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# Extreme-value theory applied to boundary estimation

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## Abstract

In this paper, the problem of nonparametric estimation of boundaries is investigated from the perspective of extreme value theory. This enables us to revisit the asymptotic theory of the popular three disposal hull estimator in a more general setting, to derive new estimators and to provide useful asymptotic results for the boundary function.

## 1 Introduction

In this paper, we consider  $x \in \mathbb{R}_+^p$ , a vector of production factors (inputs) used to produce a single quantity (output)  $y \in \mathbb{R}_+$ . The attainable production set is then defined as  $\mathbb{P} = \{(x, y) \in \mathbb{R}_+^p \times \mathbb{R}_+ \mid x \text{ can produce } y\}$ . Assumptions are usually made on this set, such as three disposability of inputs and outputs, meaning that if  $(x, y) \in \mathbb{P}$ , then  $(x', y') \in \mathbb{P}$  for any  $(x', y')$  such that  $x' \geq x$  and  $y' \leq y$ . To the extent that the efficiency of a firm is a concern, the boundary of  $\mathbb{P}$  is of interest. The efficient boundary of  $\mathbb{P}$  is the locus of optimal production plans (maximal achievable output for a given level of inputs). In our setup, the production boundary is represented by the graph of the production function  $\theta(x) = \sup\{y \mid (x, y) \in \mathbb{P}\}$ . The economic efficiency score of a firm operating at the level  $(x, y)$  is then given by the ratio  $\theta(x)/y$ .

Cazals *et al.* [12] proposed a probabilistic interpretation of the production boundary. Let  $\mathbb{P}$  be the support of the joint distribution of a random vector  $(X, Y) \in \mathbb{R}_+^p \times \mathbb{R}_+$ . The distribution function of  $(X, Y)$  can be denoted  $F(x, y)$  and  $F(\cdot|x) = F(x, \cdot)/F_X(x)$  will be used to denote the conditional distribution function of  $Y$  given  $X \leq x$ , with  $F_X(x) = F(x, \infty) > 0$ . It has been proven in [12] that

$$\theta(x) = \sup\{y \geq 0 \mid F(y|x) < 1\}$$

is a monotone non-decreasing function with  $x$ . So, for all  $x' \geq x$  with respect to the partial order,  $\theta(x') \geq \theta(x)$ . The graph of  $\theta$  is the smallest non-decreasing surface which is greater than or equal to the upper boundary of  $\mathbb{P}$ . Further, it has been shown that under the three disposability assumption,  $\theta \equiv \bar{\theta}$ , that is, the graph of  $\theta$  coincides with the production boundary.

Since  $\mathbb{P}$  is unknown, it must be estimated from a sample of i.i.d. firms  $\mathcal{X}_n = \{(X_i, Y_i) \mid i = 1, \dots, n\}$ . The *three disposal hull* (3DH) is  $\hat{\mathbb{P}}_{3\text{DH}} = \{(x, y) \in \mathbb{R}_+^{p+1} \mid y \leq Y_i, x \geq X_i, i = 1, \dots, n\}$ . The resulting 3DH estimator of  $\theta(x)$  is

$$\hat{\theta}_1(x) = \sup\{y \geq 0 \mid \hat{F}(y|x) < 1\} = \max_{i, X_i \leq x} Y_i,$$

where  $\hat{F}(y|x) = \hat{F}_n(x, y)/\hat{F}_X(x)$  with  $\hat{F}_n(x, y) = (1/n) \sum_{i=1}^n \mathbb{I}(X_i \leq x, Y_i \leq y)$  and  $\hat{F}_X(x) = \hat{F}_n(x, \infty)$ . This estimator represents the lowest monotone step function covering all of the data points  $(X_i, Y_i)$ . To summarize, under regularity conditions, the 3DH estimator  $\hat{\theta}_1(x)$  is consistent and converges to a Weibull distribution with some unknown parameters.

In this paper, we first analyze the properties of the 3DH estimator from an extreme value theory perspective. First, we provide the necessary and sufficient condition for the 3DH estimator to converge in distribution and we specify the asymptotic distribution with the appropriate rate of convergence. We also provide a limit theorem for moments in a general framework. Second, we show how the unknown parameter  $\rho_x > 0$ , involved in the necessary and sufficient extreme value conditions, is linked to the dimension  $p + 1$  of the data and to the shape parameter  $\beta > -1$  of the joint density: in the general setting where  $p \geq 1$  and  $\beta = \beta_x$  may depend on  $x$ , we obtain, under a convenient regularity condition, the general convergence rate  $n^{-1/\rho_x} = n^{-1/(\beta_x + p + 1)}$  of the 3DH estimator  $\hat{\theta}_1(x)$ . Third, we suggest a strongly consistent and asymptotically normal estimator of the unknown parameter  $\rho_x$  of the asymptotic Weibull distribution of  $\hat{\theta}_1(x)$ . This also answers the important question of how to estimate the shape parameter  $\beta_x$  of the joint density of  $(X, Y)$  when it approaches the boundary of the support  $\mathbb{P}$ . This branch of statistics has become very active these past ten years, the main contributions to this domain are listed below:

- Theoretical issues: [69, 65, 67, 27, 34, 20, 33, 35, 32, 5, 28, 13, 53, 26, 4, 1, 86, 85],
- Quantile regression: [58, 19, 70]
- Application to finance: [10, 3, 37, 11, 60, 84],
- Edge estimation [55, 54],
- Frontier estimation [23, 68, 66, 64, 82, 79, 81, 76, 16, 14, 15, 9, 2, 41, 8, 63, 17, 74, 73, 21, 83, 62, 22, 50, 38, 61, 24, 25, 72, 6, 77, 12, 57, 59, 71, 56, 45, 47, 18, 31, 75, 78, 80, 40],
- Boundary estimation [46, 49, 48, 44, 43, 42, 51, 30, 52, 7, 36, 39].

By construction, the 3DH estimator is very non-robust to extremes. Recently, Aragon *et al.* [2] constructed an original estimator of  $\theta(x)$ , which is more robust than  $\hat{\theta}_1(x)$ , but which keeps the same limiting Weibull distribution as  $\hat{\theta}_1(x)$  under the restrictive condition  $\beta = 0$ . In this paper, we provide further insights and generalize their main result. We also suggest attractive estimators of  $\theta(x)$

converging to a normal distribution, which appear to be robust to outliers. The paper is organized as follows. Section 2 presents the main results of the paper. Section 3 concludes the paper, with proofs deferred for the Appendix.

## 2 Main results

From now on, we assume that  $x \in \mathbb{R}_+^p$  such that  $F_X(x) > 0$  and will denote by  $\theta_\alpha(x)$  and  $\hat{\theta}_\alpha(x)$ , respectively, the  $\alpha$ -quantiles of the distribution function  $F(\cdot|x)$  and its empirical version  $\hat{F}(\cdot|x)$ ,

$$\theta_\alpha(x) = \inf\{y \geq 0 \mid F(y|x) \geq \alpha\} \quad \text{and} \quad \hat{\theta}_\alpha(x) = \inf\{y \geq 0 \mid \hat{F}(y|x) \geq \alpha\}$$

with  $\alpha \in ]0, 1]$ . When  $\alpha \uparrow 1$ , the conditional quantile  $\theta_\alpha(x)$  tends to  $\theta_1(x)$ , which coincides with the boundary function  $\theta(x)$ . Likewise,  $\hat{\theta}_\alpha(x)$  tends to the 3DH estimator  $\hat{\theta}_1(x)$  of  $\theta(x)$  as  $\alpha \uparrow 1$ .

We first derive the following interesting results on the problem of convergence in distribution of suitably normalized maxima  $b_n^{-1}(\hat{\theta}_1(x) - \theta(x))$ . We will denote by  $\Gamma(\cdot)$  the gamma function.

**Theorem 2.1** (i) *If there exist  $b_n > 0$  and some non-degenerate distribution function  $G_x$  such that*

$$b_n^{-1}(\hat{\theta}_1(x) - \theta(x)) \xrightarrow{d} G_x, \quad (1)$$

*then  $G_x(y)$  coincides with  $\Psi_{\rho_x}(y) = \exp\{-(-y)^{\rho_x}\}$  with support  $]-\infty, 0]$  for some  $\rho_x > 0$ .*

(ii) *There exists  $b_n > 0$  such that  $b_n^{-1}(\hat{\theta}_1(x) - \theta(x))$  converges in distribution if and only if*

$$\lim_{t \rightarrow \infty} \{1 - F(\theta(x) - 1/tz \mid x)\} / \{1 - F(\theta(x) - 1/t \mid x)\} = z^{-\rho_x} \quad \text{for all } z > 0 \quad (2)$$

*(regular variation with exponent  $-\rho_x$ , notation  $1 - F(\theta(x) - \frac{1}{t} \mid x) \in RV_{-\rho_x}$ ).*

*In this case, the norming constants  $b_n$  can be chosen as  $b_n = \theta(x) - \theta_{1-(1/nF_X(x))}(x)$ .*

(iii) *Given (2),  $\lim_{n \rightarrow \infty} \mathbb{E}\{b_n^{-1}(\theta(x) - \hat{\theta}_1(x))\}^k = \Gamma(1 + k\rho_x^{-1})$  for all integers  $k \geq 1$  and*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left[ \frac{\hat{\theta}_1(x) - \mathbb{E}(\hat{\theta}_1(x))}{\{\text{Var}(\hat{\theta}_1(x))\}^{1/2}} \leq y \right] \\ &= \Psi_{\rho_x}[\{\Gamma(1 + 2\rho_x^{-1}) - \Gamma^2(1 + \rho_x^{-1})\}^{1/2}y - \Gamma(1 + \rho_x^{-1})]. \end{aligned}$$

Since the function  $t \mapsto F_X(x)[1 - F(\theta(x) - \frac{1}{t} \mid x)] \in RV_{-\rho_x}$  (regularly varying in  $t \rightarrow \infty$ ) by (2), this function can be represented as  $t^{-\rho_x} L_x(t)$  with  $L_x(\cdot) \in RV_0$  ( $L_x$  being slowly varying) and so the extreme value condition (2) holds if and only if we have the following representation:

$$F_X(x)[1 - F(y|x)] = L_x(\{\theta(x) - y\}^{-1})(\theta(x) - y)^{\rho_x} \quad \text{as } y \uparrow \theta(x). \quad (3)$$

In the particular case where  $L_x(\{\theta(x) - y\}^{-1}) = \ell_x$  is a strictly positive function in  $x$ , it is shown in the next corollary that  $b_n \sim (n\ell_x)^{-1/\rho_x}$ . From now on, a random variable  $W$  is said to follow the distribution Weibull( $1, \rho_x$ ) if  $W^{\rho_x}$  is exponential with parameter 1.

**Corollary 2.1** *Given (3) or, equivalently, (2) with  $L_x(\{\theta(x) - y\}^{-1}) = \ell_x > 0$ , we have*

$$(n\ell_x)^{1/\rho_x} (\theta(x) - \hat{\theta}_1(x)) \xrightarrow{d} \text{Weibull}(1, \rho_x) \quad \text{as } n \rightarrow \infty.$$

We assume the differentiability of the functions  $\ell_x, \rho_x$  with  $\rho_x > p$  and  $\theta(x)$  in order to ensure the existence of the joint density near its support boundary. We distinguish between three different behaviors of this density at the boundary point  $(x, \theta(x)) \in \mathbb{R}^{p+1}$  based on how the value of  $\rho_x$  compares to the dimension  $(p+1)$ : when  $\rho_x > p+1$ , the joint density decays to zero at a speed of power  $\rho_x - (p+1)$  of the distance from the boundary; when  $\rho_x = p+1$ , the density has a sudden jump at the boundary; when  $\rho_x < p+1$ , the density increases toward infinity at a speed of power  $\rho_x - (p+1)$  of the distance from the boundary. The case  $\rho_x \leq p+1$  corresponds to sharp or fault-type boundaries.

As an immediate consequence, we obtain the convergence in distribution of the 3DH estimator with the convergence rate  $n^{-1/(\beta+2)}$ . In the other particular case where the joint density is strictly positive on the boundary, we achieve the best rate of convergence  $n^{-1/(p+1)}$ . We answer the important question of how to estimate the shape parameter  $\beta_x$  or, equivalently, the regular variation exponent  $\rho_x$  in (2).

As an immediate consequence of Theorem 2.1(iii), we obtain

$$\begin{aligned} \mathbb{E}\{\theta(x) - \hat{\theta}_1(x)\}^k &= k\{\beta_x + p + 1\}^{-1} \{n\ell_x\}^{-k/(\beta_x + p + 1)} \Gamma(k\{\beta_x + p + 1\}^{-1}) \\ &\quad + o(n^{-k/(\beta_x + p + 1)}). \end{aligned} \quad (4)$$

The result (4) also reflects the well-known curse of dimensionality from which the 3DH estimator  $\hat{\theta}_1(x)$  suffers as the number  $p$  of inputs-usage increases, as pointed out earlier in the particular case where  $\beta_x = 0$ .

By an appropriate choice of  $\alpha$  as a function of  $n$ , Aragon *et al.* [2] have shown that  $\hat{\theta}_\alpha(x)$  estimates the full boundary  $\theta(x)$  itself and converges to the same Weibull distribution as the 3DH  $\hat{\theta}_1(x)$ . The next theorem provides further insights and generalizes their main result.

**Theorem 2.2** (i) *If  $b_n^{-1}(\hat{\theta}_1(x) - \theta(x)) \xrightarrow{d} G_x$ , then for any fixed integer  $k \geq 0$ ,*

$$b_n^{-1}(\hat{\theta}_{1-k/(n\hat{F}_X(x))}(x) - \theta(x)) \xrightarrow{d} H_x \quad \text{as } n \rightarrow \infty$$

for the distribution function  $H_x(y) = G_x(y) \sum_{i=0}^k (-\log G_x(y))^i / i!$ .

- (ii) Suppose that the upper bound of the support of  $Y$  is finite. If  $b_n^{-1}(\hat{\theta}_1(x) - \theta(x)) \xrightarrow{d} G_x$ , then  $b_n^{-1}(\hat{\theta}_{\alpha_n}(x) - \theta(x)) \xrightarrow{d} G_x$  for all sequences  $\alpha_n \rightarrow 1$  satisfying  $nb_n^{-1}(1 - \alpha_n) \rightarrow 0$ .

When  $\hat{\theta}_1(x)$  converges in distribution, the estimator  $\hat{\theta}_{\alpha_n}(x)$ , for  $\alpha_n := 1 - k/n\hat{F}_X(x) < 1$  (that is,  $k = 1, 2, \dots$ , in Theorem 2.2(i)), estimates  $\theta(x)$  itself and also converges in distribution, with the same scaling, but a different limit distribution (here,  $nb_n^{-1}(1 - \alpha_n) \xrightarrow{\text{a.s.}} \infty$ ). To recover the same limit distribution as the 3DH estimator, it suffices to require that  $\alpha_n \rightarrow 1$  rapidly so that  $nb_n^{-1}(1 - \alpha_n) \rightarrow 0$ . This extends the main result of Aragon *et al.* ([2], Theorem 4.3), where the convergence rate achieves  $n^{-1/(p+1)}$  under the restrictive assumption that the density of  $(X, Y)$  is strictly positive on the boundary. Note, also, that the estimate  $\hat{\theta}_{\alpha_n}$  does not envelop all of the data points providing a robust alternative to the 3DH boundary  $\hat{\theta}_1$ ; see [24] for an analysis of its quantitative and qualitative robustness properties.

The important question of how to estimate  $\rho_x$  from the multivariate random sample  $\mathcal{X}_n$  is very similar to the problem of estimating the so-called *extreme value index*, which is based on a sample of *univariate* random variables. An attractive estimation method has been proposed, which can be easily adapted to our conditional approach: let  $k = k_n$  be a sequence of integers tending to infinity and let  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . A Pickands-type estimate of  $\rho_x$  can be derived as

$$\hat{\rho}_x = \log 2 \left( \log \frac{\hat{\theta}_{1-(2k-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(4k-1)/(n\hat{F}_X(x))}(x)}{\hat{\theta}_{1-(k-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k-1)/(n\hat{F}_X(x))}(x)} \right)^{-1}.$$

The following result is particularly important since it allows the hypothesis  $\rho_x > 0$  to be tested and will later be employed to derive asymptotic confidence intervals for  $\theta(x)$ .

**Theorem 2.3** (i) If (2) holds,  $k_n \rightarrow \infty$  and  $k_n/n \rightarrow 0$ , then  $\hat{\rho}_x \xrightarrow{p} \rho_x$ .

- (ii) If (2) holds,  $k_n/n \rightarrow 0$  and  $k_n/\log \log n \rightarrow \infty$ , then  $\hat{\rho}_x \xrightarrow{\text{a.s.}} \rho_x$ .

- (iii) Assume that  $U(t) := \theta_{1-1/(tF_X(x))}(x)$ ,  $t > \frac{1}{F_X(x)}$ , has a positive derivative and that there exists a positive function  $A(\cdot)$  such that for  $z > 0$ ,  $\lim_{t \rightarrow \infty} \{(tz)^{1+1/\rho_x} U'(tz) - t^{1+1/\rho_x} U'(t)\} / A(t) = \pm \log(z)$ , for either choice of the sign ( $\Pi$ -variation, which will in the sequel be denoted by:  $\pm t^{1+1/\rho_x} U'(t) \in \Pi(A)$ ). Then,

$$\sqrt{k_n}(\hat{\rho}_x - \rho_x) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\rho_x)), \quad (5)$$

with asymptotic variance  $\sigma^2(\rho_x) = \rho_x^2(2^{1-2/\rho_x} + 1) / \{(2^{-1/\rho_x} - 1) \log 4\}^2$ , for  $k_n \rightarrow \infty$  satisfying  $k_n = o(n/g^{-1}(n))$ , where  $g^{-1}$  is the generalized inverse function of  $g(t) = t^{3+2/\rho_x} \{U'(t)/A(t)\}^2$ .

(iv) If, for some  $\kappa > 0$  and  $\delta > 0$ , the function  $\{t^{\rho_x-1}F'(\theta(x) - \frac{1}{t} \mid x) - \delta\} \in RV_{-\kappa}$ , then (5) holds with  $g(t) = t^{3+2/\rho_x}\{U'(t)/(t^{1+1/\rho_x}U'(t) - [\delta F_X(x)]^{-1/\rho_x}(\rho_x)^{1/\rho_x-1})\}^2$ .

Note that the second order regular variation conditions (iii) and (iv) of Theorem 2.3 are difficult to check in practice, which makes the theoretical choice of the sequence  $\{k_n\}$  a hard problem. In practice, in order to choose a reasonable estimate  $\hat{\rho}_x(k_n)$  of  $\rho_x$ , one can construct the plot of  $\hat{\rho}_x$ , consisting of the points  $\{(k, \hat{\rho}_x(k)), 1 \leq k < n\hat{F}_X(x)/4\}$ , and select a value of  $\rho_x$  at which the obtained graph looks stable. This technique is known as the *Pickands plot* in the univariate extreme value literature.

We can also easily adapt the well-known moment estimator for the index of a univariate extreme value distribution to our conditional setup. Define

$$M_n^{(j)} = \frac{1}{k} \sum_{i=0}^{k-1} (\log \hat{\theta}_{1-i/(n\hat{F}_X(x))}(x) - \log \hat{\theta}_{1-k/(n\hat{F}_X(x))}(x))^j$$

for each  $j = 1, 2$  and  $k = k_n < n$ .

We can then define the moment-type estimator for the conditional regular-variation exponent  $\rho_x$  as

$$\tilde{\rho}_x = -\left\{M_n^{(1)} + 1 - \frac{1}{2}[1 - (M_n^{(1)})^2/M_n^{(2)}]^{-1}\right\}^{-1}.$$

**Theorem 2.4** (i) If (2) holds,  $k_n/n \rightarrow 0$  and  $k_n \rightarrow \infty$ , then  $\tilde{\rho}_x \xrightarrow{p} \rho_x$ .

(ii) If (2) holds,  $k_n/n \rightarrow 0$  and  $k_n/(\log n)^\delta \rightarrow \infty$  for some  $\delta > 0$ , then  $\tilde{\rho}_x \xrightarrow{a.s.} \rho_x$ .

(iii) If  $\pm t^{1/\rho_x}\{\theta(x) - U(t)\} \in \Pi(B)$  for some positive function  $B$ , then  $\sqrt{k_n}(\tilde{\rho}_x - \rho_x)$  has, asymptotically, a normal distribution with mean zero and variance

$$\rho_x(2 + \rho_x)(1 + \rho_x)^2 \left\{ 4 - 8 \frac{(2 + \rho_x)}{(3 + \rho_x)} + \frac{(11 + 5\rho_x)(2 + \rho_x)}{(3 + \rho_x)(4 + \rho_x)} \right\}$$

for  $k_n \rightarrow \infty$  satisfying  $k_n = o(n/g^{-1}(n))$ , where  $g(t) = t^{1+2/\rho_x}[\{\log \theta(x) - \log U(t)\}/B(t)]^2$ .

The next theorem enables the construction of confidence intervals for  $\theta(x)$  and for high quantile-type boundaries  $\theta_{1-p_n/F_X(x)}(x)$  when  $p_n \rightarrow 0$  and  $np_n \rightarrow \infty$ .

**Theorem 2.5** (i) Suppose that  $F(\cdot|x)$  has a positive density  $F'(\cdot|x)$  such that  $F'(\theta(x) - \frac{1}{t} \mid x) \in RV_{1-\rho_x}$ . Then,

$$\frac{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \theta_{1-p_n/F_X(x)}(x)}{\sqrt{2k_n} \frac{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x)}} \xrightarrow{d} \mathcal{N}(0, V_1(\rho_x)),$$

where  $V_1(\rho_x) = \rho_x^{-2} 2^{1-2/\rho_x} / (2^{-1/\rho_x} - 1)^2$ , provided that  $p_n \rightarrow 0$ ,  $np_n \rightarrow \infty$  and  $k_n = \lceil np_n \rceil$ .

(ii) Suppose that the conditions of Theorem 2.3(iii) or (iv) hold, and define

$$\begin{aligned}\hat{\theta}_1^*(x) &:= (2^{1/\hat{\rho}_x} - 1)^{-1} \{ \hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x) \} \\ &\quad + \hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x).\end{aligned}$$

Then, putting  $V_2(\rho_x) = 3\rho_x^{-2}2^{-1-2/\rho_x}/(2^{-1/\rho_x} - 1)^6$ , we have

$$\sqrt{2k_n} \frac{\hat{\theta}_1^*(x) - \theta(x)}{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x)} \xrightarrow{d} \mathcal{N}(0, V_2(\rho_x)).$$

(iii) Suppose that the conditions of Theorem 2.3(iii) or (iv) hold, and define

$$\begin{aligned}\tilde{\theta}_1^*(x) &:= (2^{1/\rho_x} - 1)^{-1} \{ \hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x) \} \\ &\quad + \hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x).\end{aligned}$$

Then, putting  $V_3(\rho_x) = \rho_x^{-2}2^{-2/\rho_x}/(2^{-1/\rho_x} - 1)^4$ , we have

$$\begin{aligned}&\sqrt{2k_n} \frac{\tilde{\theta}_1^*(x) - \theta(x)}{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x)} \\ &\xrightarrow{d} \mathcal{N}(0, V_3(\rho_x)), \\ &\{ \hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x) \} / \left\{ \frac{n}{2k_n} U' \left( \frac{n}{2k_n} \right) \right\}^{(6)} \\ &\xrightarrow{p} \rho_x (1 - 2^{-1/\rho_x}).\end{aligned}$$

Note that Theorem 2.5(ii) is still valid if the estimate  $\hat{\rho}_x$  is replaced by the true value  $\rho_x$ , up to a change of the asymptotic variance. It is easy to see that  $V_2(\rho_x) \geq V_3(\rho_x)$  and so the estimator  $\hat{\theta}_1^*(x)$  of  $\theta(x)$  is asymptotically more efficient than  $\tilde{\theta}_1^*(x)$ . We also conclude from (6) that  $\tilde{\theta}_1^*(x)$  and  $\hat{\theta}_1^*(x)$  have the same rate of convergence, namely  $nU'(\frac{n}{2k_n})/(2k_n)^{3/2}$ . In the particular case where  $L_x(\{\theta(x) - y\}^{-1}) = \ell_x$  in (3), we have  $U'(\frac{n}{2k_n}) = \frac{1}{\rho_x}(\frac{1}{\ell_x})^{1/\rho_x}(\frac{2k_n}{n})^{1+1/\rho_x}$ . Note, also, that in this particular case, the condition of Theorem 2.5(i) holds, that is,  $F'(\theta(x) - \frac{1}{t} \mid x) = \frac{\ell_x \rho_x}{F_X(x)}(\frac{1}{t})^{\rho_x-1} \in RV_{1-\rho_x}$ . However, the conditions of Theorem 2.3(iii) and (iv) do not hold since both functions  $t^{1+1/\rho_x}U'(t) = \frac{1}{\rho_x}(\frac{1}{\ell_x})^{1/\rho_x}$  and  $t^{\rho_x-1}F'(\theta(x) - \frac{1}{t} \mid x) = \frac{\ell_x \rho_x}{F_X(x)}$  are constant in  $t$ . Nevertheless, the conclusions of Theorem 2.3(iii) and (iv) hold in this case for all sequences  $k_n \rightarrow \infty$  satisfying  $\frac{k_n}{n} \rightarrow 0$ . The same is true for the conclusion of Theorem 2.5(ii).

**Theorem 2.6** *If the condition of Corollary 2.1 holds,  $k_n \rightarrow \infty$  and  $k_n/n \rightarrow 0$  as  $n \rightarrow \infty$ , then*

$$\begin{aligned}&\{\rho_x k_n^{1/2}/(k_n/n\ell_x)^{1/\rho_x}\} [\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) + (k_n/n\ell_x)^{1/\rho_x} - \theta(x)] \\ &\xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty.\end{aligned}$$



The optimization of the asymptotic mean-squared error of  $\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x)$  is not an appropriate criteria for selecting the optimal  $k_n$  since the resulting value of  $k_n$  does not depend on  $n$ .

We shall now construct asymptotic confidence intervals for both  $\theta(x)$  and  $\theta_{1-p_n/F_X(x)}(x)$ , using the sums  $M_n^{(1)}$  and  $M_n^{(2)}$ .

**Theorem 2.7** (i) *Under the conditions of Theorem 2.5(i),*

$$\sqrt{k_n} \frac{\hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x) - \theta_{1-p_n/F_X(x)}(x)}{M_n^{(1)} \hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x)} \xrightarrow{d} \mathcal{N}(0, V_4(\rho_x)),$$

where  $V_4(\rho_x) = (1 + 1/\rho_x)^2$ , provided that  $p_n \rightarrow 0$ ,  $np_n \rightarrow \infty$  and  $k_n = [np_n]$ .

(ii) *Suppose that the conditions of Theorem 2.4(iii) hold and that  $U(\cdot)$  has a regularly varying derivative  $U' \in RV_{-\rho_x}$ . Define the moment estimator  $\hat{\theta}(x) = \hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x) \{1 + M_n^{(1)}(1 + \tilde{\rho}_x)\}$ . Then,*

$$\sqrt{k_n} \frac{\hat{\theta}(x) - \theta(x)}{M_n^{(1)}(1 + 1/\tilde{\rho}_x) \hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x)} \xrightarrow{d} \mathcal{N}(0, V_5(\rho_x)),$$

$$V_5(\rho_x) = \rho_x^2 \left[ \frac{\rho_x}{(2 + \rho_x)} + \rho_x(2 + \rho_x) \left\{ 4 - 8 \frac{(2 + \rho_x)}{(3 + \rho_x)} + \frac{(11 + 5\rho_x)(2 + \rho_x)}{(3 + \rho_x)(4 + \rho_x)} \right\} - \frac{4\rho_x}{(3 + \rho_x)} \right].$$

We consider the case where the support boundary is linear. We choose  $(X, Y)$  uniformly distributed over the region  $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ . In this case (see, for example, [24]), it is easy to see that  $\theta(x) = x$  and  $F_X(x)[1 - F(y|x)] = (\theta(x) - y)^2$  for all  $0 \leq y \leq \theta(x)$ . Thus,  $L_x(\cdot) = \ell_x = 1$  and  $\rho_x = 2$  for all  $x$ . Therefore, the conclusions of all Theorems 2.1–2.6 hold.

We now choose a nonlinear monotone upper boundary given by the Cobb–Douglas model  $Y = X^{1/2} \exp(-U)$ , where  $X$  is uniform on  $[0, 1]$  and  $U$ , independent of  $X$ , is exponential with parameter  $\lambda = 3$  (see, for example, [24]). Here, the boundary function is  $\theta(x) = x^{1/2}$  and the conditional distribution function is  $F(y|x) = 3x^{-1}y^2 - 2x^{-3/2}y^3$  for  $0 < x \leq 1$  and  $0 \leq y \leq \theta(x)$ . It is then easily seen that the extreme value condition (2) or, equivalently, (3) holds with  $\rho_x = 2$  and  $L_x(z) = F_X(x)[3\theta(x) - \frac{2}{z}]/[\theta(x)]^3$  for all  $x \in ]0, 1]$  and  $z > 0$ .

### 3 Concluding remarks

In our approach, we provide the necessary and sufficient condition for the 3DH estimator  $\hat{\theta}_1(x)$  to converge in distribution, we specify its asymptotic distribution with the appropriate convergence rate and provide a limit theorem for moments in a general framework. We also provide further insights and generalize the main result of [2] on robust variants of the 3DH estimator, and we

provide strongly consistent and asymptotically normal estimators  $\hat{\rho}_x$  and  $\tilde{\rho}_x$  of the unknown conditional tail index  $\rho_x$  involved in the limit law of  $\hat{\theta}_1(x)$ . Moreover, when the joint density of  $(X, Y)$  decreases to zero or increases toward infinity at a speed of power  $\beta_x > -1$  of the distance from the boundary, as is often assumed in the literature, we answer the question of how  $\rho_x$  is linked to the data dimension  $p + 1$  and to the shape parameter  $\beta_x$ . The quantity  $\beta_x \neq 0$  describes the rate at which the density tends to infinity (in the case  $\beta_x < 0$ ) or to 0 (in the case  $\beta_x > 0$ ) at the boundary. When  $\beta_x = 0$ , the joint density is strictly positive on the boundary. We establish that  $\rho_x = \beta_x + (p + 1)$ .

We propose new extreme-value-based boundary estimators  $\hat{\theta}_1^*(x)$ ,  $\tilde{\theta}_1^*(x)$  and  $\hat{\theta}(x)$ , which are asymptotically normally distributed and provide useful asymptotic confidence bands for the monotone boundary function  $\theta(x)$ . These estimators have the advantage of not being limited to a bi-dimensional support and benefit from their explicit and easy formulations, which is not the case for estimators defined by optimization problems, such as local polynomial estimators. Their asymptotic normality is derived under quite natural and general extreme value conditions, without Lipschitz conditions on the boundary and without recourse to assumptions either on the marginal distribution of  $X$  or on the conditional distribution of  $Y$  given  $X = x$ , as is often the case in both statistical and econometrics literature on boundary estimation. The study of the asymptotic properties of the different estimators considered in the present paper is easily carried out by relating them to a simple dimensionless random sample and then applying standard extreme value theory (for example, [29]).

It should be clear that the monotonicity constraint on the boundary is the main difference with most of the existing approaches in the statistical literature. Indeed, the joint support of a random vector  $(X, Y)$  is often described in the literature as the set  $\{(x, y) \mid y \leq \theta(x)\}$ , where the graph of  $\theta$  is interpreted as its upper boundary. As a matter of fact, the function of interest,  $\theta$ , in our approach is the smallest monotone non-decreasing function which is greater than or equal to the boundary function  $\theta$ . To our knowledge, only the estimators 3DH and DEA estimate the quantity  $\theta$ . Of course,  $\theta$  coincides with  $\theta$  when the boundary curve is monotone, but the construction of estimators of the end-point  $\theta(x)$  of the conditional distribution of  $Y$  given  $X = x$  requires a smoothing procedure, which is not the case when the distribution of  $Y$  is conditioned by  $X \leq x$ .

We illustrate how the large-sample theory applies in practice by carrying out some Monte Carlo experiments. Good estimates of  $\theta(x)$  and  $\rho_x$  may require a large sample of the order of several thousand. Theoretically selecting the optimal extreme conditional quantiles  $\hat{\theta}_{\alpha(k_n(x))}$  for estimating  $\theta(x)$  and/or  $\rho_x$  is a difficult question that is worthy of future research. Here, we suggest a simple automatic data-driven method that provides a reasonable choice of the sequence  $\{k_n(x)\}$  for large samples.

The empirical study reveals that the simultaneous estimation of the tail index and of the boundary function requires large sample sizes to provide sensible results. The moment estimators of  $\rho_x$  and of  $\theta(x)$  sometimes provide better estimations than the Pickands estimates and sometimes not. When considering

bias and MSE,  $\hat{\theta}(x)$  and  $\tilde{\rho}_x$  provide more accurate estimations, but when the sample size is large enough,  $\hat{\theta}_1^*(x)$  and  $\hat{\rho}_x$  significantly improve and even seem to outperform the moment estimators. As far as the inference on  $\rho_x$  is concerned,  $\tilde{\rho}_x$  also provides quite reliable confidence intervals, but  $\hat{\rho}_x$  provides more satisfactory results for sufficiently large samples. However, when inference about the boundary function itself is concerned, the moment estimator provides very poor results compared with the Pickands estimator.

On the other hand, the performance of the estimator  $\tilde{\theta}_1^*(x)$ , computed when  $\rho_x$  is known, is quite remarkable, even compared with the popular 3DH. The confidence intervals for  $\theta(x)$  are very easy to compute and have quite good coverages. In addition, the results are quite stable with respect to the choice of the ‘smoothing’ parameter  $k_n(x)$ . As shown in our illustrations, the estimates also have the advantage of being robust to extreme values. This suggests, even if  $\rho_x$  is unknown, the use of a plug-in version of  $\tilde{\theta}_1^*(x)$  for making inference on  $\theta(x)$ : here, in a first step, we estimate  $\rho_x$  (using the moment estimator, unless  $N_x$  is large enough), then we use the asymptotic results for  $\tilde{\theta}_1^*(x)$ , as if  $\rho_x$  was known. A sensible practice is not to restrict the first step to one procedure, but rather to check that both Pickands and moment estimators point toward similar conclusions.

## Appendix: Proofs

**Proof of Theorem 2.1** Let  $Z^x = Y\mathbb{I}(X \leq x)$  and  $F_x(\cdot) = \{1 - F_X(x)[1 - F(\cdot|x)]\}\mathbb{I}(\cdot \geq 0)$ . It can be easily seen that  $\mathbb{P}(Z^x \leq y) = F_x(y)$  for any  $y \in \mathbb{R}$ . Therefore,  $\{Z_i^x = Y_i\mathbb{I}(X_i \leq x), i = 1, \dots, n\}$  is an i.i.d. sequence of random variables with common distribution function  $F_x$ . Moreover, it is easy to see that the right end-point of  $F_x$  coincides with  $\theta(x)$  and that  $\max_{i=1, \dots, n} Z_i^x$  coincides with  $\hat{\theta}_1(x)$ . Thus, assertion (i) follows from the Fisher–Tippett theorem. It is well known that the normalized maxima  $b_n^{-1}(\hat{\theta}_1(x) - \theta(x)) \xrightarrow{d} G$  (that is,  $F_x$  belongs to the domain of attraction of  $G = \Psi_{\rho_x}$ ) if and only if

$$\bar{F}_x(\theta(x) - 1/t) \in RV_{-\rho_x}, \quad (7)$$

where  $\bar{F}_x = 1 - F_x$ . This necessary and sufficient condition is equivalent to (2). In this case, the norming constant  $b_n$  can be taken to be equal to  $\theta(x) - \inf\{y \geq 0 \mid F_x(y) \geq 1 - \frac{1}{n}\} = \theta(x) - \inf\{y \geq 0 \mid F(y|x) \geq 1 - \frac{1}{nF_X(x)}\}$ , which gives assertion (ii). For assertion (iii), since (7) holds and  $\mathbb{E}[|Z^x|^k] = F_X(x)\mathbb{E}(Y^k|X \leq x) \leq \theta(x)^k$ , it is immediate that  $\lim_{n \rightarrow \infty} \mathbb{E}\{b_n^{-1}(\hat{\theta}_1(x) - \theta(x))\}^k = (-1)^k \Gamma(1 + k/\rho_x)$ .

**Proof of Corollary 2.1** Following the proof of Theorem 2.1, we can set  $b_n = \theta(x) - F_x^{-1}(1 - \frac{1}{n})$ , where  $F_x^{-1}(t) = \inf\{y \in ]0, \theta(x)], F_x(y) \geq t\}$  for all  $t \in ]0, 1]$ . It follows from (3) that  $F_x^{-1}(t) = \theta(x) - ((1 - t)/\ell_x)^{1/\rho_x}$  as  $t \uparrow 1$  and so  $b_n = (1/n\ell_x)^{1/\rho_x}$  for all  $n$  sufficiently large.

For the next proofs, we need the following lemma whose proof is quite easy and is thus omitted.

**Lemma 1** *Let  $Z_{(1)}^x \leq \dots \leq Z_{(n)}^x$  be the order statistics generated by the random variables  $Z_1^x, \dots, Z_n^x$ :*

- (i) *If  $\hat{F}_X(x) > 0$ , then  $\hat{\theta}_{1-k/(n\hat{F}_X(x))}(x) = Z_{(n-k)}^x$  for each  $k \in \{0, 1, \dots, n\hat{F}_X(x) - 1\}$ .*
- (ii) *For any fixed integer  $k \geq 0$ , we have  $\hat{\theta}_{1-k/(n\hat{F}_X(x))}(x) = Z_{(n-k)}^x$  as  $n \rightarrow \infty$ , with probability 1.*
- (iii) *For any sequence of integers  $k_n \geq 0$  such that  $k_n/n \rightarrow 0$  as  $n \rightarrow \infty$ , we have*

$$\hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x) = Z_{(n-k_n)}^x \quad \text{as } n \rightarrow \infty, \text{ with probability 1.}$$

**Proof of Theorem 2.2** (i) Since  $\theta(x) = F_x^{-1}(1)$  and  $\hat{\theta}_1(x) = Z_{(n)}^x$  for all  $n \geq 1$ , we have  $(\hat{\theta}_1(x) - \theta(x)) = (Z_{(n)}^x - F_x^{-1}(1))$ . Hence, if  $b_n^{-1}(\hat{\theta}_1(x) - \theta(x)) \xrightarrow{d} G_x$ , then  $b_n^{-1}(Z_{(n)}^x - F_x^{-1}(1))$  converges to the same distribution  $G_x$ . Therefore,  $b_n^{-1}(Z_{(n-k)}^x - F_x^{-1}(1)) \xrightarrow{d} H_x$  for any integer  $k \geq 0$ , where  $H_x(y) = G_x(y) \sum_{i=0}^k (-\log G(y))^i / i!$ . Finally, since  $Z_{(n-k)}^x \stackrel{\text{a.s.}}{=} \hat{\theta}_{1-k/(n\hat{F}_X(x))}(x)$  as  $n \rightarrow \infty$ , in view of Lemma 1(ii), we obtain  $b_n^{-1}(\hat{\theta}_{1-k/(n\hat{F}_X(x))}(x) - F_x^{-1}(1)) \xrightarrow{d} H_x$ . (ii) Writing  $b_n^{-1}(\hat{\theta}_\alpha(x) - \theta(x)) = b_n^{-1}(\hat{\theta}_\alpha(x) - \hat{\theta}_1(x)) + b_n^{-1}(\hat{\theta}_1(x) - \theta(x))$ , it suffices to find an appropriate sequence  $\alpha = \alpha_n \rightarrow 1$  such that  $b_n^{-1}(\hat{\theta}_{\alpha_n}(x) - \hat{\theta}_1(x)) \xrightarrow{d} 0$ . Aragon *et al.* [2] (see equation (20)) showed that  $|\hat{\theta}_\alpha(x) - \hat{\theta}_1(x)| \leq (1 - \alpha)n\hat{F}_X(x)F_Y^{-1}(1)$ , with probability 1, for any  $\alpha > 0$ . It thus suffices to choose  $\alpha = \alpha_n \rightarrow 1$  such that  $nb_n^{-1}(1 - \alpha_n) \rightarrow 0$ .

**Proof of Theorem 2.3** (i) Let  $\gamma_x = -1/\rho_x$  in (7). The Pickands estimate of the exponent of variation  $\gamma_x < 0$  is then given by  $\hat{\gamma}_x := (\log 2)^{-1} \log\{(Z_{(n-k+1)}^x - Z_{(n-2k+1)}^x)/(Z_{(n-2k+1)}^x - Z_{(n-4k+1)}^x)\}$ . Under (2), Condition (7) holds and so there exists  $b_n > 0$  such that  $\lim_{n \rightarrow \infty} \mathbb{P}[b_n^{-1}(Z_{(n)}^x - \theta(x)) \leq y] = \Psi_{-1/\gamma_x}(y)$ . Since this limit is unique only up to affine transformations, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[c_n^{-1}(Z_{(n)}^x - d_n) \leq y] = \Psi_{-1/\gamma_x}(-\gamma_x y - 1) = \exp\{-(1 + \gamma_x y)^{-1/\gamma_x}\}$$

for all  $y \leq 0$ , where  $c_n = -\gamma_x b_n$  and  $d_n = \theta(x) - b_n$ . Thus, condition (1.1) from Dekkers and de Haan [29] holds. Therefore,  $\hat{\gamma}_x \xrightarrow{p} \gamma_x$  if  $k_n \rightarrow \infty$  and  $\frac{k_n}{n} \rightarrow 0$ , in view of [29], Theorem 2.1. This gives the weak consistency of  $\hat{\rho}_x$  since  $\hat{\gamma}_x \stackrel{\text{a.s.}}{=} -1/\hat{\rho}_x$  as  $n \rightarrow \infty$ , in view of Lemma 1(iii).

(ii) Likewise, if  $\frac{k_n}{n} \rightarrow 0$  and  $\frac{k_n}{\log \log n} \rightarrow \infty$ , then  $\hat{\gamma}_x \xrightarrow{\text{a.s.}} \gamma_x$  via [29], Theorem 2.2, and so  $\hat{\rho}_x \xrightarrow{\text{a.s.}} \rho_x$ .

(iii) We have  $U(t) = \inf\{y \geq 0 \mid \frac{1}{1-F_x(y)} \geq t\}$ , which corresponds to the inverse function  $(1/(1-F_x))^{-1}(t)$ . Since  $\pm t^{1-\gamma_x} U'(t) \in \Pi(A)$  with  $\gamma_x = -1/\rho_x < 0$ , it follows from [29] (see Theorem 2.3) that  $\sqrt{k_n}(\hat{\gamma}_x - \gamma_x) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\gamma_x))$  with  $\sigma^2(\gamma_x) = \gamma_x^2(2^{2\gamma_x+1} + 1)/\{2(2^{\gamma_x} - 1) \log 2\}^2$  for  $k_n \rightarrow \infty$  satisfying  $k_n = o(n/g^{-1}(n))$ , where  $g(t) := t^{3-2\gamma_x} \{U'(t)/A(t)\}^2$ . By using the fact that  $\sqrt{k_n}(\hat{\rho}_x - \rho_x) \stackrel{\text{a.s.}}{=} \sqrt{k_n}(-\frac{1}{\hat{\gamma}_x} + \frac{1}{\gamma_x})$  as  $n \rightarrow \infty$ , in view of Lemma 1(iii) and applying the delta method, we conclude that  $\sqrt{k_n}(\hat{\rho}_x - \rho_x) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\rho_x))$  with asymptotic variance  $\sigma^2(\rho_x) = \sigma^2(\gamma_x)/\gamma_x^4$ .

(iv) Under the regularity condition, we have  $\pm\{t^{-1-1/\gamma_x} F'_x(\theta(x) - \frac{1}{t}) - \delta F_X(x)\} \in RV_{-\kappa}$ . The conclusion then follows immediately from Theorem 2.5 of [29] in conjunction with Lemma 1(iii).

**Proof of Theorem 2.4** We have, by Lemma 1(iii), that for each  $j = 1, 2$ ,

$$M_n^{(j)} = (1/k) \sum_{i=0}^{k-1} (\log Z_{(n-i)}^x - \log Z_{(n-k)}^x)^j \quad \text{as } n \rightarrow \infty, \text{ with probability 1;} \quad (8)$$

$-1/\tilde{\rho}_x$  then coincides almost surely, for all  $n$  large enough, with the well-known moment estimator  $\tilde{\gamma}_x$  of the index defined in (7) by  $\gamma_x = -1/\rho_x$ . Hence, Theorem 2.4(i) and (ii) follow from the weak and strong consistency of  $\tilde{\gamma}_x$ . Likewise, Theorem 2.4(iii) follows.

**Proof of Theorem 2.5** (i) Under the regularity condition, the distribution function  $F_x$  of  $Z^x$  has a positive derivative  $F'_x(y) = F_X(x)F'(y|x)$  for all  $y > 0$  such that  $F'_x(\theta(x) - \frac{1}{t}) \in RV_{1+1/\gamma_x}$ . Therefore, according to [29] (see Theorem 3.1),

$$\sqrt{2k_n} \frac{Z_{(n-k_n+1)}^x - F_x^{-1}(1-p_n)}{Z_{(n-k_n+1)}^x - Z_{(n-2k_n+1)}^x}$$

is asymptotically normal with mean zero and variance  $2^{2\gamma_x+1}\gamma_x^2/(2^{\gamma_x}-1)^2$ . We conclude by using the facts that  $F_x^{-1}(1-p_n) = \theta_{1-p_n/F_X(x)}(x)$  and

$$\begin{aligned} & \sqrt{2k_n} \frac{Z_{(n-k_n+1)}^x - F_x^{-1}(1-p_n)}{Z_{(n-k_n+1)}^x - Z_{(n-2k_n+1)}^x} \\ & \stackrel{\text{a.s.}}{=} \sqrt{2k_n} \frac{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - F_x^{-1}(1-p_n)}{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x)} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(ii) We have  $\hat{\theta}_1^*(x) \stackrel{\text{a.s.}}{=} \frac{Z_{(n-k_n+1)}^x - Z_{(n-2k_n+1)}^x}{2^{-\hat{\gamma}_x} - 1} + Z_{(n-k_n+1)}^x$  as  $n \rightarrow \infty$ . Following [29], Theorem 3.2,

$$\frac{\sqrt{2k_n}(\hat{\theta}_1^*(x) - \theta(x))}{Z_{(n-k_n+1)}^x - Z_{(n-2k_n+1)}^x}$$

is then asymptotically normal with mean zero and variance  $3\gamma_x^2 2^{2\gamma_x-1}/(2^{\gamma_x}-1)^6$ .

(iii) Let  $E_{(1)} \leq \dots \leq E_{(n)}$  be the order statistics of i.i.d. exponential variables  $E_1, \dots, E_n$ . Then,  $\{Z_{(n-k+1)}^x\}_{k=1}^n \stackrel{d}{=} \{U(e^{E_{(n-k+1)}})\}_{k=1}^n$ . Writing  $V(t) := U(e^t)$ , we obtain

$$\begin{aligned}
& \sqrt{2k_n} \left\{ \frac{1}{2^{-\gamma_x} - 1} + \frac{Z_{(n-k_n+1)}^x - \theta(x)}{Z_{(n-k_n+1)}^x - Z_{(n-2k_n+1)}^x} \right\} \\
& \stackrel{d}{=} \sqrt{2k_n} \left\{ \frac{1}{2^{-\gamma_x} - 1} + \frac{V(E_{(n-k_n+1)}) - \theta(x)}{V(E_{(n-k_n+1)}) - V(E_{(n-2k_n+1)})} \right\} \\
& = \left[ -\sqrt{2k_n} \left\{ \frac{V(\infty) - V(\log n/(2k_n))}{V'(\log n/(2k_n))} + \frac{1}{\gamma_x} \right\} \right. \\
& \quad + \sqrt{2k_n} \left\{ \frac{V(E_{(n-k_n+1)}) - V(E_{(n-2k_n+1)})}{2^{\gamma_x} V'(E_{(n-2k_n+1)})} - \frac{1 - 2^{-\gamma_x}}{\gamma_x} \right\} \frac{2^{\gamma_x}}{1 - 2^{\gamma_x}} \frac{V'(E_{(n-2k_n+1)})}{V'(\log n/(2k_n))} \\
& \quad \left. - \frac{\sqrt{2k_n}}{\gamma_x} \left\{ \frac{V'(E_{(n-2k_n+1)})}{V'(\log n/(2k_n))} - 1 - \gamma_x \frac{V(E_{(n-k_n+1)}) - V(\log n/(2k_n))}{V'(\log n/(2k_n))} \right\} \right] \\
& \quad \times \frac{V'(\log n/(2k_n))}{V(E_{(n-k_n+1)}) - V(E_{(n-2k_n+1)})}.
\end{aligned}$$

The first term on the right-hand side tends to zero as established by Dekkers and de Haan ([29], Proof of Theorem 3.2). The second term converges in distribution to  $\mathcal{N}(0, 1) \times \frac{2^{\gamma_x}}{1 - 2^{\gamma_x}}$ , in view of Lemma 3.1 and [29], Corollary 3.1. The third term converges in probability to  $\frac{\gamma_x}{2^{\gamma_x} - 1}$  by the same Corollary 3.1. This ends the proof of (iii), in conjunction with the fact that

$$\begin{aligned}
& \sqrt{2k_n} \frac{\tilde{\theta}_1^*(x) - \theta(x)}{\hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x) - \hat{\theta}_{1-(2k_n-1)/(n\hat{F}_X(x))}(x)} \\
& = \sqrt{2k_n} \left\{ \frac{1}{2^{-\gamma_x} - 1} + \frac{Z_{(n-k_n+1)}^x - \theta(x)}{Z_{(n-k_n+1)}^x - Z_{(n-2k_n+1)}^x} \right\} \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

with probability 1.

**Proof of Theorem 2.6** Write  $\bar{F}_x(y) := F_X(x)[1 - F(y|x)]$  and  $F_x(y) := 1 - \bar{F}_x(y)$  for all  $y \geq 0$ . Let  $R_x(y) := -\log\{\bar{F}_x(y)\}$  for all  $y \in [0, \theta(x)[$  and let  $E_{(n-k_n+1)}$  be the statistic of order  $n - k_n + 1$  generated by  $n$  independent standard exponential random variables.  $Z_{(n-k_n+1)}^x$  then has the same distribution as  $R_x^{-1}[E_{(n-k_n+1)}]$ , where  $R_x^{-1}(t) := \inf\{y \geq 0 \mid R_x(y) \geq t\} = \inf\{y \geq 0 \mid$

$F_x(y) \geq 1 - e^{-t}\} := F_x^{-1}(1 - e^{-t})$ . Hence,

$$\begin{aligned} & Z_{(n-k_n+1)}^x - F_x^{-1}\left(1 - \frac{k_n}{n}\right) \\ & \stackrel{d}{=} R_x^{-1}[E_{(n-k_n+1)}] - R_x^{-1}\left[\log\left(\frac{n}{k_n}\right)\right] \\ & = \left[E_{(n-k_n+1)} - \log\left(\frac{n}{k_n}\right)\right](R_x^{-1})' \left[\log\left(\frac{n}{k_n}\right)\right] \\ & \quad + \frac{1}{2} \left[E_{(n-k_n+1)} - \log\left(\frac{n}{k_n}\right)\right]^2 (R_x^{-1})''[\delta_n], \end{aligned}$$

provided that  $E_{(n-k_n+1)} \wedge \log(n/k_n) < \delta_n < E_{(n-k_n+1)} \vee \log(n/k_n)$ . By the regularity condition (3), we have that  $R_x^{-1}(t) = \theta(x) - (e^{-t}/\ell_x)^{1/\gamma_x}$  for all  $t$  large enough. Therefore, for all  $n$  sufficiently large,

$$\begin{aligned} & \{\rho_x k_n^{1/2}/(k_n/n\ell_x)^{1/\rho_x}\} [Z_{(n-k_n+1)}^x - F_x^{-1}(1 - k_n/n)] \\ & \stackrel{d}{=} k_n^{1/2} [E_{(n-k_n+1)} - \log(n/k_n)] \\ & \quad - \{\rho_x k_n^{1/2}/2\rho_x\} [E_{(n-k_n+1)} - \log(n/k_n)]^2 \exp\{-[\delta_n - \log(n/k_n)]/\rho_x\}. \end{aligned}$$

Since  $k_n^{1/2}[E_{(n-k_n+1)} - \log(n/k_n)] \xrightarrow{d} \mathcal{N}(0, 1)$  and  $|\delta_n - \log(n/k_n)| \leq |E_{(n-k_n+1)} - \log(n/k_n)| \xrightarrow{p} 0$  as  $n \rightarrow \infty$ , we obtain  $\{\rho_x k_n^{1/2}/(k_n/n\ell_x)^{1/\rho_x}\} [Z_{(n-k_n+1)}^x - F_x^{-1}(1 - k_n/n)] \xrightarrow{d} \mathcal{N}(0, 1)$  as  $n \rightarrow \infty$ . Since  $F_x^{-1}(t) = \theta(x) - ((1-t)/\ell_x)^{1/\rho_x}$  for all  $t < 1$  large enough, we have  $\theta(x) - F_x^{-1}(1 - \frac{k_n}{n}) = (k_n/n\ell_x)^{1/\rho_x}$  for all  $n$  sufficiently large. Thus,  $\{\rho_x k_n^{1/2}/(k_n/n\ell_x)^{1/\rho_x}\} \times [Z_{(n-k_n+1)}^x + (k_n/n\ell_x)^{1/\rho_x} - \theta(x)] \xrightarrow{d} \mathcal{N}(0, 1)$  as  $n \rightarrow \infty$ . We conclude by using the fact that  $Z_{(n-k_n+1)}^x \stackrel{\text{a.s.}}{=} \hat{\theta}_{1-(k_n-1)/(n\hat{F}_X(x))}(x)$  as  $n \rightarrow \infty$ .

**Proof of Theorem 2.7** (i) As shown in the proof of Theorem 2.5(i), we have  $F'_x(\theta(x) - \frac{1}{t}) \in RV_{1+1/\gamma_x}$ . Then, we get

$$\sqrt{k_n} \{Z_{(n-k_n)}^x - F_x^{-1}(1 - p_n)\} / M_n^{(1)} Z_{(n-k_n)}^x \xrightarrow{d} \mathcal{N}(0, V_4(-1/\gamma_x)).$$

The proof is completed by simply using the fact that  $F_x^{-1}(1 - p_n) = \theta_{1-p_n/(F_X(x))}(x)$  and  $Z_{(n-k_n)}^x \stackrel{\text{a.s.}}{=} \hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x)$  as  $n \rightarrow \infty$ .

(ii) Since  $Z_{(n-k_n)}^x \stackrel{\text{a.s.}}{=} \hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x)$  and  $\tilde{\gamma}_x \stackrel{\text{a.s.}}{=} -1/\tilde{\rho}_x$  as  $n \rightarrow \infty$ , we have  $\hat{\theta}(x) \stackrel{\text{a.s.}}{=} Z_{(n-k_n)}^x M_n^{(1)}(1 - 1/\tilde{\gamma}_x) + Z_{(n-k_n)}^x$  as  $n \rightarrow \infty$ . It is then easy to see from (8) that  $\hat{\theta}(x)$  coincides almost surely, for all  $n$  large enough, with the end-point estimator  $\hat{x}_n^*$  of  $F_x^{-1}(1)$ . It is also easy to check that  $U(t) = (1/(1 - F_x))^{-1}(t)$  and  $\gamma_x = -1/\rho_x < 0$ . We then have  $\sqrt{k_n} \{\hat{x}_n^* - F_x^{-1}(1)\} / M_n^{(1)} Z_{(n-k_n)}^x (1 - \tilde{\gamma}_x) \xrightarrow{d} \mathcal{N}(0, V_5(-1/\gamma_x))$ , which gives the desired convergence in distribution of Theorem 2.7(ii) since  $F_x^{-1}(1) = \theta(x)$ ,  $\hat{x}_n^* \stackrel{\text{a.s.}}{=} \hat{\theta}(x)$ ,  $\tilde{\gamma}_x \stackrel{\text{a.s.}}{=} -1/\tilde{\rho}_x$  and  $Z_{(n-k_n)}^x \stackrel{\text{a.s.}}{=} \hat{\theta}_{1-k_n/(n\hat{F}_X(x))}(x)$  as  $n \rightarrow \infty$ .

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