sum_{j \geq 1} \varphi(j) 2^{jd} < +\infty$, where d is given by Proposition 1.9, then T is of weak type (p_0, p_0) , and therefore of strong type (p, p) for all $p_0 < p < 2$.

3 Littlewood-Paley functionals

3.1 $L^2(\Gamma)$ -boundedness of g_β^2

In order to prove Theorem 1.13, let us introduce an extra functional.

Lemma 3.1. *Let (Γ, μ) be a weighted graph. Let P be the operator defined by (1.5). Define, for all $\beta > 0$ and all functions $f \in L^2(\Gamma)$, $g_\beta^2 f$ by*

$$g_\beta^2 f(x) = \left(\sum_{l \geq 1} b_l |(I - P^2)^\beta P^{l-1} f(x)|^2 \right)^{\frac{1}{2}}$$

where $\sum_{l \geq 1} b_l z^{l-1}$ is the Taylor series of the function $z \mapsto (1 - z)^{-2\beta}$. Then g_β^2 is $L^2(\Gamma)$ bounded. More precisely, g_β^2 is an isometry on $L^2(\Gamma)$, which means that, for all $f \in L^2(\Gamma)$,

$$\|g_\beta^2 f\|_{L^2(\Gamma)} = \|f\|_{L^2(\Gamma)}.$$

Proof. Since $\|P\|_2 \leq 1$, by spectral theory, P can be written as

$$P = \int_{-1}^1 \lambda dE(\lambda).$$

It follows that for all $l \geq 1$, one has

$$(I - P^2)^\beta P^{l-1} = \int_{-1}^1 (1 - \lambda^2)^\beta \lambda^{l-1} dE(\lambda)$$

so that, for all $f \in L^2(\Gamma)$ and $l \geq 1$,

$$\|(I - P^2)^\beta P^{l-1} f\|_{L^2}^2 = \int_{-1}^1 (1 - \lambda^2)^{2\beta} \lambda^{2(l-1)} dE_{f,f}(\lambda).$$

The L^2 -norm of $g_\beta^2 f$ can be now computed as

$$\begin{aligned} \|g_\beta^2 f\|_{L^2}^2 &= \sum_{l \geq 1} b_l \|(I - P^2)^\beta P^{l-1} f\|_{L^2}^2 \\ &= \int_{-1}^1 (1 - \lambda^2)^{2\beta} \sum_{l \geq 1} b_l \lambda^{2(l-1)} dE_{f,f}(\lambda) \\ &= \int_{-1}^1 dE_{f,f}(\lambda) \\ &= \|f\|_{L^2}^2 \end{aligned}$$

where the third line is a consequence of the definition of b_l . □

Lemma 3.2. *Let (Γ, μ) be a weighted graph satisfying (LB). Then g_β and \tilde{g}_β are $L^2(\Gamma)$ -bounded.*

Proof. Since Γ satisfies (LB), -1 is not in the L^2 spectrum of P (see for instance Lemma 1.3 in [14]). Therefore there exists $a > -1$ such that

$$P = \int_a^1 \lambda dE(\lambda).$$

Proceeding as in the proof of the Lemma 3.1, we obtain

$$\begin{aligned} \|g_\beta f\|_{L^2}^2 &= \int_a^1 (1-\lambda)^{2\beta} \sum_{l \geq 1} l^{2\beta-1} \lambda^{2(l-1)} dE_{f,f}(\lambda) \\ &\lesssim \int_a^1 (1-\lambda)^{2\beta} \frac{1}{(1-\lambda^2)^{2\beta}} dE_{f,f}(\lambda) \\ &= \int_a^1 \frac{1}{(1+\lambda)^{2\beta}} dE_{f,f}(\lambda) \\ &\lesssim \|f\|_{L^2}^2 \end{aligned}$$

where, for the second line, we use Lemma B.1.

For \tilde{g}_β , just notice that, by definition of ∇ ,

$$\|\tilde{g}_\beta f\|_{L^2} = \|g_{\beta+\frac{1}{2}} f\|_{L^2}.$$

□

3.2 $L^p(\Gamma)$ -boundedness of g_β , $2 < p < +\infty$

The proof of the L^p -boundedness of g_β for $p > 2$ is based on the following Lemma and Theorem 2.10. The idea of the proof comes from Theorem 1.16 in [5].

Lemma 3.3. *Let (Γ, μ) be a weighted graph satisfying (DV), (LB) and (DUE).*

For all $n \in \mathbb{N}^$, there exists a constant $C_n > 0$ such that, for all balls $B = B(x_0, r)$ of Γ , all $j \geq 2$ and all f supported in $C_j(B)$, one has*

$$\left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} \leq C_n 2^{j(\frac{\beta}{2} - 2n)} \left(\frac{V(B)}{V(2^j B)} \right)^{\frac{1}{2}} \|f\|_{L^2}$$

Proof. First fix $n \in \mathbb{N}^*$. Denote by η the only integer such that $\eta + 1 \geq \beta > \eta \geq 0$. We use the fact that

$$(I - P)^{\beta-1-\eta} = \sum_{k \geq 0} a_k P^k$$

where $\sum a_k z^k$ is the Taylor series of the function $(1-z)^{\beta-\eta-1}$. Note that the equality holds on $L^2(\Gamma)$ by spectral theory and (1.6). Moreover, notice that if β is an integer, then $a_k = \delta_0(k)$.

By the generalized Minkowski inequality, we get

$$\left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} \leq \sum_{k \geq 0} a_k \left(\sum_{l \geq 1} l^{2\beta-1} \left\| (I - P)^{1+\eta} P^{k+l-1} (I - P^{r^2})^n f \right\|_{L^2(B)}^2 \right)^{\frac{1}{2}}.$$

We divide the sequel of the proof in 3 steps.

1- Estimate of the inner term

Notice that $I - P^{r^2} = (I - P) \sum_{s=0}^{r^2-1} P^s$. Then, we get

$$\left\| (I - P)^{1+\eta} P^{k+l-1} (I - P^{r^2})^n f \right\|_{L^2(B)} \leq r^{2n} \sup_{s \in [0, nr^2]} \left\| (I - P)^{1+\eta+n} P^{k+l+s-1} f \right\|_{L^2(B)}$$

We now estimate the terms $\|(I - P)^{1+\eta} P^{k+l+s-1} f\|_{L^2(B)}$. For $0 \leq s \leq nr^2$, since f is supported in $C_j(B)$ and by Remark 2.2, one has,

$$\begin{aligned} & \|(I - P)^{1+n+\eta} P^{k+l-1+s} f\|_{L^2(B)} \\ & \lesssim \frac{1}{(l+k+s)^{1+\eta+n}} \exp\left(-c \frac{(2^j-1)^2 r^2}{l+k+s}\right) \|f\|_{L^2(C_j(B))} \\ & \lesssim \frac{1}{(l+k+s)^{1+\eta+n}} \exp\left(-c \frac{4^j r^2}{l+k+s}\right) \|f\|_{L^2(C_j(B))} \\ & \lesssim 2^{\frac{jd}{2}} \left(\frac{V(B)}{V(2^j B)}\right)^{\frac{1}{2}} \frac{1}{(l+k+s)^{1+\eta+n}} \exp\left(-c \frac{4^j r^2}{l+k+s}\right) \|f\|_{L^2} \end{aligned}$$

where the first line follows from (GT_2) and Cauchy-Schwarz and the third one from (DV) . Consequently, we obtain

$$\begin{aligned} & \|(I - P)^{1+\eta} P^{k+l-1} (I - P^{r^2})^n f\|_{L^2(B)} \\ & \lesssim r^{2n} 2^{\frac{jd}{2}} \sup_{s \in [0, nr^2]} \left[\frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{1+\eta+n}} \right] \left(\frac{V(B)}{V(2^j B)}\right)^{\frac{1}{2}} \|f\|_{L^2} \\ & \leq r^{2n} l^{-\eta} 2^{\frac{jd}{2}} \sup_{s \in [0, nr^2]} \left[\frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{1+n}} \right] \left(\frac{V(B)}{V(2^j B)}\right)^{\frac{1}{2}} \|f\|_{L^2}. \end{aligned} \tag{3.1}$$

2- Reverse Hölder estimates

According to Proposition C.2 below, the set of sequences $\{A_l^{k,r,j}, k \in \mathbb{N}, r \in \mathbb{N}^*, j \geq 2\}$, where

$$A_l^{k,r,j} = l^{\beta-\eta} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{1+n}} \right\},$$

is included in

$$E_M = \left\{ (a_l)_{l \geq 1}, \forall l \in \mathbb{N}^*, 0 \leq a_l \leq M \sum_{k \in \mathbb{N}^*} \frac{1}{k} a_k \right\}$$

for some $M > 0$. Therefore, Lemma C.1 below yields

$$\begin{aligned} V(B)^{-\frac{1}{2}} \left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} & \lesssim r^{2n} 2^{\frac{jd}{2}} V(2^j B)^{-\frac{1}{2}} \|f\|_{L^2} \sum_{k \geq 0} a_k \left(\sum_{l \geq 1} \frac{1}{l} (A_l^{k,r,j})^2 \right)^{\frac{1}{2}} \\ & \lesssim r^{2n} 2^{\frac{jd}{2}} V(2^j B)^{-\frac{1}{2}} \|f\|_{L^2} \sum_{k \geq 0} a_k \sum_{l \geq 1} \frac{1}{l} A_l^{k,r,j}. \end{aligned}$$

3- End of the calculus

Note, thanks to Lemma B.1, that, when β is not an integer,

$$\begin{aligned} \sum_{k=0}^{m-1} a_k (m-k)^{\beta-1-\eta} & \lesssim 1 + \sum_{k=1}^{m-1} k^{\eta-\beta} (m-k)^{\beta-\eta-1} \\ & = 1 + \frac{1}{m} \sum_{k=1}^{m-1} \left(\frac{k}{m}\right)^{\eta-\beta} \left(1 - \frac{k}{m}\right)^{\beta-1-\eta} \\ & \xrightarrow{m \rightarrow +\infty} 1 + \int_0^1 t^{\eta-\beta} (1-t)^{\beta-1-\eta} dt < +\infty. \end{aligned}$$

The integral converges since $\eta - \beta > -1$ and $\beta - 1 - \eta > -1$. It follows that

$$\sum_{k=0}^{m-1} a_k (m-k)^{\beta-\eta-1} \lesssim 1. \quad (3.2)$$

Since $a_k = \delta_0(k)$ and $\beta - 1 - \eta = 0$ when β is an integer, the result above holds for all $\beta > 0$.

Using the expression of $A_t^{k,r,j}$, we have

$$V(B)^{-\frac{1}{2}} \left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} \lesssim V(2^j B)^{-\frac{1}{2}} r^{2n} 2^{\frac{jd}{2}} \|f\|_{L^2} \sum_{m=1}^{+\infty} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{m+s}\right)}{(m+s)^{1+n}} \right\}$$

But, for some $c' \in (0, c)$,

$$\begin{aligned} & \sum_{m=1}^{+\infty} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{m+s}\right)}{(m+s)^{1+n}} \right\} \\ &= \frac{1}{(4^j r^2)^{1+n}} \sum_{m=1}^{+\infty} \sup_{s \in [0, nr^2]} \exp\left(-c \frac{4^j r^2}{m+s}\right) \left(\frac{4^j r^2}{m+s}\right)^{1+n} \\ &\lesssim \frac{1}{(4^j r^2)^{1+n}} \sum_{m=1}^{4^j r^2} \exp\left(-c' \frac{4^j r^2}{m+nr^2}\right) + \frac{1}{(4^j r^2)^{1+n}} \sum_{m=4^j r^2+1}^{+\infty} \left(\frac{4^j r^2}{m}\right)^{1+n} \\ &\lesssim 4^{-jn} r^{-2n}. \end{aligned}$$

The proof of Lemma 3.3 is now complete. \square

Proof. The proof of the L^p -boundedness of g_β for $p > 2$ is analogous to the one found in [5], Theorem 1.16, when $2 < p < +\infty$. Let us give the argument for the completeness. We are aiming to use Theorem 2.10. It is enough to verify the validity of the assumptions (2.4) and (2.5). We choose $A_B = I - (I - P^{r^2})^n$, where r is the radius of B and $n > \frac{d}{4}$.

Proof of (2.4)

We need to check that, for all $f \in L^2$, for all $x_0 \in \Gamma$ and all balls $B \ni x_0$, one has

$$\frac{1}{V(B)^{\frac{1}{2}}} \|g_\beta (I - P^{r^2})^n f\|_{L^2(B)} \lesssim (\mathcal{M}|f|^2)^{\frac{1}{2}}(x_0).$$

We can decompose

$$f = \sum_{j \geq 1} f \mathbf{1}_{C_j(B)} =: \sum_{j \geq 1} f_j.$$

First, since g_β and $I - A_B = (I - P^{r^2})^n$ are $L^2(\Gamma)$ -bounded and by the doubling property,

$$\frac{1}{V(B)^{\frac{1}{2}}} \|g_\beta (I - P^{r^2})^n f_1\|_{L^2(B)} \lesssim \frac{1}{V(B)^{\frac{1}{2}}} \|f\|_{L^2(4B)} \lesssim (\mathcal{M}|f|^2)^{\frac{1}{2}}(x_0).$$

For $j \geq 2$, Lemma 3.3 provides:

$$\begin{aligned} \frac{1}{V(B)^{\frac{1}{2}}} \|g_\beta (I - P^{r^2})^n f_j\|_{L^2(B)} &\lesssim 2^{j(\frac{d}{2}-2n)} \frac{1}{V(2^j B)^{\frac{1}{2}}} \|f_j\|_{L^2} \\ &\lesssim 2^{j(\frac{d}{2}-2n)} (\mathcal{M}|f|^2)^{\frac{1}{2}}(x_0). \end{aligned}$$

Since $n > \frac{d}{4}$, we can sum in $j \geq 1$, which gives the result.

Proof of (2.5)

What we have to show is that, for all $m \in \llbracket 1, n \rrbracket$, all $f \in L^2(\Gamma) \cap L^\infty(\Gamma)$, all $x_0 \in \Gamma$ and all balls $B \ni x_0$, one has,

$$\|g_\beta P^{mr^2} f\|_{L^\infty(B)} \lesssim (\mathcal{M}|g_\beta f|^2)^{\frac{1}{2}}(x_0).$$

First, since $\sum_{y \in G} p(x, y)m(y) = 1$, and by the use of Cauchy-Schwarz inequality, we obtain, for all $x \in \Gamma$ and $h \in L^2(\Gamma)$,

$$\left| P^{mr^2} h(x) \right| \leq \left(P^{mr^2} |h|^2(x) \right)^{\frac{1}{2}}.$$

Hence, it follows that for all $l \geq 1$

$$\left| P^{mr^2} (I - P)^\beta P^{l-1} f(x) \right|^2 \leq P^{mr^2} |(I - P)^\beta P^{l-1} f|^2(x),$$

so that, summing up in l ,

$$\begin{aligned} (g_\beta P^{mr^2} f)(x)^2 &= \sum_{l \geq 1} l^{2\beta-1} |P^{mr^2} (I - P)^\beta P^{l-1} f(x)|^2 \\ &\leq P^{mr^2} \left(\sum_{l \geq 1} l^{2\beta-1} |(I - P)^\beta P^{l-1} f|^2 \right) (x) \\ &= P^{mr^2} (|g_\beta f|^2) (x) \\ &\lesssim \mathcal{M} (|g_\beta f|^2) (x_0), \end{aligned}$$

where the last line is due to (2.3). Here ends the proof of (2.5), and the one of the L^p -boundedness of g_β for $p \in (2, +\infty)$. □

3.3 L^p -boundedness of \tilde{g}_β , $2 \leq p < p_0$

Lemma 3.4. *Let (Γ, μ) be a weighted graph satisfying (DV), (LB) and (DUE).*

For all $n \in \mathbb{N}^$, there exists a constant C_n such that, for all balls $B = B(x_0, r)$ of Γ , all $j \geq 2$ and all f supported in $C_j(B) = 2^{j+1}B \setminus 2^j B$, we get*

$$\left\| \tilde{g}_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} \leq C_n 2^{j(\frac{d}{2} - 2n)} \left(\frac{V(B)}{V(2^j B)} \right)^{\frac{1}{2}} \|f\|_{L^2}.$$

Proof. (Lemma 3.4)

The proof is analogous to the one of Lemma 3.3, and we only indicates the main differences.

Define η as in the proof of Lemma 3.3. By the use of the generalized Minkowski inequality, we get

$$\left\| \tilde{g}_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} \leq \sum_{k \geq 0} a_k \left(\sum_{l \geq 1} l^{2\beta} \left\| \nabla (I - P)^{1+\eta} P^{k+l-1} (I - P^{r^2})^n f \right\|_{L^2(B)}^2 \right)^{\frac{1}{2}}.$$

We now distinguish the cases $\beta > 0$ (i.e. $\eta \in \mathbb{N}$) and $-\frac{1}{2} < \beta \leq 0$ (i.e. $\eta = -1$).

First case: $\beta > 0$. In this case, the proof is analogous to the one in Lemma 3.3, using (GGT_2) instead of (GT_2) .

Second case: $-\frac{1}{2} < \beta \leq 0$.

1. By (GGT_2) ,

$$\left\| \nabla P^{k+l-1} (I - P^{r^2})^n f \right\|_{L^2(B)} \lesssim 2^{\frac{jd}{2}} r^{2n} \sup_{s \in [0, nr^2]} \left[\frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{n+\frac{1}{2}}} \right] \left(\frac{V(B)}{V(2^j B)} \right)^{\frac{1}{2}} \|f\|_{L^2(C_j(B))}.$$

2. Define now $B_l^{k,r,j}$ by

$$B_l^{k,r,j} = l^{\beta+\frac{1}{2}} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{n+\frac{1}{2}}} \right\}$$

Remark C.5 below therefore shows

$$\begin{aligned} V(B)^{-\frac{1}{2}} \left\| \tilde{g}_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} &\lesssim 2^{\frac{jd}{2}} V(2^j B)^{-\frac{1}{2}} \|f\|_{L^2} r^{2n} \sum_{k \geq 0} a_k \left(\sum_{l \geq 1} \frac{1}{l} (B_l^{k,r,j})^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{jd}{2}} V(2^j B)^{-\frac{1}{2}} \|f\|_{L^2} r^{2n} \sum_{k \geq 0} a_k \sum_{l \geq 1} \frac{1}{l} B_l^{k,r,j}. \end{aligned} \quad (3.3)$$

3. Thanks to Lemma B.1, one has

$$\begin{aligned} \sum_{k=0}^{m-1} a_k (m-k)^{\beta-\frac{1}{2}} &\lesssim m^{\beta-\frac{1}{2}} + \sum_{k=1}^{m-1} k^{-\beta-1} (m-k)^{\beta-\frac{1}{2}} \\ &\lesssim \frac{1}{\sqrt{m}} \int_0^1 t^{-\beta-1} (1-t)^{\beta-\frac{1}{2}} dt, \end{aligned}$$

if $\beta \in (-\frac{1}{2}, 0)$. If $\beta = 0$, we have $a_k = \delta_0(k)$, so that, in both cases,

$$\sum_{k=0}^{m-1} a_k (m-k)^{\beta-\frac{1}{2}} \lesssim \frac{1}{\sqrt{m}}. \quad (3.4)$$

Using (3.3) and (3.4), one obtains

$$\begin{aligned} V(B)^{-\frac{1}{2}} \left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} &\lesssim 2^{\frac{jd}{2}} V(2^j B)^{-\frac{1}{2}} \|f\|_{L^2} r^{2n} \sum_{m=1}^{+\infty} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{m+s}\right)}{\sqrt{m}(m+s)^{n+\frac{1}{2}}} \right\}. \end{aligned}$$

However, one has,

$$\begin{aligned} &\sum_{m=1}^{+\infty} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{m+s}\right)}{\sqrt{m}(m+s)^{n+\frac{1}{2}}} \right\} \\ &= \frac{1}{(4^j r^2)^{n+1}} \sum_{m=1}^{+\infty} \frac{2^j r}{\sqrt{m}} \sup_{s \in [0, nr^2]} \left\{ \left(\frac{4^j r^2}{m+s} \right)^{n+\frac{1}{2}} \exp\left(-c \frac{4^j r^2}{m+s}\right) \right\} \\ &\lesssim \frac{1}{(4^j r^2)^{n+1}} \sum_{m=1}^{4^j r^2} \frac{2^j r}{\sqrt{m}} + \frac{1}{(4^j r^2)^{n+1}} \sum_{m=4^j r^2+1}^{+\infty} \left(\frac{4^j r^2}{m} \right)^{n+1} \\ &\lesssim 4^{-jn} r^{-2n}. \end{aligned}$$

It yields the desired result

$$V(B)^{-\frac{1}{2}} \left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(B)} \lesssim 2^{j(\frac{d}{2}-2n)} V(2^j B)^{-\frac{1}{2}} \|f\|_{L^2}$$

□

Proof. (L^p -boundedness of \tilde{g}_β for $2 < p < p_0$)

We use Theorem 2.10 as well. The proof of (2.4) for \tilde{g}_β is analogous to the corresponding one for g_β , by use of Lemma 3.4. Let us now check (2.5). We argue as in [3] pp 932-936, using (P_2) and (GG_{p_0}) .

We want to prove that, for all $2 < p < p_0$, there exists C_n such that for all balls $B \subset \Gamma$ of radius r , all $m \in \llbracket 0, n \rrbracket$, all functions f on Γ and $x \in B$,

$$\frac{1}{V^{\frac{1}{p}}(B)} \left\| \tilde{g}_\beta P^{2mr^2} f \right\|_{L^p(B)} \leq C_n (\mathcal{M}(|\tilde{g}_\beta f|^2))^{\frac{1}{2}}(x). \quad (3.5)$$

Let $f \in L^2(\Gamma)$. Since $P^l \equiv 1$ for all $l \in \mathbb{N}$, we may write, if $g^l = (I - P)^\beta P^{l-1} f$,

$$\nabla P^{mr^2} (I - P)^\beta P^{l-1} f = \nabla P^{mr^2} (g^l - (g^l)_{4B}).$$

Write $g^l - (g^l)_{4B} = \sum_{i \geq 1} g_i^l$ with $g_i^l = (g^l - (g^l)_{4B}) \mathbb{1}_{C_i(B)}$. For $i = 1$, Lemma 4.2 in [5] and (P_2) yield

$$\begin{aligned} \left(\sum_{l \geq 1} l^{2\beta} \left(\frac{1}{V^{\frac{1}{p}}(B)} \left\| \nabla P^{mr^2} g_1^l \right\|_{L^p(B)} \right)^2 \right)^{\frac{1}{2}} &\lesssim \frac{1}{rV(4B)^{\frac{1}{2}}} \left(\sum_{l \geq 1} l^{2\beta} \|g_1^l\|_{L^2(4B)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{V(8B)} \sum_{l \geq 1} l^{2\beta} \sum_{y \in 8B} |\nabla g_1^l(y)|^2 m(y) \right)^{\frac{1}{2}} \\ &\lesssim \mathcal{M}_2(\tilde{g}_\beta f)(x). \end{aligned}$$

For $i \geq 2$, Lemma 4.2 in [5] shows that

$$\left(\sum_{l \geq 1} l^{2\beta} \left(\frac{1}{V^{\frac{1}{p}}(B)} \left\| \nabla P^{mr^2} g_i^l \right\| \right)^2 \right)^{\frac{1}{2}} \lesssim \frac{e^{-c_4 i}}{r} \left(\frac{1}{V(2^{i+1}B)} \sum_{l \geq 1} l^{2\beta} \|g_i^l\|_{L^2(C_i(B))}^2 \right)^{\frac{1}{2}}.$$

But for all $l \geq 1$,

$$\begin{aligned} \|g_i^l\|_{L^2(C_i(B))} &\leq \|g^l - (g^l)_{4B}\|_{L^2(2^{i+1}B)} \\ &\leq \|g^l - (g^l)_{2^{i+1}B}\|_{L^2(2^{i+1}B)} + V(2^{i+1}B)^{\frac{1}{2}} \sum_{j=2}^i |(g^l)_{2^j B} - (g^l)_{2^{j+1}B}|. \end{aligned}$$

For all $j \in \llbracket 2, i \rrbracket$, (P_2) implies

$$\begin{aligned} |(g^l)_{2^j B} - (g^l)_{2^{j+1}B}| &\lesssim \frac{1}{V(2^{j+1}B)^{\frac{1}{2}}} \|g^l - (g^l)_{2^{j+1}B}\|_{L^2(2^{j+1}B)} \\ &\lesssim 2^{j+1} r \frac{1}{V(2^{j+1}B)^{\frac{1}{2}}} \|\nabla g^l\|_{L^2(2^{j+1}B)}, \end{aligned}$$

while

$$\|g^l - (g^l)_{2^{i+1}B}\|_{L^2(2^{i+1}B)} \lesssim 2^{i+1} r \|\nabla g^l\|_{L^2(2^{i+1}B)},$$

so that

$$\|g_i^l\|_{L^2(C_i(B))} \lesssim \sum_{j=2}^i 2^j r \frac{V(2^{i+1}B)^{\frac{1}{2}}}{V(2^{j+1}B)^{\frac{1}{2}}} \|\nabla g^l\|_{L^2(2^{j+1}B)}.$$

As a consequence, by the Minkowski inequality,

$$\begin{aligned} \left(\frac{1}{V(2^{i+1}B)} \sum_{l \geq 1} l^{2\beta} \|g_i^l\|_{L^2(C_i(B))}^2 \right)^{\frac{1}{2}} &\lesssim \sum_{j=2}^i 2^j r \frac{1}{V(2^{j+1}B)^{\frac{1}{2}}} \left(\sum_{l \geq 1} l^{2\beta} \|\nabla g^l\|_{L^2(2^{j+1}B)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sum_{j=2}^i 2^j r \mathcal{M}_2 \tilde{g}_\beta f(x) \\ &\lesssim 2^i r \mathcal{M}_2 \tilde{g}_\beta f(x). \end{aligned}$$

□

3.4 L^p -boundedness of g_β and \tilde{g}_β , $1 < p \leq 2$

The proof of the L^p -boundedness of g_β for $1 < p < 2$ relies on Theorem 2.11, via the following lemma:

Lemma 3.5. *Let (Γ, μ) be a weighted graph satisfying (DV) , (LB) and (DUE) .*

For all $n \in \mathbb{N}^$, there exists a constant C_n such that, for all balls $B = B(x_0, r)$ of Γ , all $j \geq 2$ and all $f \in L^1(\Gamma)$ supported in B , we get*

$$\left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(C_j(B))} \leq C_n 2^{-2jn} \frac{V(2^j B)^{\frac{1}{2}}}{V(B)} \|f\|_{L^1}.$$

Proof. The proof of Lemma 3.5 is very similar to the one of Lemma 3.3, and we will therefore be sketchy. First, we still have

$$\left\| g_\beta (I - P^{r^2})^n f \right\|_{L^2(C_j(B))} \leq \sum_{k \geq 0} a_k \left(\sum_{l \geq 1} l^{2\beta-1} \left\| (I - P)^{1+\eta} P^{k+l-1} (I - P^{r^2})^n f \right\|_{L^2(C_j(B))}^2 \right)^{\frac{1}{2}}$$

where a_k is defined as in the proof of 3.3.

1- Estimate of the inner term

Let $B = B(x_0, r)$. As in Lemma 3.3 and using (GT_2) ,

$$\begin{aligned} & \left\| (I - P)^{1+\eta} P^{k+l-1} (I - P^{r^2})^n f \right\|_{L^2(C_j(B))} \\ & \lesssim l^{-\eta} \|f\|_{L^1(B)} \sup_{s \in [0, nr^2]} \left(\frac{1}{V(x_0, \sqrt{l+k+s})^{\frac{1}{2}}} \frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{1+n}} \right) \\ & \lesssim l^{-\eta} \frac{V(2^j B)^{\frac{1}{2}}}{V(B)} \|f\|_{L^1} \sup_{s \in [0, nr^2]} \left(\frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{1+n}} \right) \end{aligned} \quad (3.6)$$

where we use for the second line the following fact, consequence of (DV)

$$\frac{V(B)}{V(x_0, \sqrt{l+k+s})} \lesssim \left(\frac{r^2}{l+k+s} \right)^{\frac{d}{2}} \lesssim \exp\left(-c \frac{4^j r^2}{l+k+s}\right).$$

2/3- Conclusion

The proof is then the same (with obvious modifications) as the proof of Lemma 3.3, using the same sequence $A_l^{k,r,j}$ as in the proof of Lemma 3.3. □

We can now conclude for the L^p -boundedness of g_β and \tilde{g}_β for $1 < p < 2$.

Proof. (L^p -boundedness and weak $(1, 1)$ type of g_β for $1 < p < 2$)

We apply Theorem 2.11. It is enough to check (2.6) and (2.7) with $g(j) = 2^{-j}$. We take $A_B = P^{r^2}$ where r is the radius of B . The inequality (2.6) is then a consequence of Lemma 3.5 for $n = 1$. For the estimate (2.7), it suffices to prove that, for all balls B of Γ , all $j \geq 1$, and all f supported in B ,

$$\|P^{r^2} f\|_{L^2(C_j(B))} \lesssim \frac{V(2^{j+1} B)^{\frac{1}{2}}}{V(B)} e^{-c4^j} \|f\|_{L^1(B)}.$$

The case $j \geq 2$ is a consequence of (GT_2) and (DV) , while the case $j = 1$ follows from (UE) and (A.10). □

Proof. (L^p -boundedness and weak $(1, 1)$ type of \tilde{g}_β)

For $\beta > 0$, the proof is the analogous to the one of the L^p -boundedness of g_β , using (GGT_2) instead of (GT_2) .

The case $\beta \in (-\frac{1}{2}, 0]$ is analogous, with minor changes identical to the corresponding case in the proof of L^p -boundedness of \tilde{g}_β for $p > 2$. □

3.5 Reverse L^p inequalities for g_β and \tilde{g}_β

Let us now end up the proof of Theorem 1.13. What remains to be proved is:

Theorem 3.6. *Let (Γ, μ) be a weighted graph satisfying (DV), (LB) and (DUE). For all $1 < p < +\infty$ and $\beta > 0$, there exist three constants $C_1, C_2, C_3 > 0$ such that*

$$\|f\|_{L^p} \leq C_1 \|g_\beta f\|_{L^p} \leq C_2 \|g_\beta^2 f\|_{L^p} \leq C_3 \|f\|_{L^p} \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma).$$

Remark 3.7. *Notice that Theorem 3.6 implies Theorem 1.13 for g_β . A statement analogous to Theorem 3.6 holds with \tilde{g}_β , with the same proof, which ends the proof of Theorem 1.13.*

Proof. By Lemma B.1, we get

$$g_\beta^2 f(x) \simeq g_\beta(I+P)^\beta f(x) \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma), \forall x \in \Gamma.$$

As a consequence of this fact and Remark 1.3, for all $p \in (1, +\infty)$, we have the inequalities

$$\|g_\beta^2 f\|_{L^p} \lesssim \|g_\beta(I+P)^\beta f\|_{L^p} \lesssim \|(I+P)^\beta f\|_{L^p} \lesssim \|f\|_{L^p}. \quad (3.7)$$

The proof will then be complete if we establish, for all $1 < p < +\infty$,

$$\|f\|_{L^p} \leq \|g_\beta^2 f\|_{L^p} \quad \forall f \in L^p(\Gamma). \quad (3.8)$$

Indeed, assume that (3.8) is established. The conjunction of (3.7) and (3.8) provide the equivalences

$$\|g_\beta^2 f\|_{L^p} \simeq \|f\|_{L^p} \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma)$$

and

$$\|g_\beta f\|_{L^p} \simeq \|f\|_{L^p} \quad \forall f \in A = \{(I+P)^\beta g, g \in L^p \cap L^2\}$$

and it is therefore enough to check that A is dense in $L^p(\Gamma)$.

To that purpose, notice that (3.7) and (3.8) also provide the equivalence $\|(I+P)^\beta f\|_{L^p(\Gamma)} \simeq \|f\|_{L^p(\Gamma)}$ for all $f \in L^2(\Gamma) \cap L^p(\Gamma)$, then for all $f \in L^p(\Gamma)$ by the L^p -boundedness of $(I+P)^\beta$ and since $L^2(\Gamma) \cap L^p(\Gamma)$ is dense in $L^p(\Gamma)$. This entails that $(I+P)^\beta$ is one-to-one on $L^p(\Gamma)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$), which implies that A is dense in $L^p(\Gamma)$.

The inequality (3.8) can be proven by duality. Actually, for all $f, h \in L^2(\Gamma)$, Lemma 3.1 shows that

$$\begin{aligned} 4 \langle f, h \rangle &= \|f+h\|_2^2 - \|f-h\|_2^2 \\ &= \|g_\beta^2(f+h)\|_2^2 - \|g_\beta^2(f-h)\|_2^2 \\ &\leq \|g_\beta^2 f + g_\beta^2 h\|_2^2 - \|g_\beta^2 f - g_\beta^2 h\|_2^2 \\ &= 4 \langle g_\beta^2 f, g_\beta^2 h \rangle. \end{aligned}$$

For the third line, notice that

$$g_\beta^2 f - g_\beta^2 h \leq g_\beta^2(f-h),$$

and interverting the roles of f and h , we obtain

$$|g_\beta^2 f - g_\beta^2 h| \leq g_\beta^2(f-h),$$

so that

$$\|g_\beta^2 f - g_\beta^2 h\|_{L^2} \leq \|g_\beta^2(f-h)\|_{L^2}.$$

Thus, if $\frac{1}{p} + \frac{1}{p'} = 1$, we have for all $f \in L^p(\Gamma) \cap L^2(\Gamma)$, $1 < p < +\infty$,

$$\begin{aligned} \|f\|_{L^p(\Gamma)} &= \sup_{\substack{h \in L^2 \cap L^{p'} \\ \|h\|_{L^{p'}} \leq 1}} \langle f, h \rangle \\ &\leq \sup_{\substack{h \in L^2 \cap L^{p'} \\ \|h\|_{L^{p'}} \leq 1}} \langle g_\beta^2 f, g_\beta^2 h \rangle \\ &\leq \|g_\beta^2 f\|_{L^p} \sup_{\substack{h \in L^2 \cap L^{p'} \\ \|h\|_{L^{p'}} \leq 1}} \|g_\beta^2 h\|_{L^{p'}} \\ &\lesssim \|g_\beta^2 f\|_{L^p} \sup_{\substack{h \in L^2 \cap L^{p'} \\ \|h\|_{L^{p'}} \leq 1}} \|h\|_{L^{p'}} \\ &= \|g_\beta^2 f\|_{L^p} \end{aligned}$$

where the third line is a consequence of Hölder inequality and the fourth one follows from the boundedness of g_β^2 on $L^{p'}(\Gamma)$. We obtain the desired result

$$\|f\|_{L^p} \lesssim \|g_\beta^2 f\|_{L^p}.$$

□

4 L^p -boundedness of \tilde{g}_0 , $1 < p < 2$

Define, for all $q \in (1, 2]$ and all functions f on Γ ,

$$\tilde{N}_q f := qf\Delta f - f^{2-q}\Delta f^q$$

and, for all functions $u_n : \mathbb{N} \times \Gamma \rightarrow \mathbb{R}$,

$$N_q u_n := qu_n[\partial_n + \Delta]u_n - u_n^{2-q}[\partial_n + \Delta]u_n^q = \tilde{N}_q u_n + qu_n\partial_n u_n - u_n^{2-q}\partial_n u_n^q.$$

Here and after, $\partial_n u_n = u_{n+1} - u_n$ for all $n \in \mathbb{N}$.

Remark 4.1. • *Dungey proved in [15] that $0 \leq \tilde{N}_q(f) \leq \frac{q}{2}|\nabla f|^2$.*

- *The Young inequality shows at once that*

$$\partial_n u_n^q \geq qu_n^{q-1}\partial_n u_n, \tag{4.1}$$

and then $N_q(u_n) \leq \tilde{N}_q(u_n)$.

- *As will be shown in Proposition 4.7 below, $N_q(P^n f) \geq 0$ for all nonnegative functions f and all $n \in \mathbb{N}$.*

We also introduce the functional

$$\tilde{g}_{0,q}f(x) = \left(\sum_{n \geq 0} N_q(P_n f)(x) \right)^{\frac{1}{2}}.$$

Theorem 4.2. *If $q \in (1, 2]$, then there exists a constant $c > 0$ such that*

$$\|\tilde{g}_{0,q}f\|_q \leq c\|f\|_q$$

for all nonnegative functions $f \in L^1 \cap L^\infty$.

Corollary 4.3. *Let (Γ, μ) be a graph satisfying (LB) and (LDV) and let $q \in (1, 2]$ Then there exists $c_q > 0$ such that*

$$\|\nabla P^n f\|_q \leq \frac{c_q}{\sqrt{n}}\|f\|_q$$

Remark 4.4. *In [15], using semigroup arguments, Dungey proved the conclusion of Corollary 4.3 under the weaker assumption that -1 does not belong to the L^2 spectrum of P .*

4.1 Proof of Theorem 4.2

The proof of this result is based on Stein's argument in [21], Chapter II, also used in Riemannian manifolds in [9] and on graphs with continuous time functionals in [15].

Let us first state the maximal ergodic theorem for Markov kernels (see [19], see also [21], Chapter IV, Theorems 6 and 9):

Lemma 4.5. *Let (X, m) be a measurable space. Assume that P is a linear operator simultaneously defined and bounded from $L^1(X)$ to itself and from $L^\infty(X)$ to itself that satisfies*

i. P is self adjoint,

ii. $\|P\|_{L^1 \rightarrow L^1} \leq 1$.

Let $f^*(x) = \sup_{n \geq 0} |P^n f(x)|$. Then there exists a constant $c_q > 0$ such that

$$\|f^*\|_q \leq c_q \|f\|_q$$

for all $q \in (1, +\infty]$.

We can now turn to the proof of Theorem 4.2.

If $u_n = P^{n-1}f$, then $[\partial_n + \Delta]u_n = 0$ and, as will be proved in Proposition 4.7 below, one has

$$N_q u_n = -u_n^{2-q} [\partial_n + \Delta] u_n^q \geq 0.$$

Consequently, we have

$$\begin{aligned} \tilde{g}_{0,q} f(x)^2 &= \sum_{n \geq 0} N_q (P^n f)(x) = - \sum_{n \geq 0} [P^n f(x)]^{2-q} [\partial_n + \Delta] ([P^n f(x)]^q) \\ &\leq -f^*(x)^{2-q} \sum_{n \geq 0} [\partial_n + \Delta] ([P^n f(x)]^q). \end{aligned}$$

It follows, with $J(x) = - \sum_{n \geq 0} [\partial_n + \Delta] ([P^n f(x)]^q) \geq 0$,

$$\begin{aligned} \|\tilde{g}_{0,q} f\|_q^q &\leq \sum_{x \in \Gamma} f^*(x)^{\frac{(2-q)q}{2}} J(x)^{\frac{q}{2}} m(x) \\ &\leq \left(\sum_{x \in \Gamma} f^*(x)^q m(x) \right)^{\frac{2-q}{2}} \left(\sum_{x \in \Gamma} J(x) m(x) \right)^{\frac{q}{2}}. \end{aligned} \quad (4.2)$$

Yet, by Lemma 4.5,

$$\left(\sum_{x \in \Gamma} f^*(x)^q m(x) \right) \lesssim \|f\|_q^q \quad (4.3)$$

and since $\sum_{x \in \Gamma} \Delta g(x) m(x) = 0$ for all $g \in L^1(\Gamma)$,

$$\begin{aligned} \sum_{x \in \Gamma} J(x) m(x) &= - \sum_{x \in \Gamma} m(x) \sum_{n \geq 0} \partial_n [P^n f(x)]^q \\ &\leq \sum_{x \in \Gamma} f(x)^q m(x) = \|f\|_q^q. \end{aligned} \quad (4.4)$$

The inequality in the last line is due to the fact that, for all $N \in \mathbb{N}$,

$$\sum_{n=0}^N \partial_n \sum_{x \in \Gamma} [P^n f(x)]^q m(x) = \|f\|_q^q - \|P^{N+1} f\|_q^q.$$

Using (4.2), (4.3) and (4.4), we thus obtain the conclusion of Theorem 4.2.

4.2 Proof of Theorem 1.14

Recall some facts proved by Dungey in [15]. Define the ‘‘averaging’’ operator A by setting

$$(Af)(x) = \sum_{y \in B(x,2)} f(y) = \sum_{y \sim x} f(y)$$

for $x \in \Gamma$ and functions $f : \Gamma \rightarrow \mathbb{R}$.

Proposition 4.6. *Suppose that (Γ, μ) satisfies property (LDV), and let $q \in (1, 2]$. There exists $c_q > 0$ such that*

$$|\nabla f|^2(x) \leq c_q A(\tilde{N}_q f)(x)$$

for all $x \in \Gamma$ and all nonnegative functions $f \in L^\infty$. Moreover, there exists $c'_q > 0$ such that

$$\|AF\|_{\frac{q}{2}} \leq c'_q \|F\|_{\frac{q}{2}} \quad (4.5)$$

for all nonnegative functions F on Γ .

Note that $\frac{q}{2} \leq 1$ in (4.5), and that we use the notation $\|F\|_r := \left(\sum_{x \in \Gamma} m(x) |F(x)|^r \right)^{\frac{1}{r}}$ for $r \in (0, 1]$.

In order to prove Theorem 1.14, we need the following result.

Proposition 4.7. *Let (Γ, μ) be a weighted graph and let $q \in (1, 2]$. Then $N_q(P^n f) \geq 0$ for all functions $0 \leq f \in L^\infty$.*

Moreover, if (Γ, μ) satisfies (LB), there exists a constant $c_q > 0$ such that

$$0 \leq \tilde{N}_q(P^n f) \leq c_q N_q(P^n f).$$

Proof. (Theorem 1.14)

Proposition 4.6 yields the pointwise estimate

$$|\nabla P^n f|^2 \lesssim A(\tilde{N}_q(P^n f))$$

for $0 \leq f \in L^\infty$, so that, by Proposition 4.7,

$$\begin{aligned} (\tilde{g}_0 f)^2 &= \sum_{n \geq 0} |\nabla P^n f|^2 \\ &\lesssim \sum_{n \geq 0} A(N_q(P^n f)) \\ &= A\left(\sum_{n \geq 0} N_q(P^n f)\right) = A(\tilde{g}_{0,q} f)^2. \end{aligned}$$

Theorem 4.2 and (4.5) provide the conclusion of Theorem 1.14 for all nonnegative functions f . We obtain then L^q -boundedness of \tilde{g}_0 by subadditivity of \tilde{g}_0 . \square

It remains to prove Proposition 4.7.

Proof. (Proposition 4.7)

Taylor expansion of the function $t \mapsto t^q$, $q \in (1, 2]$, gives

$$\begin{aligned} t^q - s^q &= qs^{q-1}(t-s) + q(q-1) \int_s^t \tau^{q-2}(t-\tau) d\tau \\ &= qs^{q-1}(t-s) + q(q-1)(t-s)^2 \int_0^1 \frac{(1-u)du}{((1-u)s+ut)^{2-q}} \end{aligned} \quad (4.6)$$

for $t, s \geq 0$ with $s \neq t$. From this expansion, one has, for $q \in (1, 2]$, $0 \leq g \in L^\infty$ and $x \in \Gamma$,

$$\begin{aligned} \tilde{N}_q(g)(x) &= \sum_{y \in \Gamma} p(x, y) m(y) [qg(x)(g(x) - g(y)) - g(x)^{2-q}(g(x)^q - g(y)^q)] \\ &= q(q-1) \sum_{y: g(y) \neq g(x)} p(x, y) m(y) (g(x) - g(y))^2 \int_0^1 \frac{(1-t)g(x)^{2-q}}{((1-t)g(x) + tg(y))^{2-q}} dt \\ &= q(q-1)g(x)^{2-q} \int_0^1 (1-t) \sum_{y: g(y) \neq g(x)} p(x, y) m(y) \frac{(g(x) - g(y))^2}{((g(x) + t(g(y) - g(x)))^{2-q}} dt. \end{aligned} \quad (4.7)$$

Let $0 \leq f \in L^\infty$ and $n \in \mathbb{N}$. Define $g := P^n f$ and notice $0 \leq g \in L^\infty$. Therefore,

$$\partial_n(P^n f)(x) = (P - I)g(x) = \sum_{y \sim x} p(x, y) m(y) (g(y) - g(x))$$

and with (4.6), one has

$$\begin{aligned} \partial_n(P^n f(x))^q - q(P^n f(x))^{q-1} \partial_n(P^n f) \\ = q(q-1)((P - I)g(x))^2 \int_0^1 \frac{(1-t)dt}{(g(x) + t(P - I)g(x))^{2-q}} \end{aligned}$$

so that

$$\begin{aligned} [\tilde{N}_q - N_q]g(x) &= g(x)^{2-q} \partial_n (P^n f(x))^q - qg(x) \partial_n (P^n f) \\ &= q(q-1)g(x)^{2-q} \int_0^1 (1-t) \frac{((P-I)g(x))^2}{(g(x) + t(P-I)g(x))^{2-q}} dt \end{aligned}$$

If $g(x) = 0$, then $N_q(g)(x) = [\tilde{N}_q - N_q]g(x) = 0$, therefore the conclusion of Proposition 4.7 holds at x . Assume now that $g(x) \neq 0$. Define, for all $y \in \Gamma$, $h(y) = \frac{g(y)-g(x)}{g(x)} \geq -1$ and, for all $t \in (0, 1)$ and all $s \in [-1, +\infty)$,

$$\mathcal{F}_t(s) = \frac{s^2}{(1+ts)^{2-q}}.$$

One has

$$\tilde{N}_q(g)(x) = q(q-1)g(x)^{2-q} \int_0^1 (1-t) \sum_{y: g(y) \neq g(x)} p(x, y) m(y) \mathcal{F}_t(h(y)) dt$$

and

$$[\tilde{N}_q - N_q]g(x) = q(q-1)g(x)^{2-q} \int_0^1 (1-t) \mathcal{F}_t \left(\sum_{y: g(y) \neq g(x)} p(x, y) m(y) h(y) \right) dt.$$

Assume for a while that it is known that \mathcal{F}_t is convex on $[-1, +\infty)$ for all $t \in (0, 1)$, and let us conclude the proof of Proposition 4.7. One has $\tilde{N}_q(g)(x) \geq [\tilde{N}_q - N_q]g(x)$, which means that $N_q(g)(x) \geq 0$. Moreover, since $p(x, x) > \epsilon$, then $\sum_{y: g(y) \neq g(x)} \frac{p(x, y) m(y)}{1-\epsilon} \leq 1$, so that

$$\begin{aligned} \sum_{y: g(y) \neq g(x)} \frac{p(x, y) m(y)}{1-\epsilon} \mathcal{F}_t(h(y)) &\geq \mathcal{F}_t \left(\sum_{y: g(y) \neq g(x)} \frac{p(x, y) m(y)}{1-\epsilon} h(y) \right) \\ &\geq (1-\epsilon)^{-q} \mathcal{F}_t \left(\sum_{y: g(y) \neq g(x)} p(x, y) m(y) h(y) \right), \end{aligned}$$

where the first inequality is due to the convexity of \mathcal{F}_t and the last one to the definition of \mathcal{F}_t . We deduce

$$[\tilde{N}_q - N_q]g(x) \leq (1-\epsilon)^{q-1} \tilde{N}_q(g)(x),$$

which means

$$\tilde{N}_q(g)(x) \leq \frac{1}{1-(1-\epsilon)^{q-1}} N_q(g)(x).$$

□

It remains to prove the following lemma

Lemma 4.8. *The function \mathcal{F}_t is convex on $[-1, +\infty)$ for all $t \in (0, 1)$.*

Proof. Let $F(x) = \frac{x^2}{(1+x)^{2-q}}$. Easy computations show that F is convex on $(-1, +\infty)$. Since, for all $t \in (0, 1)$, $\mathcal{F}_t = \frac{1}{t^2} F(tx)$, \mathcal{F}_t is convex on $(-\frac{1}{t}, +\infty) \supset [-1, +\infty)$. □

4.3 Proof of Corollary 4.3

First we will prove the following result. If $q \in (1, 2]$, $n \in \mathbb{N}^*$ with $n \geq 1$ and $0 \leq f \in L^1 \cap L^\infty$, one has

$$\|N_q^{\frac{1}{2}}(P^n f)\|_q \leq \frac{C_q}{\sqrt{n}} \|f\|_q \quad (4.8)$$

Let $u_n = P^n f$ and $J_n := -(\partial_n + \Delta)(u_n^q)$. Then

$$\begin{aligned} \|N_q^{\frac{1}{2}}(P^n f)\|_q^q &= \sum_{x \in \Gamma} m(x) N_q^{q/2}(u_n)(x) \\ &= \sum_{x \in \Gamma} m(x) u_n^{\frac{q(2-q)}{2}} J_n(x)^{q/2} \\ &\leq \left[\sum_{x \in \Gamma} m(x) u_n(x)^q \right]^{\frac{2-q}{2}} \left[\sum_{x \in \Gamma} J_n(x) m(x) \right]^{\frac{q}{2}} \end{aligned} \quad (4.9)$$

where the last step follows from Hölder inequality. Yet,

$$\sum_{x \in \Gamma} m(x) u_n(x)^q = \|P^n f\|_q^q \leq \|f\|_q^q$$

and

$$\begin{aligned} \sum_{x \in \Gamma} J_n(x) m(x) &= - \sum_{x \in \Gamma} \partial_n(u_n^q)(x) m(x) \\ &\leq -q \sum_{x \in \Gamma} m(x) u_n^{q-1}(x) \partial_n u_n(x) \\ &\leq q \|u_n\|_q^{q/q'} \|\partial_n u_n\|_q \end{aligned}$$

where the first line holds because $\sum_{x \in \Gamma} \Delta g(x) m(x) = 0$ if $g \in L^1$, the second line follows from (4.1), and the third one from Hölder inequality again (with $\frac{1}{q} + \frac{1}{q'} = 1$). Here $\|u_n\|_q \leq \|f\|_q$ while $\|\partial_n u_n\|_q = \|\Delta u_n\|_q \lesssim \frac{1}{n} \|f\|_q$ by the analyticity of P on L^q . Thus

$$\sum_{x \in \Gamma} J_n(x) m(x) \lesssim \frac{1}{n} \|f\|_q^q$$

Substitution of the last two estimates in (4.9) gives

$$\|N_q^{\frac{1}{2}}(P^n f)\|_q \lesssim \frac{1}{\sqrt{n}} \|f\|_q,$$

which ends the proof of (4.8).

Now just use Propositions 4.6 and 4.7 to get Corollary 4.3.

A Further estimates for Markov chains

A.1 Time regularity estimates

The theorem we prove here is slightly more general than (and clearly implies) Theorem 2.4.

Theorem A.1. *Let (Γ, μ) be a weighted graph satisfying (LB), (DV) and (DUE). Then, for all $j \geq 0$, there exist two constants $C_j, c_j > 0$ such that, for all $(r_1, \dots, r_j) \in \mathbb{N}^{*j}$ for all $l \geq \max_{i \leq j} r_i$ and all $x, y \in \Gamma$,*

$$|(D(r_1) \dots D(r_j) p)_l(x, y)| \leq \frac{C_j r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-c_j \frac{d^2(x, y)}{l}\right).$$

We first recall the following result (Lemma 2.1 in [14]).

Lemma A.2. *Let P be a power bounded and analytic operator in a Banach space X . For each $j \in \mathbb{N}$ and $p \in (1, +\infty)$, there exists a constant $c_j > 0$ such that*

$$\|(I - P^{r_1})(I - P^{r_2}) \dots (I - P^{r_j}) P^l\|_{p \rightarrow p} \leq c_j r_1 \dots r_j l^{-j}$$

for all $j_1, \dots, j_k \in \mathbb{N}$, all $(r_1, \dots, r_j) \in \mathbb{N}^j$ and all $l \in \mathbb{N}^*$.

Proof. let us now establish Theorem A.1. We follow closely the proof of Theorem 1.1 of [14], arguing by induction on j .

The case $j = 0$ is obvious since the result is the assumption. The case $j = 1$ and $r_1 = 1$ is the one proven by Dungey in [14] and we will just here verify that the proof for $j = 1$ can be extended to all $j \in \mathbb{N}$. Assume now that, for some $j \in \mathbb{N}$, the kernel p_l satisfies for all $(r_1, \dots, r_j) \in \mathbb{N}^{*j}$ and all $l \geq \max_i r_i$

$$|D(r_j) \dots D(r_1) p_l(x, y)| \leq \frac{C_j r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-c_j \frac{d^2(x, y)}{l}\right) \quad (\text{A.1})$$

where the constant C_j depends only of the graph Γ and j .

Let $(r_1, \dots, r_{j+1}) \in \mathbb{N}^{*(j+1)}$. We then use the abstract identity (which can easily be proved by induction on k) for all linear operators A and all $k \in \mathbb{N}$:

$$I - A = 2^{-(k+1)}(I - A^{2^{k+1}}) + \sum_{i=0}^k 2^{-(i+1)}(I - A^{2^i})^2 \quad (\text{A.2})$$

where I denotes the identity operator. Hence we have, applying (A.2) with $(Au)_l = u_{l+r_{j+1}}$,

$$D(r_{j+1}) = 2^{-(k+1)}D(2^{k+1}r_{j+1}) + \sum_{i=0}^k 2^{-(i+1)}D(2^i r_{j+1})^2$$

and if we apply this formula to $(D(r_j) \dots D(r_1)p)_l$, we obtain, for all $l \in \mathbb{N}$, $k \in \mathbb{N}$ and $x, y \in \Gamma$,

$$\begin{aligned} |D(r_{j+1}) \dots D(r_1)p_l(x, y)| &\leq 2^{-(k+1)}|D(2^{k+1}r_{j+1})D(r_j) \dots D(r_1)p_l(x, y)| \\ &+ \sum_{i=0}^k 2^{-(i+1)}|D(2^i r_{j+1})^2 D(r_j) \dots D(r_1)p_l(x, y)|. \end{aligned} \quad (\text{A.3})$$

Suppose that $0 < 2^k r_{j+1} \leq l$, hence $l + 2^{k+1}r_{j+1} \leq 3l$ and (A.1) provides the estimate

$$\begin{aligned} &|D(2^{k+1}r_{j+1})D(r_j) \dots D(r_1)p_l(x, y)| \\ &\leq |D(r_j) \dots D(r_1)p_l(x, y)| + |D(r_j) \dots D(r_1)p_{l+2^{k+1}r_{j+1}}(x, y)| \\ &\leq C_j \frac{r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-c_j \frac{d^2(x, y)}{3l}\right). \end{aligned} \quad (\text{A.4})$$

Besides, observe that

$$\begin{aligned} &|D(n)^2 D(r_j) \dots D(r_1)p_l(x, y)| \\ &= \|(I - P^n)^2 (I - P^{r_j}) \dots (I - P^{r_1}) P^l\|_{L^1(\{y\}) \rightarrow L^\infty(\{x\})} \\ &\leq \|P^{l_1}\|_{L^2 \rightarrow L^\infty(\{x\})} \|(I - P^n)^2 (I - P^{r_j}) \dots (I - P^{r_1}) P^{l_2}\|_{2 \rightarrow 2} \|P^{l_3}\|_{L^1(\{y\}) \rightarrow L^2}. \end{aligned} \quad (\text{A.5})$$

whenever $l = l_1 + l_2 + l_3$. Moreover, let us notice that for all $l_0 \in \mathbb{N}^*$ and all $z \in \Gamma$, (DUE) provides

$$\begin{aligned} \|P^{l_0}\|_{L^2 \rightarrow L^\infty(\{x\})} &= \|P^{l_0}\|_{L^1(\{x\}) \rightarrow L^2} = \left(\sum_{y \in \Gamma} [p_{l_0}(x, y)]^2 m(y) \right)^{\frac{1}{2}} \\ &= p_{2l_0}(x, x)^{\frac{1}{2}} \\ &\leq \frac{C}{V(x, \sqrt{l_0})^{\frac{1}{2}}}. \end{aligned} \quad (\text{A.6})$$

The two last results ((A.5) and (A.6)) combined with Lemma A.2 and the doubling property (DV) give, with $l_1, l_2, l_3 \sim \frac{l}{3}$

$$|D(n)^2 D(r_j) \dots D(r_1)p_l(x, y)| \leq C'_j \frac{n^2 r_1 \dots r_j}{l^{j+2} V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}}. \quad (\text{A.7})$$

Collecting estimates (A.3), (A.4) and (A.7) and using that $\sum_{i=0}^k 2^{i-1} \leq 2^k$, we obtain

$$\begin{aligned} |D(r_{j+1}) \dots D(r_1)p_l(x, y)| &\lesssim 2^{-(k+1)} \frac{r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-c_j \frac{d^2(x, y)}{3l}\right) \\ &+ \sum_{i=0}^k 2^{-(i+1)} \frac{2^{2i} r_{j+1}^2 r_j \dots r_1}{l^{j+2} V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \\ &\lesssim 2^{-(k+1)} \frac{r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-c_j \frac{d^2(x, y)}{3l}\right) \\ &+ \frac{2^k r_1 \dots r_j r_{j+1}^2}{l^{j+2} V(x, \sqrt{l})} \end{aligned}$$

for all $l \geq 1$, $k \in \mathbb{N}$ with $2^k r_{j+1} \leq l$.

We will now choose k to obtain the desired inequality. If l, j, x, y satisfy

$$l \exp\left(-c_j \frac{d^2(x, y)}{4l}\right) \geq r_{j+1}.$$

We choose k such that

$$2^k r_{j+1} \leq l \exp\left(-c_j \frac{d^2(x, y)}{4l}\right) < 2^{k+1} r_{j+1}$$

which gives

$$|D(r_{j+1}) \dots D(r_1) p_l(x, y)| \lesssim \frac{r_1 \dots r_{j+1}}{l^{j+1} V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-\frac{c_j}{12} \frac{d^2(x, y)}{l}\right).$$

In the other case, i.e. $l \exp\left(-c_j \frac{d^2(x, y)}{4l}\right) \leq r_{j+1}$, observe that by (A.1)

$$\begin{aligned} & |D(r_{j+1}) \dots D(r_1) p_l(x, y)| \\ & \leq |D(r_j) \dots D(r_1) p_l(x, y)| + |D(r_j) \dots D(r_1) p_{l+r_{j+1}}(x, y)| \\ & \leq C_j \frac{r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-c_j \frac{d^2(x, y)}{l + r_{j+1}}\right) \\ & \leq C_j \frac{r_1 \dots r_j}{l^j V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-\frac{c_j}{2} \frac{d^2(x, y)}{l}\right) \\ & \leq C_j \frac{r_1 \dots r_{j+1}}{l^{j+1} V(x, \sqrt{l})^{\frac{1}{2}} V(y, \sqrt{l})^{\frac{1}{2}}} \exp\left(-\frac{c_j}{4} \frac{d^2(x, y)}{l}\right) \end{aligned}$$

where the third line holds because $l \geq r_{j+1}$. □

A.2 Gaffney-type inequalities

This paragraph is devoted to the proof of Theorem 2.5. Actually, we establish more general versions in Theorem A.3 and Corollary A.4.

Theorem A.3. *Let (Γ, μ) be a weighted graph. Assume (LB), (DV) and (DUE). Then, for all $j \in \mathbb{N}$, there exist $c, C > 0$ such that for all $(r_1, \dots, r_j) \in \mathbb{N}^j$, for all sets $E, F \subset \Gamma$ and $x_0 \in \Gamma$ such that $\sup\{d(x_0, y), y \in F\} \leq 3d(E, F)$ and all functions f supported in F ,*

$$(i) \quad \|(I - P^{r_1}) \dots (I - P^{r_j}) P^l f\|_{L^2(E)} \leq C \frac{r_1 \dots r_j}{l^j} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F)},$$

for all $(r_1, \dots, r_j) \in \mathbb{N}^*$ and all $l \geq \max_{i \leq j} r_i$.

$$(ii) \quad \|\nabla(I - P^{r_1}) \dots (I - P^{r_j}) P^l f\|_{L^2(E)} \leq C \frac{r_1 \dots r_j}{l^{j+\frac{1}{2}}} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F)},$$

for all $(r_1, \dots, r_j) \in \mathbb{N}^*$ and all $l \geq \max_{i \leq j} r_i$.

$$(iii) \quad \|\nabla(I - P^{r_1}) \dots (I - P^{r_j}) P^l f\|_{L^2(E)} \leq C \frac{r_1 \dots r_j}{l^{j+\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^2(F)},$$

for all $(r_1, \dots, r_j) \in \mathbb{N}^*$ and all $l \geq \max_{i \leq j} r_i$.

Corollary A.4. *Let (Γ, μ) be a weighted graph satisfying (LB), (DV) and (DUE). The conclusions of Theorem A.3 still hold under any of the following assumptions on E, F, x_0 and l :*

1. $\sup\{d(x_0, y), y \in F\} \leq \sqrt{l}$,
2. $\sup\{d(x_0, x), x \in E\} \leq 3d(E, F)$,
3. $\sup\{d(x_0, x), x \in E\} \leq \sqrt{l}$.

The proof of Theorem A.3 relies on:

Lemma A.5. Let (Γ, μ) be a weighted graph satisfying (DV), (LB) and (DUE), then we have the following estimates: for all $j \in \mathbb{N}$, there exist $C_j, c_j > 0$ such that for all $(r_1, \dots, r_j) \in \mathbb{N}^{*j}$ and all $l \geq \max_{i \leq j} r_i$

$$\sum_{y \in \Gamma} |(D(r_j) \dots D(r_1)p)_l(y, x)|^2 e^{c_j \frac{d(x, y)^2}{l}} m(y) \leq C_j \frac{r_1^2 \dots r_j^2}{l^{2j} V(x, \sqrt{l})} \quad (\text{A.8})$$

and

$$\sum_{y \in \Gamma} |\nabla_y (D(r_j) \dots D(r_1)p)_l(y, x)|^2 e^{c_j \frac{d(x, y)^2}{l}} m(y) \leq C_j \frac{r_1^2 \dots r_j^2}{l^{2j+1} V(x, \sqrt{l})}. \quad (\text{A.9})$$

The proof of this Lemma is analogous to Lemmas 4 and 7 in [18], where we use the estimates in Theorem A.1 instead of the estimate (UE).

Proof. (Theorem A.3)

(i) We can assume without loss of generality that $\|f\|_{L^1} = 1$. Then

$$\begin{aligned} & \|(I - P^{r_1}) \dots (I - P^{r_j}) P_l f\|_{L^2(E)}^2 \\ &= \sum_{x \in E} m(x) \left(\sum_{z \in F} (D(r_j) \dots D(r_1)p)_l(x, z) f(z) m(z) \right)^2 \\ &\leq \sum_{x \in E} m(x) \sum_{z \in F} |(D(r_j) \dots D(r_1)p)_l(x, z)|^2 |f(z)| m(z) \\ &\leq \exp\left(-c \frac{d(E, F)^2}{l}\right) \sum_{z \in F} |f(z)| m(z) \sum_{x \in E} m(x) |D(r_j) \dots D(r_1)p_l(x, z)|^2 \exp\left(c \frac{d(x, z)^2}{l}\right) \\ &\lesssim \frac{r_1^2 \dots r_j^2}{l^{2j}} \exp\left(-c \frac{d(E, F)^2}{l}\right) \sum_{z \in F} |f(z)| m(z) \frac{1}{V(z, \sqrt{l})} \\ &\lesssim \frac{r_1^2 \dots r_j^2}{l^{2j} V(x_0, \sqrt{l})} \exp\left(-c \frac{d(E, F)^2}{l}\right) \end{aligned}$$

where, for the 4th line, we use the estimate (A.8) and, for the last line, the doubling property shows

$$\frac{V(x_0, \sqrt{l})}{V(z, \sqrt{l})} \leq \frac{V(z, \sqrt{l} + 3d(E, F))}{V(z, \sqrt{l})} \lesssim \left(1 + \frac{3d(E, F)}{\sqrt{l}}\right)^d \lesssim \exp\left(\frac{c d(E, F)^2}{2l}\right) \quad (\text{A.10})$$

which leads to the result (with a different value of c).

(ii) Similar to (i) using (A.9) instead of (A.8).

(iii) This result is a consequence of (i). In fact,

$$\begin{aligned} \|\nabla(I - P^{r_1}) \dots (I - P^{r_j}) P_l f\|_{L^2(E)} &\lesssim \frac{r_1 \dots r_j}{l^{j+\frac{1}{2}}} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F)} \\ &\leq \frac{r_1 \dots r_j}{l^{j+\frac{1}{2}}} \left(\frac{m(F)}{V(x_0, \sqrt{l})}\right)^{\frac{1}{2}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^2(F)} \\ &\lesssim \frac{r_1 \dots r_j}{l^{j+\frac{1}{2}}} e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^2(F)} \end{aligned}$$

where, for the last line, the doubling property yields

$$\frac{m(F)}{V(x_0, \sqrt{l})} \leq \frac{V(x_0, 3d(E, F))}{V(x_0, \sqrt{l})} \lesssim \left(1 + \frac{3d(E, F)}{\sqrt{l}}\right)^d \lesssim \exp\left(\frac{c d(E, F)^2}{2l}\right). \quad (\text{A.11})$$

□

Proof. (of Corollary A.4)

1. Under this assumption, the proof is analogous to the one of Theorem A.3, replacing (A.10) by

$$\frac{V(x_0, \sqrt{l})}{V(z, \sqrt{l})} \lesssim 1 \quad \forall z \in F,$$

which is provided by (DV) and (A.11) by

$$\frac{m(F)}{V(x_0, \sqrt{l})} \leq 1,$$

which is due to the fact that $F \subset B(x_0, \sqrt{l})$.

2. Decompose $F = \bigcup_{i \geq 0} F_i$, with

$$F_i = F \cap \{y \in \Gamma, 3^i d(E, F) \leq d(y, E) < 3^{i+1} d(E, F)\}.$$

Remark that, if $F_i \neq \emptyset$,

$$\sup_{y \in F_i} d(x_0, y) \leq (3 + 3^{i+1})d(E, F) \leq (3 + 3^{1-i})d(E, F_i) \leq 6d(E, F_i).$$

Let T be one of the operators involved in the left-hand sides in Theorem A.3. Let $c_T > 0$ be such that, for all (\tilde{E}, \tilde{F}) such that $\sup_{y \in \tilde{F}} d(x_0, y) \leq 6d(\tilde{E}, \tilde{F})$ and all f supported in \tilde{F} , we have

$$\|Tf\|_{L^2(\tilde{E})} \leq c_T e^{-c \frac{d(\tilde{E}, \tilde{F})^2}{l}} \|f\|_{L^1(\tilde{F})}.$$

(Remember that Theorem A.3 can be proven with constant 6 instead of 3.) Then, one has

$$\|Tf\|_{L^2(E)} \leq \sum_{i=0}^{+\infty} \|T(f \mathbf{1}_{F_i})\|_{L^2(E)} \leq c_T \sum_{i=0}^{+\infty} e^{-c \frac{d(E, F_i)^2}{l}} \|f\|_{L^1(F_i)} \leq c_T e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F)},$$

which proves the second point of the corollary (note that the above sum can be restricted to the indexes i such that $F_i \neq \emptyset$).

3. Let $R = \sup_{x \in E} d(x, x_0)$. Decompose $F = \bigcup_{i \geq 1} F_i$ with

$$F_1 = F \cap B(x_0, 4R)$$

and if $i \geq 2$

$$F_i = F \cap \{y \in \Gamma, 2^i R < d(y, x_0) \leq 2^{i+1} R\}.$$

Write

$$\|Tf\|_{L^2(E)} \leq \sum_{i=1}^{+\infty} \|T(f \mathbf{1}_{F_i})\|_{L^2(E)}$$

(where T is one of the sublinear operators of Theorem A.3 and f is supported in F). We want to estimate each $\|T(f \mathbf{1}_{F_i})\|_{L^2(E)}$. First, notice that $\sup_{x \in F^1} \{d(x, x_0)\} \leq 4R \leq 4\sqrt{l}$. Use then point 1 of Corollary A.4 to obtain

$$\begin{aligned} \|T(f \mathbf{1}_{F_1})\|_{L^2(E)} &\leq c_T e^{-c \frac{d(E, F_1)^2}{l}} \|f\|_{L^1(F_1)} \\ &\leq c_T e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F_1)}. \end{aligned}$$

Next, remark that, if $i \geq 2$ and since $d(E, F_i) \geq (2^i - 1)R$,

$$\sup_{x \in F_i} \{d(x, x_0)\} \leq 2^{i+1} R \leq \frac{2^{i+1}}{2^i - 1} d(E, F_i) \leq 3d(E, F_i).$$

Hence, using Theorem A.3, one has

$$\begin{aligned} \|T(f \mathbf{1}_{F_i})\|_{L^2(E)} &\leq c_T e^{-c \frac{d(E, F_i)^2}{l}} \|f\|_{L^1(F_i)} \\ &\leq c_T e^{-c \frac{d(E, F)^2}{l}} \|f\|_{L^1(F_i)}. \end{aligned}$$

Summing up over i yields the desired conclusion. \square

B Estimates for the Taylor coefficients of $(1 - z)^{-\beta}$

Lemma B.1. *Let $\gamma > -1$. Let $\sum_{l \geq 1} a_l z^l$ be the Taylor series of the function $\frac{1}{(1-z)^{\gamma+1}}$. We have*

$$a_l \simeq l^\gamma \quad \forall l \in \mathbb{N}^*.$$

A consequence of this result is

$$\sum_{l \geq 0} l^\gamma z^l \simeq \frac{z}{(1-z)^{\gamma+1}} \quad \forall z \in [0, 1).$$

Proof. For $|z| < 1$, the holomorphic function $\frac{1}{(1-z)^{\gamma+1}}$ is equal to its Taylor series,

$$\frac{1}{(1-z)^{\gamma+1}} = \sum_{l \geq 0} \prod_{i=1}^l \left(1 + \frac{\gamma}{i}\right) z^l \quad \forall z \in B_{\mathbb{C}}(0, 1).$$

Let us check that

$$l^\gamma \simeq \prod_{i=1}^l \left(1 + \frac{\gamma}{i}\right) \quad \forall l \geq 1. \tag{B.1}$$

Indeed, one can write

$$\begin{aligned} \ln \prod_{i=1}^l \left(1 + \frac{\gamma}{i}\right) &= \sum_{i=1}^l \ln \left(1 + \frac{\gamma}{i}\right) \\ &= \gamma \sum_{i=1}^l \frac{1}{i} + \sum_{i=1}^l \left[\ln \left(1 + \frac{\gamma}{i}\right) - \frac{\gamma}{i} \right] \end{aligned}$$

Yet, one has

$$\begin{aligned} \left| \sum_{i=1}^l \left[\ln \left(1 + \frac{\gamma}{i}\right) - \frac{\gamma}{i} \right] \right| &\leq \sum_{i=1}^{\infty} \left| \ln \left(1 + \frac{\gamma}{i}\right) - \frac{\gamma}{i} \right| \\ &\lesssim \sum_{i=1}^{\infty} \frac{\gamma^2}{i^2} < +\infty. \end{aligned}$$

Hence we get

$$\ln \prod_{i=1}^l \left(1 + \frac{\gamma}{i}\right) = \gamma \ln l + O(1),$$

which yields (B.1) by applying the exponential map.

From the last result, and since the convergence radius of the series under consideration are 1, we deduce

$$\begin{aligned} \sum_{l \geq 1} l^\gamma z^l &\simeq \sum_{l \geq 1} \prod_{i=1}^l \left(1 + \frac{\gamma}{i}\right) z^l \quad \forall z \in [0, 1) \\ &= \frac{1}{(1-z)^{\gamma+1}} - 1 \\ &= \frac{1 - (1-z)^{\gamma+1}}{(1-z)^{\gamma+1}} \\ &\simeq \frac{z}{(1-z)^{\gamma+1}}. \end{aligned}$$

□

C Reverse Hölder estimates for sequences

For all $M > 0$, define the following sets of sequences

$$E_M = \left\{ (a_n)_{n \geq 1}, \forall n, 0 \leq a_n \leq M \sum_{k \in \mathbb{N}^*} \frac{1}{k} a_k \right\}$$

and

$$\tilde{E}_M = \left\{ (a_n)_{n \geq 1}, \forall n, 0 \leq a_n \leq M \sum_{k \geq n} \frac{1}{k} a_k \right\}.$$

First, we state this obvious lemma:

Lemma C.1.

$$\left(\sum_{n \geq 1} \frac{1}{n} a_n^2 \right)^{\frac{1}{2}} \leq M^{\frac{1}{2}} \sum_{n \geq 1} \frac{1}{n} a_n \quad \forall (a_n)_n \in E_M.$$

Let $\mathcal{A} = \{(A_l^{k,r,j})_{l \in \mathbb{N}^*}, k \in \mathbb{N}, r \in \mathbb{N}^*, j \geq 2\}$, where

$$A_l^{k,r,j} = l^{\beta-\eta} \sup_{s \in [0, nr^2]} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{1+n}} \right\}.$$

The parameters β and η are chosen as in section 3 and therefore $\beta - \eta \in (0, 1]$.

Proposition C.2. *There exists $M > 0$ such that $\mathcal{A} \subset E_M$.*

In order to prove Proposition C.2, we will need the following Lemmata:

Lemma C.3. *One has the next three results:*

- i. $\tilde{E}_M \subset E_M$,
- ii. if $M > 0$ and $\{(a_n^p)_n, p \in I\}$ is a set of sequences such that for all $p \in I$, $(a_n^p)_n \in \tilde{E}_M$, then $\left(\sup_{p \in I} a_n^p \right)_n \in \tilde{E}_M$,
- iii. For a positive sequence λ , we define, for all sequences $a \in \mathbb{R}^{\mathbb{N}}$, $\rho_\lambda(a)$ by

$$[\rho_\lambda(a)]_n = \frac{\lambda_n}{\lambda_{n+1}} a_{n+1}.$$

Then, if $(\lambda_n)_n$ is non decreasing and $\left(\frac{\lambda_n}{n}\right)_n$ is non increasing, \tilde{E}_M is stable by ρ_λ .

Proof. (i) and (ii) are easy to prove. Let us check (iii).

Since $(a_n)_n \in \tilde{E}_M$, we have for all $n \in \mathbb{N}^*$

$$\begin{aligned} [\rho_\lambda(a)]_n &= \frac{\lambda_n}{\lambda_{n+1}} a_{n+1} \\ &\leq M \frac{\lambda_n}{\lambda_{n+1}} \sum_{k \geq n+1} \frac{1}{k} a_k \\ &= M \frac{\lambda_n}{\lambda_{n+1}} \sum_{k \geq n} \frac{1}{k+1} a_{k+1} \\ &= M \frac{\lambda_n}{\lambda_{n+1}} \sum_{k \geq n} \frac{1}{k} \frac{k \lambda_{k+1}}{(k+1) \lambda_k} [\rho_\lambda(a)]_k \\ &\leq M \sum_{k \geq n} \frac{1}{k} [\rho_\lambda(a)]_k \end{aligned}$$

because $\left(\frac{\lambda_k}{k}\right)_k$ is non increasing and $(\lambda_n)_n$ is non decreasing. □

Define for $c, \alpha > 0$,

$$A_c^\alpha = \left\{ (a_n)_{n \in \mathbb{N}^*}, \exists n_0 \in \mathbb{N}^* \text{ such that } \forall n < n_0, a_n \leq a_{n+1} \right. \\ \left. \text{and } \forall n \geq n_0, ca_{n_0} \left(\frac{n_0}{n} \right)^\alpha \leq a_n \leq \frac{1}{c} a_{n_0} \left(\frac{n_0}{n} \right)^\alpha \right\}.$$

Lemma C.4. For all $(a_n)_n \in A_c^\alpha$ and all $n \geq n_0$ (where n_0 is given by the def of A_c^α),

$$a_n \simeq \sum_{k \geq n} \frac{1}{k} a_k.$$

In particular, there exists M (only depending on α and c) such that $A_c^\alpha \subset \tilde{E}_M$.

Proof. One has if $(a_n)_n \in A_c^\alpha$ and $n \geq n_0$,

$$\sum_{k \geq n} \frac{1}{k} a_k \simeq a_{n_0} n_0^\alpha \sum_{k \geq n} \frac{1}{k^{\alpha+1}} \\ \simeq a_{n_0} \left(\frac{n_0}{n} \right)^\alpha \\ \simeq a_n.$$

□

We are now ready for the proof of Proposition C.2.

Proof. (of Proposition C.2)

According to Lemma C.3 and Lemma C.4, we only need to prove that

$$\mathcal{A}_0 = \left\{ \left(l^{\beta-\eta} \frac{\exp\left(-c \frac{4^j r^2}{l}\right)}{l^{1+n}} \right)_{l \in \mathbb{N}^*}, r \in \mathbb{N}^*, j \geq 2 \right\}$$

is in some A_c^α . Indeed, once we proved $\mathcal{A}_0 \subset A_c^\alpha$, Lemma C.4 implies that there exists $M > 0$ such that $\mathcal{A}_0 \subset \tilde{E}_M$. The use of Lemma C.3(iii) with $\lambda_l = l^{\beta-\eta}$ yields, since $\beta - \eta \in (0, 1]$,

$$\mathcal{A}_1 := \bigcup_{k \in \mathbb{N}} (\rho_\lambda)^k (\mathcal{A}_0) = \left\{ \left(l^{\beta-\eta} \frac{\exp\left(-c \frac{4^j r^2}{l+k}\right)}{(l+k)^{1+n}} \right)_{l \in \mathbb{N}^*}, r \in \mathbb{N}^*, k \in \mathbb{N}, j \geq 2 \right\} \subset \tilde{E}_M.$$

Lemma C.3(ii) thus provides that $\mathcal{A} \subset \tilde{E}_M$ and Lemma C.3(i) that $\mathcal{A} \subset E_M$.

It remains to prove that $\mathcal{A}_0 \subset A_c^\alpha$. The result is a consequence of the following facts.

For $\gamma \in [0, 1]$ and $n \geq 1$, the function

$$F : t \in \mathbb{R}_+ \mapsto t^\gamma \frac{\exp\left(-\frac{d}{t}\right)}{t^{n+1}}$$

satisfies

- $F(0) = 0$ and $\lim_{t \rightarrow +\infty} F(t) = 0$,
- F reaches its unique maximum at $t_0 = \frac{d}{n+1-\gamma}$,
- $\frac{e^{\gamma-n-1}}{t^{n+1-\gamma}} \leq F(t) \leq \frac{1}{t^{n+1-\gamma}}$ for all $t \geq t_0$.

□

Remark C.5. If $\beta \in (-\frac{1}{2}, 0]$ and

$$B_l^{k,r,j} = l^{\beta+\frac{1}{2}} \sup_{s \in \llbracket 0, nr^2 \rrbracket} \left\{ \frac{\exp\left(-c \frac{4^j r^2}{l+k+s}\right)}{(l+k+s)^{n+\frac{1}{2}}} \right\},$$

a careful inspection of the proof of Proposition C.2 shows that the conclusion of Proposition C.2 also holds for $B_l^{k,r,j}$.

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