Integrability methods in the time minimal coherence transfer for Ising chains of three spins

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INTEGRABILITY METHODS IN THE TIME MINIMAL COHERENCE TRANSFER FOR ISING CHAINS OF THREE SPINS

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Abstract. The objective of this article is to analyze the integrability properties of extremal solutions of Pontryagin Maximum Principle in the time minimal control of a linear spin system with Ising coupling in relation with conjugate and cut loci computations. Restricting to the case of three spins, the problem is equivalent to analyze a family of almost-Riemannian metrics on the sphere S^2, with Grushin equatorial singularity. The problem can be lifted into a SR-invariant problem on $SO(3)$, this leads to a complete understanding of the geometry of the problem and to an explicit parametrization of the extremals using an appropriate chart as well as elliptic functions. This approach is compared with the direct analysis of the Liouville metrics on the sphere where the parametrization of the extremals is obtained by computing a Liouville normal form. Finally, an algebraic approach is presented in the framework of the application of differential Galois theory to integrability.

1. **Introduction.** Over the past decade, the application of geometric optimal control techniques to the dynamics of spin systems with applications to Nuclear Magnetic Resonance (NMR) spectroscopy and quantum information processing [17] has been an intense research direction. In particular, a series of articles focus on the time optimal control of a linear chain of spins with Ising couplings [14, 23]. Using Hamiltonian Formalism and an adapted rotating frame, the control system is defined by $H = H_d + H_c$, where H_d is the internal Hamiltonian, $H_d = \sum_{i,j} J_{ij} I_{iz} I_{jz}$, with J_{ij} representing the coupling between the spins, $I_{k\alpha} = 1 \otimes ... \otimes I_\alpha \otimes ... \otimes 1$, I_α, $\alpha \in x, y, z$ being the Pauli matrix and the system is controlled by external radio-frequency pulse on resonance to each spin defining $H_c = \sum_i (u_{i1} I_{ix} + u_{i2} I_{iy})$.

Restricting to the case of three spins, the objective of this article is to provide the preliminary work to compute the optimal solutions parametrized by Pontryagin Maximum Principle. We here focus on the integrability aspects of the problem by using three different approaches.

A first point of view which already appears in the pioneering work [14], see also [9] in a different context, consists in lifting the problem on a sub-Riemannian invariant (SR-invariant) problem on $SO(3)$ that depends on a parameter k representing the

ratio of the coupling constants J_{12}, J_{23} between the spins. Such a metric is a limit case of invariant Riemannian metrics on $SO(3)$, the so-called Euler-Poinsot rigid body problem in mechanics. Using the seminal work in [13], we define a chart that identifies locally $SO(3)$ to $S^2 \times S^1$ which enlightens the geometry of the problem and leads to an explicit computation of the extremals using elliptic functions.

Another approach consists in integrating the system directly on S^2. In this context the problem is equivalent to analyze a family of $2D-$ Liouville metrics on S^2 with an equatorial singularity. The integrability properties is equivalent to the calculation of the Liouville normal form [4, 5] using the additional first integral. In our case it corresponds to the Hamiltonian of the round metrics on S^2, induced by the Casimir function on $SO(3)$. This point of view is very important to analyze the optimality properties related to the conjugate and cut loci of the metrics, indeed it is related to similar calculations on Liouville surfaces that generalizes the case of ellipsoids [10, 11].

Finally, the third approach consists in using our problem as a test-bed platform to apply the algebraic framework of Galois differential theory in integrability [18] to compute the solutions. First, the optimal control is calculated using the Jacobi elliptic functions and inserted in the equations. This reduces the computations to the integration of a time-depending linear equation whose coefficients are expressed in terms of the Jacobi elliptic functions. The Picard Vessiot extension and the associated Galois group are computed to parametrize the extremal solutions.

Different computations were done independently but the parametrizations are compared in the conclusion. Also we briefly discuss the application to the computations of the conjugate and cut loci, following the method in [7] combining geometric analysis on Liouville surfaces and numerical computations. More details can be found in [12].

2. **The mathematical model.** In this paper, we detail the presentation of the problem in the case of three spins, but the problem can be easily generalized to a chain of n spins. We follow the presentation of [14, 22, 23].

We introduce the spin $1/2$ matrices I_α related to the Pauli matrices by a $1/2$ factor. Such matrices satisfy:

$$[I_x, I_y] = \mathrm{i}I_z, \quad I_x^2 = I_y^2 = I_z^2 = 1/4.$$

The Hilbert space L of the system is the space formed by the tensor product of the three two-dimensional spin $1/2$ Hilbert space. Assuming a single input system, the Hamiltonian of the system decomposes into:

$$H = H_d + H_c$$

where

$$H_d = 2(J_{12}I_{1z}I_{2z} + J_{23}I_{2z}I_{3z}), \qquad H_c = u(t)I_{2y}.$$

We consider the time evolution of the vector $X = (x_1, x_2, x_3, x_4)^\top$ where $x_1 = \langle I_{1x} \rangle$, $x_2 = \langle 2I_{1y2z} \rangle$, $x_3 = \langle 2I_{1y2x} \rangle$, $x_4 = \langle 4I_{1y2y3z} \rangle$ with $\langle \rangle$ denoting the *expectation value*. To compute the dynamics, we introduce a 8×8 matrix $\rho \in L$, called the *density matrix*, which satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}[H, \rho].$$

Using the definition of the expectation value of a given operator: $\langle O \rangle = \text{Tr}(O\rho)$, one gets:

$$\frac{d}{dt}\langle I_{1x} \rangle = \text{Tr}(I_{1x}\frac{d\rho}{dt}) = -i\text{Tr}(I_{1x}[H,\rho]) = -i\text{Tr}([I_{1x},H]\rho).$$

Hence, we deduce that:

$$\frac{d}{dt}x_1 = -J_{12}\text{Tr}(2I_{1y}I_{2z}\rho).$$

By rescaling the time by a factor J_{12}, it becomes:

$$\frac{d}{dt}x_1 = -x_2$$

Similar computations lead to the evolution of X given by:

$$\frac{dX}{dt} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -u & 0 \\ 0 & u & 0 & -k \\ 0 & 0 & k & 0 \end{pmatrix} X, \quad \text{where } k = \frac{J_{23}}{J_{12}}.$$

The optimal control problem is to transfer in minimum time $(1,0,0,0)^\top$ to $(0,0,0,1)^\top$. It is an intermediate step to realize the transfer in minimum time from I_{1x} to I_{3x}. Indeed, it connects the first spin to the third one by controlling the second spin.

Introducing the following coordinates:

$$r_1 = x_1 \quad r_2 = \sqrt{x_2^2 + x_3^2} \quad r_3 = x_4$$

and denoting

$$\tan\alpha = x_3/x_2,$$

the system becomes:

$$\frac{d}{dt}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 & u_3 & 0 \\ -u_3 & 0 & u_1 \\ 0 & -u_1 & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{1}$$

where $r = (r_1, r_2, r_3) \in S^2$ and $u_1 = -k\sin(\alpha)$, $u_3 = -\cos(\alpha)$ are the components of the control.

In those coordinates, the minimum time problem is equivalent to determine the fastest transfer on the sphere from $(1,0,0)$ to $(0,0,1)$.

It can be written as an L^2-minimization problem as follows:

$$\frac{dr_1}{dt} = u_3 r_2, \quad \frac{dr_2}{dt} = -u_3 r_1 + u_1 r_3, \quad \frac{dr_3}{dt} = -u_1 r_2,$$

$$\min_{u(.)} \int_0^T (I_1 u_1^2(t) + I_3 u_3^2(t))dt, \quad k^2 = \frac{I_1}{I_3}.$$

The problem is equivalent to an *almost-Riemannian* problem on the sphere S^2 with a singularity at the equator $r_2 = 0$, for the corresponding metric:

$$g = \frac{dr_1^2 + k^2 dr_3^2}{r_2^2}$$

(See [1] for more details about such metrics.)

Introducing the spherical coordinates $r_2 = \cos\varphi$, $r_1 = \sin\varphi\cos\theta$, $r_3 = \sin\varphi\sin\theta$, where $\varphi = \pi/2$ corresponds to the equator, the metric g take the form

$$g = \tan^2\varphi(k^2\cos^2\theta + \sin^2\theta)\mathrm{d}\theta^2 + (\cos^2\theta + k^2\sin^2\theta)\mathrm{d}\varphi^2$$
$$+ 2(k^2 - 1)\tan\varphi\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\varphi,$$

with the associated Hamiltonian

$$H = \frac{1}{4k^2}\Big(\cot\mathrm{an}^2\varphi(\cos^2\theta + k^2\sin^2\theta)p_\theta^2 + (k^2\cos^2\theta + \sin^2\theta)p_\varphi^2$$
$$- 2(k^2 - 1)\cot\mathrm{an}\varphi\sin\theta\cos\theta p_\theta p_\varphi\Big).$$

If $k = 1$, the Hamiltonian takes the form $H = \frac{1}{2}\big(p_\varphi^2 + p_\theta^2\cot\mathrm{an}^2\varphi\big)$ and describes the *standard Grushin metric on S^2*.

3. Connection with invariant metrics on $SO(3)$ and integration.

3.1. **Lifting procedure.** A first approach to analyze the optimal control problem and parametrize the extremals consists in lifting the problem onto $SO(3)$. We introduce the matrix $R(t) = (r_{ij}(t))$ on $SO(3)$ where the third row is identified to the unit vector $r(t)$ defined previously: $r_{31} = r_1$, $r_{32} = r_2$, $r_{33} = r_3$, and we consider the right-invariant control system:

$$\frac{\mathrm{d}}{\mathrm{d}t}R^\top(t) = \begin{bmatrix} 0 & u_3 & 0 \\ -u_3 & 0 & u_1 \\ 0 & -u_1 & 0 \end{bmatrix}R^\top(t)$$

where the last column of $R^\top$ describes the evolution of the vector r. Our optimal control problem can then be stated as:

$$\min_{u(.)}\int_0^T (I_1 u_1^2(t) + I_3 u_3^2(t))\mathrm{d}t$$

for the right-invariant control system with the following boundary conditions:

$$R^\top(0) = \begin{bmatrix} & 1 \\ * & 0 \\ & 0 \end{bmatrix}, \quad R^\top(T) = \begin{bmatrix} & 0 \\ * & 0 \\ & 1 \end{bmatrix}$$

which consist in steering the third axis of the frame $R^\top$ from e_1 to e_3, where (e_i) is the canonical basis of $\mathbb{R}^3$.

Similarly, it can be transformed into a left-invariant control problem to use the geometric framework and the computations in [13] :

$$\frac{\mathrm{d}R}{\mathrm{d}t} = R\begin{bmatrix} 0 & -u_3 & 0 \\ u_3 & 0 & -u_1 \\ 0 & u_1 & 0 \end{bmatrix}, \quad \min_{u(.)}\int_0^T (I_1 u_1^2(t) + I_3 u_3^2(t))\mathrm{d}t$$

with the corresponding boundary conditions.

This defines a left-invariant SR-problem on $SO(3)$ depending upon the parameter $k^2 = I_1/I_3$. Upon an appropriate limit process $I_2 \to +\infty$, this is related to the Euler-Poinsot rigid body motion [2] :

$$\frac{\mathrm{d}R}{\mathrm{d}t} = R\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \quad \min_{u(.)}\int_0^T (I_1 u_1^2(t) + I_2 u_2^2(t) + I_3 u_3^2(t))\mathrm{d}t$$

which is well-known model for left-invariant metrics on $SO(3)$, depending on two parameters e.g. the ratios I_2/I_1, I_3/I_1. Two special cases are:

- The bi-invariant case $I_1 = I_2 = I_3$ where the geodesics are the rotations of $SO(3)$.
- The case of revolution where $I_1 = I_3$.

3.2. **Extremal equations and Integration.** The optimal solutions to our problem can be parametrized by the Pontryagin Maximum Principle [19], and thanks to the explicit formula given in [13], the solutions can be computed in both the Riemannian and the sub-Riemannian cases using elliptic functions.

We introduce:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

that satisfy the Lie brackets relations:

$$[A_1, A_2] = -A_3, \quad [A_1, A_3] = A_2, \quad [A_2, A_3] = -A_1.$$

Consider now the following optimal control problem on $SO(3)$:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sum_{i=1}^{3} u_i \vec{A}_i(R), \quad \min_{u(.)} \frac{1}{2} \int_0^T \sum_{i=1}^{3} I_i u_i^2(t) \mathrm{d}t$$

where the I_i's are the principal momenta of inertia of the body. The extremal equation will be derived using appropriate coordinates. Let λ be an element of $T_R^* SO(3)$ and denote $H_i = \lambda(\vec{A}_i(R))$, $i = 1, 2, 3$ the symplectic lift on the vector fields $\vec{A}_i$. The pseudo-Hamiltonian associated to the problem takes the form:

$$H = \sum_{i=1}^{3} u_i H_i - \frac{1}{2} \sum_{i=1}^{3} I_i u_i^2.$$

The extremal control is computed using the relation $\dfrac{\partial H}{\partial u_i} = 0$, and we obtain

$$u_i = \frac{H_i}{I_i} \quad i = 1, 2, 3.$$

Plugging this expression for the u_i into H, we get the Hamiltonian :

$$H_n = \frac{1}{2} \Big(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} \Big)$$

The SR-case is obtained by setting $u_2 = 0$ which corresponds to take $I_2 \to +\infty$, and leads to the Hamiltonian :

$$H_n = \frac{1}{2} \Big(\frac{H_1^2}{I_1} + \frac{H_3^2}{I_3} \Big).$$

The evolution of the vector $H = (H_1, H_2, H_3)$ is given by the Euler equation:

$$\frac{\mathrm{d}H_i}{\mathrm{d}t} = \mathrm{d}H_i(\vec{H_n}) = \{H_i, H_n\}$$

where $\{,\}$ denotes the Poisson bracket. Using the relation between Poisson and Lie brackets: $\{H_i, H_j\} = \lambda([A_i, A_j])$, we obtain the *Euler-equation*:

- Riemannian case

$$\frac{\mathrm{d}H_1}{\mathrm{d}t} = H_2 H_3 \big(\frac{1}{I_3} - \frac{1}{I_2}\big), \frac{\mathrm{d}H_2}{\mathrm{d}t} = H_1 H_3 \big(\frac{1}{I_1} - \frac{1}{I_3}\big), \frac{\mathrm{d}H_3}{\mathrm{d}t} = H_1 H_2 \big(\frac{1}{I_2} - \frac{1}{I_1}\big). \quad (2)$$

- SR-case ($I_2 \to +\infty$)

$$\frac{\mathrm{d}H_1}{\mathrm{d}t} = \frac{H_2 H_3}{I_3}, \quad \frac{\mathrm{d}H_2}{\mathrm{d}t} = H_1 H_3 \big(\frac{1}{I_1} - \frac{1}{I_3}\big), \quad \frac{\mathrm{d}H_3}{\mathrm{d}t} = -\frac{H_1 H_2}{I_1}. \quad (3)$$

The extremals equations are a classical example of (super) integrable system which is a consequence of the following proposition [13].

Proposition 1. *For each invariant Hamiltonian H_n on $SO(3)$, the extremal system is integrable by quadratures using the four first-integrals: the Hamiltonian H_n and the Hamiltonian lifts of the right-invariant vector fields $A_i R$.*

To provide details on the quadratures we introduce the following.

3.2.1. *Distinguished chart.* Each element $R \in SO(3)$ is represented on a chart U by the following element (r, Φ_1) of $S^2 \times S^1$ where :

- r is the third row of the matrix R,
- Φ_1, Φ_2, Φ_3 are the Euler-angles computed with the convention:

$$R = \exp(\Phi_1 A_3) \circ \exp(\Phi_2 A_2) \circ \exp(\Phi_3 A_3).$$

3.2.2. *Useful formulas.* We recall the following:

Hamiltonian using Euler-angles. Expressed in terms of the Euler angles the Hamiltonian for the Euler Poinsot rigid body motion takes the form:

$$\begin{aligned}
H_n = &\frac{1}{2I_1}\Big(p_2 \sin \Phi_3 - \frac{\cos \Phi_3}{\sin \Phi_2}(p_1 - p_3 \cos \Phi_2)\Big)^2 + \frac{1}{2I_3}p_3^2 \\
&+ \frac{1}{2I_2}\Big(p_2 \cos \Phi_3 + \frac{\sin \Phi_3}{\sin \Phi_2}(p_1 - p_3 \cos \Phi_2)\Big)^2
\end{aligned}$$

where p_i is the canonical impulse associated to Φ_i, $i = 1, 2, 3$. Observe that Φ_1 is a cyclic variable. As previously, the SR-case is obtained by taking $I_2 \to +\infty$.

In both cases we have the following crucial proposition [13].

Proposition 2.

1. The angles Φ_2 and Φ_3 can be obtained from the relations:

$$H_1 = -|H| \sin \Phi_2 \cos \Phi_3, \quad H_2 = |H| \sin \Phi_2 \sin \Phi_3, \quad H_3 = |H| \cos \Phi_2$$

2. We have, with $r = (r_1, r_2, r_3) \in S^2$ (third row of R):

$$r_1 = -\sin \Phi_2 \cos \Phi_3 = \frac{H_1}{|H|}, \quad r_2 = \sin \Phi_2 \sin \Phi_3 = \frac{H_2}{|H|}, \quad r_3 = \cos \Phi_2 = \frac{H_3}{|H|}.$$

While the Euler equation can be integrated using H_n and the Casimir function $G^2 = H_1^2 + H_2^2 + H_3^2$, the angle Φ_1 can be computed by quadrature using [13].

Proposition 3. *In the invariant case, Φ_1 is solution of the equation*

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}t} = |H| \frac{H_1 \frac{\partial H_n}{\partial H_1} + H_2 \frac{\partial H_n}{\partial H_2}}{H_1^2 + H_2^2}$$

This leads to an uniform integration procedure in the invariant case using elliptic functions that we detail in the SR-case.

3.2.3. *Integration of the Euler equation in the SR-case.* We fix the level set for the Hamiltonian $H_n = \frac{1}{2}\left(\frac{H_1^2}{I_1} + \frac{H_3^2}{I_3}\right) = \frac{1}{2}$, and we introduce the angle β as follows:

$$\cos\beta = \frac{H_1}{\sqrt{I_1}}, \quad \sin\beta = \frac{H_3}{\sqrt{I_3}}.$$

Using the Euler equation, we deduce that β is solution of the pendulum equation:

$$\ddot{\beta} = \frac{1}{2}\sin 2\beta\left(\frac{I_1 - I_3}{I_1 I_3}\right).$$

We introduce $\nu = 2\beta$, and we obtain the equation:

$$\dot{\nu}^2 + 2\frac{I_1 - I_3}{I_1 I_3}\cos\nu = C$$

We can assume $I_3 > I_1$ and $I_3 = 1$ and use [16]. We define

$$\omega^2 = \frac{I_3 - I_1}{I_1 I_3} > 0,$$

and according to the constant C we have two types of generic solutions.

- **Oscillating solutions**

$$\sin\beta = m\,\mathrm{sn}(\omega t, m)$$
$$\cos\beta = \mathrm{dn}(\omega t, m)$$

where m is the modulus defined by C and we denote $4K(m)$ the period of sn,

$$K(m) = \int_0^{\pi/2} \frac{d\Phi}{\sqrt{1 - m^2\sin^2\Phi}}$$

and the control is given by :

$$u_1 = \frac{H_1}{I_1}, \quad u_3 = \frac{H_3}{I_3}$$

- **Rotating solutions** we have :

$$\sin\beta = \mathrm{sn}(\omega t/m, m)$$
$$\cos\beta = \mathrm{cn}(\omega t/m, m)$$

We can deduce the following proposition.

Proposition 4. *Setting* $\omega^2 = \frac{1}{k^2} - 1$, *the components of the control are given by:*
- *In the oscillating case:*

$$u_1 = \frac{1}{k}\mathrm{dn}(\omega t, m), \quad u_3 = m\,\mathrm{sn}(\omega t, m)$$

- *In the rotating case:*

$$u_1 = k\,\mathrm{cn}(\omega t/m, m), \quad u_3 = \mathrm{sn}(\omega t/m, m)$$

To compute Φ_1, we need the elliptic integral of the third kind

$$\Pi(u, a, k) = \int_0^u \frac{dt}{(1 - at^2)\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}}$$

See [16] for the result in the Riemannian case and [8] in the SR-case.

4. **Direct integration on** S^2 **using Liouville theory.**

4.1. **2D-Riemannian metrics whose geodesics flows are integrable by means of linear and quadratic integrals in momenta.** We first recall some results from [4, 5].

4.1.1. *Linear Case.* Let g be a real analytic Riemannian metric $g(x,y) = a(x,y)\mathrm{d}x^2 + 2b(x,y)\mathrm{d}x\mathrm{d}y + c(x,y)\mathrm{d}y^2$ on a surface M and assume that the extremal flow possesses a non zero linear (in momenta) integral F. Then, there exists local coordinates u and v in which the metric has the polar form $\mathrm{d}u^2 + m(u)\mathrm{d}v^2$ and p_v is a first integral (Clairaut relation). This case is called *the case of revolution.*

4.1.2. *Liouville case.* If the metric g admits an additional first integral F quadratic in momenta, the surface M is called a *Liouville surface.* This case is more intricate, and we present in details the algorithm to compute a normal form to integrate the extremal flow. First, we introduce *isothermal coordinates* (x,y) such that the metric takes the form:

$$g = \lambda(x,y)(\mathrm{d}x^2 + \mathrm{d}y^2).$$

If we denote p_x, p_y the adjoint variables, the first integral is given by:

$$F = b_1(x,y)p_x^2 + 2b_2(x,y)p_x p_y + b_3(x,y)p_y^2,$$

where the functions $b_i(x,y)$ are analytic. Let us now consider the function

$$R(z) = b_1 - b_3 + 2ib_2, \qquad z = x + iy.$$

According to [4] this mapping R is holomorphic.

Every diffeomorphism $x = \varphi(u,v), y = \psi(u,v)$ which preserves the isothermal form and the orientation satisfies the Cauchy-Riemann relations:

$$\varphi_u = \psi_v, \qquad \varphi_v = -\psi_u$$

and the mapping

$$\phi : w = u + iu \to z = x + iy$$

is holomorphic. We denote $D = (\varphi_u \psi_v - \psi_u \varphi_v)^{-1} = (\varphi_u^2 + \psi_u^2)^{-1}$, and we have:

$$p_x = D(p_u \psi_v - p_v \psi_u), \qquad p_y = D(-p_u \varphi_v + p_v \varphi_u)^{-1}.$$

Expressing F using the (u,v) coordinates, we obtain:

$$F(u,v) = p_u^2 b_1'^2(u,v) + 2p_u p_v b_2'(u,v) + p_v^2 b_3'(u,v).$$

An easy computation provides:

$$S = (b_1' - b_3' + 2ib_2') = D^2(\varphi_u - i\psi_u)^2(b_1 - b_3 + 2ib_2) = (\varphi_u + i\psi_u)^{-2}(b_1 - b_3 + 2ib_2),$$

where $\phi' = (\varphi_u + i\psi_u)$. We choose the change of coordinates such that $S = 1$. Hence, we must solve the equation

$$\varphi_u + i\psi_u = \sqrt{R(z)}. \tag{4}$$

In the new coordinates, the metric takes the *Liouville normal form*

$$g(u,v) = (f(u) + g(v))(\mathrm{d}u^2 + \mathrm{d}v^2).$$

To integrate, we use [5], Theorem 6.5.

Theorem 4.1.

1. *The equations of the extremals for the Liouville metric can be written as:*

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \pm\frac{\sqrt{f(u) + a}}{\sqrt{g(v) - a}}.$$

2. The extremals themselves are defined by the relation:

$$\int \frac{\mathrm{d}u}{\sqrt{f(u)+a}} \pm \int \frac{\mathrm{d}v}{\sqrt{g(v)-a}} = c.$$

4.2. **Computations of the Liouville normal form.** The Hamiltonian of the metric g can be written as:

$$H = H_0 + k^* H'^2, \qquad k^* = k^2 - 1,$$

where H_0 is the Hamiltonian of the Grushin case $(k = 1)$:

$$H_0 = \frac{1}{2}(p_\varphi^2 + p_\theta^2 \cotan^2\varphi)$$

and

$$H' = p_\varphi \cos\theta - p_\theta \cotan\varphi \sin\theta.$$

We can interpret S^2 as the homogeneous space $SO(3)/SO(2)$ where $SO(2)$ is the Lie subgroup leaving e_3 invariant. In this interpretation, the Casimir function $|G|^2 = H_1^2 + H_2^2 + H_3^2$ corresponds to the bi-invariant case. On the homogeneous space, this defines the round sphere with constant curvature $+1$ whose metric in spherical coordinates takes the form:

$$g = \mathrm{d}\varphi^2 + \sin^2\varphi \mathrm{d}\theta^2$$

and corresponds to the Hamiltonian:

$$F = p_\varphi^2 + \frac{p_\theta^2}{\sin^2\varphi}.$$

A direct computation provides the following proposition.

Proposition 5. *We have:*

$$\{H_0, F\} = \{H', F\} = 0$$

which implies that $\{H, F\} = 0$ for each k^*.

4.2.1. *Integration in the Grushin case.* This situation corresponds to a case of revolution and the integration is standard. The metric is already in the polar form and in our problem it is interesting to interpret the Grushin case as a deformation of the round case using the following homotopy:

$$g_\lambda = \mathrm{d}\varphi^2 + m_\lambda(\varphi)\mathrm{d}\theta^2,$$

where $m_\lambda(\varphi) = \sin^2\varphi/(1 - \lambda\sin^2\varphi)$, $\lambda \in [0, 1]$. Indeed, it allows us to use the geometric framework developed in [6].

Except for the meridian solutions, the $\varphi-$variable is periodic and denoting $\psi = \frac{\pi}{2} - \varphi$, the evolution of ψ is given by:

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = \frac{\cos^2\psi - p_\theta^2(1 - \lambda\cos^2\psi)}{\cos^2\psi}.$$

We denote $X = \sin\psi$ and X_+ and X_- the positive and negative roots of:

$$1 + p_\theta^2(\lambda - 1) = X^2(1 + \lambda p_\theta^2).$$

Introducing Y as $X = X_+ Y$, we have the following proposition.

Proposition 6.

1. *The period of ψ is given by:*

$$T(p_\theta) = \frac{2\pi}{\sqrt{1 + \lambda p_\theta^2}}.$$

2. *The ψ-variable in the normalized coordinates is given by:*

$$\arcsin Y(t) = (1 + \lambda p_\theta^2)^{1/2} t.$$

The θ-variable is integrated using :

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = p_\theta \frac{1 - \lambda(1 - \sin^2 \psi)}{1 - \sin^2 \psi}$$

and we get

$$\theta(t) = \frac{p_\theta}{\sqrt{1 + \lambda p_\theta^2}\sqrt{1 - X_+^2}} \arctan\left(\sqrt{1 - X_+^2} \tan\left(t\sqrt{1 + \lambda p_\theta^2}\right)\right) - \lambda p_\theta t.$$

4.2.2. *Integration $k \neq 1$.* The metric is

$$g = \lambda(x, y)(\mathrm{d}x^2 + \mathrm{d}y^2),$$

where $\lambda(x, y) = \dfrac{1}{r_2^2} = \dfrac{1}{1 - x^2 - y^2/k^2}$ and the dual variables are related by
$p_\varphi = \cos\varphi(p_x \cos\theta + kp_y \sin\theta)$, $p_\theta = \sin\varphi(-p_x \sin\theta + kp_y \cos\theta)$.

Moreover :

$$\cos\varphi = \sqrt{1 - x^2 - y^2/k^2} \qquad\qquad \sin\varphi = \sqrt{x^2 + y^2/k^2}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2/k^2}} \qquad\qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2/k^2}}$$

Hence in the isothermal coordinates (x, y) :

$$F = (1 - x^2)p_x^2 - 2xyp_xp_y + (k^2 - y^2)p_y^2$$

and using the notation of section 4.1.2, one has :

$$R(z) = 1 - k^2 - z^2$$

The solution of (4) is

$$w = \Phi(z) = \arctan\frac{z}{\sqrt{1 - k^2 - z^2}} + C$$

Hence

$$\tan w = \frac{z}{\sqrt{1 - k^2 - z^2}} \Rightarrow z^2 = (1 - k^2)\sin^2 w.$$

Case k<1 We take

$$z = \sqrt{1 - k^2}\sin\omega \quad x = \sqrt{1 - k^2}\sin u \cosh v \quad y = \sqrt{1 - k^2}\cos u \sinh v.$$

The dual variables are given by :

$$p_x = \frac{p_u \cos u \cosh v + p_v \sin u \sinh v}{\sqrt{1 - k^2}(\cos^2 u + \sinh^2 v)}, \quad p_y = \frac{p_v \cos u \cosh v - p_u \sin u \sinh v}{\sqrt{1 - k^2}(\cos^2 u + \sinh^2 v)}$$

Hence we have

$$F = (c + 1)p_u^2 + cp_v^2 \quad c = \frac{(k^2 - 1)\sinh^2 v + k^2}{(1 - k^2)(\cos^2 u \sinh^2 v)}$$

Therefore we have the following proposition :

Proposition 7. *In the case $k < 1$, the Liouville form is given by :*

$$f(u) = \frac{2k^2}{-1 - k^2 + (k^2 - 1)\cos 2u} \quad g(v) = \frac{2k^2}{1 + k^2 + (k^2 - 1)\cosh 2v}$$

Case k>1 Similarly using $z = i\sqrt{k^2 - 1}\sin w$, we get :

$$f(u) = -\frac{2k^2}{1 + k^2 + (k^2 - 1)\cos 2u} \quad g(v) = -\frac{2k^2}{-1 - k^2 + (k^2 - 1)\cosh 2v}$$

In both cases, the integration of the geodesics using Theorem 4.1 gives us elliptic integrals.

5. **Algebraic techniques.** In this section we present the techniques to integrate the equations using differential Galois techniques, see [3, 18] for an introduction.

5.1. **Step 1 : Computation of the control.** To illustrate the techniques it is sufficient to consider one case. Hence for simplicity we assume the following normalizations : $I_3 = 1$ and $I_1 > I_3$.

The components of the control are given in proposition 4.

5.2. **Step 2 : Computation of the position.** One must integrate the time depending linear equation :

$$\frac{d}{dt}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 & u_3 & 0 \\ -u_3 & 0 & u_1 \\ 0 & -u_1 & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{5}$$

There are two possible controls

$$u_3 = m\,\mathrm{sn}(\omega t, m), \quad u_1 = k^{-1}\mathrm{dn}(\omega t, m), \tag{6}$$

$$u_3 = \mathrm{sn}(\omega t/m, m), \quad u_1 = k^{-1}\mathrm{cn}(\omega t/m, m), \tag{7}$$

We first begin with the controls (6). This is a system of differential equations, and its coefficients belong to the function field $K = \mathbb{C}(\mathrm{sn}(\omega t, m), \mathrm{cn}(\omega t, m), \mathrm{dn}(\omega t, m))$. This field is a differential field, i.e. for any $f \in K$, $\partial_t f \in K$. Let us consider the resolvent matrix of equation (5), and the differential field extension $L = K(a_{1,1}, \ldots, a_{3,3})$ generated by the entries of the resolvent matrix.

Definition 5.1. See ([21],1.42) We call L the Picard-Vessiot field of equation (5). We say that equation (5) is solvable if there exist a tower of field extensions $K_0 = K \subset K_1 \subset \cdots \subset K_n = L$ such that for all $i = 0 \ldots n - 1$

- $K_{i+1} = K_i(a)$ where a is algebraic over K_i.
- $K_{i+1} = K_i(a)$ where $\partial_t a \in K_i$.
- $K_{i+1} = K_i(a)$ where $\frac{\partial_t a}{a} \in K_i$.

The Galois group of equation (5) is the group of differential automorphisms of L stabilizing K, i.e. automorphisms σ of L such that $\partial_t \sigma = \sigma \partial_t$ and $\sigma_{|K} = \mathrm{id}$, see ([21],1.25,1.26,1.27) for details.

Proposition 8. *The Galois group G of equation (5) is isomorphic to a Lie subgroup of $GL_3(\mathbb{C})$. The equation (5) is solvable if and only if G is virtually solvable, i.e. its identity component is solvable.*

Until now, all these concepts are theoretical. But we can classify all Lie subgroup of $GL_3(\mathbb{C})$ and make some test to know if the Galois group is a virtually solvable one, this has been done in all generality for third order operators in [20]. The first thing to notice is that equation (5) conserves the Euclidean norm. Indeed, $r_1^2 + r_2^2 + r_3^2$ is constant because the matrix associated to the differential system (5) is in $so(3)$. So the Galois group G is in fact a subgroup of the (complex) group $SO_3(\mathbb{C})$.

The Lie subgroups of $SO_3(\mathbb{C})$ are well known. All strict subgroups of $SO_3(\mathbb{C})$ are finite or stabilize one axis. So the Galois group is either finite, either a finite extension of $SO_2(\mathbb{C})$, or $SO_3(\mathbb{C})$. All these groups are virtually solvable, except the last one $SO_3(\mathbb{C})$ which is simple. So, if we want to try to solve this equation, the first thing to do is to know if G stabilize one axis.

Theorem 5.2. *If $m \notin \{-1, 0, 1\}$, $k \notin \{-1, 1\}$, $m^2 k^2 - k^2 - m^2 \neq 0$ then the Galois group of equation* (5) *with controls* (6) *is isomorphic to $\mathbb{C}^*$.*

Proof. We begin by a variable change $z = sn(wt, m)$. The system now becomes

$$w\sqrt{1-z^2}\sqrt{1-m^2z^2}r' = \begin{bmatrix} 0 & mz & 0 \\ -mz & 0 & k^{-1}\sqrt{1-m^2z^2} \\ 0 & -k^{-1}\sqrt{1-m^2z^2} & 0 \end{bmatrix} r$$

The coefficients are now in $K_0 = \mathbb{C}(\sqrt{1-z^2}, \sqrt{1-m^2z^2})$, which is an algebraic field and so will be easier to manipulate. We denote abusively again L the Picard Vessiot field of this system. We now use the cyclic vector method to build a third order differential equation for which r_1 is solution: for any $i \in \mathbb{N}$, $\partial_t^i r_1$ is a linear form in r_1, r_2, r_3. Thus these linear forms for $i = 0 \ldots 3$ are not independent. So there exist a linear combination of $\partial_t^i r_1$, $i = 0 \ldots 3$ equal to zero. The relation between these linear forms gives a differential equation for $r_1(z)$ (taking into account the relation between parameters $w^2 = 1/k^2 - 1$)

$$\begin{aligned} &z^2(k^2-1)(m^2z^2-1)(m^2z^4-m^2z^2-z^2+1)r_1'''+ \\ &z(3m^4z^6 - 4m^2z^4 - 2m^2z^2 + z^2 + 2)(k^2-1)r_1''+ \\ &(-k^2m^2z^4 + 3k^2m^2z^2 - 3m^2z^2 - 2k^2 + z^2 + 2)r_1' + k^2m^2z^3r_1 = 0 \end{aligned} \tag{8}$$

Let us remark the following

- We can express r_2, r_3 as linear combinations in K_0 of derivatives of r_1. Thus L is generated by the solutions of (8) and their derivatives. So L is also the Picard Vessiot field of equation (8).
- The equation (8) has now rational coefficients due to simplifications. The relation $w^2 = 1/k^2 - 1$ allowed to only use the parameters m, k.

Now the expsols routine of package DEtools of Maple find $r_1(z) = \sqrt{1 - m^2z^2}$ as solution (which is dn before the variable change). So there is a solution in K_0, and this implies that the Galois group stabilize an axis (and is equal to id on this axis). So we already know that the Galois group is a subgroup of $O_2(\mathbb{C})$, and thus that our equation is solvable. We can now reduce the order of the equation, posing $r_1(z) = \sqrt{1 - m^2z^2} \int f(z)dz$. We obtain a second order differential equation, on

which we use the Kovacic algorithm [15] to obtain the solutions for r_1 :

$$r_1(z) = c_1\sqrt{1 - m^2z^2}+$$

$$c_2\sqrt{1 - m^2z^2}\int \frac{z}{m}\sqrt{\frac{k^2m^2 + m^2z^2 - k^2 - m^2}{(1 - m^2z^2)^3(1 - z^2)}}e^{\int \frac{k\sqrt{k^2m^2-k^2-m^2}\sqrt{(1-z^2)(1-m^2z^2)}}{\sqrt{1-k^2}(z^2-1)(k^2m^2+m^2z^2-k^2-m^2)}\mathrm{d}z}\mathrm{d}z+$$

$$c_3\sqrt{1 - m^2z^2}\int \frac{z}{m}\sqrt{\frac{k^2m^2 + m^2z^2 - k^2 - m^2}{(1 - m^2z^2)^3(1 - z^2)}}e^{-\int \frac{k\sqrt{k^2m^2-k^2-m^2}\sqrt{(1-z^2)(1-m^2z^2)}}{\sqrt{1-k^2}(z^2-1)(k^2m^2+m^2z^2-k^2-m^2)}\mathrm{d}z}\mathrm{d}z$$

$$(9)$$

This implies that the integral

$$\int \frac{\sqrt{(1 - z^2)(1 - m^2z^2)}}{(z^2 - 1)(k^2m^2 + m^2z^2 - k^2 - m^2)}\mathrm{d}z$$

belongs to L. This is an integral over an element of K_0, and this integral is an elliptic integral, [16]. So it does not belong to K_0, and thus the Galois group G is of dimension at least 1. Now the only two remaining possibilities for G are $SO_2(\mathbb{C}) \simeq \mathbb{C}^*$ or $O_2(\mathbb{C})$. In the first case, the Galois group would be connected, and so no algebraic extensions would be necessary to express the solutions. This is indeed the case, as we can put the square root $\sqrt{k^2m^2 + m^2z^2 - k^2 - m^2}$ inside the integral in the exponential. So expressing the solutions only uses an exponential integral of an element of K_0. So $G \simeq \mathbb{C}^*$.

$\square$

Remark that the Galois group we computed is over the base field K_0. However, the third order differential equation for r_1 has rational coefficients. So it makes sense to compute the Galois group over $\mathbb{C}(z)$. Over $\mathbb{C}(z)$, $\sqrt{(1 - z^2)(1 - m^2z^2)}$ is an extension of degree 2, and so the Galois group is $G \simeq D_\infty = O_2(\mathbb{C})$.

5.3. **Step 3 Simplifications of the solution.** This expression is Liouvillian (finitely many integrals, exponentials and algebraic extensions), but is not the optimal one. Indeed, the Galois group has dimension one. This suggests that only one integral symbol should be necessary to write the solution. So a simple expression with no iterated integrals exists. Let us find it. As the Galois group over $\mathbb{C}(z)$ is D_∞, there exists a basis of solutions of the form

$$R_1^t(z) = c_1\sqrt{1 - m^2z^2} + c_2Fe^{\int \sqrt{G}\mathrm{d}z} + c_3Fe^{-\int \sqrt{G}\mathrm{d}z} \tag{10}$$

with $F'/F, G \in \mathbb{C}(z)$. We first compute the symmetric power 2 of equation (8), i.e. the linear differential equation whose solutions are the products of two solutions of equation (8). We remark that this symmetric power should have F^2 in its space of solutions. We only have to compute the vector space of hyperexponential solutions to find "candidates" for F. The solutions are $\alpha + \beta z^2$. Thus F should be of the form $F = \sqrt{\alpha + \beta z^2}$.

We can now inject $Fe^{\int \sqrt{G}\mathrm{d}z}$ in equation (8). This produces a large non linear differential equation in $G(z)$. But $\sqrt{G(z)}$ also appears in the equation, and we can act the multivaluation of the square root $\sqrt{G(z)} \longrightarrow -\sqrt{G(z)}$. This gives us another equation, and after simplification, we manage to obtain a linear equation

for G

$$2(6\alpha m^2 z^6 - 3\alpha m^2 z^4 + 3\beta m^2 z^4 - 4\alpha z^4 + \alpha z^2 - \beta z^2 - 2\beta)(m^2 z^2 - 1)(k^2 - 1)G(z)+$$

$$3z(m^2 z^2 - 1)^2(\alpha z^2 + \beta)(z^2 - 1)(k^2 - 1)G'(z)+$$

$$2z^2(\alpha z^2 + \beta)^{-2}(\beta m^2 + \alpha)(3\alpha^2 k^2 m^2 z^4 + 3\alpha\beta k^2 m^2 z^4 - 3\alpha^2 m^2 z^4 - 3\alpha\beta m^2 z^4 -$$

$$2\alpha^2 k^2 z^2 + \alpha^2 z^4 - 2\alpha\beta k^2 z^2 + 2\alpha^2 z^2 + \alpha\beta k^2 + 4\alpha\beta z^2 + \beta^2 k^2 - \alpha\beta) = 0 \tag{11}$$

Solving this equation with dsolve, we obtain only one solution in $\mathbb{C}(z)$

$$G(z) = z^2 \frac{(\alpha k^2 + \alpha z^2 + \beta k^2 - \alpha)(\beta m^2 + \alpha)}{(\alpha z^2 + \beta)^2(m^2 z^2 - 1)(k^2 - 1)(z^2 - 1)}$$

Injecting this in the original non linear equation gives

$$\frac{(m^2 z^2 - 1)^2(\alpha z^2 + \beta)^4(\alpha k^2 + \beta k^2 - \alpha)}{zm^2(\alpha k^2 + \alpha z^2 + \beta k^2 - \alpha)^2)} = 0$$

Thus $\alpha k^2 + \beta k^2 - \alpha = 0$. Thus with $\alpha = k^2, \beta = 1 - k^2$ the equation is satisfied. So the solutions for r_1 are

$$r_1 = c_1 \sqrt{1 - m^2 z^2} + c_2 \sqrt{k^2 z^2 + 1 - k^2} e^{\int \frac{z^2 k}{k^2 z^2 - k^2 + 1} \sqrt{\frac{m^2 k^2 - k^2 - m^2}{(1-k^2)(1-m^2 z^2)(1-z^2)}} dz}$$

$$+ c_3 \sqrt{k^2 z^2 + 1 - k^2} e^{-\int \frac{z^2 k}{k^2 z^2 - k^2 + 1} \sqrt{\frac{m^2 k^2 - k^2 - m^2}{(1-k^2)(1-m^2 z^2)(1-z^2)}} dz} \tag{12}$$

To conclude, let us do the same for the second equation.

Theorem 5.3. *If $m \notin \{-1, 0, 1\}$, $k \notin \{-1, 1\}$, $m^2 k^2 - k^2 - 1 \neq 0$ then the Galois group of equation* (5) *with controls* (7) *is isomorphic to $\mathbb{C}^*$.*

Proof. The proof is similar to the previous one for controls (6). We produce a third order differential equation for r_1. This time, the expsols routine finds $\sqrt{1 - z^2}$. Again, we search solutions under the form $c_1\sqrt{1 - z^2} + c_2 F e^{\int \sqrt{G} dz} + c_3 F e^{-\int \sqrt{G} dz}$, and we find

$$r_1 = c_1 \sqrt{1 - z^2} + c_2 \sqrt{k^2 m^2 z^2 + 1 - k^2} e^{\int \frac{z^2 k m^2}{k^2 m^2 z^2 - k^2 + 1} \sqrt{\frac{k^2 m^2 - k^2 + 1}{(k^2 - 1)(1 - z^2)(1 - m^2 z^2)}} dz}$$

$$+ c_3 \sqrt{k^2 m^2 z^2 + 1 - k^2} e^{-\int \frac{z^2 k m^2}{k^2 m^2 z^2 - k^2 + 1} \sqrt{\frac{k^2 m^2 - k^2 + 1}{(k^2 - 1)(1 - z^2)(1 - m^2 z^2)}} dz} \tag{13}$$

$\square$

5.4. **Step 4 Special values.** For some special values of m, k, the Galois group simplifies. These exceptional parameter values can be seen as singularities or confluences on the formula of r_1. Let us make an analysis for controls (6) (the other one is similar).

The main case appear when the integral in the exponential identically vanishes, i.e. $k^2 m^2 - k^2 - m^2 = 0$. There is one singularity for $k = 0$, but which is present in the original system and so is excluded. The other problematic cases are confluences: The elliptic integral simplifies, and it is no longer guaranteed that the exponential of it is transcendental. These are the cases $k^2 = 1$ and $m^2 = 1$.

Proposition 9. *Let $E = \{\pm\sqrt{1 - s^2}, s \in \mathbb{Q}\}$. The Galois group of* (5) *over K_0 is*

- id if $k^2 = 1$.
- $\mathbb{C}^*$ if $m^2 = 1$, $1/k \notin E$.
- $\mathbb{C}^*$ if $m = 0$, $k \notin E$.
- $\mathbb{Z}/n\mathbb{Z}$ if $m^2 = 1$, $1/k \in E$.
- $\mathbb{Z}/n\mathbb{Z}$ if $m = 0$, $k \in E$.
- $\mathbb{C}^+$ if $k^2 m^2 - k^2 - m^2 = 0$ and $k^2 \neq 1$.
- $\mathbb{C}^*$ in other cases.

Proof. For $k^2 m^2 - k^2 - m^2 = 0$, the solutions can be obtained using a limiting process noting $k^2 m^2 - k^2 - m^2 = \epsilon$ and making a series expansion in ϵ

$$\sqrt{k^2 z^2 - k^2 + 1}, \ \sqrt{k^2 z^2 - k^2 + 1} \int \frac{z^2}{\sqrt{1 - z^2}(k^2 z^2 - k^2 + 1)^{3/2}} \mathrm{d}z$$

$$\sqrt{k^2 z^2 - k^2 + 1} \left(\left(\int \frac{z^2}{\sqrt{1 - z^2}(k^2 z^2 - k^2 + 1)^{3/2}} \mathrm{d}z \right)^2 + \frac{k^2 - 1}{k^4 (k^2 z^2 - k^2 + 1)} \right)$$

The Galois group is then additive instead of multiplicative, $G \simeq \mathbb{C}^+$. If $k^2 = 1$, the equation becomes a first order equation, whose solutions are $c_1 \sqrt{1 - m^2 z^2}$, so in K_0, thus $G = id$.

If $m^2 = 1$, the dsolve command returns an expression in which the only possibly transcendental term is $((z-1)/(z+1))^{\frac{k}{2\sqrt{k^2 - 1}}}$. This term is algebraic if and only if $1/k \in E$ (and so $G \simeq \mathbb{Z}/n\mathbb{Z}$). If $m = 0$, the integral is no longer elliptic, and is given by

$$-\frac{1}{2} \ln \left(\sqrt{1 - k^2} + \frac{z}{\sqrt{z^2 - 1}} \right) + \frac{1}{2} \ln \left(\sqrt{1 - k^2} - \frac{z}{\sqrt{z^2 - 1}} \right) + \frac{1}{\sqrt{1 - k^2}} \ln \left(z + \sqrt{z^2 - 1} \right)$$

The exponential of this expression is algebraic if and only if $k \in E$ (and so $G \simeq \mathbb{Z}/n\mathbb{Z}$). $\qquad\square$

5.5. Coming back to the initial variable.
The algebraic solution for the controls (6) (corresponding to $c_1 = 1, c_2 = 0, c_3 = 0$) gives

$$r = (\mathrm{dn}(\omega t, m), -wm \, \mathrm{cn}(\omega t, m), m/k \, \mathrm{sn}(\omega t, m))$$

The solution for $c_1 = 0, c_2 = 1, c_3 = 0$ is given by

$$r_1(t) = \sqrt{k^2 \mathrm{sn}(\omega t, m)^2 + 1 - k^2} \, e^{\frac{1}{k} \sqrt{\frac{k^2 m^2 - k^2 - m^2}{1 - k^2}} \left(\omega t - \Pi \left(\mathrm{sn}(\omega t, m), \frac{k^2}{k^2 - 1}, m \right) \right)}$$

where Π denotes the elliptic integral of the third kind. The formula in the case $c_1 = 0$, $c_2 = 0$, $c_3 = 1$ is similar up to a sign.

In the special case $k^2 m^2 - k^2 - m^2 = 0$, only an elliptic integral of the second kind is necessary, and in the even more special case of finite Galois groups, all the solutions are algebraic in z, and thus are algebraic in $\cos(\omega t), \sin(\omega t)$ or $\exp(\omega t)$. Note also that with the modulus $m \in]0, 1]$, $k^2 m^2 - k^2 - m^2 \neq 0$.

6. Conclusion.

6.1. Comparison of the integrability methods and geometric applications.
Integrability methods

Euler angles and proposition 2 determine the third row of the matrix $R(t) \in SO(3)$ using Jacobi elliptic functions (solutions of the pendulum equation) and a further quadrature gives the elliptic integral Π to parametrize the angle Φ_1 (the formulas in 3.2.1 gives exponential). The computation of all the rows of the matrix $R(t)$ is associated to the computation of the Picard Vessiot extension. The method of section 5 consists in reducing the computation to a third order linear differential equation which is reparametrized using Jacobi elliptic functions to produce a specific

algebraic solution corresponding to the third row of the matrix. Using this specific solution the third order equation is reduced to a second order equation which is solved using Kovacic algorithm. Concerning Liouville approach and theorem 4.1 the explicit computations of the integral $\int (f(u) + a)^{-1/2} \mathrm{d}u$ and $\int (g(v) - a)^{-1/2} \mathrm{d}v$ superposes the Jacobi elliptic functions and the elliptic integral of the third kind.
Geometric applications

The geometric applications on the computations are the following. The Liouville approach is used to determine the conjugate and cut loci which is achieved in the next section. The Picard Vessiot extension is an important step to compute the conjugate and cut loci for the SR-invariant metrics on $SO(3)$.

6.2. **Optimality problem.** Our work was a preliminary study to a complete analysis of the optimality problem which can be handled using the technical framework introduced in [7] combining geometric analysis and numerical simulations.

We use the following concepts. On the almost-Riemannian surface (M, g), the cut point along a geometric curve γ, projection of an extremal solution of the Maximum Principle, emanating from $q_0 \in M$ is the first point where it ceases to be minimizing and we denote $C_{cut}(q_0)$ the set of such points forming the cut locus. The first conjugate point is the point where it ceases to be minimizing among the set of geodesics C^1-close to γ emanating from q_0 and we denote $C(q_0)$ the set of such points, forming the conjugate locus.

The computations of the conjugate and cut loci on surfaces is a very difficult problem and only recently [10] was proved the Jacobi conjecture : the conjugate locus of a non-umbilical point on ellipsoids has exactly four cusps. A consequence being that the cut locus is a segment. Also in [11] the result was extended to a wider class of Liouville surfaces which possess such simple cut and conjugate loci.

To generalize such computations in our case we can use two different techniques. First of all, we can make an explicit computations of the conjugate loci using our extremal parametrization or numerical simulations [7]. Secondly we can try to relate the simple structure of the conjugate loci to some monotonicity properties of Gaussian curvature or parametrized integrals related to extremals.

Also in our case, one must take into account the existence of equatorial Grushin singularities of the metric.

We briefly present the method to determine the conjugate and cut loci of an one parameter family of Liouville metrics.

First, we have the following lemma.

Lemma 6.1. *The family of metrics g depending upon the parameter k have a discrete symmetry group generated by the two reflexions : $H(\varphi, p_\varphi)) = H(\pi - \varphi, -p_\varphi)$ (reflexion with respect to the equator) and $H(\theta, p_\theta) = H(-\theta, -p_\theta)$ (reflexion with respect to the meridian).*

The next step is to use the Grushin singularity resolution described in [6]. (This result is valid for every k.)

Proposition 10. *Near the equator point q_0 identified to 0, the conjugate and cut loci for the metric restricted to a neighborhood of 0 can be computed using the local model : $\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{x^2}$ where $x = 0$ is the equator. The cut locus is a segment $[-\epsilon, \epsilon]$ minus 0 while the conjugate locus is formed by four symmetric curves of the form $x = cy^2$, minus 0, and tangential to the meridian identified to $\theta_0 = 0$.*

Note in particular that this computation allows to determine by continuation the conjugate and cut loci at an equatorial point. In particular we have, see also [9] for a similar result.

Proposition 11. *For every k, the cut locus of an equatorial point is the equator minus this point.*

Proof. A simple computation shows that the Gaussian curvature in each hemisphere is strictly negative. Hence there is no conjugate point for a geodesic starting from the equatorial point before returning to the equator. Due to the reflectional symmetry with respect to the equator two geodesics starting from an equatorial point intersect with same length when returning to the equator. This proves the result. $\square$

Finally, the simple structure of the conjugate and cut loci in the Grushin case is well known. Roughly spoken it can be simply deduced from the convexity of the period mapping $p_\theta \to T(p_\theta) = \frac{2\pi}{\sqrt{1+p_\theta^2}}$ given in proposition 6, see [6].

Proposition 12. *In the Grushin case we have :*
1. *The cut and conjugate loci of a pole is the antipodal point.*
2. *The cut locus of an equatorial point q_0 is the whole equator minus this point, while the conjugate locus has a double heart shape, with four meridional singularities, two at q_0 and two cusps on the opposite meridian.*
3. *The conjugate locus of a point not a pole nor on the equator has only four cusps and the cut locus is a simple segment on the antipodal parallel.*

The problem is to generalize this simple description to the case $k \neq 1$. This is the analog on the ellipsoid to generalize the case of revolution to a general ellipsoid.

From the theoretical point of view the main steps are the following :

Step 1 : Liouville metrics on the sphere are classified using different notions of equivalence (Liouville equivalence, equivalence of geodesic flow or isometry of the metrics) see [5] and the relation between the case of ellipsoids and Euler-Poinsot rigid body motion is well understood.

Step 2 : The characterization of the metrics on the sphere satisfying the Jacobi conjecture on the ellipsoids : "The conjugate locus of a generic point has only four cusps and the cut locus is a segment" is the object of intense research activities [10, 11] but the verification of the conditions is always a complicated task even in the case of revolution.

Due to this difficulty we adopt here a different point of view coming from [7] applied to the case of spins and to be compared to a similar analysis on the ellipsoid.
Geometric framework

Ellipsoids The general case is obtained by gluing the oblate and prolate cases and the conjugate and cut loci are numerically determined using this analysis. For a non umbilical point the conjugate locus has four cusps and the cut locus is a segment. For an umbilical point they shrink to a single point.

Spin case The sphere is interpreted by gluing the two hemispheres at the equator associated to the Grushin singularity of the metric.

The Grushin case $k = 1$ is described in Proposition.12 and corresponds to the oblate case for an ellipsoid (outside the equator).

For the case $k \neq 1$ the first step is to compute the set η of points where the first integral F is proportional to the Hamiltonian H. For a general ellipsoid we
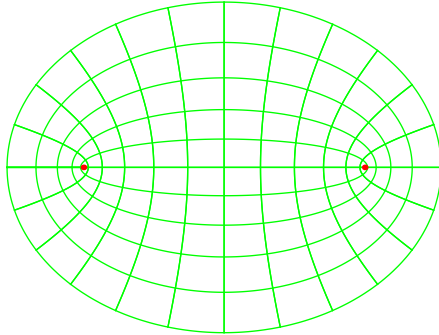
FIGURE 1. Level sets of Liouville coordinates and the two points
of η on the north hemisphere. (Projection on the (x,y)-plane.)

have $\#\eta = 4$ and such points are umbilical points. In the spin cases we have again
$\#\eta = 4$, those points being solutions of $R(z) = 0$.

We represent on fig.1 such points in each hemisphere in relation with the level
sets of the Liouville coordinates.

Numerically we evaluate the conjugate and cut loci of a point in η which shrinks
into a single point, see fig.2.

We compute numerically the conjugate and cut loci of a point not on the equator
nor in η. It is represented on fig.3 confirming the simple structure of the conjugate
and cut loci.

6.3. **Link with conjugate and cut loci computation for an invariant Rie-
mannian on sub-Riemannian metrics on** $SO(3)$**.** Note also that a side effect
of our computations is a first step towards the computation of conjugate and cut
loci for the metrics on $SO(3)$, which is an important challenge. Also it can be
generalized to the case of simple groups of dimension 3.

6.4. **Extension to the case of** 4 **spins.** In this case, we introduce the coordinates
$x_1 = \langle I_{1x} \rangle$, $x_2 = \langle 2I_{1y}I_{2z} \rangle$, $x_3 = \langle 2I_{1y}I_{2x} \rangle$, $x_4 = \langle 4I_{1y}I_{2y}I_{3z} \rangle$, $x_5 = \langle 4I_{1y}I_{2y}I_{3x} \rangle$,
$x_6 = \langle 8I_{1y}I_{2y}I_{3y}I_{4z} \rangle$, and the vector $X = (x_1, \ldots, x_6)^\top$.

Controlling the two spins 2 and 3, one gets the system :

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -u & 0 & 0 & 0 \\ 0 & u & 0 & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & 0 & -v & 0 \\ 0 & 0 & 0 & v & 0 & -k_5 \\ 0 & 0 & 0 & 0 & k_5 & 0 \end{bmatrix} X$$

where $k_3 = \frac{J_{23}}{J_{12}}$, $k_5 = \frac{J_{34}}{J_{12}}$.

Using the notations :

$$x_1 = r_1, \qquad x_2 = r_2 \cos \alpha_2, \qquad x_3 = r_2 \sin \alpha_2,$$
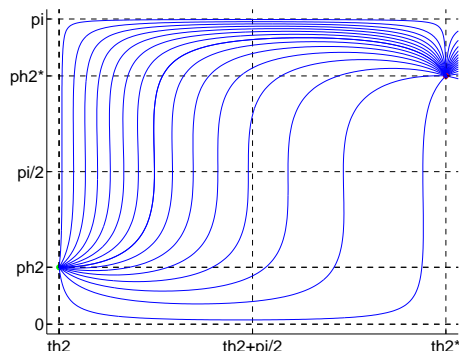$$x_4 = r_3 \cos \alpha_3, \qquad x_5 = r_3 \sin \alpha_3, \qquad x_6 = r_4.$$

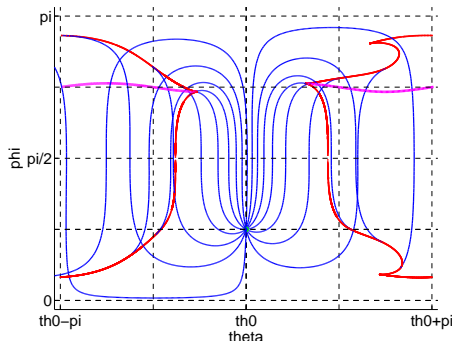FIGURE 2. Conjugate and cut loci of a point in η. (Spherical coordinates.) Here we use the value $k = 1.2$.

FIGURE 3. Conjugate and cut loci of a generic point. (Spherical coordinates.) We set $k = 1.1$ and we start from $(\pi/4, \pi/4)$.

We get the system

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u_1 \begin{bmatrix} -r_2 \\ r_1 \\ 0 \\ 0 \end{bmatrix} + k_3 u_2 \begin{bmatrix} 0 \\ -r_3 \\ r_2 \\ 0 \end{bmatrix} + k_5 u_3 \begin{bmatrix} 0 \\ 0 \\ -r_4 \\ r_3 \end{bmatrix}$$

with

$$u_1 = \cos \alpha_2, \quad u_2 = \sin \alpha_2 \cos \alpha_3, \quad u_3 = \sin \alpha_3.$$

From the control parametrization, the control set is the surface $M = \{u_1^2 + u_2^2 + u_3^2 - u_1^2 u_3^2 = 1\}$. The control constraint can be written :

$$v\left(r, \frac{\mathrm{d}r}{\mathrm{d}t}\right) = g_2\left(r, \frac{\mathrm{d}r}{\mathrm{d}t}\right) + g_4\left(r, \frac{\mathrm{d}r}{\mathrm{d}t}\right) = 1$$

with g_2, g_4 respectively quadratic and quartic with respect to the speed.

The optimal control problem is a time minimal transfer. The candidate as minimizers can be computed using the maximum principle. One can observe that the optimal problem is not a SR-problem unless we consider small controls and the control constraints can be approximated with $g_4 = 0$. In this case the problem is associated to an invariant SR-problem on $SO(4)$. An interesting question in this context is to generalize the integrability issues.

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