Gauss-compatible Galerkin schemes for time-dependent Maxwell equations

Martin Campos Pinto, Eric Sonnendrücker

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Abstract

In this article we propose a unified analysis for conforming and non-conforming finite element methods that provides a partial answer to the problem of preserving discrete divergence constraints when computing numerical solutions to the time-dependent Maxwell system. In particular, we formulate a compatibility condition relative to the preservation of genuinely oscillating modes that takes the form of a generalized commuting diagram, and we show that compatible schemes satisfy convergence estimates leading to long-time stability with respect to stationary solutions. These findings are applied by specifying compatible formulations for several classes of Galerkin methods, such as the usual curl-conforming finite elements and the centered discontinuous Galerkin (DG) scheme. We also propose a new conforming/non-conforming Galerkin (Conga) method where fully discontinuous solutions are computed by embedding the general structure of curl-conforming finite elements into larger DG spaces. In addition to naturally preserving one of the Gauss laws in a strong sense, the Conga method is both spectrally correct and energy conserving, unlike existing DG discretizations where the introduction of a dissipative penalty term is needed to avoid the presence of spurious modes.

1   Introduction

Preserving a discrete version of the Gauss laws has always been an important issue in the development of numerical schemes for the time-dependent Maxwell equations. At the continuous level indeed, an important property of the Ampere and Faraday equations

$$\partial_t \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{B} = -\boldsymbol{J}$$
$$\partial_t \boldsymbol{B} + \mathbf{curl}\, \boldsymbol{E} = 0 \tag{1.1}$$

is to preserve the divergence constraints on the fields

$$\mathrm{div}\, \boldsymbol{E} = \rho$$
$$\mathrm{div}\, \boldsymbol{B} = 0 \tag{1.2}$$

provided they are satisfied at initial time and the sources ρ and $\boldsymbol{J}$ satisfy the so-called continuity equation

$$\partial_t \rho + \mathrm{div}\, \boldsymbol{J} = 0. \tag{1.3}$$

However when numerical approximations are involved this may not be true.

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In that regard, the different classes of Galerkin approximations do not come with equal properties. For instance curl-conforming finite element methods present a rather favorable situation, in that they usually preserve the Gauss laws in a natural finite element sense, see e.g., [42, 16]. This is not the case with non-conforming methods such as finite volume or Discontinuous Galerkin (DG) schemes where it is known that on general meshes the most natural discrete version of the divergence constraints cannot hold [26]. A practical consequence of that theoretical weakness is the development of small errors in the computed electromagnetic field which accumulate to large deviations for long simulation times [37, 43].

When the sources themselves are obtained by a numerical treatment, as is the case with particle simulations of the Vlasov-Maxwell system, such numerical artifacts are often designated as a lack of charge conservation. Indeed the growing inconsistencies that one observes in the fields may be related, through the Gauss laws, to a lack of charge separation: particles tend to stick together, which is normally prevented by Gauss's law. Nevertheless, part of the problem lies in the discretization of the Maxwell equations themselves, as can be seen in e.g. [43] where such unphysical behavior appears with analytical sources that do satisfy the balance equation (1.3).

In order to make DG – and other schemes lacking a proper discrete charge conservation – physically acceptable, one usually resorts to correction techniques such as projection methods or divergence cleaning methods based on generalized Lagrange multiplier formulations of the Maxwell equations [30, 36, 37]. This works well in many cases, in particular when the investigated problem is close to electrostatic. However it introduces an artificial non locality in the numerical scheme which can have disastrous effects in some situations, for example in laser plasma interaction problems where these non localities can trigger an instability before the laser hits the plasma.

We note that the preservation of the Gauss law for time dependent problems is strongly related to the non existence of spurious eigenvalues for the discrete curl curl operator, which has triggered a lot of efforts in the applied mathematics community, see e.g. [19, 10, 6, 9]. The problem is that when Gauss's law is not explicitly applied, the eigenspace corresponding to the zero eigenvalue is infinite dimensional and not compact. Good numerical methods generally avoid mixing the eigenspaces corresponding to vanishing and non vanishing eigenvalues, but not all of them do.

Be it for the time dependent or the eigenvalue problem, solutions have been developed using appropriate exact sequences of discrete spaces along with commuting projections [11, 20, 22, 42, 2]. In addition to classical finite element spaces, this framework has been extended to spline finite elements in [15]. By construction, these solutions are restricted to conforming methods.

In this article we extend these results to non-conforming methods and propose a unified analysis. Our findings are twofold. First we formulate a compatibility condition for semi-discrete schemes with sources that takes the form of a generalized commuting diagram, and we show that methods verifying this condition are long-time stable with respect to exact stationary solutions, which solves most of the large deviation problems described in the literature without resorting to divergence cleaning techniques. Specifically, by applying this analysis to conforming and non-conforming Galerkin methods we describe several approximation operators for the sources that make the usual curl-conforming finite element method and the centered DG scheme compatible in the above sense.

Second, we propose a new conforming/non-conforming Galerkin (Conga) method that closely follows the structure of the curl-conforming finite element schemes but uses fully discontinuous spaces just as standard DG schemes. In contrast to existing DG discretizations where the introduction of a dissipative penalty term is necessary to avoid the presence of spurious modes [27, 44, 14] the Conga method is shown to be both spectrally correct and energy conserving, just as curl-conforming finite element schemes. Moreover, it naturally preserves one of the Gauss laws in a strong sense.

2

The outline is as follows. In Section 2 we express the time-dependent Maxwell system $\partial_t U - \mathcal{A}U = -F$ in the abstract setting of Hilbert complexes with exact sequences of differential operators, and we ask ourselves in which sense a numerical method of the form $\partial_t U_h - \mathcal{A}_h U_h = -F_h$ should be compatible with the Gauss laws. By studying first the homogeneous case $F = 0$ we observe that in this abstract setting the Gauss laws is naturally interpreted as a constraint relative to the existence of stationary modes, for which it is straightforward to derive a discrete analog that can be imposed on the numerical solutions.

In Section 3 we then extend this constraint to the case with sources by formulating a compatibility condition of the form $F_h = \Pi_h F$ with $\Pi_h \mathcal{A} = \mathcal{A}_h \hat{\Pi}_h$ and we show that compatible schemes are long-time stable with respect to exact stationary solutions. In this abstract setting we then provide compatible formulations for several conforming and non-conforming Galerkin methods, including a novel Conga method that is shown to be spectrally correct in Section 4. Finally, compatible operators for the current approximation are provided in Section 5 for several explicit semi-discretizations of the 3d Maxwell equations. Numerical results using an elementary test-case in 2d validate the effectiveness of the proposed approach.

2 An abstract setting for time-dependent Maxwell problems

Because our analysis relies on the spectral properties of the Maxwell evolution operator and that of its approximation by mixed methods, we reformulate the problem in the abstract setting of Hilbert complexes, following the framework and notation from Ref. [3].

2.1 Exact sequences in Hilbert complexes

We consider a Hilbert complex $(W, d) = (W^l, d^l)_{l=0,\dots,n}$ consisting of a finite sequence of Hilbert spaces W^l with closed and densely-defined linear operators $d^l : W^l \to W^{l+1}$ that satisfy

$$d^l d^{l-1} = 0. \tag{2.1}$$

In particular, the range of d^{l-1} is contained in the domain of d^l, denoted V^l. We further assume that the sequence (V, d) is *exact*, in the sense that the range of d^{l-1} actually coincides with the null space of d^l, which we denote by $\mathfrak{Z}^l$. Thus, we have

$$\mathfrak{Z}^l := \ker d^l = d^{l-1} V^{l-1}. \tag{2.2}$$

In our applications the W^l's will be L^2 spaces, and the d^l's will correspond to differential operators such as $-\operatorname{grad}$, curl or div. To reformulate the time-dependent Maxwell system in this abstract setting, following [3] we let d_{l+1}^* be the adjoint of d^l: it is a closed, densely-defined operator from W^{l+1} to W^l and its domain is denoted V_{l+1}^*. For instance if d^l is the unbounded operator $\mathbf{curl} : L^2(\Omega)^3 \to L^2(\Omega)^3$ with domain $V^l = H(\mathbf{curl}; \Omega)$, its adjoint d_{l+1}^* will also be a curl operator but with domain $V_{l+1}^* = H_0(\mathbf{curl})$, see e.g. [41, Sec. 0]. And the reverse situation is also possible since the closed, densely-defined d^l coincides with $(d_{l+1}^*)^*$. In particular, we have

$$\langle d_{l+1}^* v, u \rangle = \langle v, d^l u \rangle, \qquad v \in V_{l+1}^*, \ u \in V^l. \tag{2.3}$$

Here we use the same notation $\langle \cdot, \cdot \rangle$ for the scalar products in any of the (L^2) spaces W^l, and accordingly we shall use $\|\cdot\|$ for the corresponding norms. Applying (2.3) to $u = d^{l-1}\varphi$ with $\varphi \in V^{l-1}$, we infer from (2.1) that d_{l+1}^* maps V_{l+1}^* to V_l^* and

$$d_l^* d_{l+1}^* = 0. \tag{2.4}$$

Using the density of V^{l-1} in W^{l-1} we then see that $d_l^* u = 0$ holds iff $\langle u, d^{l-1}\varphi \rangle = 0$ for all $\varphi \in V^{l-1}$, hence (2.2) yields

$$\ker d_l^* = \mathfrak{Z}^{l\,\perp_W} \tag{2.5}$$

where the $\perp_W$ exponent denotes an orthogonal complement in the proper space from the complex W, here W^l. Note that in the sequel we will use the $\perp$ exponent alone to denote an orthonal complement in (the proper space from) the complex V, see e.g. (2.6) or (2.13) below. We next verify the following estimates.

Lemma 2.1 (Poincaré inequalities)**.** *There is a constant $c_P = c_P(V, d)$ such that*

$$\|u\| \le c_P \|d^l u\|, \qquad u \in \mathfrak{Z}^{l\,\perp_W} \cap V^l =: \mathfrak{Z}^{l\,\perp} \tag{2.6}$$

and

$$\|w\| \le c_P \|d_{l+1}^* w\|, \qquad w \in \mathfrak{Z}^{l+1} \cap V_{l+1}^*. \tag{2.7}$$

Proof. The first estimate is given in [3, Eq. (16)]. It is obtained by observing first that $\mathfrak{Z}^{l\,\perp}$ is the orthogonal complement of $\mathfrak{Z}^l$ in the Hilbert space V^l equipped with the scalar product $\langle u, v \rangle_V := \langle u, v \rangle + \langle du, dv \rangle$, and second that d^l defines a bounded bijection between the Hilbert spaces $(\mathfrak{Z}^{l\,\perp}, \langle \cdot, \cdot \rangle_V)$ and $(\mathfrak{Z}^{l+1}, \langle \cdot, \cdot \rangle)$, the latter being closed according to (2.2). Estimate (2.6) then follows by Banach's bounded inverse theorem. To obtain estimate (2.7) we finally consider $w \in \mathfrak{Z}^{l+1} \cap V_{l+1}^*$ and let $\bar{u} \in \mathfrak{Z}^{l\,\perp}$ be such that $d^l \bar{u} = w$. The definition of the adjoint operator gives then

$$\|d_{l+1}^* w\| = \sup_{u \in V^l} \frac{\langle w, d^l u \rangle}{\|u\|} \ge \frac{\langle w, d^l \bar{u} \rangle}{\|\bar{u}\|} = \frac{\|w\| \|d^l \bar{u}\|}{\|\bar{u}\|} \ge c_p^{-1} \|w\|$$

where the last inequality is (2.6). $\qquad \square$

Our analysis will use some of the properties of the inverse K of the abstract Hodge Laplacian operator $L = d^{k-1} d_k^* + d_{k+1}^* d^k$ corresponding to a particular index k that will be chosen so that both d^k and its adjoint d_{k+1}^* correspond to curl operators. To do so we will assume in Section 4 that the inclusion from the dense intersection $V^k \cap V_k^*$ in W^k is compact. According to [3, Sec. 3] we then know that K is compact and selfadjoint, as an operator from W^k to itself.

The space discretization methods studied in this article will involve finite-dimensional subcomplexes (V_h, d) of (V, d), i.e., sequences of conforming discrete spaces satisfying $V_h^l \subset V^l$ and $d^l V_h^l \subset V_h^{l+1}$. These spaces will serve either as *primary* discretization spaces in conforming methods, or as *auxiliary* discretization spaces in the considered non-conforming methods. It is well-known (see, e.g., [12, 28, 3]) that the quality of a mixed method based on the subcomplex (V_h, d) strongly relies on the existence of projection operators π_h^l mapping some conforming domains $\bar{V}^l \subset V^l$ to the discrete spaces V_h^l, such that the following diagram commutes

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{\; d^{l-1} \;} & \bar{V}^l & \xrightarrow{\; d^l \;} & \bar{V}^{l+1} & \xrightarrow{\; d^{l+1} \;} & \cdots \\
 & & \Big\downarrow{\scriptstyle \pi_h^l} & & \Big\downarrow{\scriptstyle \pi_h^{l+1}} & & \\
\cdots & \xrightarrow{\; d^{l-1} \;} & V_h^l & \xrightarrow{\; d^l \;} & V_h^{l+1} & \xrightarrow{\; d^{l+1} \;} & \cdots
\end{array}
\tag{2.8}
$$

in the sense that

$$d^l \pi_h^l = \pi_h^{l+1} d^l \qquad \text{holds on } \bar{V}^l. \tag{2.9}$$

Here the projection operators may be bounded in V or in W, but for some applications they do not need to be so, and the $\bar{V}^l$'s can be proper subspaces of the V^l's. Either way, it can be verified that if the diagram (2.8) commutes then the discrete subcomplex inherits the exactness of (V, d), in the sense that

$$\mathfrak{Z}_h^l := \mathfrak{Z}^l \cap V_h^l = \ker \left(d^l |_{V_h^l} \right) = d^{l-1} V_h^{l-1}. \tag{2.10}$$

An important tool in the study of mixed methods is the adjoint $d_{l+1,h}^* : V_h^{l+1} \to V_h^l$ of the operator d^l restricted to V_h^l. It is characterized by

$$\langle d_{l+1,h}^* v, u \rangle = \langle v, d^l u \rangle, \qquad v \in V_h^{l+1}, \ u \in V_h^l. \tag{2.11}$$

Using (2.1) we then find $\langle d_{l,h}^* d_{l+1,h}^* v, \varphi \rangle = \langle d_{l+1,h}^* v, d^{l-1} \varphi \rangle = \langle v, d^l d^{l-1} \varphi \rangle = 0$ for all $v \in V_h^{l+1}$ and $\varphi \in V_h^{l-1}$. Thus,

$$d_{l,h}^* d_{l+1,h}^* = 0. \tag{2.12}$$

Finally, one easily derives from (2.10) a characterization of the adjoint kernel,

$$\mathfrak{Z}_h^{l\perp} := \mathfrak{Z}_h^{l\perp_W} \cap V_h^l = \ker d_{l,h}^*. \tag{2.13}$$

2.2 The abstract Maxwell evolution system

In the above setting, an abstract version of the time-dependent Maxwell system can be given in a form that is standard in the literature, see e.g., [1, 21]. It reads

$$\begin{cases} \partial_t U - \mathcal{A} U = -F \\ \qquad U(0) = U^0 \in \mathcal{V}, \end{cases} \tag{2.14}$$

where $\mathcal{A}$ is the linear operator defined from $\mathcal{W} := W^k \times W^{k+1}$ to itself by

$$\mathcal{A} = \begin{pmatrix} 0 & -d_{k+1}^* \\ d^k & 0 \end{pmatrix}, \quad \text{with dense domain} \quad \mathcal{V} := D(\mathcal{A}) = V^k \times V_{k+1}^*. \tag{2.15}$$

Indeed in the applications k will be chosen such that both d^k and its adjoint d_{k+1}^* are curl operators. The source $F = (f, g)^t$ corresponds to a generalized current density. In Section 5 we will consider either terms of the form $F = (0, J)^t$ corresponding to a strong formulation of the Ampere equation (in which case U represents $(B, E)^t$), or terms of the form $F = (J, 0)^t$ that correspond to a strong formulation of the Faraday equation (in which case U represents $(E, -B)^t$).

From the properties of d^k and d_{k+1}^* we can see that $\mathcal{A}$ is closed and skew-symmetric. Indeed for any $U = (u, v)^t$ and $U' = (u', v')^t$ in $\mathcal{V}$, (2.3) gives

$$\langle \mathcal{A} U, U' \rangle = \langle d^k u, v' \rangle - \langle d_{k+1}^* v, u' \rangle = \langle u, d_{k+1}^* v' \rangle - \langle v, d^k u' \rangle = \langle U, -\mathcal{A} U' \rangle. \tag{2.16}$$

Since d^k is closed and densely-defined one can actually infer from the skew-symmetry that $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ have the same domain, hence $\mathcal{A}^* = -\mathcal{A}$. In particular, $\mathcal{A}$ generates a contraction semigroup of class $\mathcal{C}_0$ ([46, Sec. IX.8]) and the following result is a direct application of Corollaries 2.2 p. 106 and 2.5 p. 107 in [40].

Lemma 2.2. *If $F \in \mathcal{C}^1([0, T]; \mathcal{W})$, then the Cauchy problem (2.14)-(2.15) has a unique solution $U \in \mathcal{C}^0([0, T[; \mathcal{V})$ that is continuously differentiable on $]0, T[$.*

In addition to the evolution equation (2.14), the Maxwell system consists of a two-component Gauss law

$$\mathcal{D}U(t) = R(t), \quad t \geq 0, \qquad \text{with} \qquad \mathcal{D} = \begin{pmatrix} d_k^* & 0 \\ 0 & d^{k+1} \end{pmatrix}. \tag{2.17}$$

Here $\mathcal{D}$ corresponds to a two-components divergence operator and $R = (\theta, \rho)^t \in \mathcal{W}$ corresponds to a generalized charge density. In particular, (2.17) implies that $U(t)$ belongs to the domain of $\mathcal{D}$, $V_k^* \times V^{k+1}$. From the cochain (2.1) and chain (2.4) relations one sees that $\mathcal{D}\mathcal{A} = 0$, hence (2.17) actually amounts to a property being satisfied by the data. Namely,

$$\begin{cases} F \in \mathcal{C}^0([0,T]; D(\mathcal{D})) \\ U^0 \in D(\mathcal{D}) = V_k^* \times V^{k+1} \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t R + \mathcal{D}F = 0 \\ \mathcal{D}U^0 = R(0). \end{cases} \tag{2.18}$$

Before studying the Galerkin approximations to (2.14), we verify the following result.

Lemma 2.3. *The kernel of $\mathcal{A}$ reads*

$$\ker \mathcal{A} = \mathfrak{Z}^k \times \mathfrak{Z}^{k+1 \perp_W} \tag{2.19}$$

and its orthogonal complement in $\mathcal{W} = W^k \times W^{k+1}$ coincides with the range of $\mathcal{A}$,

$$\mathrm{Im}(\mathcal{A}) = (\ker \mathcal{A})^{\perp_{\mathcal{W}}} = \mathfrak{Z}^{k \perp_W} \times \mathfrak{Z}^{k+1}. \tag{2.20}$$

Proof. The identity (2.19) simply follows from (2.5) and the definition of $\mathfrak{Z}^k$. To prove (2.20) we can show that the range of $\mathcal{A}$ is closed. The claimed identity will then follow from the closed range theorem and the fact that $\mathcal{A}^* = -\mathcal{A}$. Thus, let $(u_n, v_n)^t$ be a sequence in $\mathcal{V}$ such that $(u_n', v_n')^t := \mathcal{A}(u_n, v_n)^t$ converges to some U' in $\mathcal{W}$. Setting $\bar{u}_n := (I - P_{\mathfrak{Z}^k})u_n$ and $\bar{v}_n := P_{\mathfrak{Z}^{k+1}}v_n$ we then infer from (2.2) and (2.5) that

$$\begin{pmatrix} -d_{k+1}^* \bar{v}_n \\ d^k \bar{u}_n \end{pmatrix} = \begin{pmatrix} -d_{k+1}^* v_n \\ d^k u_n \end{pmatrix} = \begin{pmatrix} u_n' \\ v_n' \end{pmatrix} \qquad \text{and} \qquad \begin{cases} \bar{u}_n \in \mathfrak{Z}^{k \perp} \\ \bar{v}_n \in V_{k+1}^* \cap \mathfrak{Z}^{k+1}. \end{cases}$$

The convergence of the sequence $(\bar{u}_n, \bar{v}_n)^t$ to some $\bar{U}$ (in $\mathcal{W}$) follows then from the Poincaré inequalities (2.6)-(2.7), and since $\mathcal{A}$ is closed $(\bar{U}, U')$ is in the graph of $\mathcal{A}$. Hence $\mathrm{Im}(\mathcal{A})$ is closed and the identity follows. $\qquad \square$

2.3 Conforming Galerkin approximations and the Gauss law

To derive Galerkin approximations to the abstract Maxwell evolution system (2.14), we write the latter in variational form using (2.3), as

$$\begin{cases} \langle \partial_t u, z \rangle + \langle v, d^k z \rangle = -\langle f, z \rangle & z \in V^k \\ \langle \partial_t v, w \rangle - \langle d^k u, w \rangle = -\langle g, w \rangle & w \in V_{k+1}^*. \end{cases} \tag{2.21}$$

Here the standard Galerkin approach (see, e.g., [31, 42, 45]) defines (u_h, v_h) in $\mathcal{C}^1(]0, T[; V_h^k \times V_h^{k+1})$, solution to

$$\begin{cases} \langle \partial_t u_h, z \rangle + \langle v_h, d^k z \rangle = -\langle f_h, z \rangle & z \in V_h^k \\ \langle \partial_t v_h, w \rangle - \langle d^k u_h, w \rangle = -\langle g_h, w \rangle & w \in V_h^{k+1}, \end{cases} \tag{2.22}$$

with proper approximations (f_h, g_h) and (u_h^0, v_h^0) to the source terms and initial data. Clearly, the embedding $V_h^k \subset V^k$ is required for (2.22) to be well defined, and for this reason the method is

said *conforming*. As for v_h, we note that other spaces could be chosen, and indeed in Refs. [32, 33] similar schemes are considered, where v_h is sought for in a fully discontinuous space. For additional examples and literature on conforming approximations, see e.g. [28, 12].

Here, the motivation for taking v_h (and w) in V_h^{k+1} comes from the fact that the method preserves a strong discrete version of the second Gauss law in (2.17). Indeed, if the discrete data v_h^0 and g_h are in V_h^{k+1} and if they satisfy a strong discrete version of the second equations in (2.18), namely

$$\begin{cases} \partial_t \rho_h + d^{k+1} g_h = 0 \\ d^{k+1} v_h^0 = \rho_h(0) \end{cases}$$

for some approximation ρ_h to ρ, then $\partial_t v_h$ is also in V_h^{k+1}, and using (2.1) we find

$$d^{k+1} v_h(t) = \rho_h(t), \quad t \geq 0. \tag{2.23}$$

As for the first Gauss law in (2.17), it can only be satisfied in a weak sense since $\partial_t u_h$ has no reason to be in V_k^*. Specifically, the first equation from (2.22) reads

$$\partial_t u_h + d_{k+1,h}^* v_h = -f_h$$

where $d_{k+1,h}^* : V_h^{k+1} \to V_h^k$ is the adjoint of $d^k|_{V_h^k}$, see (2.11). Thus if the data u_h^0 and f_h satisfy

$$\begin{cases} \partial_t \theta_h + d_{k,h}^* f_h = 0 \\ d_{k,h}^* u_h^0 = \theta_h(0) \end{cases}$$

for some approximation θ_h to θ, then using (2.12) we obtain

$$d_{k,h}^* u_h(t) = \theta_h(t), \quad t \geq 0. \tag{2.24}$$

Now, as for any weak equation, the question arises as to whether (2.24) should be considered *too weak* to be numerically relevant. A partial answer to that question is obtained by observing that (2.24) amounts to satisfying

$$\langle u_h, d^{k-1}\varphi \rangle = \langle \theta_h, \varphi \rangle \qquad \text{for all } \varphi \in V_h^{k-1}. \tag{2.25}$$

In the applications d^{k-1} will be a $-\operatorname{grad}$ operator, hence (2.24) is a standard finite element version of the Gauss law involving H^1 test functions. In particular, if $u_h = d^{k-1}\phi_h$ is an electric field deriving from a discrete potential $\phi_h \in V_h^{k-1}$, then (2.24) simply means that ϕ_h solves a conforming Galerkin approximation of the abstract Poisson equation $(d^{k-1})^* d^{k-1}\phi_h = \theta_h$, which in itself can be considered as numerically relevant.

2.4 The fundamental homogeneous Gauss law

To give another argument that validates the relevance of the above discrete Gauss laws, we rewrite the conforming approximation (2.22) as an evolution equation

$$\begin{cases} \partial_t U_h - \mathcal{A}_h U_h = -F_h \\ \qquad U_h(0) = U_h^0 \in \mathcal{V}_h, \end{cases} \tag{2.26}$$

where the skew-symmetric operator

$$\mathcal{A}_h := \begin{pmatrix} 0 & -d_{k+1,h}^* \\ d^k & 0 \end{pmatrix} \quad \text{maps} \quad \mathcal{V}_h := V_h^k \times V_h^{k+1} \quad \text{to itself.} \tag{2.27}$$

7

The discrete Gauss laws (2.23)-(2.24) read then

$$\mathcal{D}_h U_h = R_h \qquad \text{with} \qquad \mathcal{D}_h = \begin{pmatrix} d^*_{k,h} & 0 \\ 0 & d^{k+1} \end{pmatrix} \tag{2.28}$$

and here $R_h = (\theta_h, \rho_h)^t$ is a generalized discrete charge density. The preservation of this two-component Gauss law by the conforming scheme is then expressed by the relation $\mathcal{D}_h \mathcal{A}_h = 0$. Parallel to (2.26), one also finds *non-conforming* Galerkin approximations to (2.14), of the form

$$\begin{cases} \partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = -\tilde{F}_h \\ \tilde{U}_h(0) = \tilde{U}_h^0 \in \tilde{\mathcal{V}}_h, \end{cases} \tag{2.29}$$

where $\tilde{\mathcal{A}}_h$ is a skew-symmetric operator approximating $\mathcal{A}$ on a non-conforming space of the form $\tilde{\mathcal{V}}_h = \tilde{V}_h^k \times \tilde{V}_h^{k+1}$ with $\tilde{V}_h^k \not\subset V^k$. For example in TE 2d centered-flux DG schemes [27, 24], the discontinuous scalar magnetic field $\tilde{v}_h \equiv B_z$ is strongly divergence-free by construction and hence belongs to the conforming space V^{k+1}, whereas the electric field $\tilde{u}_h \equiv (E_x, E_y)$ belongs to some non-conforming approximation of V^k. One can then show (see, e.g., [24, Prop. 3.7]) that $\tilde{u}_h$ satisfies a weak Gauss law involving the same test functions as in (2.25). Thus, in the source-free case a two-component constraint of the form

$$\tilde{\mathcal{D}}_h \tilde{U}_h = 0 \tag{2.30}$$

holds with essentially the same discrete divergence operator than in (2.28), extended to $\tilde{\mathcal{V}}_h$. Now, because $\tilde{u}_h$ belongs to a presumably larger space than in the conforming case, one intuitively feels that in order to be numerically relevant, a weak Gauss law should also involve a larger space of test functions.

An algebraic argument supporting the relevance of (2.28) compared to (2.30) can then be obtained by observing that in the homogeneous case $(R, F = 0)$, the Gauss law amounts to a constraint relative to the *oscillating modes* of the solutions to the Maxwell equation $\partial_t U = \mathcal{A}U$.

Definition 2.4 (oscillating modes). Since $\mathcal{A}$ is skew-symmetric, all its eigenvalues are imaginary and correspond to solutions that oscillate. But not all are genuinely oscillating: the orthogonal decomposition

$$\mathcal{W} = \ker \mathcal{A} \oplus (\ker \mathcal{A})^{\perp_{\mathcal{W}}}$$

corresponds to a separation between the stationary solutions $(\partial_t U = 0)$ and those which really oscillate $(\partial_t U = i\omega U, \omega \neq 0)$. Only the latter will be said *oscillating*.

Specifically, we observe that in addition to the inclusion $\text{Im}(\mathcal{A}) \subset \ker \mathcal{D}$ equivalent to $\mathcal{D}\mathcal{A} = 0$ which expresses the preservation of the Gauss law $\mathcal{D}U = 0$ by the Maxwell equation, what we actually have is an identity: using (2.5), (2.19) and (2.20) we find indeed $\text{Im}(\mathcal{A}) = (\ker \mathcal{A})^{\perp_{\mathcal{W}}} = \mathfrak{Z}^{k \perp_{\mathcal{W}}} \times \mathfrak{Z}^{k+1} = \ker \mathcal{D}$. Thus in the absence of sources, the Gauss law (2.17) is equivalently rewritten as

$$U(t) \in (\ker \mathcal{A})^{\perp_{\mathcal{W}}}, \qquad t \geq 0,$$

which states that $U(t)$ can only be composed of oscillating modes. Since the discrete evolution operator is also assumed skew-symmetric, it is straightforward to formulate an analog property at the discrete level.

Definition 2.5. We say that a scheme of the form (2.29) satisfies the *fundamental homogeneous Gauss law* if the solutions to the homogeneous equation $(F_h = 0)$ are only composed of oscillating modes (see Def. 2.4) for the discrete evolution operator $\tilde{\mathcal{A}}_h$, i.e., if they verify

$$\tilde{U}_h(t) \in (\ker \tilde{\mathcal{A}}_h)^{\perp_{\mathcal{W}}}, \qquad t \geq 0. \tag{2.31}$$

8

We note that in addition to being a natural property to expect from the discrete solutions, (2.31) is related to a classical condition in the study of spectrally correct methods for the Maxwell eigenvalue problem, see e.g., [19, 22]. Turning back to the relevance of the discrete Gauss laws, we observe that:

(i) In the conforming case (2.26)-(2.27) the discrete operators satisfy a relation analog to the continuous one, namely

$$\ker \mathcal{D}_h = \mathbf{3}_h^{k\,\perp} \times \mathbf{3}_h^{k+1} = (\ker \mathcal{A}_h)^{\perp_{\mathcal{W}}} \cap \mathcal{V}_h = \operatorname{Im}(\mathcal{A}_h). \tag{2.32}$$

As a consequence, in the absence of sources the discrete Gauss law $\mathcal{D}_h U_h = 0$ is *strong enough* to guarantee the fundamental Gauss law in the sense of Definition 2.5.

(ii) In the non-conforming case (2.29) where the discontinuous space $\tilde{\mathcal{V}}_h$ may be much larger, there is no reason why we should have $\ker \tilde{\mathcal{D}}_h \subset (\ker \tilde{\mathcal{A}}_h)^{\perp_{\mathcal{W}}}$. Therefore the discrete Gauss law $\tilde{\mathcal{D}}_h \tilde{U}_h = 0$ is a priori *too weak* to guarantee the fundamental Gauss law in the sense of Definition 2.5.

In the remainder of the paper we will design non-conforming Galerkin methods of the form (2.29) that satisfy the fundamental homogeneous Gauss law, although no divergence constraint stronger than (2.30) is specified. And when sources are present, we will look for schemes that are *compatible* with the property stated in Definition 2.5, in the sense that they should satisfy

$$U(t) \in (\ker \mathcal{A})^{\perp_{\mathcal{W}}}, \quad t \geq 0 \quad \implies \quad \tilde{U}_h(t) \in \ker(\tilde{\mathcal{A}}_h)^{\perp_{\mathcal{W}}}, \quad t \geq 0. \tag{2.33}$$

In Section 3 below we shall give a somehow stronger definition for compatible schemes. But before doing so we observe that since $\tilde{\mathcal{A}}_h$ is assumed skew-symmetric, solutions to (2.29) clearly satisfy

$$\partial_t \tilde{U}_h + \tilde{F}_h \in (\ker \tilde{\mathcal{A}}_h)^{\perp_{\mathcal{W}}}.$$

In particular, it is easily seen that satisfying property (2.33) essentially relies on the data approximation. An obvious sufficient property reads indeed

$$U^0, F \in (\ker \mathcal{A})^{\perp_{\mathcal{W}}} \quad \implies \quad \tilde{U}_h^0, \tilde{F}_h \in (\ker \tilde{\mathcal{A}}_h)^{\perp_{\mathcal{W}}}. \tag{2.34}$$

3 Compatible Galerkin approximations

In this section we propose an abstract compatibility property for generic approximations to (2.14), based either on conforming or on non-conforming spaces.

3.1 The Gauss law meets commuting diagrams

Following our observation that for skew-symmetric Galerkin approximations, being compatible with the fundamental Gauss law in the sense of (2.33) is essentially a matter of proper data approximation, we make the latter part explicit and consider semi-discrete schemes of the form

$$\begin{cases} \partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = -\Pi_h F \\ \qquad \tilde{U}_h(0) = \tilde{U}_h^0 \in \tilde{\mathcal{V}}_h. \end{cases} \tag{3.1}$$

Here,

- $\tilde{\mathcal{V}}_h$ is a discrete subspace of $\mathcal{W} = W^k \times W^{k+1}$ but not necessarily a subset of either $V^k \times V^{k+1}$ or $\mathcal{V} = V^k \times V_{k+1}^*$, the domain of $\mathcal{A}$;

9

- $\tilde{\mathcal{A}}_h : \tilde{\mathcal{V}}_h \to \tilde{\mathcal{V}}_h$ is a skew-symmetric bounded operator approximating $\mathcal{A}$;

- Π_h is a projection on $\tilde{\mathcal{V}}_h$ that may not be $\mathcal{W}$-bounded but has a dense domain in $\mathcal{W}$.

Thus, the conforming method corresponds to taking $\tilde{\mathcal{V}}_h = \mathcal{V}_h$ and $\tilde{\mathcal{A}}_h = \mathcal{A}_h$ as in (2.27), i.e.,

$$\mathcal{A}_h := \begin{pmatrix} 0 & -d^*_{k+1,h} \\ d^k & 0 \end{pmatrix} : \mathcal{V}_h \to \mathcal{V}_h \quad \text{with} \quad \mathcal{V}_h := V^k_h \times V^{k+1}_h. \tag{3.2}$$

Non-conforming methods are obtained with spaces $\tilde{\mathcal{V}}_h$ that consist of totally discontinuous functions, typically piecewise polynomials with no continuity constraint between neighboring elements. Note that the semi-group arguments used in Section 2.2 guarantee the well-posedness of (3.1).

We now specify the condition (2.34) for a scheme of the form (3.1). According to Lemma 2.3 and the identity $\text{Im}(\mathcal{A}) = (\ker \mathcal{A})^{\perp_\mathcal{W}}$, what we may require is essentially that the data approximation maps the range of the continuous evolution operator $\mathcal{A}$ into that of the discrete one $\tilde{\mathcal{A}}_h$. In other terms, for any $F = \mathcal{A}Z$ with $Z \in \mathcal{V}$ there should be a discrete field $Z_h \in \tilde{\mathcal{V}}_h$ such that $\Pi_h F = \Pi_h \mathcal{A}Z = \tilde{\mathcal{A}}_h Z_h$, at least when Z is sufficiently smooth. And because $\tilde{\mathcal{A}}_h$ is meant to be an approximation to $\mathcal{A}$, it is natural to ask that Z_h be an approximation to Z.

In order to derive error estimates we formalize the above property in terms of an auxiliary approximation operator $\hat{\Pi}_h$.

Definition 3.1 (Gauss-compatible schemes). A scheme of the form (3.1) is said *compatible* on a subspace $\hat{\mathcal{V}}$ of $\mathcal{V} = D(\mathcal{A})$ if there exists an approximation operator $\hat{\Pi}_h : \hat{\mathcal{V}} \to \tilde{\mathcal{V}}_h$ that converges pointwise to the identity as $h \to 0$, and for which

$$\begin{array}{ccc}
\hat{\mathcal{V}} & \xrightarrow{\ \ \mathcal{A}\ \ } & \mathcal{A}\hat{\mathcal{V}} \\
{\scriptstyle \hat{\Pi}_h} \downarrow & & \downarrow {\scriptstyle \Pi_h} \\
\tilde{\mathcal{V}}_h & \xrightarrow{\ \ \tilde{\mathcal{A}}_h\ \ } & \tilde{\mathcal{V}}_h
\end{array} \tag{3.3}$$

is a commuting diagram, in the sense that $\Pi_h \mathcal{A} = \tilde{\mathcal{A}}_h \hat{\Pi}_h$ holds on $\hat{\mathcal{V}}$.

Interestingly enough, we observe that although (3.3) has been primarily derived to preserve the oscillating nature of the solutions at the discrete level (see Def. 2.4), it also yields a compatibility property relative to *stationary* solutions. Indeed, if the continuous source is associated to an exact steady state, i.e., if $F = \mathcal{A}\bar{U}$ with $\bar{U} \in \hat{\mathcal{V}}$, then its approximation $\Pi_h F$ is associated to the steady state $\hat{\Pi}_h \bar{U} \approx \bar{U}$.

Remark 3.2. In fact, the commuting diagram (3.3) implies two key embeddings for the involved projection operators. Specifically, we have

$$\Pi_h((\ker \mathcal{A})^{\perp_\mathcal{W}} \cap \mathcal{A}\hat{\mathcal{V}}) \subset (\ker \tilde{\mathcal{A}}_h)^{\perp_\mathcal{W}} \quad \text{and} \quad \hat{\Pi}_h(\ker \mathcal{A} \cap \hat{\mathcal{V}}) \subset \ker \tilde{\mathcal{A}}_h. \tag{3.4}$$

Following the discussion in Section 2.4, these relations can be interpreted by saying that "Π_h maps the continuous divergence-free fields into discrete divergence-free fields", and "$\hat{\Pi}_h$ maps the continuous curl-free fields into discrete curl-free fields".

It is then easily verified that compatible schemes enjoy the following properties.

Theorem 3.3. *If the semi-discrete scheme* (3.1) *is compatible, then*

(i) *when associated to the initial data* $\tilde{U}^0_h = \Pi_h U^0$, *it satisfies Property* (2.34) *(on $\mathcal{A}\hat{\mathcal{V}}$) and hence the fundamental homogeneous Gauss law ;*

10

(ii) if the solution U to the Cauchy problem (2.14) is in $\mathcal{C}^0([0,T[;\hat{\mathcal{V}})$, we have a $\mathcal{W}$-error bound

$$\|(\tilde{U}_h - \hat{\Pi}_h U)(t)\| \leq \|\tilde{U}_h^0 - \hat{\Pi}_h U^0\| + \int_0^t \|(\Pi_h - \hat{\Pi}_h)\partial_t U(s)\| \, \mathrm{d}s, \qquad t \geq 0 \qquad (3.5)$$

(iii) and an additional estimate for the discrete derivatives (again for $t \geq 0$)

$$\|(\tilde{\mathcal{A}}_h \tilde{U}_h - \Pi_h \mathcal{A}U)(t)\| \leq \|\tilde{\mathcal{A}}_h \tilde{U}_h^0 - \Pi_h \mathcal{A}U^0\| + \int_0^t \|\tilde{\mathcal{A}}_h(\Pi_h - \hat{\Pi}_h)\partial_t U(s)\| \, \mathrm{d}s. \qquad (3.6)$$

Proof. Property *(i)* readily follows from the first embedding in (3.4) and the observation that (2.34) is a sufficient condition for (2.33). To show *(ii)* and *(iii)* we next observe that U satisfies

$$\Pi_h \partial_t U = \Pi_h \mathcal{A}U - \Pi_h F = \tilde{\mathcal{A}}_h \hat{\Pi}_h U - \Pi_h F,$$

hence for the discrete solution we have

$$\partial_t(U_h - \hat{\Pi}_h U) = (\partial_t U_h - \Pi_h \partial_t U) + (\Pi_h - \hat{\Pi}_h)\partial_t U = \tilde{\mathcal{A}}_h(U_h - \hat{\Pi}_h U) + (\Pi_h - \hat{\Pi}_h)\partial_t U.$$

Applying $\tilde{\mathcal{A}}_h$ to the latter and using the commuting diagram (3.3) further gives

$$\partial_t(\tilde{\mathcal{A}}_h U_h - \Pi_h \mathcal{A}U) = \tilde{\mathcal{A}}_h(\tilde{\mathcal{A}}_h U_h - \Pi_h \mathcal{A}U) + \tilde{\mathcal{A}}_h(\Pi_h - \hat{\Pi}_h)\partial_t U,$$

and (3.5)-(3.6) follows since $\tilde{\mathcal{A}}_h$ generates a contraction semi-group. $\qquad\square$

From Theorem 3.3 we readily derive a long-time stability result.

Corollary 3.4. *If $\bar{U} \in \hat{\mathcal{V}}$ is a steady state solution to (2.14), the discrete solution U_h satisfies (for all $t \geq 0$)*

$$\|\tilde{U}_h(t) - \hat{\Pi}_h \bar{U}\| \leq \|\tilde{U}_h^0 - \hat{\Pi}_h \bar{U}\| \quad \text{and} \quad \|\tilde{\mathcal{A}}_h \tilde{U}_h(t) - \Pi_h \mathcal{A}\bar{U}\| \leq \|\tilde{\mathcal{A}}_h \tilde{U}_h^0 - \Pi_h \mathcal{A}\bar{U}\|.$$

Remark 3.5. A priori error estimates leading to long-time stability results with respect to steady-state solutions have already been obtained for conforming schemes, see e.g., [32, 33, 31]. One interest of our estimates is that they can be applied with similar success to non-conforming schemes, where we are not aware of such long-time stability results.

Remark 3.6. We may emphasize that Theorem 3.3 and its corollary involve no assumptions on the discrete operator $\tilde{\mathcal{A}}_h$, other than the skew-symmetry and the existence of the commuting diagram (3.3). In particular the above estimates do not rely on any of the properties of the discrete complexes described in Section 2.

3.2 The case of conforming Galerkin approximations

Before designing non-conforming compatible schemes, we first show that the conforming Galerkin method (2.26)-(2.27) is compatible when equipped with a source approximation $F_h = \Pi_h F$ that involves a projection π_h^{k+1} from the commuting diagram (2.8).

Theorem 3.7 (Compatibility of the conforming Galerkin method)**.** *If $\tilde{\mathcal{A}}_h = \mathcal{A}_h$ is the conforming operator (3.2) defined on $\tilde{\mathcal{V}}_h = \mathcal{V}_h$, the scheme (3.1) equipped with*

$$\Pi_h = \begin{pmatrix} P_{V_h^k} & 0 \\ 0 & \pi_h^{k+1} \end{pmatrix} : (W^k \times \bar{V}^{k+1}) \to \mathcal{V}_h \qquad (3.7)$$

is compatible. In particular, it satisfies the commuting diagram (3.3) with

$$\hat{\Pi}_h = \begin{pmatrix} \pi_h^k & 0 \\ 0 & P_{V_h^{k+1}} \end{pmatrix} : \hat{\mathcal{V}} \to \mathcal{V}_h \qquad \text{where} \quad \hat{\mathcal{V}} := \bar{V}^k \times V_{k+1}^*. \qquad (3.8)$$

Remark 3.8. Here $\hat{\Pi}_h$ is obviously defined on $\bar{V}^k \times W^k$, but for the diagram (3.3) to commute $\mathcal{A}$ must be defined on $\hat{\mathcal{V}}$, i.e., we need $\hat{\mathcal{V}} \subset \mathcal{V} = V^k \times V_{k+1}^*$, hence (3.8).

Proof. With the above operators, the compatibility relation (3.3) reads

$$\begin{cases} \pi_h^{k+1} d^k = d^k \pi_h^k & \text{on } \bar{V}^k \\ P_{V_h^k} d_{k+1}^* = d_{k+1,h}^* P_{V_h^{k+1}} & \text{on } V_{k+1}^*. \end{cases} \tag{3.9}$$

The first relation is simply (2.9). Using the properties (2.3) and (2.11) of the continuous and discrete adjoints of d^k, we then compute for $w \in V_{k+1}^*$ and $z \in V_h^k$

$$\langle P_{V_h^k} d_{k+1}^* w, z \rangle = \langle d_{k+1}^* w, z \rangle = \langle w, d^k z \rangle = \langle P_{V_h^{k+1}} w, d^k z \rangle = \langle d_{k+1,h}^* P_{V_h^{k+1}} w, z \rangle.$$

$\square$

We end this section with a classical observation. Since d^k maps V_h^k to V_h^{k+1} and π_h^{k+1} maps into V_h^{k+1}, the second equation from $\partial_t U_h - \mathcal{A}_h U_h = -\Pi_h F$ defined with (3.2) and (3.7) holds in a strong sense in V_h^{k+1}. Thus, an equivalent formulation of the compatible conforming approximation is (writing $U_h = (u_h, v_h)^t \in V_h^k \times V_h^{k+1}$)

$$\begin{cases} \langle \partial_t u_h, \varphi_h \rangle + \langle v_h, d^k \varphi_h \rangle = -\langle f, \varphi_h \rangle, & \text{for } \varphi_h \in V_h^k \\ \partial_t v_h - d^k u_h = -\pi_h^{k+1} g & (\text{in } V_h^{k+1}). \end{cases} \tag{3.10}$$

3.3 Conforming / non-conforming Galerkin (Conga) approximations

We now turn to the problem which first motivated this work, namely the design of compatible schemes based on discontinuous spaces $\tilde{\mathcal{V}}_h$ for which an explicit time discretization does not require to invert a global mass matrix. Although it is possible to make standard DG schemes Gauss-compatible by equipping them with proper approximation operators for the data, as will be seen in Sections 3.4 and 5.4 below, in this section we begin by describing an approximation method that aims at preserving the computational structure of the conforming scheme (3.10), but in the framework of discontinuous function spaces. In this new scheme the fields are sought in a product space made of conforming and non-conforming functions, i.e.,

$$\tilde{\mathcal{V}}_h := \tilde{V}_h^k \times V_h^{k+1} \qquad \text{with} \qquad V_h^k \subset \tilde{V}_h^k \not\subset V^k.$$

Here the fact that the second space (V_h^{k+1}) is conforming should not be a concern: indeed our new scheme will preserve the fact that, just as in (3.10), the second equation holds strongly in V_h^{k+1}. Hence it will involve no mass matrix in that space. Specifically, the Conga scheme is based on a projection operator

$$\mathcal{P}_h^k : \tilde{V}_h^k \to V_h^k \subset \tilde{V}_h^k \tag{3.11}$$

seen as a bounded operator from the non-conforming space $\tilde{V}_h^k \subset W^k$ to itself. We ask that it satisfies a moment preserving property,

$$\langle (I - \mathcal{P}_h^k) u, z \rangle = 0, \qquad z \in M_h^k, \tag{3.12}$$

with spaces M_h^k that have a dense union in W^k for $h \to 0$, so that the adjoint operator satisfies

$$(\mathcal{P}_h^k)^* P_{\tilde{V}_h^k} v \to v \qquad \text{as} \qquad h \to 0 \tag{3.13}$$

12

for all $v \in W^k$. We then define the Conga evolution operator as

$$\tilde{\mathcal{A}}_h := \begin{pmatrix} 0 & -(\mathcal{P}_h^k)^* d_{k+1,h}^* \\ d^k \mathcal{P}_h^k & 0 \end{pmatrix} : \tilde{\mathcal{V}}_h \to \tilde{\mathcal{V}}_h \qquad \text{with} \qquad \tilde{\mathcal{V}}_h := \tilde{V}_h^k \times V_h^{k+1}, \qquad (3.14)$$

indeed this will guarantee that the second equation of $\partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = -\Pi_h F$ holds strongly in V_h^{k+1}. Since $(\mathcal{P}_h^k)^* d_{k+1,h}^*$ is the adjoint of $d^k \mathcal{P}_h^k$ seen as an operator from $\tilde{V}_h^k$ to V_h^{k+1}, we easily check that $\tilde{\mathcal{A}}_h$ is skew-symmetric and bounded on $\tilde{\mathcal{V}}_h$.

As for the data approximation, we consider

$$\Pi_h := \begin{pmatrix} (\mathcal{P}_h^k)^* P_{\tilde{V}_h^k} & 0 \\ 0 & \pi_h^{k+1} \end{pmatrix} : (W^k \times \bar{V}^{k+1}) \to \tilde{\mathcal{V}}_h, \qquad (3.15)$$

where $\pi_h^{k+1} : \bar{V}^{k+1} \to V_h^{k+1}$ is a projection operator that satisfies a commuting diagram of the form (2.8). We then have the following result.

Theorem 3.9 (Compatibility of the Conga method)**.** *The method* (3.1) *defined by* (3.14) *and* (3.15) *satisfies the compatibility relation* (3.3) *with the same projection operator* $\hat{\Pi}_h$ *and domain* $\hat{\mathcal{V}} = \bar{V}^k \times V_{k+1}^*$ *as in Theorem 3.7.*

Proof. Given (3.14) and (3.15), the claimed relation (3.3) now reads

$$\begin{cases} \pi_h^{k+1} d^k = d^k \mathcal{P}_h^k \pi_h^k & \text{on } \bar{V}^k \\ (\mathcal{P}_h^k)^* P_{\tilde{V}_h^k} d_{k+1}^* = (\mathcal{P}_h^k)^* d_{k+1,h}^* P_{V_h^{k+1}} & \text{on } V_{k+1}^*. \end{cases} \qquad (3.16)$$

Here the first equality follows from the commuting diagram (2.8) and the fact that π_h^k maps into V_h^k, where $\mathcal{P}_h^k = I$. We next infer from the embedding $V_h^k \subset \tilde{V}_h^k$ that $(P_{\tilde{V}_h^k} - P_{V_h^k})\mathcal{P}_h^k = 0$, as well as the adjoint identity

$$(\mathcal{P}_h^k)^* (P_{\tilde{V}_h^k} - P_{V_h^k}) = 0,$$

hence the second equality in (3.16) follows from the second one in (3.9). $\qquad \square$

We may now give an explicit form for the Conga scheme. Using (3.15), the source $F = (f, g)$ is approximated by $\Pi_h F = ((\mathcal{P}_h^k)^* P_{\tilde{V}_h^k} f, \pi_h^{k+1} g)$ and the first term gives

$$\langle (\mathcal{P}_h^k)^* P_{\tilde{V}_h^k} f, \tilde{\varphi}_h \rangle = \langle P_{\tilde{V}_h^k} f, \mathcal{P}_h^k \tilde{\varphi}_h \rangle = \langle f, \mathcal{P}_h^k \tilde{\varphi}_h \rangle \quad \text{for } \tilde{\varphi}_h \in \tilde{V}_h^k.$$

Next, using the embedding $d^k V_h^k \subset V_h^{k+1}$ we see that the second equation from $\partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = -\Pi_h F$ holds strongly in V_h^{k+1}, as announced. It follows that the compatible Conga scheme reads (with $\tilde{U}_h = (\tilde{u}_h, \tilde{v}_h)^t \in \tilde{V}_h^k \times V_h^{k+1}$)

$$\begin{cases} \langle \partial_t \tilde{u}_h, \tilde{\varphi}_h \rangle + \langle \tilde{v}_h, d^k \mathcal{P}_h^k \tilde{\varphi}_h \rangle = -\langle f, \mathcal{P}_h^k \tilde{\varphi}_h \rangle & \text{for } \tilde{\varphi}_h \in \tilde{V}_h^k \\ \partial_t \tilde{v}_h - d^k \mathcal{P}_h^k \tilde{u}_h = -\pi_h^{k+1} g & (\text{in } V_h^{k+1}). \end{cases} \qquad (3.17)$$

Remark 3.10 (Conga as an intermediate method). Given a non-conforming space $\tilde{V}_h^k \not\subset V^k$, we observe that the Conga method is an intermediate approach in that it allows to switch from the conforming Galerkin method to non-conforming ones, just by changing the conforming projection $\mathcal{P}_h^k$. In particular if $\mathcal{P}_h^k$ is defined as the orthogonal projection on V_h^k, then (3.17) is equivalent to the conforming scheme (3.1)-(3.2). Indeed, using the fact that $(I - (\mathcal{P}_h^k)^*)\tilde{u}_h$ is always a constant in (3.17) we find that the Conga solution reads $(\tilde{u}_h, \tilde{v}_h) = (u_h + \tilde{w}_h, v_h)$ where (u_h, v_h) is the conforming solution and $\tilde{w}_h = (I - P_{V_h^k})\tilde{u}_h = (I - P_{V_h^k})\tilde{u}_h^0$.

In Section 5.3 we will describe some operators $\mathcal{P}_h^k$ that can be applied locally, so that the resulting Conga scheme can be implemented using only sparse matrices.

3.4 The case of discontinuous Galerkin approximations

Non-dissipative DG approximations to (2.14) can be cast into the form (3.1). In 3d they correspond to taking the same discontinuous space $\tilde{V}_h$ for both $\tilde{V}_h^k$ and $\tilde{V}_h^{k+1}$, and to setting

$$\tilde{\mathcal{A}}_h := \begin{pmatrix} 0 & -(\tilde{d}_h)^* \\ \tilde{d}_h & 0 \end{pmatrix} : \tilde{\mathcal{V}}_h \to \tilde{\mathcal{V}}_h \quad \text{with} \quad \tilde{\mathcal{V}}_h := \tilde{V}_h \times \tilde{V}_h \tag{3.18}$$

where $\tilde{d}_h : \tilde{V}_h \to \tilde{V}_h$ is the corresponding approximation to the differential (curl) operator d^k. In Section 5.4 we will show that in the case of unpenalized centered DG schemes it is possible to rewrite this discrete operator as

$$\tilde{d}_h = P_{\tilde{V}_h} d^k \mathcal{P}_h^k \tag{3.19}$$

where $\mathcal{P}_h^k$ is a projection mapping $\tilde{V}_h$ on an *auxiliary conforming space* $V_h^k \subset V^k$ that is not necessarily a subspace of $\tilde{V}_h$, and a similar result holds for $(\tilde{d}_h)^*$. It will then be possible to build a data approximation operator Π_h that makes the semi-discrete DG scheme (3.1) Gauss-compatible. Indeed, if $\tilde{\pi}_h^l$, $l = k, k+1$, are operators mapping on $V_h^k \cap \tilde{V}_h$ and $\tilde{V}_h$ respectively, such that

$$\begin{array}{ccc}
\bar{V}^k & \xrightarrow{\ d^k\ } & \bar{V}^{k+1} \\
\tilde{\pi}_h^k \downarrow & & \downarrow \tilde{\pi}_h^{k+1} \\
V_h^k \cap \tilde{V}_h & \xrightarrow{\ d^k\ } & \tilde{V}_h
\end{array} \tag{3.20}$$

is a commuting diagram, then we have (on the domain $\bar{V}^k$ of $\tilde{\pi}_h^k$)

$$\tilde{d}_h \tilde{\pi}_h^k = P_{\tilde{V}_h} d^k \mathcal{P}_h^k \tilde{\pi}_h^k = P_{\tilde{V}_h} d^k \tilde{\pi}_h^k = P_{\tilde{V}_h} \tilde{\pi}_h^{k+1} d^k = \tilde{\pi}_h^{k+1} d^k, \tag{3.21}$$

by using (3.19), the fact that $\mathcal{P}_h^k$ is a projection on V_h^k, the above commuting diagram and the fact that $\tilde{\pi}_h^{k+1}$ maps into $\tilde{V}_h$. A similar identity holds for $(\tilde{d}_h)^*$, hence the following result (see Th. 5.7 below for a specific statement).

Theorem 3.11 (Compatibility in the DG case). *If the operators $\tilde{\pi}_h^k$ and $\tilde{\pi}_h^{k+1}$ are such that the diagram (3.20) commutes, then the method (3.1) defined with the centered DG evolution operator (3.18) and complemented with*

$$\Pi_h = \begin{pmatrix} \tilde{\pi}_h^{k+1} & 0 \\ 0 & \tilde{\pi}_h^{k+1} \end{pmatrix} : (\bar{V}^{k+1} \times \bar{V}^{k+1}) \to \tilde{\mathcal{V}}_h \tag{3.22}$$

is compatible. In particular, it satisfies the commuting diagram (3.3) with

$$\hat{\Pi}_h = \begin{pmatrix} \tilde{\pi}_h^k & 0 \\ 0 & \tilde{\pi}_h^k \end{pmatrix} : \hat{\mathcal{V}} \to \tilde{\mathcal{V}}_h \qquad \text{where} \quad \hat{\mathcal{V}} := \bar{V}^k \times \bar{V}^k. \tag{3.23}$$

4 Spectral correctness of the non-conforming Conga scheme

In this section we show that the eigenmodes of the Conga operator $\tilde{\mathcal{A}}_h$ defined in (3.14) converge towards the continuous ones, provided that

(H1) the dense intersection $V^k \cap V_k^*$ in W^k is compact,

(H2) there exists a cochain projection π_h (i.e., projection operators for which the diagram (2.8) commutes) that is uniformly W-bounded with respect to h,

(H3) the conforming projection $\mathcal{P}_h^k$ involved in $\tilde{\mathcal{A}}_h$ satisfies (3.11)-(3.13).

In addition to being of interest per se, we note that the spectral correctness of a discrete operator is a fundamental property for the approximation of the associated evolution equation, see, e.g., [27, 8].

For the subsequent analysis we let $\tilde{\mathfrak{Z}}_h^k$ be the null space of $d^k \mathcal{P}_h^k : \tilde{V}_h^k \to V_h^{k+1}$. Note that

$$\tilde{\mathfrak{Z}}_h^k := \ker(d^k \mathcal{P}_h^k) = \ker(d^k|_{V_h^k}) \oplus \ker \mathcal{P}_h^k = \mathfrak{Z}_h^k \oplus (I - \mathcal{P}_h^k)\tilde{V}_h^k. \tag{4.1}$$

We henceforth use a short notation for the following orthogonal projections,

$$P^\perp := P_{\mathfrak{Z}^{k\perp_W}}, \qquad P_h^\perp := P_{\mathfrak{Z}_h^{k\perp}} \quad \text{and} \quad \tilde{P}_h^\perp := P_{\tilde{\mathfrak{Z}}_h^{k\perp}}.$$

4.1 Characterization of the continuous eigenmodes

To begin our analysis we characterize the eigenvalues and eigenvectors of $\mathcal{A}$ in terms of those of the compact operator K defined in [3, Sec. 3], see Section 2.1. Specifically, it will be convenient to restrict K to the orthogonal complement of $\mathfrak{Z}^k$. Thus we set

$$G := KP^\perp \tag{4.2}$$

and we observe from [3, Eq. (19)] that for any $u \in W^k$, Gu is the unique element of $\mathfrak{Z}^{k\perp}$ (see (2.6)) that satisfies

$$\langle d^k Gu, d^k z \rangle = \langle u, z \rangle, \qquad z \in \mathfrak{Z}^{k\perp}. \tag{4.3}$$

In particular, we see that $G = P^\perp K P^\perp$ is a compact and selfadjoint operator from W^k to itself. As such it has a countable set of nonnegative eigenvalues, each of finite multiplicity, that accumulate only at 0. We denote by

$$0 < \lambda_1 \le \lambda_2 \le \dots \tag{4.4}$$

the inverses of its *positive* eigenvalues, each one being repeated according to its multiplicity. From the density of V^k we infer that $\ker G = \mathfrak{Z}^k$, hence the complement space $\mathfrak{Z}^{k\perp_W}$ admits an orthonormal basis $(e_i)_{i \ge 1}$ of eigenvectors corresponding to the λ_i's. We denote by $\mathfrak{E}_i$ the one-dimensional space spanned by e_i.

Proposition 4.1. *We have the following results.*

(i) *The eigenvalues of $\mathcal{A}$ are of the form $i\omega$ with $\omega \in \mathbb{R}$.*

(ii) *The kernel of $\mathcal{A}$ reads*

$$\ker \mathcal{A} = \mathfrak{Z}^k \times \mathfrak{Z}^{k+1\perp_W}.$$

(iii) *Given $\omega \ne 0$, $U = (u, v)^t$ is an eigenvector of $\mathcal{A}$ associated to $i\omega$ iff*

$$U \in (\ker \mathcal{A})^{\perp_W} \qquad \text{and} \qquad \begin{cases} Gu = \omega^{-2}u \\ v = (i\omega)^{-1}d^k u. \end{cases}$$

15

Proof. Assertions *(i)* and *(ii)* are direct consequences of the skew-symmetry (2.16) and of (2.5). Using again the skew-symmetry of $\mathcal{A}$, we then observe that U is an eigenvector associated to the eigenvalue $i\omega \neq 0$ iff $U \in (\ker \mathcal{A})^{\perp_W} \cap \mathcal{V}$ satisfies $i\omega U = \mathcal{A}U$, which is equivalent (writing $U = (u, v)^t$) to

$$
\begin{cases}
u \in \mathfrak{Z}^{k\,\perp_W} \cap V^k = \mathfrak{Z}^{k\,\perp} \\
i\omega u = -d^*_{k+1}v \\
i\omega v = d^k u
\end{cases}
\iff
\begin{cases}
u \in \mathfrak{Z}^{k\,\perp} \\
\langle \omega^2 u, z \rangle = \langle d^k u, d^k z \rangle, \quad z \in \mathfrak{Z}^{k\,\perp} \\
i\omega v = d^k u
\end{cases}
$$

so that *(iii)* follows from the characterization (4.3) of G. $\qquad\square$

4.2   Characterization of the conforming discrete eigenmodes

In Cor. 3.17 of [3], it is shown that under the assumptions (H1)-(H2) listed at the beginning of Section 4 the operator $K_h : V_h^k \to V_h^k$ characterized by

$$
\langle d^*_{k,h} K_h u, d^*_{k,h} z \rangle + \langle d^k K_h u, d^k z \rangle = \langle u, z \rangle, \qquad z \in V_h^k, \tag{4.5}
$$

is a convergent approximation to K, in the sense that

$$
\|K - K_h P_{V_h^k}\|_{\mathcal{L}(W^k, W^k)} \to 0 \qquad \text{as} \qquad h \to 0. \tag{4.6}
$$

Since K_h is selfadjoint and obviously compact, this norm convergence is equivalent to the convergence of the discrete eigenmodes towards the continuous ones, in a sense that will soon be recalled, and corresponds to [5, Def. 2.1]. We shall then characterize the eigenmodes of the conforming operator $\mathcal{A}_h$ in terms of those of K_h. As above, what we are actually interested in is the restriction of K_h to the complement of $\mathfrak{Z}_h^k$ in V_h^k. Thus we set

$$
G_h := K_h P_h^\perp \tag{4.7}
$$

and taking $u \in \mathfrak{Z}_h^{k\,\perp}$ in (4.5) yields $\langle d^*_{k,h} G_h u, d^*_{k,h} z \rangle = \langle d^*_{k,h} K_h u, d^*_{k,h} z \rangle = 0$ for all $z \in \mathfrak{Z}_h^k$. Now, since $\ker d^*_{k,h} = \mathfrak{Z}_h^{k\,\perp}$ this also holds for all $z \in V_h^k$, hence we can take $z = G_h u$ which shows that $G_h u \in \mathfrak{Z}_h^{k\,\perp}$. It follows that for any $u \in W^k$, $G_h u$ is the unique element of $\mathfrak{Z}_h^{k\,\perp}$ that satisfies

$$
\langle d^k G_h u, d^k z \rangle = \langle u, z \rangle, \qquad z \in \mathfrak{Z}_h^{k\,\perp}. \tag{4.8}
$$

Thus $G_h = P_h^\perp K_h P_h^\perp$ is a compact and selfadjoint operator from W^k to itself, and as such its eigenvalues are nonnegative and of finite multiplicity. We denote by

$$
0 < \lambda_{1,h} \leq \lambda_{2,h} \leq ... \leq \lambda_{N_h^\perp, h}, \tag{4.9}
$$

$N_h^\perp = \dim \mathfrak{Z}_h^{k\,\perp}$, the inverses of the positive eigenvalues of G_h, each one being repeated according to its multiplicity. Similarly as for the continuous case we see that $\mathfrak{Z}_h^{k\,\perp}$ admits an orthonormal basis $(e_{i,h})_{i\geq 1}$ of eigenvectors corresponding to the $\lambda_{i,h}$'s, and we denote by $\mathfrak{E}_{i,h}$ the one-dimensional space spanned by $u_{i,h}$.

We then have the following proposition that characterizes the eigenvalues and eigenvectors of the conforming operator (2.27) in terms of those of G_h.

Proposition 4.2. *We have the following results.*

  (i) The eigenvalues of $\mathcal{A}_h$ are of the form $i\omega$ with $\omega \in \mathbb{R}$.

(ii) The kernel of $\mathcal{A}_h$ reads

$$\ker \mathcal{A}_h = \mathfrak{Z}_h^{k+1\,\perp} \times \mathfrak{Z}_h^k. \tag{4.10}$$

(iii) Given $\omega \neq 0$, $U = (v, u)^t$ is an eigenvector of $\mathcal{A}_h$ associated to $i\omega$ iff

$$\begin{cases} G_h u = \omega^{-2} u \\ v = (i\omega)^{-1} d^k u. \end{cases}$$

Proof. Assertions *(i)* and *(ii)* are direct consequences of the skew-symmetry of $\mathcal{A}_h$ and of (2.10), (2.13). Using again the skew-symmetry of $\mathcal{A}_h$, we then observe that $U = (v, u)^t$ is an eigenvector associated to the eigenvalue $i\omega \neq 0$ iff

$$\begin{cases} u \in (\mathfrak{Z}_h^k)^{\perp_W} \cap V_h^k = \mathfrak{Z}_h^{k\,\perp} \\ i\omega v = d^k u \\ i\omega u = -d_{k+1,h}^* v \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} u \in \mathfrak{Z}_h^{k\,\perp} \\ i\omega v = d^k u \\ \langle \omega^2 u, z \rangle = \langle d^k u, d^k z \rangle, \quad z \in \mathfrak{Z}_h^{k\,\perp} \end{cases}$$

so that *(iii)* follows from the characterization (4.8) of G_h. $\qquad\square$

4.3 Characterization of the Conga eigenmodes

Turning to the non-conforming case we let $\tilde{G}_h : W^k \to \tilde{\mathfrak{Z}}_h^{k\,\perp}$ be the operator characterized by

$$\langle d^k \mathcal{P}_h^k \tilde{G}_h u, d^k \mathcal{P}_h^k z \rangle = \langle u, z \rangle, \qquad z \in \tilde{\mathfrak{Z}}_h^{k\,\perp}. \tag{4.11}$$

It is easily verified that this operator is compact and selfadjoint. In fact, in Lemma 4.4 below we shall specify the link between $\tilde{G}_h$ and its conforming counterpart G_h. Similarly as for the conforming case, we denote by

$$0 < \tilde{\lambda}_{1,h} \leq \tilde{\lambda}_{2,h} \leq \ldots \leq \tilde{\lambda}_{\tilde{N}_h^\perp, h}, \tag{4.12}$$

$\tilde{N}_h^\perp = \dim \tilde{\mathfrak{Z}}_h^{k\,\perp}$, the inverses of its positive eigenvalues, each one being repeated according to its multiplicity. We let $(\tilde{e}_{i,h})_{i \geq 1}$ denote an orthonormal basis of eigenvectors for $\tilde{\mathfrak{Z}}_h^{k\,\perp}$, corresponding to the $\tilde{\lambda}_{i,h}$'s, and we denote by $\tilde{\mathfrak{E}}_{i,h}$ the one-dimensional space spanned by $\tilde{u}_{i,h}$. We may then characterize the eigenmodes of the non-conforming operator (3.14) in terms of those of $\tilde{G}_h$.

Proposition 4.3. *We have the following results.*

(i) The eigenvalues of $\tilde{\mathcal{A}}_h$ are of the form $i\omega$ with $\omega \in \mathbb{R}$.

(ii) The kernel of $\tilde{\mathcal{A}}_h$ reads

$$\ker \tilde{\mathcal{A}}_h = \mathfrak{Z}_h^{k+1\,\perp} \times \tilde{\mathfrak{Z}}_h^k. \tag{4.13}$$

(iii) Given $\omega \neq 0$, $U = (v, u)^t$ is an eigenvector of $\tilde{\mathcal{A}}_h$ associated to $i\omega$ iff

$$\begin{cases} \tilde{G}_h u = \omega^{-2} u \\ v = (i\omega)^{-1} d^k \mathcal{P}_h^k u. \end{cases}$$

Proof. Assertion *(i)* follows from (2.13) and the skew-symmetry of $\tilde{\mathcal{A}}_h$. Next we observe that $(\mathcal{P}_h^k)^* d_{k+1,h}^*$ is the adjoint of $d^k \mathcal{P}_h^k : \tilde{V}_h^k \to V_h^{k+1}$ which range is $d^k \mathcal{P}_h^k \tilde{V}_h^k = d^k \mathcal{P}_h^k \tilde{V}_h^k = \mathfrak{Z}_h^{k+1}$ thanks to (3.11) and (2.10). Therefore we have

$$\ker((\mathcal{P}_h^k)^* d_{k+1,h}^*) = \mathfrak{Z}_h^{k+1\,\perp_W} \cap V_h^{k+1} = \mathfrak{Z}_h^{k+1\,\perp},$$

17

and $\tilde{\mathfrak{Z}}_h^k$ is the null space of $d^k\mathcal{P}_h^k$, hence *(ii)*. Using again the skew-symmetry of $\tilde{\mathcal{A}}_h$, we observe that $U = (v,u)^t$ is an eigenvector associated to the eigenvalue $i\omega \neq 0$ iff

$$\begin{cases} u \in (\tilde{\mathfrak{Z}}_h^k)^{\perp_W} \cap \tilde{V}_h^k = \tilde{\mathfrak{Z}}_h^{k\perp} \\ i\omega v = d^k\mathcal{P}_h^k u \\ i\omega u = -(\mathcal{P}_h^k u)^* d_{k+1,h}^* v \end{cases} \iff \begin{cases} u \in \tilde{\mathfrak{Z}}_h^{k\perp} \\ i\omega v = d^k\mathcal{P}_h^k u \\ \langle \omega^2 u, z \rangle = \langle d^k\mathcal{P}_h^k u, d^k\mathcal{P}_h^k z \rangle, \quad z \in \tilde{\mathfrak{Z}}_h^{k\perp} \end{cases}$$

so that *(iii)* follows from the characterization (4.11) of $\tilde{G}_h$. $\qquad\square$

We end this section by providing an expression of the non-conforming operator $\tilde{G}_h$ in terms of the conforming one.

Lemma 4.4. *The operators G_h and $\tilde{G}_h$, characterized by (4.8) and (4.11), satisfy (on W^k)*

$$\tilde{G}_h = \tilde{P}_h^\perp G_h \tilde{P}_h^\perp. \tag{4.14}$$

Proof. The identity clearly holds on $(\tilde{\mathfrak{Z}}_h^{k\perp})^{\perp_W} = (I - \tilde{P}_h^\perp)W^k$. We consider then $v \in \tilde{\mathfrak{Z}}_h^{k\perp}$ and denote $\tilde{v} := \tilde{P}_h^\perp G_h \tilde{P}_h^\perp v = \tilde{P}_h^\perp G_h v$. Given w in $\tilde{\mathfrak{Z}}_h^{k\perp}$ (or even in $\tilde{V}_h^k$), we compute

$$\begin{aligned} \langle d^k\mathcal{P}_h^k \tilde{v}, d^k\mathcal{P}_h^k w \rangle &= \langle \tilde{P}_h^\perp G_h v, (d^k\mathcal{P}_h^k)^* d^k\mathcal{P}_h^k w \rangle \\ &= \langle G_h v, (d^k\mathcal{P}_h^k)^* d^k\mathcal{P}_h^k w \rangle \\ &= \langle d^k\mathcal{P}_h^k G_h v, d^k\mathcal{P}_h^k w \rangle \\ &= \langle d^k G_h v, d^k\mathcal{P}_h^k w \rangle \\ &= \langle d^k G_h v, d^k P_h^\perp \mathcal{P}_h^k w \rangle \\ &= \langle v, P_h^\perp \mathcal{P}_h^k w \rangle \\ &= \langle v, \mathcal{P}_h^k w \rangle \\ &= \langle v, w \rangle. \end{aligned}$$

Here the respective equalities use (i) the definition of $(d^k\mathcal{P}_h^k)^*$ as the adjoint of $d^k\mathcal{P}_h^k : \tilde{V}_h^k \to V_h^{k+1}$, (ii) the fact that its range is in $\tilde{\mathfrak{Z}}_h^{k\perp}$, (iii) the embeddings $\mathfrak{Z}_h^{k\perp} \subset V_h^k \subset \tilde{V}_h^k$, see (3.11), (iv) the identity $\mathcal{P}_h^k = I$ on V_h^k, (v) the observation (used with $\bar{w} = \mathcal{P}_h^k w$) that

$$d^k\bar{w} = d^k\big(P_h^\perp \bar{w} + (I - P_h^\perp)\bar{w}\big) = d^k P_h^\perp \bar{w}, \qquad \bar{w} \in V_h^k, \tag{4.15}$$

(vi) the characterization (4.8) of G_h, (vii) the fact that $(I - P_h^\perp)\bar{w} \in \mathfrak{Z}_h^k \subset \tilde{\mathfrak{Z}}_h^k$ hence orthogonal to v, and (viii) the fact that $(I - \mathcal{P}_h^k)w$ is in $\tilde{\mathfrak{Z}}_h^k$, hence also orthogonal to v. The result $\tilde{v} = \tilde{G}_h v$ follows from the characterization (4.11) of $\tilde{G}_h$. $\qquad\square$

4.4 Convergence of the discrete eigenmodes

We are now in position to show that both the conforming and non-conforming discrete eigenmodes converge towards the continuous ones in the sense of [5, Def. 2.1], that we now recall. For any positive integer N we let $m(N)$ denote the dimension of the space generated by the eigenspaces of the first *distinct* eigenvalues (4.4). Thus $\lambda_{m(1)}, \ldots, \lambda_{m(N)}$ are the first N distinct eigenvalues and $\mathfrak{E}_1 + \cdots + \mathfrak{E}_{m(N)}$ is the space spanned by the associated eigenspaces: it does not depend on the choice of the eigenbasis $(u_i)_{i \geq 1}$. With these notations we say that the discrete eigenmodes $(\lambda_{i,h}, e_{i,h})_{i \geq 1}$ converge to the continuous ones $(\lambda_i, e_i)_{i \geq 1}$ if, for any given $\varepsilon > 0$ and $N \geq 1$, there exists a mesh parameter $h_0 > 0$ such that for all $h \leq h_0$ we have

$$\max_{1 \leq i \leq m(N)} |\lambda_i - \lambda_{i,h}| \leq \varepsilon \qquad \text{and} \qquad \text{gap}\left(\sum_{1=i}^{m(N)} \mathfrak{E}_i, \sum_{1=i}^{m(N)} \mathfrak{E}_{i,h}\right) \leq \varepsilon, \tag{4.16}$$

where the gap between two spaces is classically defined as

$$\text{gap}(\mathfrak{E}, \mathfrak{F}) := \max \left(\sup_{\substack{u \in \mathfrak{E} \\ \|u\| \leq 1}} \inf_{v \in \mathfrak{F}} \|u - v\|, \sup_{\substack{v \in \mathfrak{F} \\ \|v\| \leq 1}} \inf_{u \in \mathfrak{E}} \|u - v\| \right). \tag{4.17}$$

Obviously the same applies to the non-conforming eigenmodes $(\tilde{\lambda}_{i,h}, \tilde{e}_{i,h})_{i \geq 1}$ as well. A key result in the perturbation theory of linear operators (see, e.g. [5] or [3]) is that, for eigenmode problems corresponding to the compact selfadjoint operators G and G_h (resp. $\tilde{G}_h$), the above convergence holds if the operators G_h (resp. $\tilde{G}_h$) converge to G in $\mathcal{L}(W^k, W^k)$.

It is well known that this convergence holds in the conforming case, see e.g., [4, 35, 20]. In this abstract setting this is essentially a consequence of the uniform convergence (4.6). Specifically, the following result holds.

Theorem 4.5. *The operator $G_h : W^k \to \mathfrak{Z}_h^{k\perp} \subset W^k$ defined by (4.8) satisfies*

$$\|G - G_h\|_{\mathcal{L}} \to 0 \tag{4.18}$$

in the operator norm $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\mathcal{L}(W^k, W^k)}$.

Proof. Using the embedding $\mathfrak{Z}_h^{k\perp} \subset V_h^k$ and Definition (4.7) gives $G_h = K_h P_{V_h^k} P_h^\perp$. Using next (4.2), i.e. $G = KP^\perp$, we decompose

$$\|G - G_h\|_{\mathcal{L}} \leq \|(K - K_h P_{V_h^k}) P_h^\perp\|_{\mathcal{L}} + \|K(P^\perp - P_h^\perp)\|_{\mathcal{L}}$$

$$\leq \|K - K_h P_{V_h^k}\|_{\mathcal{L}} + \|(P^\perp - P_h^\perp)K\|_{\mathcal{L}}$$

where the second inequality uses the unit bound on orthogonal projections and the fact that a bounded operator and its adjoint have the same norm (here all the operators are selfadjoint). By right composition with the compact operator K the pointwise convergence of $P^\perp - P_h^\perp$ to 0 that is proved in Lemma 4.7 yields the norm convergence of $(P^\perp - P_h^\perp)K$ to 0. The limit (4.18) follows then from (4.6). $\square$

Corollary 4.6. *The conforming discrete eigenmodes $(\lambda_{i,h}, u_{i,h})_{i \geq 1}$ converge to the continuous ones $(\lambda_i, u_i)_{i \geq 1}$ in the sense of (4.16).*

In Theorem 4.8 and Corollary 4.9 below we will show that a similar convergence holds true in the non-conforming case. We begin with an auxiliary lemma.

Lemma 4.7. *The distance between the closed subspaces $\mathfrak{Z}_h^{k\perp}$ and $\mathfrak{Z}^{k\perp_W}$ is estimated by the bounds*

$$\|(I - P_h^\perp)P^\perp v\| \leq \|(I - P_{V_h^k})P^\perp v\| \tag{4.19}$$

and

$$\|P_h^\perp(I - P^\perp)v\| \leq \|(I - \pi_h^k)(I - P^\perp)v\| \tag{4.20}$$

for all $v \in W^k$. In particular, we have

$$(P^\perp - P_h^\perp)v \to 0 \quad as \quad h \to 0. \tag{4.21}$$

Proof. Let $v^\perp \in \mathfrak{Z}^{k\perp_W}$: we have $P_{V_h^k}v^\perp \in V_h^k$ and $\langle P_{V_h^k}v^\perp, w \rangle = \langle v, w \rangle = 0$ for $w \in \mathfrak{Z}_h^k \subset \mathfrak{Z}^k$. Thus,

$$P_{V_h^k}\mathfrak{Z}^{k\perp_W} \subset \mathfrak{Z}_h^{k\perp}$$

19

and estimate (4.19) follows, indeed

$$\|(I - P_h^\perp)v^\perp\| = \inf_{v_h^\perp \in \mathfrak{Z}_h^{k\perp}} \|v^\perp - v_h^\perp\| \le \|(I - P_{V_h^k})v^\perp\|.$$

Next we observe from the commuting diagram (2.9) that π_h^k maps $\mathfrak{Z}^k$ to $\mathfrak{Z}_h^k$. Since $(I - P^\perp)W^k = (\mathfrak{Z}^{k\perp_W})^{\perp_W} = \mathfrak{Z}^k$ this yields $\pi_h^k(I - P^\perp)W^k \subset \mathfrak{Z}_h^k$, hence we have

$$\|P_h^\perp(I - P^\perp)v\|^2 = \langle (I - P^\perp)v, P_h^\perp(I - P^\perp)v \rangle = \langle (I - \pi_h^k)(I - P^\perp)v, P_h^\perp(I - P^\perp)v \rangle$$

which gives (4.20) with a Cauchy-Schwarz inequality. The pointwise convergence (4.21) is then easily inferred from the fact that π_h^k and $P_{V_h^k}$ are projections on the discrete spaces V_h^k which union $\cup_{h\to 0} V_h^k$ is assumed dense in W^k. $\qquad\square$

We are now in position to establish a uniform convergence result for the operator $\tilde{G}_h$ characterizing the Conga eigenmodes.

Theorem 4.8. *The operator $\tilde{G}_h : W^k \to \tilde{\mathfrak{Z}}_h^{k\perp} \subset W^k$ defined by (4.11) satisfies*

$$\|G - \tilde{G}_h\|_{\mathcal{L}(W^k, W^k)} \to 0 \qquad as \ \ h \to 0. \tag{4.22}$$

Proof. Using Lemma 4.4 and the arguments in the proof of Theorem 4.5, we write

$$\|G_h - \tilde{G}_h\|_{\mathcal{L}} \le \|(I - \tilde{P}_h^\perp)G_h\|_{\mathcal{L}} + \|\tilde{P}_h^\perp G_h(I - \tilde{P}_h^\perp)\|_{\mathcal{L}} \le 2\|(I - \tilde{P}_h^\perp)G_h\|_{\mathcal{L}}$$

and using the fact that G_h maps into $\mathfrak{Z}_h^{k\perp}$, we continue with

$$\|(I - \tilde{P}_h^\perp)G_h\|_{\mathcal{L}} = \|(I - \tilde{P}_h^\perp)P_h^\perp G_h\|_{\mathcal{L}} \le \|(I - \tilde{P}_h^\perp)P_h^\perp G\|_{\mathcal{L}} + \|G - G_h\|_{\mathcal{L}}.$$

We next observe that $(\mathcal{P}_h^k)^*$ maps $\mathfrak{Z}_h^{k\perp}$ into $\tilde{\mathfrak{Z}}_h^{k\perp}$. Hence

$$\|(I - \tilde{P}_h^\perp)P_h^\perp Gv\| = \inf_{\tilde{v}' \in \tilde{\mathfrak{Z}}_h^{k\perp}} \|P_h^\perp Gv - \tilde{v}'\| \le \|(I - (\mathcal{P}_h^k)^*)P_h^\perp Gv\|$$

Here the operator $(\mathcal{P}_h^k)^*$ can be replaced by $(\mathcal{P}_h^k)^* P_{\tilde{V}_h^k}$ which is assumed to converge pointwise to I, see (3.13). Thus, $(I - \tilde{P}_h^\perp)P_h^\perp Gv$ converges pointwise to 0, which by right composition with the compact operator G leads to a norm convergence,

$$\|(I - \tilde{P}_h^\perp)P_h^\perp G\|_{\mathcal{L}} \to 0 \qquad as \ \ h \to 0.$$

The proof then follows from (4.18). $\qquad\square$

Corollary 4.9. *The non-conforming discrete eigenmodes $(\tilde{\lambda}_{i,h}, \tilde{u}_{i,h})_{i\ge 1}$ converge to the continuous ones $(\lambda_i, u_i)_{i\ge 1}$ in the sense of (4.16).*

Remark 4.10. To our knowledge, the Conga method is the first non-conforming method that is shown to be energy-conserving and spectrally correct. For instance the centered DG method is known to have a large number of spurious eigenvalues [27, 44], and penalized DG schemes are spectrally correct [14, 13] but they dissipate energy.

5   Application to the 3d Maxwell equations

To apply the above analysis to the Maxwell equations in a bounded domain Ω of $\mathbb{R}^3$ with metallic boundary conditions, we can take

$$
H^1(\Omega) \xrightarrow{\ \mathbf{grad}\ } H(\mathbf{curl};\Omega) \xrightarrow{\ \mathbf{curl}\ } H(\mathrm{div}\,;\Omega) \xrightarrow{\ \mathrm{div}\ } L^2(\Omega)
\tag{5.1}
$$

for the primal domain complex $V^0 \to V^1 \to V^2 \to V^3$ and

$$
H_0^1(\Omega) \xrightarrow{\ \mathbf{grad}\ } H_0(\mathbf{curl};\Omega) \xrightarrow{\ \mathbf{curl}\ } H_0(\mathrm{div}\,;\Omega) \xrightarrow{\ \mathrm{div}\ } L^2(\Omega)
\tag{5.2}
$$

for the dual one $V_3^* \to V_2^* \to V_1^* \to V_0^*$, or vice-versa. Here we assume that Ω is a bounded and simply-connected Lipschitz domain of $\mathbb{R}^3$, so that the above de Rham complexes are exact sequences, see e.g. [47, Sec. 3.2]. In each case we define the Hilbert spaces as $W^0 = W^3 = L^2(\Omega)$ and $W^1 = W^2 = L^2(\Omega)^3$, and we take $k = 1$ so that both d^1 and its adjoint d_2^* are curl operators. Using bold-face fonts to distinguish vector-valued fields, the evolution problem (2.14)-(2.15) reads then

$$
\begin{cases} \partial_t \boldsymbol{E} - \mathbf{curl}\,\boldsymbol{B} = -\boldsymbol{J} \\ \partial_t \boldsymbol{B} + \mathbf{curl}\,\boldsymbol{E} = 0 \end{cases} \quad \text{with} \quad \begin{cases} \boldsymbol{E}^0 \in H_0(\mathbf{curl};\Omega) \\ \boldsymbol{B}^0 \in H(\mathbf{curl};\Omega) \end{cases}
\tag{5.3}
$$

and Lemma 2.2 states that for $\boldsymbol{J} \in \mathcal{C}^1([0,T]; L^2(\Omega)^3)$ there exists a unique solution $(\boldsymbol{E}, \boldsymbol{B})$ in $\mathcal{C}^0([0,T[; H_0(\mathbf{curl};\Omega) \times H(\mathbf{curl};\Omega))$ that is continuously differentiable on $]0,T[$. As for the Gauss law (2.17), it reads

$$
\begin{cases} \mathrm{div}\,\boldsymbol{E}(t) = \rho \\ \mathrm{div}\,\boldsymbol{B}(t) = 0 \end{cases} \quad \text{with} \quad \begin{cases} \boldsymbol{E}(t) \in H(\mathrm{div}\,;\Omega) \\ \boldsymbol{B}(t) \in H_0(\mathrm{div}\,;\Omega) \end{cases}
\tag{5.4}
$$

and the continuity equation verified by the sources is just (1.3).

Although it makes no difference on the continuous problem whether one takes (5.1) or (5.2) for the primal complex, on the conforming Galerkin approximation (2.22) it leads to two different methods. As will be seen in Section 5.2, the first choice leads to a strong discretization of the Ampere equation with natural boundary conditions (i.e., in the discrete spaces), whereas the second choice leads to a strong discretization of the Faraday equation with essential boundary conditions.

5.1   Exact sequences of conforming finite element spaces

To build conforming approximations of the above complexes, we assume that Ω is partitioned by a regular family of conforming simplicial meshes $(\mathcal{T}_h)_{h>0}$. By $\mathcal{F}_h$ and $\mathcal{E}_h$ we denote the sets of faces and edges of the mesh, and we assume that the latter are oriented by some arbitrary choice of unit vectors $\boldsymbol{n}_f$ and $\boldsymbol{\tau}_e$, respectively normal to the faces $f \in \mathcal{F}_h$ and tangent to the edges $e \in \mathcal{E}_h$. Following [2, Sec. 3.5] we can then choose between several sequences of standard finite element spaces. These sequences are based on the piecewise polynomial spaces $\mathbb{P}_r \Lambda^l(\Omega, \mathcal{T}_h)$ and $\mathbb{P}_r^- \Lambda^l(\Omega, \mathcal{T}_h)$ that consist of differential l-forms of maximal degree r. For the sake of completeness we recall the correspondences given in [2, Table 5.2], and specify our notations. Note that we often write $X(\mathcal{T}_h) = \{u \in L^2(\Omega) : u|_T \in X(T),\ T \in \mathcal{T}_h\}$ to denote functions spaces with given piecewise structure X, as in (5.10) below. Thus,

- $\mathbb{P}_r\Lambda^0(\Omega, \mathcal{T}_h)$ corresponds to the continuous "Lagrange" elements

$$\mathcal{L}_r(\Omega, \mathcal{T}_h) := \mathbb{P}_r(\mathcal{T}_h) \cap \mathcal{C}(\Omega), \tag{5.5}$$

  where $\mathbb{P}_r(T)$ contains the polynomials of maximal degree $\leq r$ on $T \in \mathcal{T}_h$

- $\mathbb{P}_r\Lambda^1(\Omega, \mathcal{T}_h)$ corresponds to the second-kind Nédélec elements

$$\mathcal{N}_r^{\mathrm{II}}(\Omega, \mathcal{T}_h) := \mathbb{P}_r(\mathcal{T}_h)^3 \cap H(\mathbf{curl}; \Omega) \tag{5.6}$$

- $\mathbb{P}_r^-\Lambda^1(\Omega, \mathcal{T}_h)$ corresponds to the first-kind Nédélec elements

$$\begin{cases} \mathcal{N}_{r-1}^{\mathrm{I}}(\Omega, \mathcal{T}_h) := \mathcal{N}_{r-1}^{\mathrm{I}}(\mathcal{T}_h) \cap H(\mathbf{curl}; \Omega) \\ \text{with } \mathcal{N}_{r-1}^{\mathrm{I}}(T) := \mathbb{P}_{r-1}(T)^3 \oplus (\boldsymbol{x} \wedge \mathbb{P}_{r-1}^{\mathrm{hom}}(T)^3) \end{cases} \tag{5.7}$$

  where $\mathbb{P}_{r-1}^{\mathrm{hom}}$ is the space of homogeneous polynomials of degree $r-1$

- $\mathbb{P}_r\Lambda^2(\Omega, \mathcal{T}_h)$ corresponds to the Brezzi-Douglas-Marini elements (also called second-kind $H(\mathrm{div})$ Nédélec space),

$$\mathcal{BDM}_r(\Omega, \mathcal{T}_h) := \mathbb{P}_r(\mathcal{T}_h)^3 \cap H(\mathrm{div}; \Omega) \tag{5.8}$$

- $\mathbb{P}_r^-\Lambda^2(\Omega, \mathcal{T}_h)$ corresponds to the Raviart-Thomas elements (also called first-kind $H(\mathrm{div})$ Nédélec spaces),

$$\begin{cases} \mathcal{RT}_{r-1}(\Omega, \mathcal{T}_h) := \mathcal{RT}_{r-1}(\mathcal{T}_h) \cap H(\mathrm{div}; \Omega) \\ \text{with } \mathcal{RT}_{r-1}(T) := \mathbb{P}_{r-1}(T)^3 \oplus (\boldsymbol{x}\mathbb{P}_{r-1}^{\mathrm{hom}}(T)) \end{cases} \tag{5.9}$$

- $\mathbb{P}_r\Lambda^3(\Omega, \mathcal{T}_h)$ corresponds to the fully discontinuous elements

$$\mathbb{P}_r(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in \mathbb{P}_r(T), \ T \in \mathcal{T}_h\}. \tag{5.10}$$

Using the above spaces we can then invoke [3, Th. 5.9] which says that for each of the following sequences of discrete spaces $V_h^0 \to V_h^1 \to V_h^2 \to V_h^3$,

$$\begin{cases} \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{N}_{p-1}^{\mathrm{II}}(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{BDM}_{p-2}(\Omega, \mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathbb{P}_{p-3}(\mathcal{T}_h) \\[2ex] \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{N}_{p-1}^{\mathrm{II}}(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{RT}_{p-2}(\Omega, \mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathbb{P}_{p-2}(\mathcal{T}_h) \\[2ex] \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{N}_{p-1}^{\mathrm{I}}(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{BDM}_{p-1}(\Omega, \mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathbb{P}_{p-2}(\mathcal{T}_h) \\[2ex] \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{grad}} \mathcal{N}_{p-1}^{\mathrm{I}}(\Omega, \mathcal{T}_h) \xrightarrow{\mathbf{curl}} \mathcal{RT}_{p-1}(\Omega, \mathcal{T}_h) \xrightarrow{\mathrm{div}} \mathbb{P}_{p-1}(\mathcal{T}_h) \end{cases} \tag{5.11}$$

there exists a sequence of projection operators $\pi_h^l : V^l \to V_h^l$, $l = 0, \cdots, 3$, on the primal sequence (5.1) that is uniformly L^2-bounded with respect to h and for which (2.8) is a commuting diagram (with $\bar{V}^l = V^l$). Note that if one defines the primal spaces V^l with essential boundary conditions as in (5.2), then the boundary conditions need to be incorporated in the definition of the finite element spaces as well, by forcing the classical degrees of freedom associated to boundary faces to vanish.

5.2 Compatible conforming finite elements

According to Theorem 3.7 we know that compatible conforming Galerkin methods can be obtained by approximating the sources with a generalized projection Π_h of the form (3.7), based on two projection operators π_h^1 and π_h^2 for which (2.8) is a commuting diagram. Here we could take the L^2-bounded projection operators just mentionned, which existence is guaranteed by [3, Th. 5.9]. However, because these operators may not be simple to implement we can use instead standard finite element interpolations which also verify a commuting diagram property, even though they are defined on smaller spaces $\bar{V}^l$ of smooth functions. Let us denote these projection operators by $\pi_h^{\mathbf{curl}} : \bar{V}^1 \to V_h^1$ and $\pi_h^{\mathrm{div}} : \bar{V}^2 \to V_h^2$. For instance if the last sequence from (5.11) is taken for

$$V_h^0 \xrightarrow{d^0 = -\mathbf{grad}} V_h^1 \xrightarrow{d^1 = \mathbf{curl}} V_h^2 \xrightarrow{d^2 = \mathrm{div}} V_h^3$$

(5.12)

(with essential or natural boundary conditions), we can define

$$\begin{cases} \pi_h^{\mathbf{curl}} : H^2(\Omega)^3 \to V_h^1 = \mathcal{N}_{p-1}^{\mathrm{I}}(\Omega, \mathcal{T}_h) \\ \pi_h^{\mathrm{div}} : H^1(\Omega)^3 \to V_h^2 = \mathcal{RT}_{p-1}(\Omega, \mathcal{T}_h) \end{cases}$$

(5.13)

as the standard Nédélec and Raviart-Thomas interpolations. For a precise definition see, e.g., (2.5.49)-(2.5.32) and (2.5.26)-(2.5.10) in Ref. [7], where they are respectively denoted Σ_h and Π_h. Error estimates are then available: we have

$$\begin{cases} \|(I - \pi_h^{\mathbf{curl}})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m, & 2 \le m \le p \\ \|(I - \pi_h^{\mathrm{div}})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m, & 1 \le m \le p \end{cases}$$

(5.14)

where $|\cdot|_m$ denotes the usual $H^m(\Omega)$ semi-norm, see, e.g., Propositions 2.5.7 and 2.5.4 in [7]. Similar estimates hold for the orthogonal projection operators, namely

$$\|(I - P_{V_h^1})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m \qquad \text{and} \qquad \|(I - P_{V_h^2})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m, \qquad 0 \le m \le p.$$

(5.15)

In particular, the projection operators corresponding to (3.7) and (3.8), i.e.,

$$\Pi_h = \begin{pmatrix} P_{V_h^1} & 0 \\ 0 & \pi_h^{\mathrm{div}} \end{pmatrix} : \left(L^2(\Omega)^3 \times H^1(\Omega)^3\right) \to V_h^1 \times V_h^2$$

and

$$\hat{\Pi}_h = \begin{pmatrix} \pi_h^{\mathbf{curl}} & 0 \\ 0 & P_{V_h^2} \end{pmatrix} : \hat{\mathcal{V}} \to V_h^1 \times V_h^2 \qquad \text{with} \quad \hat{\mathcal{V}} := H^2(\Omega)^3 \times H(\mathbf{curl}; \Omega)$$

satisfy the following a priori estimates,

$$\begin{cases} \|(I - \Pi_h)U\| \le ch^m |U|_m, & 1 \le m \le p \\ \|(I - \hat{\Pi}_h)U\| \le ch^m |U|_m, & 2 \le m \le p. \end{cases}$$

(5.16)

With the above material, Theorem 3.7 yields two compatible conforming schemes depending whether one takes (5.1) or (5.2) for the primal complex. The first choice corresponds to a strong discretization of the Ampere equation: the discrete spaces (5.12) are defined as one of the sequences from (5.11) with natural boundary conditions, and the scheme computes the solution $(\boldsymbol{B}_h, \boldsymbol{E}_h) \in \mathcal{C}^1(]0, T[; V_h^1 \times V_h^2)$ to

$$\begin{cases} \langle \partial_t \boldsymbol{B}_h, \boldsymbol{\varphi}^\mu \rangle + \langle \boldsymbol{E}_h, \mathbf{curl}\, \boldsymbol{\varphi}^\mu \rangle = 0 & \boldsymbol{\varphi}^\mu \in V_h^1 \subset H(\mathbf{curl}; \Omega) \\ \langle \partial_t \boldsymbol{E}_h, \boldsymbol{\varphi}^\varepsilon \rangle - \langle \mathbf{curl}\, \boldsymbol{B}_h, \boldsymbol{\varphi}^\varepsilon \rangle = -\langle \pi_h^{\mathrm{div}}\, \boldsymbol{J}, \boldsymbol{\varphi}^\varepsilon \rangle & \boldsymbol{\varphi}^\varepsilon \in V_h^2 \subset H(\mathrm{div}\,; \Omega) \end{cases}$$

(5.17)

and the embedding $\mathbf{curl}\, V_h^1 \subset V_h^2$ allows to rewrite the second equation as

$$\partial_t \boldsymbol{E}_h - \mathbf{curl}\, \boldsymbol{B}_h = -\pi_h^{\mathrm{div}} \boldsymbol{J} \qquad (\text{in } V_h^2).$$

Moreover if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one may set $(\boldsymbol{B}_h, \boldsymbol{E}_h)(0) := (P_{V_h^1} \boldsymbol{B}^0, \pi_h^{\mathrm{div}} \boldsymbol{E}^0)$ as specified in Theorem 3.3.

The second choice corresponds to a strong discretization of the Faraday equation: again the sequence of discrete spaces (5.12) is defined as one of those from (5.11), now with essential boundary conditions, and the scheme computes the solution $(\boldsymbol{E}_h, \boldsymbol{B}_h) \in \mathcal{C}^1(]0, T[; V_h^1 \times V_h^2)$ to

$$\begin{cases} \langle \partial_t \boldsymbol{E}_h, \boldsymbol{\varphi}^\varepsilon \rangle - \langle \boldsymbol{B}_h, \mathbf{curl}\, \boldsymbol{\varphi}^\varepsilon \rangle = -\langle \boldsymbol{J}, \boldsymbol{\varphi}^\varepsilon \rangle & \boldsymbol{\varphi}^\varepsilon \in V_h^1 \subset H_0(\mathbf{curl}; \Omega) \\[2mm] \langle \partial_t \boldsymbol{B}_h, \boldsymbol{\varphi}^\mu \rangle + \langle \mathbf{curl}\, \boldsymbol{E}_h, \boldsymbol{\varphi}^\mu \rangle = 0 & \boldsymbol{\varphi}^\mu \in V_h^2 \subset H_0(\mathrm{div}\,; \Omega). \end{cases} \tag{5.18}$$

Here the embedding $\mathbf{curl}\, V_h^1 \subset V_h^2$ allows to rewrite the second equation as

$$\partial_t \boldsymbol{B}_h + \mathbf{curl}\, \boldsymbol{E}_h = 0 \qquad (\text{in } V_h^2).$$

And again if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one can follow the statement from Theorem 3.3 and set $(\boldsymbol{E}_h, \boldsymbol{B}_h)(0) := (P_{V_h^1} \boldsymbol{E}^0, \pi_h^{\mathrm{div}} \boldsymbol{B}^0)$.

Finally, for any of the above methods the error bound (3.5) and the stability result apply. For instance in the case where the finite element spaces are defined as the last sequence from (5.11), using the a priori estimate (5.16) one obtains

$$\|(\boldsymbol{E}_h - \boldsymbol{E}, \boldsymbol{B}_h - \boldsymbol{B})(t)\| \le ch^m \left(|(\boldsymbol{E}^0, \boldsymbol{B}^0)|_m + |(\boldsymbol{E}, \boldsymbol{B})(t)|_m + \int_0^t |\partial_t(\boldsymbol{E}, \boldsymbol{B})(s)|_m \, \mathrm{d}s \right) \tag{5.19}$$

for $2 \le m \le p$ and a constant c independent of h and $t \ge 0$.

Remark 5.1. Since π_h^{div} never appears in the scheme (5.18), we may equivalently define Π_h and $\hat{\Pi}_h$ with the L^2-bounded operators π_h^l from [3, Th. 5.9] instead of $\pi_h^{\mathbf{curl}}$ and π_h^{div}. Now, as these satisfy $\|(I - \pi_h^l)\boldsymbol{u}\| \le ch^m \|\boldsymbol{u}\|_m$ for $0 \le m \le p$, this shows that in Estimate (5.19) we can take the same values for m.

As pointed out in Remark 3.5, a priori error estimates leading to long-time stability results are well-known for conforming schemes, see e.g., [32, 33, 31]. In the following sections we will see that the abstract analysis of Section 3 can be applied to non-conforming schemes, where we are not aware of long-time stability results.

5.3 Compatible conforming/non-conforming Galerkin (Conga) schemes

We may now construct Conga schemes based on non-conforming spaces $\tilde{V}_h^1 \not\subset V^1$ that are Gauss-compatible in the sense of Definition 3.1. Following Section 3.3, the main ingredients are:

- an exact sequence of *conforming* spaces (5.12) satisfying $V_h^1 \subset \tilde{V}_h^1$

- a projection operator on the conforming space V_h^1,

$$\mathcal{P}_h^1 : \tilde{V}_h^1 \to V_h^1, \tag{5.20}$$

that preserves spaces of moments with dense union in $L^2(\Omega)^3$, see (3.13).

Since every conforming space listed in (5.11) has the form $V_h^1 = X^1(\mathcal{T}_h) \cap V^1$, discontinuous spaces containing their conforming counterparts can be taken as $\tilde{V}_h^1 := X^1(\mathcal{T}_h)$, but standard spaces of piecewise polynomials with sufficiently large degrees also can be used. We shall then construct the projection operator $\mathcal{P}_h^1$ by averaging locally the standard finite element interpolation, as follows. To each of the above spaces $X^1(T)$ are classically associated d^1 (i.e., curl-conforming) degrees of freedom, either of "volume" type which involve integrals over single elements $T \in \mathcal{T}_h$, or "interface" type which may involve integrals over faces and edges. Then, for some $\boldsymbol{u} \in \tilde{V}_h^1$ we may define the projection $\mathcal{P}_h^1 \boldsymbol{u}$ by assigning the values of its degrees of freedom: either to those of $\boldsymbol{u}$ if they are of volume type, or to those of an average trace of $\boldsymbol{u}$ if they are of interface type. The resulting operator is simple to implement in a finite element code, and it is local.

To build a projection on the first-kind Nédélec space $V_h^1 = \mathcal{N}_{p-1}^I(\Omega, \mathcal{T}_h)$ for instance, we may use the fact that the local element $X^1(T) = \mathcal{N}_{p-1}^I(T)$ is equipped with degrees of freedom of volume, face and edge type, see [38] or [25, Sec. III-5.3],

$$\begin{cases} \mathcal{M}_{\mathrm{vol}}(\boldsymbol{u}) := \{\int_T \boldsymbol{u} \cdot \boldsymbol{\pi} \; : \; \boldsymbol{\pi} \in \mathbb{P}_{p-3}(T)^3, \; T \in \mathcal{T}_h\} \\ \mathcal{M}_{\mathrm{face}}(\boldsymbol{u}) := \{\int_f (\boldsymbol{u} \wedge \boldsymbol{n}_f) \cdot \boldsymbol{\pi} \; : \; \boldsymbol{\pi} \in \mathbb{P}_{p-2}(f)^2, \; f \in \mathcal{F}_h\} \\ \mathcal{M}_{\mathrm{edge}}(\boldsymbol{u}) := \{\int_e (\boldsymbol{u} \cdot \boldsymbol{\tau}_e) \pi \; : \; \pi \in \mathbb{P}_{p-1}(e), \; e \in \mathcal{E}_h\}. \end{cases} \tag{5.21}$$

These linear forms are unisolvent in the sense that when restricted to some element T they characterize the functions of $X^1(T)$, and they are V^1-conforming in the sense that an element of $X^1(\mathcal{T}_h)$ is in $V^1 = H(\mathbf{curl}; \Omega)$ if an only if its one-sided traces over any mesh interface define the same degrees of freedom. Thus, given a smooth $\boldsymbol{u}$ the relations $\mathcal{M}_{\mathrm{vol}}(\boldsymbol{u}_h - \boldsymbol{u}) = \{0\}$, $\mathcal{M}_{\mathrm{face}}(\boldsymbol{u}_h - \boldsymbol{u}) = \{0\}$ and $\mathcal{M}_{\mathrm{edge}}(\boldsymbol{u}_h - \boldsymbol{u}) = \{0\}$ define a unique interpolate $\boldsymbol{u}_h \in V_h^1$, and as previously said, for a piecewise smooth $\boldsymbol{u}$ we can average the multivalued traces. Namely, we define $\mathcal{P}_h^1 : \tilde{V}_h^1 \to V_h^1$ by

$$\mathcal{M}_{\mathrm{vol}}(\mathcal{P}_h^1 \boldsymbol{u} - \boldsymbol{u}) = \{0\}, \quad \mathcal{M}_{\mathrm{face}}(\mathcal{P}_h^1 \boldsymbol{u} - \{\boldsymbol{u}\}_f) = \{0\}, \quad \mathcal{M}_{\mathrm{edge}}(\mathcal{P}_h^1 \boldsymbol{u} - \{\boldsymbol{u}\}_e) = \{0\} \tag{5.22}$$

with $\{\boldsymbol{u}\}_f := \frac{1}{2}(\boldsymbol{u}|_{T^-} + \boldsymbol{u}|_{T^+})|_f$ and similarly for $\{\boldsymbol{u}\}_e$ on the edges.

Proposition 5.2. *Let $\tilde{V}_h^1$ be a piecewise polynomial space containing the curl-conforming $V_h^1 = \mathcal{N}_{p-1}^I(\Omega, \mathcal{T}_h)$, say $\tilde{V}_h^1 := \mathbb{P}_p(\mathcal{T}_h)^3$. On this space the projection operator (5.22) is uniformly L^2-bounded with respect to h,*

$$\|\mathcal{P}_h^1 \boldsymbol{u}\| \le c\|\boldsymbol{u}\|, \qquad \boldsymbol{u} \in \tilde{V}_h^1, \tag{5.23}$$

and it satisfies the moment preserving property (3.12) with

$$M_h^1 = \mathbb{P}_{p-3}(\mathcal{T}_h)^3. \tag{5.24}$$

Moreover, seen as an operator on $\tilde{V}_h^1$, it has an adjoint $(\mathcal{P}_h^1)^* : \tilde{V}_h^1 \to \tilde{V}_h^1$ such that

$$\|(I - (\mathcal{P}_h^1)^* P_{\tilde{V}_h^1})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m, \qquad 0 \le m \le p - 2. \tag{5.25}$$

Proof. Relation (5.24) readily follows from the definition (5.21) of the volume dofs. As for Estimate (5.23), it is easily obtained with classical arguments: Denoting

$$F_T : \boldsymbol{x} \mapsto \boldsymbol{x}_T + B_T \boldsymbol{x}$$

the affine transformation that maps some fixed reference element $\hat{T}$ onto T, we let

$$\Phi_T : \boldsymbol{u} \mapsto h^{\frac{1}{2}} B_T^t (\boldsymbol{u} \circ F_T) \tag{5.26}$$

so that [25, Lemma III-5.5] reads $\Phi_T(\mathcal{N}_{p-1}^I(T)) = \mathcal{N}_{p-1}^I(\hat{T})$. Here the $h^{\frac{1}{2}}$ scaling is such that $\|\Phi_T \boldsymbol{u}\|_{L^2(\hat{T})} \sim \|\boldsymbol{u}\|_{L^2(T)}$ holds with constants independent of h, T and $\boldsymbol{u}$, assuming the shape

regularity of the mesh. A localized version of $\mathcal{P}_h^1$ is then defined as follows. We let $\mathcal{T}_h(T) := \{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$ be the patch of elements around T and define $\bar{r}_T : \mathcal{C}^0(\mathcal{T}_h(T)) \to \mathcal{N}_{p-1}^1(T)$ by the following relations which, among the degrees of freedom from (5.21) only involve those that are defined on the element T and its boundary:

$$\mathcal{M}_{T,\text{vol}}(\bar{r}_T \boldsymbol{u} - \boldsymbol{u}) = \{0\}, \quad \mathcal{M}_{T,\text{face}}(\bar{r}_T \boldsymbol{u} - \{\boldsymbol{u}\}_f) = \{0\}, \quad \mathcal{M}_{T,\text{edge}}(\bar{r}_T \boldsymbol{u} - \{\boldsymbol{u}\}_e) = \{0\}.$$

In particular, we have $\bar{r}_T \boldsymbol{u} = (\mathcal{P}_h^1 \boldsymbol{u})|_T$ and the above observation yields

$$\|\mathcal{P}_h^1 \boldsymbol{u}\|_{L^2(T)} = \|\bar{r}_T \boldsymbol{u}\|_{L^2(T)} \sim \|\Phi_T(\bar{r}_T \boldsymbol{u})\|_{L^2(\hat{T})}. \tag{5.27}$$

Using the same arguments than those leading to [25, Eq. III-(5.35)] and the fact that we have $B_T^t \{\boldsymbol{u}\}_f \circ F_T = \{B_T^t(\boldsymbol{u} \circ F_T)\}_{F_T^{-1}(f)}$ on the faces of T and similarly for the edges, one then finds $\Phi_T(\bar{r}_T \boldsymbol{u}) = \bar{r}_{\hat{T}}(\Phi_T \boldsymbol{u})$ where in the latter projection the averaging involves the values of $\Phi_T \boldsymbol{u}$ on the cells $F_T^{-1}(T')$, $T' \in \mathcal{T}_h(T)$. A straightforward computation using (5.21) gives then

$$\|\Phi_T(\bar{r}_T \boldsymbol{u})\|_{L^2(\hat{T})} = \|\bar{r}_{\hat{T}}(\Phi_T \boldsymbol{u})\|_{L^2(\hat{T})} \lesssim \max_{T' \in \mathcal{T}_h(T)} \|\Phi_T \boldsymbol{u}\|_{L^\infty(F_T^{-1}(T'))} \tag{5.28}$$

with a constant depending on p. Using the equivalence of norms over the finite dimensional space $\mathbb{P}_p(F_T^{-1}(T'))^3$ and the same scaling argument as in (5.27) we compute next

$$\|\Phi_T \boldsymbol{u}\|_{L^\infty(F_T^{-1}(T'))} \sim \|\Phi_T \boldsymbol{u}\|_{L^2(F_T^{-1}(T'))} \sim \|\boldsymbol{u}\|_{L^2(T')}, \quad \boldsymbol{u} \in \tilde{V}_h^1.$$

Estimate (5.23) follows by summing over $T \in \mathcal{T}_h$ and by using the fact that $\|P_{\tilde{V}_h^1}\|_{\mathcal{L}(L^2(\Omega)^3)} \leq 1$. Finally, the error estimate (5.25) follows from the fact that the operator

$$(\mathcal{P}_h^1)^* P_{\tilde{V}_h^1} : L^2(\Omega)^3 \to \tilde{V}_h^1$$

preserves the polynomials of degree less than $p-2$ and that it is uniformly L^2-bounded with respect to h, as the adjoint of $P_{\tilde{V}_h^1} \mathcal{P}_h^1$. $\qquad \square$

In particular, if π_h^{div} is as in (5.13) then the projection corresponding to (3.15),

$$\Pi_h = \begin{pmatrix} (\mathcal{P}_h^1)^* P_{\tilde{V}_h^1} & 0 \\ 0 & \pi_h^{\text{div}} \end{pmatrix} : \left(L^2(\Omega)^3 \times H^1(\Omega)^3\right) \to \tilde{V}_h^1 \times V_h^2$$

satisfies the following a priori estimate derived from (5.14) and (5.25),

$$\|(I - \Pi_h)U\| \leq c h^m |U|_m, \quad 1 \leq m \leq p - 2. \tag{5.29}$$

Applying Theorem 3.9 we then obtain two compatible non-conforming Conga schemes, depending whether one takes (5.1) or (5.2) for the primal complex. As in the conforming case, the first choice corresponds to a strong discretization of the Ampere equation: again the conforming finite element spaces (5.12) are defined as one of the sequences from (5.11), using natural boundary conditions, and the non-conforming space $\tilde{V}_h^1$ must contain its conforming counterpart V_h^1. The scheme computes then the unique solution $(\tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{E}}_h) \in \mathcal{C}^1(]0, T[; \tilde{V}_h^1 \times V_h^2)$ to

$$\begin{cases} \langle \partial_t \tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{\varphi}}^\mu \rangle + \langle \tilde{\boldsymbol{E}}_h, \mathbf{curl}\, \mathcal{P}_h^1 \tilde{\boldsymbol{\varphi}}^\mu \rangle = 0 & \tilde{\boldsymbol{\varphi}}^\mu \in \tilde{V}_h^1 \not\subset H(\mathbf{curl}; \Omega) \\ \langle \partial_t \tilde{\boldsymbol{E}}_h, \boldsymbol{\varphi}^\varepsilon \rangle - \langle \mathbf{curl}\, \mathcal{P}_h^1 \tilde{\boldsymbol{B}}_h, \boldsymbol{\varphi}^\varepsilon \rangle = -\langle \pi_h^{\text{div}} \boldsymbol{J}, \boldsymbol{\varphi}^\varepsilon \rangle & \boldsymbol{\varphi}^\varepsilon \in V_h^2 \subset H(\text{div}; \Omega). \end{cases} \tag{5.30}$$

Here the embedding $\mathbf{curl}\, V_h^1 \subset V_h^2$ allows to rewrite the second equation as

$$\partial_t \tilde{\boldsymbol{E}}_h - \mathbf{curl}\,\mathcal{P}_h^1 \tilde{\boldsymbol{B}}_h = -\pi_h^{\mathrm{div}}\,\boldsymbol{J} \qquad (\text{in } V_h^2)$$

and if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one can set $(\tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{E}}_h)(0) := ((\mathcal{P}_h^1)^* P_{\tilde{V}_h^1}\boldsymbol{B}^0, \pi_h^{\mathrm{div}}\,\boldsymbol{E}^0)$, see Theorem 3.3.

As for the second choice, it corresponds to a strong discretization of the Faraday equation just as in the conforming case: again the sequence of conforming discrete spaces (5.12) is taken among those from (5.11), using essential boundary conditions. The non-conforming $\tilde{V}_h^1$ must then contain its conforming counterpart V_h^1, and the scheme computes the unique solution $(\tilde{\boldsymbol{E}}_h, \tilde{\boldsymbol{B}}_h) \in \mathcal{C}^1(]0, T[; \tilde{V}_h^1 \times V_h^2)$ to

$$\begin{cases} \langle \partial_t \tilde{\boldsymbol{E}}_h, \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle - \langle \tilde{\boldsymbol{B}}_h, \mathbf{curl}\,\mathcal{P}_h^1 \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle = -\langle \boldsymbol{J}, \mathcal{P}_h^1 \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle & \tilde{\boldsymbol{\varphi}}^\varepsilon \in V_h^1 \not\subset H_0(\mathbf{curl}; \Omega) \\ \langle \partial_t \tilde{\boldsymbol{B}}_h, \boldsymbol{\varphi}^\mu \rangle + \langle \mathbf{curl}\,\mathcal{P}_h^1 \tilde{\boldsymbol{E}}_h, \boldsymbol{\varphi}^\mu \rangle = 0 & \boldsymbol{\varphi}^\mu \in V_h^2 \subset H_0(\mathrm{div}\,; \Omega) \end{cases} \qquad (5.31)$$

where the expression for the source follows from the discussion in Section 3.3. Here the embedding $\mathbf{curl}\, V_h^1 \subset V_h^2$ allows to rewrite the second equation as

$$\partial_t \tilde{\boldsymbol{B}}_h + \mathbf{curl}\,\mathcal{P}_h^1 \tilde{\boldsymbol{E}}_h = 0 \qquad (\text{in } V_h^2).$$

And again, to be compatible with the fundamental homogeneous Gauss law one can apply Theorem 3.3 and set $(\tilde{\boldsymbol{E}}_h, \tilde{\boldsymbol{B}}_h)(0) := ((\mathcal{P}_h^1)^* P_{V_h^1}\boldsymbol{E}^0, \pi_h^{\mathrm{div}}\,\boldsymbol{B}^0)$.

Remark 5.3 (Coupling with particle methods). It is interesting to note that in the Conga scheme (5.31) the proposed discretization of the source is well suited to the case where the Maxwell solver is coupled with a point particle scheme. Indeed, in a time-discrete setting where the sources need to be averaged over each time step in order to satisfy a discrete continuity equation, the resulting particle current takes the form of weighted Dirac measures supported on the individual trajectories and we know that the product of the latter against the curl-conforming finite element function $\mathcal{P}_h^1 \tilde{\boldsymbol{\varphi}}^\varepsilon$ is well defined, see [16, Lemma 3.1]. This is not be case in general when the product is taken against $\tilde{\boldsymbol{\varphi}}^\varepsilon$, because some particles may be following the mesh interfaces where the functions of $\tilde{V}_h^1$ are fully discontinuous.

Finally, for any of the above methods the error bound (3.5) and the stability result apply. For instance in the case where the conforming finite element spaces are defined as the last sequence from (5.11) and the conforming projection operator $\mathcal{P}_h^1$ is given by (5.22), using the a priori estimates (5.16) and (5.29) one obtains

$$\|(\tilde{\boldsymbol{E}}_h - \boldsymbol{E}, \tilde{\boldsymbol{B}}_h - \boldsymbol{B})(t)\| \leq ch^m \left(|(\boldsymbol{E}^0, \boldsymbol{B}^0)|_m + |(\boldsymbol{E}, \boldsymbol{B})(t)|_m + \int_0^t |\partial_t(\boldsymbol{E}, \boldsymbol{B})(s)|_m \, \mathrm{d}s \right) \qquad (5.32)$$

for $1 \leq m \leq p - 2$ and a constant c independent of h and $t \geq 0$.

Remark 5.4. Arguing as in Remark 5.1 we can see that for the scheme (5.31) Estimate (5.32) actually holds for $0 \leq m \leq p - 2$.

The following result derives from the spectral analysis in Section 4.

Theorem 5.5. *Let $p \geq 3$. If the non-conforming space $\tilde{V}_h^1$ contains its conforming counterpart $V_h^1 = \mathcal{N}_{p-1}^{\mathrm{I}}(\Omega, \mathcal{T}_h)$, then the evolution operators $\tilde{\mathcal{A}}_h$ corresponding to the schemes (5.30) and (5.31), with $\mathcal{P}_h^1$ defined by (5.22), are spectrally correct: their eigenmodes converge to those of the exact Maxwell system in the sense of (4.16). In particular, these two Conga schemes are free of spurious discrete eigenvalues and eigenvectors.*

27

Proof. Let us verify that Assumptions (H1)-(H3) listed at the beginning of Section 4 hold. First, the Conga schemes (5.30) and (5.31) correspond to taking (5.2) for the primal (resp. dual) complex and (5.2) for the dual (resp. primal) one, so that the intersection $V^1 \cap V_1^*$ is the classical space $X_T = H(\mathbf{curl}; \Omega) \cap H_0(\text{div}; \Omega)$, resp. $X_N = H(\text{div}; \Omega) \cap H_0(\mathbf{curl}; \Omega)$. These spaces are known to be dense and compactly embedded in $W^1 = L^2(\Omega)^3$, see e.g., [34, Cor. 3.49], so that (H1) is satisfied. Next, (H2) holds according to [3, Th. 5.9], as recalled in Section 5.1. Finally (H3) holds for $\mathcal{P}_h^1$ the projection (5.22) on the auxiliary conforming space V_h^1 contained in $\tilde{V}_h$, indeed for $p \geq 3$ the preserved moments (5.24) form a dense union in $L^2(\Omega)^3$, which yields (3.13). Thus Corollary 4.9 applies: the eigenmodes of the Conga operator $\tilde{A}_h$, (3.14), converge to the exact ones and the spectral correctness follows. $\qquad\square$

5.4 Compatible DG schemes

When built on conforming triangulations $\mathcal{T}_h$ of Ω, DG approximations to the time-dependent Maxwell equation (5.3) typically involve spaces of piecewise polynomial functions with no continuity requirements across inter-element faces, such as

$$\tilde{V}_h := \mathbb{P}_{p_{\mathrm{dg}}}(\mathcal{T}_h)^3 \quad \text{with} \quad p_{\mathrm{dg}} \in \mathbb{N}. \tag{5.33}$$

The semi-discrete centered DG scheme reads then (see, e.g., [24, Eq. (22)]): find $(\tilde{E}_h, \tilde{B}_h) \in \mathcal{C}^1(]0, T[; \tilde{V}_h \times \tilde{V}_h)$ the unique solution to

$$\begin{cases} \langle \partial_t \tilde{E}_h, \tilde{\varphi}^\varepsilon \rangle - \langle \tilde{B}_h, \mathbf{curl}_h \tilde{\varphi}^\varepsilon \rangle + \langle \{\tilde{B}_h\}, [\tilde{\varphi}^\varepsilon] \rangle_{\mathcal{F}_h} = -\langle \tilde{J}_h, \tilde{\varphi}^\varepsilon \rangle & \tilde{\varphi}^\varepsilon \in \tilde{V}_h \\ \langle \partial_t \tilde{B}_h, \tilde{\varphi}^\mu \rangle + \langle \tilde{E}_h, \mathbf{curl}_h \tilde{\varphi}^\mu \rangle - \langle \{\tilde{E}_h\}, [\tilde{\varphi}^\mu] \rangle_{\mathcal{F}_h^{\mathrm{int}}} = 0 & \tilde{\varphi}^\mu \in \tilde{V}_h \end{cases} \tag{5.34}$$

with given initial data and approximated source $\tilde{J}_h \in \tilde{V}_h$. Here $\mathbf{curl}_h$ is the piecewise operator given by $(\mathbf{curl}_h \boldsymbol{u})|_T := \mathbf{curl}(\boldsymbol{u}|_T)$ for all $T \in \mathcal{T}_h$, and

$$\{\boldsymbol{u}\}_f := \frac{1}{2}(\boldsymbol{u}|_{T^-} + \boldsymbol{u}|_{T^+})|_f \quad \text{and} \quad [\boldsymbol{u}]_f := (\boldsymbol{n}^- \wedge \boldsymbol{u}|_{T^-} + \boldsymbol{n}^+ \wedge \boldsymbol{u}|_{T^+})|_f, \qquad f \in \mathcal{F}_h^{\mathrm{int}},$$

denote the average and tangential jump on an internal face f shared by two elements $T^\pm$ (for which $\boldsymbol{n}^\pm$ are the outward unit vectors). On a boundary face they are defined as

$$\{\boldsymbol{u}\}_f := \boldsymbol{u} \quad \text{and} \quad [\boldsymbol{u}]_f := \boldsymbol{n} \wedge \boldsymbol{u}, \qquad f \in \mathcal{F}_h \setminus \mathcal{F}_h^{\mathrm{int}}.$$

To provide a compatible DG scheme we need to recast (5.34) as an evolution equation of the form (3.1) and identify proper projection operators $\Pi_h, \hat{\Pi}_h$ following the lines of Section 3.4. Thus, we let $\tilde{d}_h : \tilde{V}_h \to \tilde{V}_h$ be the DG approximation to the operator $\mathbf{curl} : H(\mathbf{curl}; \Omega) \to L^2(\Omega)^3$,

$$\langle \tilde{d}_h \boldsymbol{u}, \boldsymbol{v} \rangle := \langle \boldsymbol{u}, \mathbf{curl}_h \boldsymbol{v} \rangle - \langle \{\boldsymbol{u}\}, [\boldsymbol{v}] \rangle_{\mathcal{F}_h}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \tilde{V}_h,$$

and similarly let $\tilde{d}_{h,0} : \tilde{V}_h \to \tilde{V}_h$ be the approximation to the adjoint operator $\mathbf{curl} : H_0(\mathbf{curl}; \Omega) \to L^2(\Omega)^3$,

$$\langle \tilde{d}_{h,0} \boldsymbol{u}, \boldsymbol{v} \rangle := \langle \boldsymbol{u}, \mathbf{curl}_h \boldsymbol{v} \rangle - \langle \{\boldsymbol{u}\}, [\boldsymbol{v}] \rangle_{\mathcal{F}_h^{\mathrm{int}}}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \tilde{V}_h,$$

so that the DG evolution operator involved in (5.34) reads

$$\tilde{A}_h := \begin{pmatrix} 0 & \tilde{d}_h \\ -\tilde{d}_{h,0} & 0 \end{pmatrix} : \tilde{\mathcal{V}}_h \to \tilde{\mathcal{V}}_h \quad \text{with} \quad \tilde{\mathcal{V}}_h := \tilde{V}_h \times \tilde{V}_h. \tag{5.35}$$

We can verify that $\tilde{A}_h$ is indeed skew-symmetric: using local Green formulas

$$\langle \mathbf{curl}(\boldsymbol{u}|_T), \boldsymbol{v} \rangle_T = \langle \boldsymbol{u}, \mathbf{curl}(\boldsymbol{v}|_T) \rangle_T - \langle \boldsymbol{u}|_T, \boldsymbol{n}^T \wedge \boldsymbol{v}|_T \rangle_{\partial T}, \qquad T \in \mathcal{T}_h$$

and the identity $\langle [\boldsymbol{u}], \{\boldsymbol{v}\}\rangle_f + \sum_{T=T^\pm(f)} \langle \boldsymbol{u}|_T, \boldsymbol{n}^T \wedge \boldsymbol{v}|_T \rangle_f = \langle \{\boldsymbol{u}\}, [\boldsymbol{v}]\rangle_f$ valid for $f \in \mathcal{F}_h^{\mathrm{int}}$, we find that $\langle \tilde{d}_h \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, \tilde{d}_{h,0}\boldsymbol{v}\rangle$ holds for all $\boldsymbol{u}, \boldsymbol{v} \in \tilde{V}_h$, hence

$$(\tilde{d}_h)^* = \tilde{d}_{h,0}.$$

In Section 3.4 it was announced that a compatible DG scheme could be designed by defining V_h^1 as an *auxiliary* curl-conforming space associated with a projection $\mathcal{P}_h^1 : \tilde{V}_h \to V_h^1$ for which a relation like (3.19) holds for both $\tilde{d}_h$ and $(\tilde{d}_h)^*$. Note that unlike in the Conga case, here the space V_h^1 does not need to be a subset of $\tilde{V}_h$. The following result specifies this construction for the DG space (5.33).

Lemma 5.6. *On the DG space* $\tilde{V}_h := \mathbb{P}_{p_{\mathrm{dg}}}(\mathcal{T}_h)^3$, *let*

$$\mathcal{P}_h^1 : \tilde{V}_h \to V_h^1 := \mathcal{N}_{p_{\mathrm{dg}}+1}^{\mathrm{I}}(\Omega; \mathcal{T}_h) \subset H(\mathbf{curl}; \Omega) \tag{5.36}$$

be the averaged finite element interpolation operator defined as in (5.22) *using the degrees of freedom from* (5.21) *with* $p = p_{\mathrm{dg}} + 2$. *Similarly, we define*

$$\mathcal{P}_{h,0}^1 : \tilde{V}_h \to V_{h,0}^1 := \mathcal{N}_{p_{\mathrm{dg}}+1}^{\mathrm{I}}(\Omega; \mathcal{T}_h) \cap H_0(\mathbf{curl}; \Omega) \tag{5.37}$$

by forcing the boundary degrees of freedom to vanish in the curl-conforming Nédélec space. Then the DG curl operators involved in the centered scheme (5.34) *read*

$$\tilde{d}_h = P_{\tilde{V}_h} \, \mathbf{curl} \, \mathcal{P}_h^1 \qquad and \qquad \tilde{d}_{h,0} = P_{\tilde{V}_h} \, \mathbf{curl} \, \mathcal{P}_{h,0}^1. \tag{5.38}$$

Proof. For $\boldsymbol{u}, \boldsymbol{v} \in \tilde{V}_h$, applying local Green formulas we compute

$$\langle P_{\tilde{V}_h} \, \mathbf{curl} \, \mathcal{P}_h^1 \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \mathbf{curl} \, \mathcal{P}_h^1 \boldsymbol{u}, \boldsymbol{v}\rangle = \sum_{T \in \mathcal{T}_h} \langle \mathcal{P}_h^1 \boldsymbol{u}, \mathbf{curl}(\boldsymbol{v}|_T)\rangle_T + \langle \boldsymbol{n}^T \wedge \mathcal{P}_h^1 \boldsymbol{u}, \boldsymbol{v}|_T\rangle_{\partial T}$$

$$= \sum_{T \in \mathcal{T}_h} \langle \boldsymbol{u}, \mathbf{curl}(\boldsymbol{v}|_T)\rangle_T + \langle \boldsymbol{n}^T \wedge \{\boldsymbol{u}\}, \boldsymbol{v}|_T\rangle_{\partial T}$$

$$= \langle \boldsymbol{u}, \mathbf{curl}_h \, \boldsymbol{v}\rangle - \langle \{\boldsymbol{u}\}, [\boldsymbol{v}]\rangle_{\mathcal{F}_h} = \langle \tilde{d}_h \boldsymbol{u}, \boldsymbol{v}\rangle$$

where we have used the definition of $\mathcal{P}_h^1$ in the third equality, in particular the fact that the polynomial pieces $\mathbf{curl}(\boldsymbol{v}|_T)$ and $\boldsymbol{v}|_f$ are of respective degrees $p_{\mathrm{dg}} - 1$ and p_{dg}, thus matching the degrees of freedom (5.21) with $p = p_{\mathrm{dg}} + 2$. The identity $\tilde{d}_h = P_{\tilde{V}_h} \, \mathbf{curl} \, \mathcal{P}_h^1$ easily follows, and the same computation proves the result for the homogeneous operators in (5.38). Note that here the orthogonal projection $P_{\tilde{V}_h}$ is required since $\mathbf{curl}$ does not map V_h^1 into $\tilde{V}_h$. $\qquad\square$

To conclude the construction described in Section 3.4 we must find operators $\tilde{\pi}_h^{\mathbf{curl}}$ and $\tilde{\pi}_h^{\mathrm{div}}$ that satisfy a commuting diagram and map into the proper spaces as required by (3.20). For the auxiliary conforming space V_h^1 introduced in (5.36) we have $V_h^1 \cap \tilde{V}_h = \mathbb{P}_{p_{\mathrm{dg}}}(\mathcal{T}_h)^3 \cap H(\mathbf{curl}; \Omega) = \mathcal{N}_{p_{\mathrm{dg}}}^{\mathrm{II}}(\Omega, \mathcal{T}_h)$ so that we can define

$$\tilde{\pi}_h^{\mathbf{curl}} : H^2(\Omega)^3 \to \mathcal{N}_{p_{\mathrm{dg}}}^{\mathrm{II}}(\Omega, \mathcal{T}_h) \quad and \quad \tilde{\pi}_h^{\mathrm{div}} : H^1(\Omega)^3 \to \mathcal{BDM}_{p_{\mathrm{dg}}-1}(\Omega, \mathcal{T}_h)$$

as the interpolation operators corresponding to the second-kind Nédélec elements defined in [39, Sec. 3.1 and 2.1]. By construction, these operators map into the proper spaces as required by (3.20), and they can also be used for the homogeneous auxiliary space $V_{h,0}^1$ introduced in (5.37). Specifically, we have

$$\tilde{\pi}_h^{\mathbf{curl}} : \bar{V}^1 \to \mathbb{P}_{p_{\mathrm{dg}}}(\mathcal{T}_h)^3 \cap H(\mathbf{curl}; \Omega) = V_h^1 \cap \tilde{V}_h \quad for \quad \bar{V}^1 := H^2(\Omega)^3$$

and
$$\tilde{\pi}_h^{\mathbf{curl}} : \bar{V}_0^1 \to \mathbb{P}_{p_{\mathrm{dg}}}(\mathcal{T}_h)^3 \cap H_0(\mathbf{curl};\Omega) = V_{h,0}^1 \cap \tilde{V}_h \quad \text{for} \quad \bar{V}_0^1 := \bar{V}^1 \cap H_0(\mathbf{curl};\Omega)$$

as for $\tilde{\pi}_h^{\mathrm{div}}$, it clearly maps $\bar{V}^2 := H^1(\Omega)^3$ into $\tilde{V}_h$. Finally, Prop. 2 from [39] gives

$$\tilde{\pi}_h^{\mathrm{div}} \, \mathbf{curl} = \mathbf{curl} \, \tilde{\pi}_h^{\mathbf{curl}} \qquad \text{on both } \bar{V}^1 \text{ and } \bar{V}_0^1,$$

so that (3.20) is satisfied in both cases. In particular, using (5.38) we can compute as in (3.21),

$$\tilde{d}_h \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \, \mathbf{curl} \, \mathcal{P}_h^1 \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \, \mathbf{curl} \, \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \tilde{\pi}_h^{\mathrm{div}} \, \mathbf{curl} = \tilde{\pi}_h^{\mathrm{div}} \, \mathbf{curl} \tag{5.39}$$

on $\bar{V}^1 = H^2(\Omega)^3$, and similarly for the homogeneous operators,

$$\tilde{d}_{h,0} \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \, \mathbf{curl} \, \mathcal{P}_{h,0}^1 \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \, \mathbf{curl} \, \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \tilde{\pi}_h^{\mathrm{div}} \, \mathbf{curl} = \tilde{\pi}_h^{\mathrm{div}} \, \mathbf{curl} \tag{5.40}$$

on $\bar{V}_0^1 = \bar{V}^1 \cap H_0(\mathbf{curl};\Omega)$. This leads to the following compatibility result.

Theorem 5.7 (Compatible DG methods)**.** *In the case where $\tilde{\mathcal{A}}_h$ is the centered DG evolution operator* (5.35), *the scheme* (3.1) *complemented with the projection*

$$\Pi_h = \begin{pmatrix} \tilde{\pi}_h^{\mathrm{div}} & 0 \\ 0 & \tilde{\pi}_h^{\mathrm{div}} \end{pmatrix} : \left(H^1(\Omega)^3 \times H^1(\Omega)^3\right) \to \tilde{\mathcal{V}}_h := \tilde{V}_h \times \tilde{V}_h \tag{5.41}$$

is compatible. In particular, it satisfies the commuting diagram (3.3) *with*

$$\hat{\Pi}_h = \begin{pmatrix} \tilde{\pi}_h^{\mathbf{curl}} & 0 \\ 0 & \tilde{\pi}_h^{\mathbf{curl}} \end{pmatrix} : \hat{\mathcal{V}} \to \tilde{\mathcal{V}}_h \qquad \text{with} \quad \hat{\mathcal{V}} := \bar{V}_0^1 \times \bar{V}^1. \tag{5.42}$$

Proof. Given (5.35) and (5.41)-(5.42), relation (3.3) amounts to (5.39)-(5.40). $\qquad\square$

In more classical terms, what the above result states is that the DG scheme

$$\begin{cases} \langle \partial_t \tilde{\boldsymbol{E}}_h, \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle - \langle \tilde{\boldsymbol{B}}_h, \mathbf{curl}_h \, \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle + \langle \{\tilde{\boldsymbol{B}}_h\}, [\tilde{\boldsymbol{\varphi}}^\varepsilon] \rangle_{\mathcal{F}_h} = -\langle \tilde{\pi}_h^{\mathrm{div}} \boldsymbol{J}, \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle & \tilde{\boldsymbol{\varphi}}^\varepsilon \in \tilde{V}_h \\[2mm] \langle \partial_t \tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{\varphi}}^\mu \rangle + \langle \tilde{\boldsymbol{E}}_h, \mathbf{curl}_h \, \tilde{\boldsymbol{\varphi}}^\mu \rangle - \langle \{\tilde{\boldsymbol{E}}_h\}, [\tilde{\boldsymbol{\varphi}}^\mu] \rangle_{\mathcal{F}_h^{\mathrm{int}}} = 0 & \tilde{\boldsymbol{\varphi}}^\mu \in \tilde{V}_h \end{cases}$$

is compatible in the sense of Def. 3.1. To apply next the error bound (3.5) we use

$$\begin{cases} \|(I - \tilde{\pi}_h^{\mathbf{curl}})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m, & 2 \le m \le p_{\mathrm{dg}} + 1 \\[2mm] \|(I - \tilde{\pi}_h^{\mathrm{div}})\boldsymbol{u}\| \le ch^m |\boldsymbol{u}|_m, & 1 \le m \le p_{\mathrm{dg}} \end{cases} \tag{5.43}$$

derived from [39, Prop. 1 and 3], which yield

$$\|(\tilde{\boldsymbol{E}}_h - \boldsymbol{E}, \tilde{\boldsymbol{B}}_h - \boldsymbol{B})(t)\| \le ch^m \left(|(\boldsymbol{E}^0, \boldsymbol{B}^0)|_m + |(\boldsymbol{E}, \boldsymbol{B})(t)|_m + \int_0^t |\partial_t(\boldsymbol{E}, \boldsymbol{B})(s)|_m \, \mathrm{d}s \right) \tag{5.44}$$

for $2 \le m \le p_{\mathrm{dg}}$ and a constant c independent of h and $t \ge 0$. We observe that the convergence order is the same than in [24, Th. 3.5], but the time dependence is improved. Indeed in (5.44) the upper bound is a constant for steady state solutions.

5.5 Numerical validation in 2d

To validate the relevance of the proposed approach we now run 2d versions of some of the above schemes on a test-case proposed in [29, 23] to study the numerical charge conservation properties and also considered in [43] to assess the long-time stability of DG solvers with hyperbolic field correction. Here the Transverse Electric (TE) mode of the normalized Maxwell system is posed in a metallic cavity $\Omega = [0, 1]^2$ with a current density given by

$$\boldsymbol{J}(t, x, y) = (\cos(t) - 1) \begin{pmatrix} \pi \cos(\pi x) + \pi^2 x \sin(\pi y) \\ \pi \cos(\pi y) + \pi^2 y \sin(\pi x) \end{pmatrix} - \cos(t) \begin{pmatrix} x \sin(\pi y) \\ y \sin(\pi x) \end{pmatrix}$$

and the initial data is $\boldsymbol{E}^0 = 0$ and $B^0 = 0$ so as to approximate the exact solution

$$\begin{cases} \boldsymbol{E}(t, x, y) = \sin(t) \begin{pmatrix} x \sin(\pi y) \\ y \sin(\pi x) \end{pmatrix} \\ B(t, x, y) = (\cos(t) - 1)(\pi y \cos(\pi x) - \pi x \cos(\pi y)). \end{cases}$$

In Figure 1 the curves show the time evolution of the L^2 norm of numerical $\boldsymbol{E}_h$ fields obtained as follows. In the left panel a 2d version of the standard finite element method (5.18) is used with curl-conforming elements $V_h^1 = \mathcal{N}_2^\mathrm{I}(\Omega, \mathcal{T}_h)$, whereas the center and right panels show a 2d version of the Conga scheme (5.31) with a DG space $\tilde{V}_h^1 = \mathbb{P}_3(\mathcal{T}_h)^2$ for the electric field and $V_h^1 = \mathcal{N}_2^\mathrm{I}(\Omega, \mathcal{T}_h) \subset \tilde{V}_h^1$ for the auxiliary space, using a conforming projection $\mathcal{P}_h^1$ defined as an averaged Nédélec interpolation similarly as in (5.22). In the center panel the current density is then approximated by an L^2 projection $\tilde{\boldsymbol{J}}_h = P_{\tilde{V}_h^1}\boldsymbol{J}$ which corresponds to replacing in (5.31) the products $\langle \boldsymbol{J}, \mathcal{P}_h^1 \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle$ by the standard ones $\langle \boldsymbol{J}, \tilde{\boldsymbol{\varphi}}^\varepsilon \rangle$. Finally in the right panel a compatible current approximation $\tilde{\boldsymbol{J}}_h = (\mathcal{P}_h^1)^* P_{\tilde{V}_h^1}\boldsymbol{J}$ is used, corresponding to (5.31). (For a detailed presentation of theses schemes in 2d and for more numerical results we refer to the forthcoming articles [17, 18].) Here all the runs implement the same leap-frog time scheme, and in each panel the norm of the exact solution is plotted with dashed lines for comparison. These results clearly demonstrate the importance of discretizing the sources in a compatible way.
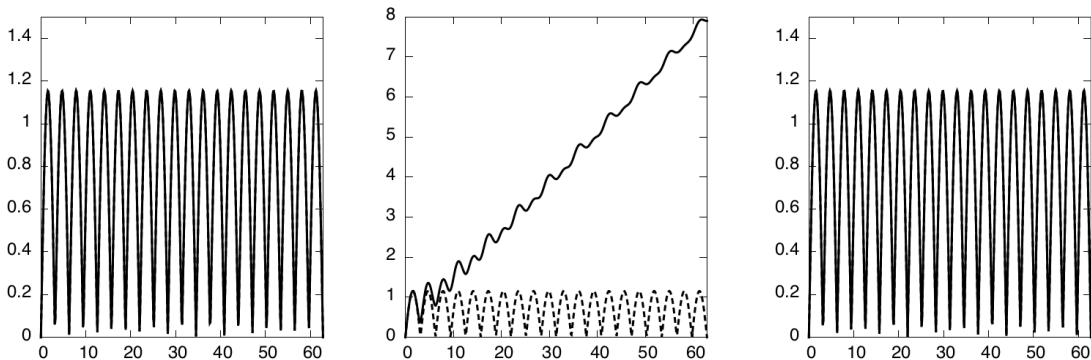

Figure 1: Time evolution of the L^2 norm of electric fields computed with 2d versions of some of the above schemes: in the left panel a standard curl-conforming finite element method corresponding to (5.18), in the center panel a Conga scheme (5.31) using a standard L^2 projection for the current density, and in the right panel the same Conga scheme using the compatible approximation for the source corresponding to the actual right hand side in (5.31).

6 Conclusion

In this work we have formulated a generic compatibility property for energy-preserving approximations to the time-dependent Maxwell equations with sources. This property takes the form of a two-component commuting diagram and derives from a spectral interpretation of the homogeneous Gauss law that is also related to the study of spurious modes in the numerical approximation to the Maxwell eigenvalue problem.

We have shown that semi-discrete schemes satisfying this compatibility property are long-time stable with respect to steady state solutions in both L^2 and energy norm, which solves the problem of the large deviations developed by certain classes of numerical schemes on long simulation times.

In addition, we have introduced a new mixed method called conforming/non-conforming Galerkin (Conga), defined as a relaxation of the standard conforming approximation. Like DG schemes this method can be implemented using only local discrete operators, and it preserves some of the structural benefits of conforming approximations such as the spectral correctness of its discrete evolution operator, and the ability to preserve one Gauss law in a strong sense.

Finally, we have described several approximation operators for the sources that make Galerkin methods Gauss-compatible in the above sense, be it for the standard conforming Galerkin method, for centered DG schemes or for this new Conga method.

Acknowledgments

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