

# On the value of randomization

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# Introduction

Actual taxes may be random when the tax base is not fully known by the tax authority:

There may be administration errors, or successful tax evasion with imperfect auditing.

Customers often buy goods and services whose quality and/or price depend on future contingencies:

Quality of a journey is random when strikes, equipment malfunction or unavailable aircrafts.

Online travel intermediaries offer non-refundable 'opaque' low price hotel rooms (some characteristics are revealed after payment).

Both quality and price may be random when firms use overselling or overbooking combined with consolation rewards.

In most economic models, concave objectives on convex sets lead to nonrandom choices.

But asymmetric information and self-selection in principal agent models introduce nonconvexities.

This paper studies whether a deterministic optimum can locally be

# Intuitions

A random tax system may be optimal when taxpayers differ in their attitudes towards risk

Tax randomization enables the government to separate agents who otherwise would have been treated equally.

Consider an economy with two types of agents:

1. One is skilled and displays a high level of risk aversion,
2. The other is unskilled and has a lower level of risk aversion.

Assume that redistribution favors the unskilled (poor) agent.

If neither skills nor risk aversions are publicly observed, the skilled would pretend to be unskilled if they faced too high a tax.

With a random tax on the unskilled workers, the risk averse skilled types are discouraged to pretend being unskilled.

This may yield a social gain possibly overcoming the efficiency loss due to risk bearing on the unskilled.

The most general results on the usefulness of random taxation appear in [?].

In the airline case, exposure to risk discourages business men (who want to be on time) to buy second class tickets.

## A rudimentary principal-agent setup

The principal faces two agents  $i = 1, 2$ .

Utility of agent  $i$  is  $u_i(\cdot)$  defined on two goods.

The VNM utility index  $u_i(\cdot)$  is increasing and concave.

A 'contract' specifies the quantities of each good.

Ex ante utility of  $i$  receiving a random contract  $\tilde{z}_i$  is  $\mathbb{E}u_i(\tilde{z}_i)$ .

The principal chooses a profile  $(\tilde{z}_1, \tilde{z}_2)$  which maximizes  $\mathbb{E}u_1(\tilde{z}_1)$ .

Preferences are private information to the agent.

If too much is given to type 1, type 2 will pretend to be of type 1.

Incentive compatibility requires  $\mathbb{E}u_2(\tilde{z}_2) \geq \mathbb{E}u_2(\tilde{z}_1)$ .

Other constraints, e.g., feasibility, is irrelevant to the argument.

## Local optimality

Let  $(z_1^*, z_2^*)$  be the optimal deterministic contract.

This contract maximizes  $u_1(z_1)$  s.t.  $u_2(z_2) \geq u_2(z_1)$ .

The associated Lagrangian is  $\mathcal{L}(z) = u_1(z_1) + \lambda_{21} [u_2(z_2) - u_2(z_1)]$ .

First-order conditions: An interior optimum  $z^*$  satisfies  $\nabla_z \mathcal{L}(z^*) = 0$ .

Second-order conditions involve:

1. the  $4 \times 4$  Hessian  $H = \nabla_z^2 \mathcal{L}(z^*)$  at  $z^*$ ,
2. the tangent space to the active constraints at  $z^*$ ,  
i.e., all  $z$  s.t.  $\nabla_z [u_2(z_2^*) - u_2(z_1^*)]' z = 0$ .

Local optimality requires that  $z' H z \leq 0$  for all  $z$  in this space.

Here there is one binding constraint whereas  $z$  is a  $4 \times 1$  vector.

One eigenvalue of the  $4 \times 4$  Hessian  $H$  must be non positive.

Local optimality is consistent with the other eigenvalues being positive.

# Nonconcave Lagrangian

The Hessian is  $H = \nabla_z^2 u_1(z_1) + \lambda_{21} [\nabla_z^2 u_2(z_2) - \nabla_z^2 u_2(z_1)]$ .

Since the utility indices are concave,  $\nabla_z^2 u_i(z_i)$  is negative definite.

A nonconcave Lagrangian is more likely to obtain when:

1. The optimum  $z^*$  is far from the first-best optimum.
2. Agent 2 is strongly risk-averse when she mimicks agent 1.

Of course all these quantities are determined endogenously.

They must be compatible with optimality of the deterministic program.

Consider now a profile of random contracts  $(\tilde{z}_1, \tilde{z}_2)$ .

Say that  $\tilde{z}_i$  is distributed according to cdf  $F_i$  on support  $Z_i$ .

The optimal random contract maximizes  $\mathbb{E}u_1(\tilde{z}_1)$  s.t.  $\mathbb{E}u_2(\tilde{z}_2) \geq \mathbb{E}u_2(\tilde{z}_1)$ .

The associated Lagrangian is  $\tilde{\mathcal{L}}(z) = \mathbb{E}u_1(\tilde{z}_1) + \lambda_{21} [\mathbb{E}u_2(\tilde{z}_2) - \mathbb{E}u_2(\tilde{z}_1)]$ .

This Lagrangian is crucial in assessing whether randomness may be useful.

## Useful randomization: Step 1

Consider a local maximum ( $z^*$ ) among the nonrandom contracts.

Consider deviations ( $d\tilde{z}$ ) s.t. IC is binding at ( $z^* + d\tilde{z}$ ).

The change in the objective is

$$\mathbb{E}u_1(z_1^* + d\tilde{z}_1) - u_1(z_1^*) = \tilde{\mathcal{L}} - \mathcal{L}.$$

The initial deterministic optimum satisfies the first-order conditions.

Thus the reform yields at most a second-order change to the objective:

$$\tilde{\mathcal{L}} - \mathcal{L} = \frac{1}{2}\mathbb{E}d\tilde{z}'Hd\tilde{z} + o(\|d\tilde{z}\|^2).$$



## Useful randomization: Step 2

Consider a deviation from  $z^*$  taking two values:

$$d\tilde{z}^1 = tv + \beta(t), \quad d\tilde{z}^2 = -tv + \beta(t),$$

drawn independently with equal probability.

This deviation has two parts:

1. A deterministic part  $\beta(t)$  ensuring that the constraint is binding
2. A lottery with zero mean in some direction  $v$  of the Hessian.

The small positive scalar  $t$  measures the scale of the change.

## Useful randomization: Step 3

From Steps 1 and 2,  $\mathbb{E}u_1(z_1^* + d\tilde{z}_1) - u_1(z_1^*)$  equals

$$\frac{1}{2} \sum_{i=1}^2 \frac{1}{2} [(-1)^i t v + \beta(t)]' H [(-1)^i t v + \beta(t)] + o(\|d\tilde{z}\|^2).$$

It can be shown that  $\beta(t)$  is  $O(t^2)$  at most: For  $t \simeq 0$ ,

$$\mathbb{E}u_1(z_1^* + d\tilde{z}_1) - u_1(z_1^*) \simeq \frac{t^2}{2} v' H v + o(t^2).$$

Taking  $v$  an eigenvector of  $H$  yields our main result:

**Proposition 1.** Consider a nonrandom optimum among the nonrandom contracts. The objective can be improved upon through local random contracts if the Hessian  $H$  of the nonrandom Lagrangian has at least one positive eigenvalue.

# General statements 1: Deterministic problem

Consider the deterministic constrained optimization problem:

$$\max_{x \in \mathbb{R}^M} \{f(x) \mid g_n(x) \geq 0 \text{ for } n = 1, \dots, N\}.$$

The functions  $f(\cdot)$  and  $g_n(\cdot)$  are twice continuously differentiable.

The Lagrangian  $\mathcal{L}(x, \lambda)$  is  $f(x) + \lambda'g(x)$ , where  $\lambda \in \mathbb{R}_+^N$ .

1. There exists  $\lambda^* \geq 0$  s.t.  $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$ , and the complementary slackness conditions  $\lambda_n^* g_n(x^*) = 0$  hold for all  $n$ .
2. The Hessian  $H = \nabla_x^2 \mathcal{L}(x^*, \lambda^*)$  is s.t.  $x'Hx \leq 0$  for all  $x$  such that  $\nabla g_n(x^*)'x = 0$  for all  $n$  with  $g_n(x^*) = 0$ .

## General statements 2: Random problem

Consider the random optimization problem:

$$\max_{\tilde{x}} \{ \mathbb{E}f(\tilde{x}) \mid \mathbb{E}g_n(\tilde{x}) \geq 0 \text{ for } n = 1, \dots, N \},$$

where  $\tilde{x}$  is a random variable with values in  $\mathbb{R}^M$ .

1. If  $H$  is negative definite, then  $x^*$  is a local maximum of the random problem.
2. If  $H$  has a positive eigenvalue, then there is a vector  $v$  such that  $v'Hv > 0$ , and  $\mathbb{E}f(x^* + \tilde{h}(t)) > f(x^*)$  for the random deviation  $\tilde{h}(t)$  equal to  $tv + \beta(t)$  and to  $-tv + \beta(t)$  with equal probabilities, for small enough  $t$  different from 0, and  $\beta(t)$  satisfying

$$\nabla g_n(x^*)' \beta(t) + \frac{1}{2} t^2 v' \nabla^2 g_n(x^*) v = o(t^2),$$

for all the active constraints.

# Remarks

All our results would apply if there were also equality constraints.

1. Qualification of the (binding) constraints
2. Zero eigenvalues

## A taxation example

Two types of agents,  $n_1$  of type 1 and  $n_2$  of type 2,  $n_1 + n_2 = 1$ .

Type 1 agents are 'disabled' and do not supply any labor.

They consume  $c_1$  and get a utility  $u_1(c_1)$ , with  $u_1$  increasing and concave.

Type 2 consumes  $c_2$  and produces  $y_2$  goods.

Their utility is  $u_2(c_2) - v_2(y_2)$ ,  $u_2' > 0$ ,  $u_2'' < 0$ , and  $v_2(0) = 0$ ,  $v_2', v_2'' > 0$ .

The government chooses  $(c_1, c_2, y_2)$  which maximizes

$$a_1 n_1 u_1(c_1) + a_2 n_2 [u_2(c_2) - v_2(y_2)]$$

subject to

$$n_1 c_1 + n_2 c_2 \leq n_2 y_2 \quad \text{and} \quad u_2(c_2) - v_2(y_2) \geq u_2(c_1).$$

When  $a_2/a_1$  is low enough, the incentive constraint binds at the optimum. Then, the optimum is s.t.

$$n_1 c_1 + n_2 c_2 = n_2 y_2,$$

$$u_2(c_2) - v_2(y_2) = u_2(c_1),$$

and the first-order condition

$$u_2'(c_2) = v_2'(y_2)$$

holds.

One can check that the solution is always a local maximum.

Can we improve upon the deterministic optimum through random taxes?

The Hessian is

$$\begin{pmatrix} a_1 n_1 u_1''(c_1) - \lambda u_2''(c_1) & 0 & 0 \\ 0 & (a_2 n_2 + \lambda) u_2''(c_2) & 0 \\ 0 & 0 & -(a_2 n_2 + \lambda) v_2''(y_2) \end{pmatrix}.$$

There is a profitable local random deviation iff  $a_1 n_1 u_1''(c_1) - \lambda u_2''(c_1) > 0$ .



Proposition 2. A necessary and sufficient condition for the existence of an open interval of values of social weights where the deterministic second-best optimum is locally dominated by a random allocation is

$$\frac{r_1^A(c_1^*)}{r_2^A(c_2^*)} \left( 1 + \frac{n_1 u_2'(c_2^*)}{n_2 u_2'(c_1^*)} \right) < 1.$$

where  $r_i^A(c) = -u_i''(c)/u_i'(c)$  is the coefficient of absolute risk aversion of type  $i$  when consuming  $c$ .

1. The positive eigenvalue is associated with  $c_1 \Rightarrow$  random disability allowance  $c_1$ .
2. The incentive constraint yields  $u_2(c_2^*) > u_2(c_1^*) \Rightarrow c_2^* > c_1^*$ . Hence, useful randomization if  $r_1^A(c_1^*) < n_2 r_2^A(c_2^*)$ , i.e., 2 is much more risk averse than 1.

## Example 2: Monopoly pricing

The principal produces a commodity in different qualities.

One good of quality  $q$  costs  $c(q)$ , with  $c(0) = 0$ , and  $c', c'' > 0$ .

Each agent buys at most one good.

A type  $i$  agent buying a quality  $q$  good at price  $p$  has utility  $v_i(\theta_i q - p)$ .

By convention valuations increase with  $i$ ,  $\theta_i < \theta_{i+1}$  for all  $i \leq l - 1$ .

The problem of the seller is to choose a profile  $(\tilde{p}_i, \tilde{q}_i)$  which maximizes

$$\sum_{i=1}^l n_i \mathbb{E}[\tilde{p}_i - c(\tilde{q}_i)]$$

s.t.

$$\mathbb{E}v_i(\theta_i \tilde{q}_i - \tilde{p}_i) \geq 0 \quad \text{for all } i$$

$$\mathbb{E}v_i(\theta_i \tilde{q}_i - \tilde{p}_i) \geq \mathbb{E}v_i(\theta_i \tilde{q}_j - \tilde{p}_j) \quad \text{for all } i, j.$$

# Deterministic optimum

The single crossing condition is satisfied.

If  $q_i \uparrow$  with  $i$ , then IR of type 1 and neighboring downward IC bind.

The interior optimum satisfies

$$n_i c'(q_i) = N_i \theta_i - N_{i+1} \theta_{i+1} \quad \text{with } N_i = \sum_{j=i}^I n_j \text{ for } i \leq I,$$

and  $N_{I+1}$  has been set to zero.

In the sequel, we assume that:

1.  $N_i \theta_i > N_{i+1} \theta_{i+1}$  for all  $i$
2. The sequence  $(N_i \theta_i - N_{i+1} \theta_{i+1}) / n_i$  is increasing with  $i$
3.  $c'(0) = 0$  and  $c'(\infty) = \infty$ .

## Random deviations

The Hessian  $H$  is a diagonal matrix whose  $i$ th diagonal entry is

$$\frac{\partial^2 \mathcal{L}}{\partial q_i^2} = -n_i c''(q_i) - N_i \theta_i^2 r_i^A + N_{i+1} \theta_{i+1}^2 r_{i+1}^A,$$

where  $r_i^A$  is the coefficient of absolute risk aversion of type  $i$  at the deterministic optimum.

**Proposition 3.** Suppose that  $H$  is of full rank at the deterministic optimum. It is worthwhile to locally randomize the quality designed for type  $i$  consumers if and only if

$$N_{i+1} \theta_{i+1}^2 r_{i+1}^A > N_i \theta_i^2 r_i^A + n_i c''(q_i).$$

$N_{i+1} = 0 \Rightarrow$  quality offered to the highest type is nonrandom.

More generally the risk aversions of the consumers matter as expected.

$\uparrow c'' \Rightarrow$  the seller is reluctant to randomize quality.