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# Optimal Strokes for Driftless Swimmers: A General Geometric Approach

Thomas Chambrion <sup>\*</sup>   Laetitia Giraldi <sup>†</sup>   Alexandre Munnier <sup>‡</sup>

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## Abstract

This paper presents a unified geometric approach for the optimization of the shape deformations of the so-called driftless swimmers. The class of driftless swimmers includes, among other, isolated swimmers in an infinite 3D Stokes flow (case of micro-swimmers in viscous fluids) or isolated swimmers in an infinite 2D or 3D potential flow. A general framework is introduced, in which five usual nonlinear optimization problems related with the maximization of the traveled distance with a constrained energy consumption) are stated.

We prove the existence of regular minimizers under generic controllability assumptions. The results are illustrated with an in-depth study of the isolated swimmer with two degrees of freedom in a 2D potential flow.

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# 1 Introduction

Understanding the mechanics of swimming has been an issue in Mathematical Physics for a long time. Aside from improving the academic understanding of locomotion in fluid, this interest growth from the observation that fish and aquatic mammals evolved swimming capabilities far superior to those achieved by human technology and consequently provide an attractive model for the design of biomimetic robots. Significant contributions to this matter are due to Taylor [20], J. Lighthill [13], E.M. Purcell [19] and T. Y. Wu [23].

Among the many models available in the literature, let us focus on those for which the Reynolds number of the fluid is either very low or very high. The main interest of these cases lies in the fact that the dynamics governing the fluid-swimmer system are simple enough to allow theoretical results to be proved. These two cases are usually referred to as “driftless models”. The first one for which the fluid is assumed to be very viscous is called “resistive model”. It is relevant for microswimmers (like microorganisms) and consists in neglecting the inertial effects in the modeling. The second one, called “reactive model”, is obtained by neglecting rather the viscous forces and is supposed to be relevant for swimmers with elongated bodies (like eels). Surprisingly, the dynamics are very close for both cases and their study fall under the same general abstract framework.

The well-posedness of the system of equations for these models was established for instance in [11], [8] and [15] in a Stokesian flow and in [9] in a perfect fluid. The controllability is addressed in [9] and [15] where the authors prove the generic controllability of 3D driftless swimmers in a perfect and Stokesian flow respectively. An earlier result was established in [5] and next generalized in [6], in which 3D three or four- sphere mechanisms are shown to be controllable. The controllability of the same three-spheres mechanisms is shown to be improved by the presence of boundaries in the fluid domain in [6, 12].

To our knowledge, still few theoretical studies have been conducted about optimal swimming (although more numerical approach of the problem are many). In [16], J. Loheac et al. are interested in optimizing the swimming of a 3D slightly deformable sphere in order to minimize its displacement time. In [4], the authors describe an algorithm allowing optimizing the strokes for a three-sphere swimmer, based on the theory of calculus of variations.

## 1.1 Contribution

The aim of this work is to provide a general framework to study optimal controllability of driftless swimmers. In particular, every aforementioned paper falls within this framework. After recalling minimal hypotheses ensuring the controllability of the system under consideration, we shall focus

on the study of optimal strokes i.e. periodic shape changes. More precisely, we will be interested in the following points:

- Existence of optimal strokes, minimizing or maximizing various cost functionals (related to the energy of the system, the efficiency, the time). As in [16], constraints on the state of the system are taken into account (for instance, the deformations are required to be *not too large*).
- Qualitative properties of the optimal strokes (or, differently stated, of the corresponding optimal controls). In particular, we will show how the optimal controls corresponding to different cost functionals can actually be deduced one from the others.
- Regularity and monotony of the value functions (does the cost increase along with the covered distance?).

Most of the proofs rely on the following arguments:

- The analyticity of the system;
- The Riemannian and sub-Riemannian underlying structure.

For pedagogical purposes, the results will be applied to the toy model introduced in [9] and [18]. The main interest of this model, dealing with a swimmer in a 2D potential flow, is that the governing equations, although not trivial, can be made fully explicit by means of complex calculus.

## 1.2 Outline and Main Achievements

Section 2 is devoted to the definition and the abstract analysis of the optimization problems. An abstract framework is introduced (Section 2.1), and classical sufficient conditions for controllability are recalled (Section 2.2). After stating 5 classical optimization problems (Section 2.3), we show that every one admits minimizers or maximizers (Section 2.4). We prove that most of these problems are actually equivalent (for instance, it is completely equivalent to minimize the time, as in [16] and to minimize the efficiency as in [5]). Then we focus on two value functions: the first one associates with every covered distance the minimal cost necessary to achieve this displacement and the other associates with every given cost, the maximum distance that can be covered with no greater cost. Among others properties, we prove the continuity and study the monotony of these functions (Section 2.5).

In Section 3, we show that two important cases of locomotion in a fluid (namely the locomotion of a single swimmer in an infinite extent of fluid at infinite and zero Reynolds number) fit within the framework introduced in Section 2.1.

In Section 4, we restrict our study to the case where the shape manifold  $\mathcal{S}$  is 2 dimensional. We show that the stroke optimization problem turns into an isoperimetric problem on the shape manifold. Numerical simulations are presented to illustrate our results.

For the ease of the reader, we present two short surveys at the end of the paper. The first one deals with Riemannian geometry (Appendix A) while the second one presents the Orbit theorem (Appendix B).

## 2 Seeking of optimal strokes

### 2.1 Abstract Framework and Notation

We introduce in this subsection the general framework of our study. In the sequel, we call *swimmer* any 5-uple  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$ , where:

- $(\mathcal{S}, \mathbf{g})$  is a  $N$ -dimensional ( $N \geq 1$ ), connected, analytic manifold endowed with an analytic Riemannian structure  $\mathbf{g}$ . Every element  $\mathbf{s}$  of  $\mathcal{S}$  stands for a possible shape of the swimmer. The shape changes of the swimmer over a time interval  $[0, T]$  will be described by a function  $\mathbf{s} : [0, T] \mapsto \mathbf{s}(t) \in \mathcal{S}$ .
- The metric  $\mathbf{g}$  will be used to measure the cost required to achieve this shape change. The cost of a shape change  $\mathbf{s} : [0, T] \mapsto \mathbf{s}(t) \in \mathcal{S}$  could be, for instance, the length of the curve parameterized by the function  $\mathbf{s}$ , i.e.

$$\int_0^T \sqrt{\mathbf{g}_{\mathbf{s}(t)}(\dot{\mathbf{s}}(t), \dot{\mathbf{s}}(t))} dt, \quad (1a)$$

or something more energy-like, usually called the action:

$$\frac{1}{2} \int_0^T \mathbf{g}_{\mathbf{s}(t)}(\dot{\mathbf{s}}(t), \dot{\mathbf{s}}(t)) dt. \quad (1b)$$

- The reference shape  $\mathbf{s}^\dagger$  is a point of  $\mathcal{S}$  which could be thought of as the natural shape of the swimmer, when it is at rest for instance. It will be the starting point for every shape change we will consider.
- The mapping  $\mathbf{Q}^{\mathcal{S}} : T\mathcal{S} \rightarrow \mathbb{R}^n$  is an analytic vector valued 1-form. It accounts for the physical constraints that every shape change has to satisfy to physically make sense. Let us be more specific:

**Definition 2.1** *An admissible shape change is any absolutely continuous curve  $\mathbf{s} : [0, T] \rightarrow \mathcal{S}$ , with essentially bounded first derivative, and which satisfies for almost every time,*

$$\mathbf{Q}_{\mathbf{s}(t)}^{\mathcal{S}} \dot{\mathbf{s}}(t) = 0. \quad (2)$$

This last identity means that for a given shape (i.e. a given element of  $\mathcal{S}$ ), not every direction on  $\mathcal{S}$  is admissible. For instance, by self-deforming, the swimmer will not be allowed to modify the position of its center of mass.

Among admissible shape changes, we will mostly focus on strokes:

**Definition 2.2** *An admissible shape change  $\mathbf{s} : [0, T] \mapsto \mathcal{S}$  will be termed a stroke if  $\mathbf{s}(0) = \mathbf{s}(T)$ .*

- We are only interested in the motion of the swimmer in one given direction. The displacement in this direction is measured thanks to the analytic differential 1-form  $\mathcal{L}$  on  $\mathcal{S}$ . When undergoing the admissible shape change  $\mathbf{s} : [0, T] \rightarrow \mathcal{S}$ , the displacement of the swimmer is given by:

$$\int_0^T \mathcal{L}_{\mathbf{s}(t)}(\dot{\mathbf{s}}(t)) dt. \quad (3)$$

Most of our results will rest on the following elementary but fundamental observation:

**Remark 2.1** *The constraint (2) as well as the quantities (1a) and (3) are time reparameterization invariant. They depend only on the oriented curve  $\Gamma \subset \mathcal{S}$ , a parameterization of which being  $\mathbf{s} : [0, T] \rightarrow \mathcal{S}$ .*

The famous *Scallop Theorem* (see for instance [19]) can be seen as a straightforward consequence of this remark. Indeed, it states that if the shape change is nothing more than a parameterization back and forth of a curve on  $\mathcal{S}$ , then the resulting displacement is null.

## 2.2 Geometric controllability assumptions

Let us consider an abstract swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  as described in Section 2.1, and denote, for every  $\mathbf{s} \in \mathcal{S}$ :

$$\ker \mathbf{Q}^{\mathcal{S}} = \Delta_{\mathbf{s}}^{\mathcal{S}} \subset T_{\mathbf{s}}\mathcal{S}.$$

Assume that the dimension of  $\Delta_{\mathbf{s}}^{\mathcal{S}}$  is not always zero (otherwise, it would mean that there is no shape change satisfying the self-propelled constraints (2)). For every  $\mathbf{s} \in \mathcal{S}$ , we denote by  $\{\mathbf{X}_1(\mathbf{s}), \dots, \mathbf{X}_p(\mathbf{s})\}$  ( $p > 0$ ) a spanning set of  $\Delta_{\mathbf{s}}^{\mathcal{S}}$  where the vector fields  $\mathbf{s} \mapsto \mathbf{X}_j(\mathbf{s}) \in T_{\mathbf{s}}\mathcal{S}$  ( $1 \leq j \leq p$ ) are assumed to be analytic. We denote  $\mathcal{X} := \{\mathbf{X}_j, j = 1, \dots, p\} \subset \mathfrak{X}(\mathcal{S})$  and we shall call  $\mathcal{X}$  an analytic spanning family of the distribution  $\Delta^{\mathcal{S}}$ . Notice that is general, the spanning family  $\mathcal{X}$  cannot be required to be a basis of  $\Delta_{\mathbf{s}}^{\mathcal{S}}$  at every point  $\mathbf{s}$  of  $\mathcal{S}$  because the analytic vector fields  $\mathbf{X}_j$  cannot be prevented from vanishing at some point of the manifold (see for instance the example in Subsection 3.5).

**Proposition 2.1** *Any absolutely continuous function with essentially bounded derivatives  $\mathbf{s} : [0, T] \rightarrow \mathcal{S}$  is an admissible shape change if and only if it is solution (in the sense of Carathéodory) of a Cauchy problem*

$$\dot{\mathbf{s}}(t) = \sum_{j=1}^p u_j(t) \mathbf{X}_j(\mathbf{s}(t)) \quad (t > 0), \quad (4a)$$

$$\mathbf{s}(0) = \mathbf{s}^\dagger, \quad (4b)$$

for some  $\mathbf{u} = (u_1, \dots, u_p) \in L^\infty([0, T], \mathbb{R}^p)$ .

**Proof.** The proof is elementary: For every admissible shape change, the function  $\mathbf{u} = (u_1, \dots, u_p) \in L^\infty([0, T], \mathbb{R}^p)$  gives the coordinates of  $\dot{\mathbf{s}}$  in the spanning family  $\mathcal{X}(\mathbf{s})$ . ■

System (4) allows associating with every measurable function  $\mathbf{u} \in L^\infty([0, T], \mathbb{R}^p)$  an admissible shape change, at least for times small enough.

We define  $\mathcal{M}$  as being the analytic  $(N + 1)$ -dimensional manifold  $\mathcal{S} \times \mathbb{R}$ . Then, we introduce the projectors  $\pi_{\mathcal{S}}$  and  $\pi_{\mathbb{R}}$  by:

$$\begin{aligned} \pi_{\mathcal{S}} : \quad \mathcal{M} &\rightarrow \mathcal{S} & \text{and} & & \pi_{\mathbb{R}} : \quad \mathcal{M} &\rightarrow \mathbb{R} \\ (\mathbf{s}, r) &\mapsto \pi_{\mathcal{S}}(\boldsymbol{\xi}) = \mathbf{s} & & & (\mathbf{s}, r) &\mapsto \pi_{\mathbb{R}}(\boldsymbol{\xi}) = r. \end{aligned}$$

On  $\mathcal{M}$ , we define the analytic vectors fields:

$$\mathbf{Z}_j(\boldsymbol{\xi}) := \begin{pmatrix} \mathbf{X}_j(\pi_{\mathcal{S}}\boldsymbol{\xi}) \\ \mathcal{L}_{\mathbf{s}}\mathbf{X}_j(\pi_{\mathcal{S}}\boldsymbol{\xi}) \end{pmatrix}, \quad (j = 1, \dots, p), \quad (5)$$

we denote  $\mathcal{Z} := \{\mathbf{Y}_j, j = 1, \dots, p\} \subset \mathfrak{X}(\mathcal{M})$  and we define the distribution

$$\Delta_{\boldsymbol{\xi}}^{\mathcal{M}} = \text{span } \mathcal{Z}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathcal{M}.$$

System (4) and the dynamics (31) can now be gathered as a unique dynamical system on  $\mathcal{M}$ :

$$\dot{\boldsymbol{\xi}}(t) = \sum_{j=1}^p u_j(t) \mathbf{Z}_j(\boldsymbol{\xi}(t)) \quad (t > 0), \quad (6a)$$

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}^\dagger \quad (6b)$$

where  $\boldsymbol{\xi}^\dagger = (\mathbf{s}^\dagger, 0)$ .

**Remark 2.2** *The choice of the initial condition  $(\mathbf{s}^\dagger, 0)$  (and not  $(\mathbf{s}^\dagger, r_0)$  for some  $r_0 \neq 0$ ) is physically irrelevant, since the vector fields  $\mathbf{Z}_j$  ( $1 \leq j \leq p$ ), and hence the dynamics (6), do not depend upon the  $\mathbb{R}$  component of the variable  $\boldsymbol{\xi}$ .*



**Definition 2.3** For every positive time  $T$ , every swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  and every analytic spanning family  $\mathcal{X}$  of  $\Delta^{\mathcal{S}}$ , we denote by  $\mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(T)$  the set of all the controls  $\mathbf{u} = (u_j)_{1 \leq j \leq p} \in L^\infty([0, T], \mathbb{R}^p)$  for which the solution of (4) (and hence of (6)) is defined on  $[0, T]$ .

For any given control  $\mathbf{u} \in \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(T)$ , we denote

$$t \in [0, T] \mapsto \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \mathbf{u}) \in \mathcal{M},$$

the solution to (6) with control  $\mathbf{u}$ .

**Remark 2.3** According to this notation, we have:

$$\pi_{\mathbb{R}} \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(T, \mathbf{u}) = \int_0^T \mathcal{L}_{\pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \mathbf{u})} \frac{d}{dt} \pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \mathbf{u}) dt.$$

One hypothesis required in order to ensure that the swimmer is controllable is that  $\mathcal{X}$  is bracket generating on  $\mathcal{S}$ . Observe that this condition does not depend on the particular choice of the spanning family  $\mathcal{X}$  but only on the distribution  $\Delta^{\mathcal{S}}$  and hence on the vector valued 1-form  $\mathbf{Q}^{\mathcal{S}}$ . It can be easily verified that if  $\mathcal{X}$  and  $\mathcal{X}'$  are two smooth spanning families of  $\Delta^{\mathcal{S}}$ , then for every  $\mathbf{s} \in \mathcal{S}$ :

$$\text{Lie}_{\mathbf{s}} \mathcal{X} = \text{Lie}_{\mathbf{s}} \mathcal{X}' = \text{Lie}_{\mathbf{s}} \Delta^{\mathcal{S}}.$$

Taking into account this observation, we define:

**Hypothesis 1** The swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  is such that

1. The distribution  $\Delta^{\mathcal{S}}$  is bracket generating on  $\mathcal{S}$ , i.e.

$$\dim \text{Lie}_{\mathbf{s}} \Delta^{\mathcal{S}} = \dim \mathcal{S}, \quad \forall \mathbf{s} \in \mathcal{S};$$

2. There exists  $\boldsymbol{\xi} \in \mathcal{M}$  such that  $\dim \text{Lie}_{\boldsymbol{\xi}} \Delta^{\mathcal{M}} = \dim \mathcal{M}$ .

**Lemma 2.2** Hypothesis 1 leads to:

$$\dim \text{Lie}_{\boldsymbol{\xi}} \Delta^{\mathcal{M}} = \dim \mathcal{M}, \quad \forall \boldsymbol{\xi} \in \mathcal{M}.$$

**Proof.** Let  $\boldsymbol{\xi}^* = (\mathbf{s}^*, 0)$  be such that  $\dim \text{Lie}_{\boldsymbol{\xi}^*} \Delta^{\mathcal{M}} = \dim \mathcal{M}$ . As already mentioned, the choice of 0 for the  $\mathbb{R}$  component of  $\boldsymbol{\xi}^*$  is irrelevant regarding the Lie algebra  $\text{Lie}_{\boldsymbol{\xi}^*} \Delta^{\mathcal{M}} = \text{Lie}_{\boldsymbol{\xi}^*} \mathcal{Z}$  since, for every  $j = 1, \dots, p$ ,  $\mathbf{Z}_j(\boldsymbol{\xi}) = \mathbf{Z}_j(\pi_{\mathcal{S}} \boldsymbol{\xi})$ . Consider now any  $\boldsymbol{\xi} = (\mathbf{s}, r) \in \mathcal{M}$  and denote  $\mathcal{O}(\boldsymbol{\xi})$  the orbit of  $\mathcal{Z}$  through  $\boldsymbol{\xi}$ . Since  $\Delta^{\mathcal{S}}$  is bracket generating on  $\mathcal{S}$ , Rashevsky-Chow Theorem ensures that for any  $T > 0$ , there exists a control  $\mathbf{u} \in L^\infty([0, T], \mathbb{R}^p)$  such that the solution to the EDO (4) with Cauchy data  $\mathbf{s}$  is equal to  $\mathbf{s}^*$  at the final time  $T$ . Using this control in EDO (6) with Cauchy data  $\boldsymbol{\xi} = (\mathbf{s}, r)$ , we deduce that the solution reaches a point  $\tilde{\boldsymbol{\xi}}^* = (\mathbf{s}^*, r^*)$  at the time  $T$  for

some  $r^* \in \mathbb{R}$  ( $\xi$  and  $\tilde{\xi}^*$  are both in  $\mathcal{O}(\xi)$ ). But since  $\pi_S \tilde{\xi}^* = \pi_S \xi^*$ , we have the equality  $\dim \text{Lie}_{\tilde{\xi}^*} \mathcal{Z} = \dim \text{Lie}_{\xi^*} \mathcal{Z} = \dim \mathcal{M}$ . According to the Orbit Theorem, the dimension of the Lie algebra of  $\mathcal{Z}$  is constant on  $\mathcal{O}(\xi)$  and hence we have also  $\dim \text{Lie}_{\xi} \mathcal{Z} = \dim \mathcal{M}$ . The proof is now complete. ■

Assuming only, in the second point of Hypothesis 1, that the equality holds for one point of  $\mathcal{M}$  may seem a somehow useless mathematical refinement. Quite the reverse, the explicit computation of  $\text{Lie}_{\xi} \Delta^{\mathcal{M}}$  in concrete cases is often very involved and can still be hardly carry out for one particular  $\xi$  (see for instance [4, 6, 10, 12, 15]).

**Definition 2.4** *A swimmer satisfying Hypothesis 1 will be called controllable.*

This definition is justified by the following Theorem:

**Theorem 2.3** *Let  $\mathfrak{S}$  be a swimmer satisfying Hypothesis 1. Then, for every  $T > 0$ , every analytic spanning family  $\mathcal{X}$  of  $\Delta^{\mathfrak{S}}$ , every  $\xi^{\ddagger}$  in  $\mathcal{M}$  and every open, connected set  $\mathcal{O} \subset \mathcal{M}$  containing  $\xi^{\ddagger}$  and  $\xi^{\ddagger}$ , there exists a control  $\mathbf{u} \in L^{\infty}([0, T], \mathbb{R}^p)$  such that  $\xi_{\mathfrak{S}}^{\mathcal{X}}(T, \mathbf{u}) = \xi^{\ddagger}$  and  $\xi_{\mathfrak{S}}^{\mathcal{X}}(t, \mathbf{u}) \in \mathcal{O}$  for every  $t \in [0, T]$ .*

**Proof.** This is a straightforward consequence of the analytic Orbit theorem. ■

Notice that this theorem applies for the models (high and low Reynolds numbers swimmers) introduced in Section 3. For these models, controllability is ensured as soon as Hypothesis 1 is fulfilled.

Here below is a detailed analysis for the example of the isolated swimmer with two degrees of freedom in a 2D potential flow (see Subsection 3.5 for an introduction).

### 2.3 Statement of optimal problems

In this section, we address the main problems that we are interested in. A controllable swimmer  $\mathfrak{S}$  being given, and a cost being chosen (among those presented in (1)), what is the *best* possible stroke? By *best*, it is understood that the swimmer is wished to swim as far as possible with a corresponding cost as low as possible.

To be rigorously stated, the question has to be split into several closely related but not always equivalent problems:

1. What is the stroke minimizing the cost among those allowing traveling a given, fixed distance?

2. What is the stroke maximizing the travelled distance among those whose cost is not greater than a given fixed bound?

In the where case the cost is not important, we can also be interested in seeking the stroke maximizing the mean swimming velocity.

We shall conduct a detailed study on every one of these problems, focusing on the existence of optimal strokes and deriving their main properties.

To begin with, let us restrict slightly the scope of our study by introducing a new hypothesis that the swimmer has to satisfy:

**Hypothesis 2** *The swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  is such that there exists an analytic basis  $\mathcal{X} = \{\mathbf{X}_j, j = 1, \dots, p\}$  of the distribution  $\Delta^{\mathcal{S}}$ .*

**Definition 2.5** *A swimmer  $\mathfrak{S}$  satisfying Hypothesis 2 will be termed trivialized.*

Applying a Gram-Schmidt process, we can assume that for every  $\mathbf{s} \in \mathcal{S}$ , the family  $\{\mathbf{X}_j(\mathbf{s}), j = 1, \dots, p\}$  in Hypothesis 2 is an orthonormal basis (for the Riemannian scalar product  $\mathbf{g}$  of  $\mathcal{S}$ ) of  $\Delta_{\mathbf{s}}^{\mathcal{S}}$ . As already mentioned before, it is in general not possible to extract from any smooth spanning family of  $\Delta^{\mathcal{S}}$  a smooth basis on the whole manifold  $\mathcal{S}$  (see also the computations in Subsection 3.5). Nevertheless, any swimmer can be locally trivialized:

**Proposition 2.4** *Let  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  be a swimmer. Then, there exists an open connected subset  $\mathcal{S}'$  (for the topology of  $\mathcal{S}$ ) containing  $\mathbf{s}^\dagger$  such that  $\mathfrak{S}' = (\mathcal{S}', \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  is a trivialized swimmer.*

Notice in particular that any open subset of an analytic manifold is still an analytic manifold.

Let  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  be a trivialized, controllable swimmer,  $\mathcal{K}$  be a compact of  $\mathcal{S}$  containing  $\mathbf{s}^\dagger$  and  $\mathcal{X}$  be an orthonormal basis of  $\Delta^{\mathcal{S}}$ . For every  $\boldsymbol{\xi}^\dagger$  in  $\mathcal{M}$  and  $T \geq 0$ , we define

$$\begin{aligned} \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) &:= \left\{ \mathbf{u} \in \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(T) : \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(T, \mathbf{u}) = \boldsymbol{\xi}^\dagger \right\}; \\ \widehat{\mathcal{U}}_{\mathfrak{S}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) &:= \left\{ \mathbf{u} \in \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) : \|\mathbf{u}(t)\|_{\mathbb{R}^p} = 1 \ \forall t \in [0, T] \right\}; \\ \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(T) &:= \left\{ \mathbf{u} \in \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(T) : \pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \mathbf{u}) \in \mathcal{K} \ \forall t \in [0, T] \right\}; \\ \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) &:= \left\{ \mathbf{u} \in \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) : \pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \mathbf{u}) \in \mathcal{K} \ \forall t \in [0, T] \right\}; \\ \widehat{\mathcal{U}}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) &:= \left\{ \mathbf{u} \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) : \|\mathbf{u}(t)\|_{\mathbb{R}^p} = 1 \ \forall t \in [0, T] \right\}. \end{aligned}$$

The following Lemma

**Lemma 2.5** *If there exist an orthonormal basis  $\mathcal{X}$  of  $\Delta^{\mathcal{S}}$  and  $T > 0$  such that the set  $\mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\dagger)$  is empty, then it is empty for every orthonormal basis  $\mathcal{X}$  of  $\Delta^{\mathcal{S}}$  and every  $T > 0$ .*

**Proof.** Assume that for some  $\mathcal{X}$  and  $T > 0$ , the set  $\mathcal{U}_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\dagger)$  is nonempty and denote by  $\mathbf{u}$  one of its elements. Then, for any  $T' > 0$ , define:

$$\alpha = \frac{\|\mathbf{u}\|_{L^1([0,T],\mathbb{R}^p)}}{T'}; \quad (7a)$$

$$\phi(t) = \frac{1}{\alpha} \int_0^t \|\mathbf{u}(s)\|_{\mathbb{R}^p} ds, \quad t \in [0, T]; \quad (7b)$$

$$\tilde{\mathbf{u}}(t) = \alpha \frac{\mathbf{u}(\phi^{-1}(t))}{\|\mathbf{u}(\phi^{-1}(t))\|_{\mathbb{R}^p}}, \quad t \in [0, T']. \quad (7c)$$

It can be easily verified that  $\tilde{\mathbf{u}} \in \mathcal{U}_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(T', \boldsymbol{\xi}^\dagger)$ . Notice that  $t \in [0, T'] \mapsto \boldsymbol{\xi}_{\mathfrak{G}}^{\mathcal{X}}(t, \tilde{\mathbf{u}}) \in \mathcal{M}$  is nothing but a time reparameterization of the curve, also parameterized by  $t \in [0, T] \mapsto \boldsymbol{\xi}_{\mathfrak{G}}^{\mathcal{X}}(t, \mathbf{u}) \in \mathcal{M}$ . Saying that  $\mathcal{U}_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\dagger)$  is nonempty means that there exist an allowable curve on  $\mathcal{M}$ , whose projection on  $\mathcal{S}$  is contained in  $\mathcal{K}$  and which links  $\boldsymbol{\xi}^\dagger$  to  $\boldsymbol{\xi}^\dagger$ . The existence of such a curve depends neither on  $\mathcal{X}$  nor on  $T$ . ■

In every one of the problems stated below, we make the convention that the infimum of an empty set is equal to  $+\infty$  while the supremum is equal to  $-\infty$ .

**Problem 1 (Minimizing the Riemannian length)** For any  $\delta^\dagger \in \mathbb{R}$  and  $T > 0$ , set  $\boldsymbol{\xi}^\dagger = (\mathbf{s}^\dagger, \delta^\dagger)$  and determine:

$$\Phi_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T) = \inf \left\{ \int_0^T \|\mathbf{u}(t)\|_{\mathbb{R}^p} dt : \mathbf{u} \in \mathcal{U}_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) \right\}. \quad (8)$$

Notice that, since  $\mathcal{X}$  is assumed to be orthonormal, we have also:

$$\int_0^T \|\mathbf{u}(t)\|_{\mathbb{R}^p} dt = \int_0^T \sqrt{\mathbf{g}(\pi_{\mathcal{S}}\boldsymbol{\xi}_{\mathfrak{G}}^{\mathcal{X}}(t, \mathbf{u}), \pi_{\mathcal{S}}\boldsymbol{\xi}_{\mathfrak{G}}^{\mathcal{X}}(t, \mathbf{u}))} dt,$$

which is the length of the curve  $\Gamma \subset \mathcal{K}$  parameterized by  $t \in [0, T] \mapsto \pi_{\mathcal{S}}\boldsymbol{\xi}_{\mathfrak{G}}^{\mathcal{X}}(t, \mathbf{u}) \in \mathcal{S}$ . The lengths of the curves on  $\mathcal{S}$  do not depend on the parameterization, so Problem 1 is time parameterization invariant.

Modifying the cost leads to:

**Problem 2 (Minimizing the action)** For any  $\delta^\dagger \in \mathbb{R}$  and  $T > 0$ , set  $\boldsymbol{\xi}^\dagger = (\mathbf{s}^\dagger, \delta^\dagger)$  and determine:

$$\Theta_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T) = \inf \left\{ \frac{1}{2} \int_0^T \|\mathbf{u}(t)\|_{\mathbb{R}^p}^2 dt : \mathbf{u} \in \mathcal{U}_{\mathfrak{G},\mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T) \right\}. \quad (9)$$

Remark that, unlike the cost in (8), the cost in (9) is not time parameterization independent.

Before stating results about these problems, let us introduce a last related optimal problem, studied for instance in []:

**Problem 3 (Optimizing the time)** For any  $\delta^\dagger \in \mathbb{R}$ , denote  $\boldsymbol{\xi}^\dagger = (\mathbf{s}^\dagger, \delta^\dagger)$  and determine:

$$T_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger) = \inf\{T : \widehat{\mathcal{U}}_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\boldsymbol{\xi}^\dagger, T) \neq \emptyset\}.$$

**Problem 4 (Maximizing traveling distance with bounded Riemannian length)** For any  $l \geq 0$  and  $T \geq 0$ , determine:

$$\Psi_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(l, T) = \sup \left\{ \pi_{\mathbb{R}} \boldsymbol{\xi}_{\mathfrak{S}}^\mathcal{X}(T, \mathbf{u}) : \mathbf{u} \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(T), \pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}}^\mathcal{X}(T, \mathbf{u}) = \mathbf{s}^\dagger, \right. \\ \left. \text{and } \int_0^T \|\mathbf{u}(t)\|_{\mathbb{R}^p} dt \leq l \right\}. \quad (10)$$

**Problem 5 (Maximizing traveling distance with bounded action)** For any  $l \geq 0$  and  $T \geq 0$ , determine:

$$\Lambda_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(l, T) = \sup \left\{ \pi_{\mathbb{R}} \boldsymbol{\xi}_{\mathfrak{S}}^\mathcal{X}(T, \mathbf{u}) : \mathbf{u} \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(T), \pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}}^\mathcal{X}(T, \mathbf{u}) = \mathbf{s}^\dagger, \right. \\ \left. \text{and } \frac{1}{2} \int_0^T \|\mathbf{u}(t)\|_{\mathbb{R}^p}^2 dt \leq l \right\}. \quad (11)$$

Using the control  $\mathbf{u} = \mathbf{0}$  in the last two problems, we deduce that  $\Psi_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(l, T) \geq 0$  and  $\Lambda_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(l, T) \geq 0$ .

## 2.4 Firsts Properties of the Optimal Strokes

In this Subsection, we will derive properties of the optimal strokes resting on the Riemannian structure of  $\mathcal{S}$ . Then, in the following Subsection, we will introduce an make use of the sub-Riemannian structure of  $\mathcal{M}$ .

To begin with, let us focus on the firsts three problems:

**Theorem 2.6** Let  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^\mathcal{S}, \mathbf{s}^\dagger, \mathcal{L})$  be a controllable, trivialized swimmer, and  $\mathcal{K}$  be a compact of  $\mathcal{S}$  containing  $\mathbf{s}^\dagger$ . Then

1. For every  $\delta^\dagger \in \mathbb{R}$ , the quantities  $\Phi_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ ,  $\Theta_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ , and  $T_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger)$  are either all of them infinite or all of them finite, for every  $T \geq 0$  and every orthonormal basis  $\mathcal{X}$  of  $\Delta^\mathcal{S}$ .
2. If  $\mathbf{s}^\dagger \in \mathring{\mathcal{K}}$  then  $\Phi_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ ,  $\Theta_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ , and  $T_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger)$  are all of them finite, for every  $T \geq 0$ , every orthonormal basis  $\mathcal{X}$  of  $\Delta^\mathcal{S}$  and every  $\delta^\dagger \in \mathbb{R}$ .
3. If  $\Phi_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ ,  $\Theta_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ , and  $T_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger)$  are finite, then there exist minimizers or maximizers to every Problem 1, 2 and 3.
4. For every  $T \geq 0$  and every  $\delta^\dagger \in \mathbb{R}$ ,  $\Phi_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$ ,  $\Theta_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger, T)$  and  $T_{\mathfrak{S}, \mathcal{K}}^\mathcal{X}(\delta^\dagger)$  do not depend on  $\mathcal{X}$ . So from now on, we drop  $\mathcal{X}$  in the notation.

5.  $\Phi_{\mathfrak{E},\mathcal{K}}(\delta^\dagger, T)$  does not depend on  $T$ . So from now on, we drop  $T$  in the notation.

6. The following identities hold for every  $\delta^\dagger \in \mathbb{R}$ , and every  $T > 0$ :

$$\Phi_{\mathfrak{E},\mathcal{K}}(\delta^\dagger) = T_{\mathfrak{E},\mathcal{K}}(\delta^\dagger) \quad (12a)$$

$$\Theta_{\mathfrak{E},\mathcal{K}}(\delta^\dagger, T) = (1/2)(T_{\mathfrak{E},\mathcal{K}}(\delta^\dagger))^2/T \quad (12b)$$

7. Any minimizer  $\mathbf{u} \in \mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$  to Problem 2:

(a) is such that  $\|\mathbf{u}(t)\|_{\mathbb{R}^p}$  is constant at every moment;

(b) is also a minimizer to Problem 1;

(c) is proportional to a minimizer of Problem 3.

### Proof.

1. Let  $\delta^\dagger$  be given. According to Lemma 2.5, if  $\mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\dagger)$  is empty for some  $T > 0$ , then it is empty for every  $T > 0$  and therefore every quantity  $\Phi_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T)$ ,  $\Theta_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T)$ , and  $T_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger)$  is infinite. Reciprocally, if for some  $T > 0$  there exists  $\mathbf{u} \in \mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\dagger)$ , then, for every  $T > 0$ , the set  $\mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\dagger)$  is nonempty as well and  $\Phi_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T)$  and  $\Theta_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T)$  are both finite. Moreover, from the control  $\mathbf{u}$ , we can build  $\tilde{\mathbf{u}} \in \widehat{\mathcal{U}}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\|\mathbf{u}\|_{L^1([0,T],\mathbb{R}^p)}, \boldsymbol{\xi}^\dagger)$  by setting  $\alpha = 1$  in (7). We deduce that  $T_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\delta^\dagger)$  is finite and the first assertion of the Theorem is proved.

2. Denote  $\mathcal{O}_1$  the connected component of  $\mathring{\mathcal{K}}$  containing  $\mathbf{s}^\dagger$  and for every  $\delta^\dagger \in \mathbb{R}$ , take  $\mathcal{O} = \mathcal{O}_1 \times ]-\delta^\dagger - 1, \delta^\dagger + 1[$  in Theorem 2.3. The Theorem ensures that for every  $T > 0$  and every orthonormal basis  $\mathcal{X}$  of  $\Delta^{\mathcal{S}}$ , the set  $\mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$  is nonempty.

3. We prove now all the remaining points of the Theorem. Let  $\mathcal{X}$  (an orthonormal basis of  $\Delta^{\mathcal{S}}$ ),  $T > 0$  and  $\delta^\dagger \in \mathbb{R}$  be given. For any control  $\mathbf{u} \in \mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$ , denote  $\tilde{\mathbf{u}} \in \mathcal{U}_{\mathfrak{E},\mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$  the control defined in (7) with  $T' = T$ . One can easily verify that:

$$\int_0^T \|\mathbf{u}(s)\|_{\mathbb{R}^p} ds = \int_0^T \|\tilde{\mathbf{u}}(s)\|_{\mathbb{R}^p} ds \quad (13)$$

and

$$\begin{aligned} \int_0^T \|\mathbf{u}(s)\|_{\mathbb{R}^p}^2 ds &\geq \frac{1}{T} \left( \int_0^T \|\mathbf{u}(s)\|_{\mathbb{R}^p} ds \right)^2 \\ &= \int_0^T \|\tilde{\mathbf{u}}(s)\|_{\mathbb{R}^p}^2 ds, \end{aligned} \quad (14)$$

with equality in (14) if and only if  $\|\mathbf{u}(s)\|_{\mathbb{R}^p}$  is constant, i.e.  $\mathbf{u} = \tilde{\mathbf{u}}$ . So, replacing  $\mathbf{u}$  by  $\tilde{\mathbf{u}}$  does not modify the cost functional of Problem 1 and does not increase the cost functional of Problem 2. Moreover, since

$$\int_0^T \|\tilde{\mathbf{u}}(s)\|_{\mathbb{R}^p}^2 ds = \frac{1}{T} \left( \int_0^T \|\tilde{\mathbf{u}}(s)\|_{\mathbb{R}^p} ds \right)^2, \quad (15)$$

if  $(\mathbf{u}_n)_n$  is a minimizing sequence for either Problem 1 or Problem 2, then  $(\tilde{\mathbf{u}}_n)_n$  is a minimizing sequence for both Problem 1 and Problem 2. By construction, the sequence  $(\tilde{\mathbf{u}}_n)_n$  is bounded in  $L^\infty([0, T], \mathbb{R}^p)$ . Hence, up to a subsequence extraction, we can assume that  $(\tilde{\mathbf{u}}_n)_n$  weakly converges, for instance, in  $L^2([0, T], \mathbb{R}^p)$  toward  $\mathbf{u}^*$ . In particular, the following inequality holds:

$$\int_0^T \|\mathbf{u}^*(s)\|_{\mathbb{R}^p}^2 ds \leq \liminf_{n \rightarrow +\infty} \int_0^T \|\tilde{\mathbf{u}}_n(s)\|_{\mathbb{R}^p}^2 ds. \quad (16)$$

Let us verify that  $\mathbf{u}^* \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$ .

The functions  $t \in [0, T] \mapsto \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \tilde{\mathbf{u}}_n) \in \mathcal{M}$  are equi-Lipschitz continuous on  $[0, T]$ , because the vector fields  $\mathbf{Z}_i$  are analytic on the compact  $\mathcal{K}$  and hence bounded. According to Ascoli Theorem, we can assume that, up to a subsequence extraction, the sequence  $(t \mapsto \boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \tilde{\mathbf{u}}_n))_n$  converges uniformly on  $[0, T]$  toward a Lipschitz continuous function  $t \in [0, T] \mapsto \boldsymbol{\xi}^* \in \mathcal{M}$ . Furthermore, the curve  $t \in [0, T] \mapsto \pi_{\mathfrak{S}} \boldsymbol{\xi}^* \in \mathcal{S}$  is absolutely continuous, with bounded derivative and thus is an admissible shape change (with support in  $\mathcal{K}$ ). For every  $t \in [0, T]$  and every  $n \in \mathbb{N}$ , we have:

$$\boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(t, \tilde{\mathbf{u}}_n) = \boldsymbol{\xi}^\dagger + \sum_{i=1}^p \int_0^t \tilde{u}_i^n(s) \mathbf{Z}_i(\boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(s, \tilde{\mathbf{u}}_n)) ds. \quad (17)$$

Since  $\tilde{u}_i^n \rightharpoonup u_i^*$  in  $L^2([0, T], \mathbb{R})$  and  $\mathbf{Z}_i(\boldsymbol{\xi}_{\mathfrak{S}}^{\mathcal{X}}(\cdot, \tilde{\mathbf{u}}_n)) \rightarrow \mathbf{Z}_i(\boldsymbol{\xi}^*)$  uniformly on  $[0, T]$  (and hence also in  $L^2([0, T], T\mathcal{M})$ ), passing to the limit as  $n \rightarrow +\infty$  in (17) leads to:

$$\boldsymbol{\xi}^*(t) = \boldsymbol{\xi}^\dagger + \sum_{i=1}^p \int_0^t u_i^*(s) \mathbf{Z}_i(\boldsymbol{\xi}^*(s)) ds, \quad t \in [0, T].$$

We have now proved that  $\mathbf{u}^*$  is indeed a minimizer to Problems 2. Moreover, since equality in (14) holds if and only if  $\|\mathbf{u}^*(t)\|_{\mathbb{R}^p}$  is constant for every  $t \in [0, T]$ , we infer that  $\mathbf{u}^* = \tilde{\mathbf{u}}^*$ . This equality leads

to the following estimates:

$$\begin{aligned} \int_0^T \|\mathbf{u}^*(s)\|_{\mathbb{R}^p} ds &= \sqrt{T} \left( \int_0^T \|\mathbf{u}^*(s)\|_{\mathbb{R}^p}^2 ds \right)^{1/2} \\ &\leq \liminf_{n \rightarrow +\infty} \sqrt{T} \left( \int_0^T \|\tilde{\mathbf{u}}_n(s)\|_{\mathbb{R}^p}^2 ds \right)^{1/2} \\ &= \liminf_{n \rightarrow +\infty} \int_0^T \|\tilde{\mathbf{u}}_n(s)\|_{\mathbb{R}^p} ds, \end{aligned}$$

and  $\mathbf{u}^*$  is also a minimizer to Problem 1. Using this control in (15), we obtain the equality:

$$\Theta_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T) = \frac{1}{2T} (\Phi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T))^2. \quad (18)$$

Eventually, as already mentioned earlier, for every  $T > 0$  and from any control  $\mathbf{u} \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$ , we can build  $\tilde{\mathbf{u}} \in \widehat{\mathcal{U}}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\|\mathbf{u}\|_{L^1([0, T], \mathbb{R}^p)}, \boldsymbol{\xi}^\dagger)$  by setting  $\alpha = 1$  in (7). The identity (13) becomes:

$$\int_0^T \|\mathbf{u}(s)\|_{\mathbb{R}^p} ds = \int_0^{\|\mathbf{u}\|_{L^1([0, T], \mathbb{R}^p)}} ds,$$

whence we deduce that

$$T_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\delta^\dagger) = \Phi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T).$$

This equality tells us that  $\Phi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T)$  does not depend on  $T$ . We conclude the proof of the theorem by observing again that  $\Phi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\delta^\dagger, T)$  is the length of the curve on  $\mathcal{S}$  parameterized by  $t \in [0, T] \mapsto \pi_{\mathcal{S}} \boldsymbol{\xi}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(t, \mathbf{u}^*) \in \mathcal{S}$  and that this length does not depend on  $\mathcal{X}$ .

■

We address now Problems 4 and 5:

**Theorem 2.7** *Let  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  be a controllable, trivialized swimmer, and  $\mathcal{K}$  be a compact of  $\mathcal{S}$  containing  $\mathbf{s}^\dagger$ . Then*

1. *Problems 4 and 5 admit maximizers for every  $T \geq 0$ , every orthonormal basis  $\mathcal{X}$  of  $\Delta^{\mathcal{S}}$  and every  $l \geq 0$ .*
2. *Any maximizer  $\mathbf{u} \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\boldsymbol{\xi}^\dagger, T)$  to Problem 5*
  - (a) *is such that  $\|\mathbf{u}(t)\|_{\mathbb{R}^p}$  is constant at every moment;*
  - (b) *is proportional to a maximizer of Problem 4.*
3.  *$\Psi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(l, T)$  and  $\Lambda_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(l, T)$  do not depend on  $\mathcal{X}$  so we drop it in the notation.*



4.  $\Psi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(l, T)$  does not depend on  $T$  so we drop it in the notation.
5. The following identity holds for every  $T > 0$  and  $l \geq 0$ :

$$\Lambda_{\mathfrak{S}, \mathcal{K}}((1/2)l^2/T, T) = \Psi_{\mathfrak{S}, \mathcal{K}}(l). \quad (19)$$

**Proof.** To prove the existence of maximizers to Problems 4 and 5, we follow the lines of the proof of Theorem 2.6: Let  $\mathcal{X}$ ,  $l \geq 0$  and  $T \geq 0$  be given and consider first a maximizing sequence  $(\mathbf{u}_n)_n$  to Problem 5. Then notice that the renormalized and reparameterized control  $(\tilde{\mathbf{u}}_n)_n$  is actually not only a maximizing sequence to Problem 5 but also to Problem 4 with  $l' = \sqrt{lT}$ . Then, up to subsequences extractions and invoking Ascoli Theorem and the weak convergence in  $L^2([0, T], \mathbb{R}^p)$  of  $(\tilde{\mathbf{u}}_n)_n$ , we prove the existence of a common maximizer  $\mathbf{u}^* = \tilde{\mathbf{u}}^*$  to Problems 5 (with  $l$ ) and 4 (with  $l' = \sqrt{lT}$ ). Using the control  $\mathbf{u}^*$ , we also get the equality (19).

Once more, the time reparameterization invariance of Problem 4 leads to infer that  $\Psi_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}$  does not depend neither on  $\mathcal{X}$  nor  $T$ . Eventually, identity (19) ensures that  $\Lambda_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(l, T)$  does not depend on  $\mathcal{X}$ . ■

## 2.5 Further Properties of the Optimal Strokes

In order to prove further properties on Problems 1-5, we need to introduce the Sub-Riemannian structure on  $\mathcal{M}$ .

Let a trivialized swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^S, \mathbf{s}^\dagger, \mathcal{L})$  and  $\mathcal{X} := \{\mathbf{X}_j, j = 1, \dots, p\}$  an orthonormal basis of  $\Delta_{\mathfrak{S}}^S$  be given. From the analytic vector fields  $\mathbf{X}_j$  on  $\mathcal{S}$ , we build the analytic vector fields  $\mathbf{Z}_j$  ( $j = 1, \dots, p$ ) on  $\mathcal{M} = \mathcal{S} \times \mathbb{R}$  as described in (5). Then, we define

$$\mathcal{Z} = \{\mathbf{Z}_j, j = 1, \dots, p\},$$

and the distribution on  $\mathcal{M}$ :

$$\Delta_{\xi}^{\mathcal{M}} := \text{span } \mathcal{Z}(\xi) \subset T_{\xi}\mathcal{M}, \quad \xi \in \mathcal{M}.$$

We denote by  $\mathbb{Z}(\xi)$  the matrix whose column vectors are the  $\mathbf{Z}_j(\xi)$  and we introduce the Euclidean bundle  $\mathbf{U} := \mathcal{M} \times \mathbb{R}^p$  endowed with the Euclidean norm of  $\mathbb{R}^p$  and the morphism of vector bundles

$$f : (\xi, \mathbf{u}) \in \mathbf{U} \mapsto (\xi, \mathbb{Z}(\xi)\mathbf{u}) \in T\mathcal{M}.$$

Following [3, Definition 3.1], we claim that the manifold  $\mathcal{M}$  endowed with the triple  $(\mathcal{M}, \mathbf{U}, f)$  is an analytic sub-Riemannian manifold. According to Definition 3.6 from the same booklet, we define the admissible curves as

being the Lipschitz curves  $\boldsymbol{\xi} : [0, T] \mapsto \mathcal{M}$  for which there exists a control function  $\mathbf{u} \in L^\infty([0, T], \mathbb{R}^p)$  such that, for a.e.  $t \in [0, T]$ :

$$\dot{\boldsymbol{\xi}} = \mathbb{Z}(\boldsymbol{\xi})\mathbf{u}.$$

Notice that it is exactly the dynamics (6) that we are dealing with. The sub-Riemannian manifold can be equipped with the so-called *Carnot-Carathéodory distance* (see [3, Definition 3.13]) denoted by  $d(\cdot, \cdot)$ . In particular, the following identity holds for any  $\delta^\dagger \in \mathbb{R}$  and  $\boldsymbol{\xi} = (\mathbf{s}^\dagger, \delta^\dagger)$ :

$$d(\boldsymbol{\xi}^\dagger, \boldsymbol{\xi}^\ddagger) = \inf \left\{ \int_0^T \|\mathbf{u}(s)\|_{\mathbb{R}^p} ds, \mathbf{u} \in \mathcal{U}_{\mathfrak{S}}^{\mathcal{X}}(T, \boldsymbol{\xi}^\ddagger) \right\}.$$

Be aware that actually, neither  $\mathcal{X}$  nor  $T$  matters in this definition.

**Theorem 2.8** *Let  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  be a controllable, trivialized swimmer, and  $\mathcal{K}$  be a compact of  $\mathcal{S}$  such that  $\mathbf{s}^\dagger \in \mathring{\mathcal{K}}$ . Then:*

1. *For every  $l > 0$ , we have  $\Psi_{\mathfrak{S}, \mathcal{K}}(l) > 0$ . For every  $l > 0$  and  $T > 0$ , we have  $\Lambda_{\mathfrak{S}, \mathcal{K}}(l, T) > 0$ .*

2. *The functions*

$$l \in \mathbb{R}_+ \mapsto \Psi_{\mathfrak{S}, \mathcal{K}}(l) \in \mathbb{R}_+ \tag{20a}$$

*is increasing and right continuous.*

3. *For every  $T > 0$ , the function*

$$l \in \mathbb{R}_+ \mapsto \Lambda_{\mathfrak{S}, \mathcal{K}}(l, T) \in \mathbb{R}_+ \tag{20b}$$

*is increasing and right continuous.*

4. *For every maximizer to Problems 4 or 5, the constraints are saturated.*

5. *The function*

$$\delta \in \mathbb{R} \mapsto \Phi_{\mathfrak{S}, \mathcal{K}}(\delta) \in \mathbb{R} \tag{20c}$$

*is even and uniformly continuous.*

6. *For every  $T > 0$ , the function*

$$\delta \in \mathbb{R} \mapsto \Theta_{\mathfrak{S}, \mathcal{K}}(\delta, T) \in \mathbb{R} \tag{20d}$$

*is even and uniformly continuous.*

7. *For every  $l \geq 0$  and every  $T \geq 0$ :*

$$\Phi_{\mathfrak{S}, \mathcal{K}}(\Psi_{\mathfrak{S}, \mathcal{K}}(l)) = l \tag{21a}$$

$$\Theta_{\mathfrak{S}, \mathcal{K}}(\Lambda_{\mathfrak{S}, \mathcal{K}}(l, T), T) = l. \tag{21b}$$

Regarding the last point of the theorem, notice that, for every  $\delta^\dagger \in \mathbb{R}_+$ , we have  $\Psi_{\mathfrak{S},\mathcal{K}}(\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger)) \geq \delta^\dagger$  but it may happen that  $\Psi_{\mathfrak{S},\mathcal{K}}(\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger)) > \delta^\dagger$  for some  $\delta^\dagger$ . Indeed, according to the definition of  $\Psi_{\mathfrak{S},\mathcal{K}}(\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger))$ , we have:

$$\Psi_{\mathfrak{S},\mathcal{K}}(\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger)) = \max \left\{ \pi_{\mathbb{R}} \xi_{\mathfrak{S}}^{\mathcal{X}}(T, \mathbf{u}) : \mathbf{u} \in \mathcal{U}_{\mathfrak{S},\mathcal{K}}^{\mathcal{X}}(T), \pi_{\mathfrak{S}} \xi_{\mathfrak{S}}^{\mathcal{X}}(T, \mathbf{u}) = \mathbf{s}^\dagger, \int_0^T \|\mathbf{u}(s)\|_{\mathbb{R}^p} ds \leq \Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger) \right\}. \quad (22)$$

**Proof.**

1. For any  $a \geq 0$  and  $\xi \in \mathcal{M}$ , denote by  $B^{\mathcal{M}}(\xi, a)$  the sub-Riemannian ball, of radius  $a$  and centered at  $\xi$ . According to [3, Theorem 3.8], the Carnot-Carathéodory distance induces the manifold topology on  $\mathcal{M}$ . We deduce that, for every  $l > 0$ , the set  $B(\xi^\dagger, l) \cap (\mathring{\mathcal{K}} \times \mathbb{R})$  is open an nonempty (it contains  $\mathbf{s}^\dagger$ ) and hence

$$(\{\mathbf{s}\} \times \mathbb{R}) \cap B(\xi^\dagger, l) \cap (\mathring{\mathcal{K}} \times \mathbb{R})$$

contains an open set  $\{\mathbf{s}^\dagger\} \times ]-\varepsilon, \varepsilon[$  for some  $\varepsilon > 0$ . We infer that, for every  $l > 0$ , we have  $\Psi_{\mathfrak{S},\mathcal{K}}(l) > \varepsilon > 0$ . We use the relation (19) to deduce that, for every  $l > 0$  and  $T > 0$ , we also have  $\Lambda_{\mathfrak{S},\mathcal{K}}(l, T) > 0$ .

2. The function (20a) is clearly nondecreasing. Moreover, for every  $l, l' \geq 0$ , we have:

$$\Psi_{\mathfrak{S},\mathcal{K}}(l + l') \geq \Psi_{\mathfrak{S},\mathcal{K}}(l) + \Psi_{\mathfrak{S},\mathcal{K}}(l'). \quad (23)$$

Indeed, recall that every minimizer to Problem 4 is time parameterization invariant and consider the curves  $\Gamma \subset \mathcal{S}$  of length  $l$  and  $\Gamma' \subset \mathcal{S}$  of length  $l'$  corresponding to the maximizers of Problem 4 for  $l$  and  $l'$  respectively. Then denote  $\Gamma'' = \Gamma \cup \Gamma'$ . This curve is admissible, closed, of length  $l + l'$  and produces a displacement no greater than  $\Psi_{\mathfrak{S},\mathcal{K}}(l + l')$ . Inequality (23) together with the first point of the Theorem yield the increasing property of function (20a).

In order to prove the right continuity, let  $l \in \mathbb{R}_+$  be given and consider a decreasing sequence  $(l_n)_n$  converging to  $l$ . For every  $l_n$  ( $n \in \mathbb{N}$ ), denote by  $\mathbf{u}_n$  the minimizer to Problem 4 such that  $\|\mathbf{u}_n(t)\|_{\mathbb{R}^p}$  is constant for every  $t \in [0, T]$ . The sequence  $(\mathbf{u}_n)_n$  is bounded in  $L^2([0, T], \mathbb{R}^p)$  and the sequence  $(\xi_{\mathfrak{S}}^{\mathcal{X}}(\cdot, \mathbf{u}_n))_n$  is bounded in  $C^0([0, T], \mathcal{M})$ , therefore there exists a subsequence  $(l_{n_k})_k$  such that  $(\Psi_{\mathfrak{S},\mathcal{K}}(l_{n_k}))_k$  converges to  $\limsup \Psi_{\mathfrak{S},\mathcal{K}}(l_n)$  while  $(\mathbf{u}_{n_k})_k$  weakly converges in  $L^2([0, T], \mathbb{R}^p)$  to  $\mathbf{u}^*$  and  $(\xi_{\mathfrak{S}}^{\mathcal{X}}(\cdot, \mathbf{u}_{n_k}))_k$  uniformly converges to  $\xi^*$ .

On the one hand, arguing as for (17), we deduce that  $\xi^* = \xi_{\mathfrak{S}}^{\mathcal{X}}(\cdot, \mathbf{u}^*)$  and then that

$$\Psi_{\mathfrak{S},\mathcal{K}}(l_{n_k}) \rightarrow \pi_{\mathbb{R}} \xi_{\mathfrak{S}}^{\mathcal{X}}(T, \mathbf{u}^*) \quad \text{as } k \rightarrow +\infty$$

with  $\|\mathbf{u}^*\|_{L^1([0,T],\mathbb{R}^p)} \leq \liminf l_n = l$ . Therefore we obtain that:

$$\limsup \Psi_{\mathfrak{S},\mathcal{K}}(l_n) \leq \Psi_{\mathfrak{S},\mathcal{K}}(l).$$

On the other hand, since  $\Psi_{\mathfrak{S},\mathcal{K}}$  is increasing and  $l \leq l_n$  for every  $n$ , we deduce that

$$\Psi_{\mathfrak{S},\mathcal{K}}(l) \leq \Psi_{\mathfrak{S},\mathcal{K}}(l_n),$$

and the conclusion arises by taking the limit inf in both sides of the inequality.

3. The proof of this point is a straightforward consequence of the preceding point and relation (19).
4. The same reasoning as for the second point proves that the constraints are saturated in Problems 4 or 5 for the maximizers. Indeed, if the constraints were not saturated, then it would be possible to add to the optimal curve on  $\mathcal{S}$  a small loop that would produce an extra displacement (according to the first point).
5. For  $\varepsilon$  small enough, we have  $B^{\mathcal{M}}(\boldsymbol{\xi}^\dagger, \varepsilon) \subset \mathring{\mathcal{K}}$  (because the sub-Riemannian topology coincides with the manifold topology). In this case, for every  $\delta^\dagger \in \pi_{\mathbb{R}}^{-1}B^{\mathcal{M}}(\boldsymbol{\xi}^\dagger, \varepsilon)$ , we have:

$$\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger) = d(\boldsymbol{\xi}^\dagger, \boldsymbol{\xi}^\dagger).$$

According to [3, Theorem 3.18], the Carnot-Caratheodory distance is continuous for the manifold topology, so we deduce that the function (20c) is continuous in a neighborhood of 0. It is even because, for every  $\delta$  and every minimizer  $\mathbf{u}$  on  $[0, T]$ , the control  $t \mapsto \mathbf{u}(T - t)$  is a minimizer with the same Riemannian length, associated to  $-\delta$ .

Observe now that for every  $\delta^\dagger, h \in \mathbb{R}$ :

$$\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger + h) \leq \Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger) + \Phi_{\mathfrak{S},\mathcal{K}}(h),$$

whence we infer that:

$$\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger + h) - \Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger) \leq \Phi_{\mathfrak{S},\mathcal{K}}(h).$$

Writing that

$$\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger) \leq \Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger + h) + \Phi_{\mathfrak{S},\mathcal{K}}(-h),$$

and since the function is even, we finally get:

$$|\Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger + h) - \Phi_{\mathfrak{S},\mathcal{K}}(\delta^\dagger)| \leq \Phi_{\mathfrak{S},\mathcal{K}}(h),$$

which, with the continuity in 0, proves that the function is uniformly continuous on  $\mathbb{R}$ .

6. The proof of this point is a straightforward consequence of the preceding point and relation (12).
7. For every  $l \geq 0$ , we clearly have  $\Phi_{\mathfrak{S}, \mathcal{K}}(\Psi_{\mathfrak{S}, \mathcal{K}}(l)) \leq l$ . The inequality  $\Phi_{\mathfrak{S}, \mathcal{K}}(\Psi_{\mathfrak{S}, \mathcal{K}}(l)) < l$  for some  $l \geq 0$  and the existence of minimizers to Problem 1 would contradict the fact that the constraint is saturated for every maximizer of Problem 4.

■

The function (20a) has no reason to be monotone in the general case. Let us define:

**Hypothesis 3** *The swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  is such that  $\mathbf{Q}^{\mathcal{S}} = \mathbf{0}$  (there is no self-propelled constraints).*

Every swimmer satisfying (3) is called *unconstrained*. For unconstrained swimmers, every absolutely continuous curve on  $\mathcal{S}$ , with essentially bounded first derivative, is an admissible shape change. Under Hypothesis 3, we can state:

**Theorem 2.9** *Let  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  be a controllable, trivialized, unconstrained swimmer, and  $\mathcal{K}$  be a compact of  $\mathcal{S}$  such that  $\mathbf{s}^\dagger \in \overset{\circ}{\mathcal{K}}$ . Then there exists  $\varepsilon > 0$  such that the function (20a) is increasing on  $] - \varepsilon, \varepsilon[$ .*

**Proof.** Denote by  $B^{\mathcal{S}}(\mathbf{s}^\dagger, r)$  the Riemannian ball on  $\mathcal{S}$ , where the radius  $r$  is given by Lemma A.1. Let  $\varepsilon > 0$  be small enough such that  $B^{\mathcal{M}}(\xi^\dagger, \varepsilon) \subset \overset{\circ}{\mathcal{K}}$  (it is always possible because the sub-Riemannian topology coincides with the manifold topology) and  $\pi_{\mathcal{S}} B^{\mathcal{M}}(\xi^\dagger, \varepsilon) \subset B^{\mathcal{S}}(\mathbf{s}^\dagger, r)$ .

Assume now that there exist  $0 < \delta_0 < \delta_1 < \varepsilon$  such that

$$\Phi_{\mathfrak{S}, \mathcal{K}}(\delta_1) \leq \Phi_{\mathfrak{S}, \mathcal{K}}(\delta_0). \quad (24)$$

Denote  $\xi^1 = (\mathbf{s}^\dagger, \delta_1)$  and for some  $\mathcal{X} = \{\mathbf{X}_j, j = 1, \dots, p\}$  (an orthonormal basis of  $\Delta^{\mathcal{S}}$ ) and  $T > 0$ , denote by  $\mathbf{u}^1 \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(\xi^1, T)$  a control minimizing Problem 1. Introduce as well  $\gamma_1 = \pi_{\mathcal{S}} \xi(\cdot, \mathbf{u}^1)$  and  $\Gamma_1$  the curve on  $\mathcal{S}$  parameterized by  $\gamma_1$ . The following identity holds:

$$\Phi_{\mathfrak{S}, \mathcal{K}}(\delta_1) = \int_0^T \|\mathbf{u}^1(t)\|_{\mathbf{R}^p} dt = \ell(\Gamma_1).$$

According to Lemma A.1 with  $x_0 = \mathbf{s}^\dagger$  and  $\gamma = \gamma_1$ , there exists a continuous function  $\psi : [0, 1] \times [0, T] \rightarrow \mathcal{S}$  such that, for every  $s \in [0, 1]$ ,  $\psi(s, \cdot)$  is absolutely continuous with essentially bounded first derivative,  $\psi(1, \cdot) = \gamma_1$  and  $\psi(0, \cdot) = \mathbf{s}^\dagger$ . Denoting by  $\mathbf{u}^s = (u_j^s)_{1 \leq j \leq p} \in \mathcal{U}_{\mathfrak{S}, \mathcal{K}}^{\mathcal{X}}(T)$  the control such that

$$u_j^s(t) = \mathbf{g}_{\psi(s,t)}(\partial_t \psi(s,t), \mathbf{X}_j(\psi(s,t))), \quad t \in [0, T],$$

and by  $\Gamma_s$  the curve parameterized by  $t \in [0, T] \mapsto \psi(s, t) = \pi_S \boldsymbol{\xi}(t, \mathbf{u}^s)$  ( $s \in [0, 1]$ ), we have, for every  $s \in [0, 1]$ :

$$\Phi_{\mathfrak{E}, \mathcal{K}}(\Xi(s)) \leq \int_0^T \|\mathbf{u}^s(t)\|_{\mathbb{R}^p} dt = \ell(\Gamma_s).$$

where we have set:

$$\Xi : s \in [0, 1] \mapsto \pi_{\mathbb{R}} \boldsymbol{\xi}_{\mathfrak{E}}^{\mathcal{X}}(T, \mathbf{u}^s) \in \mathbb{R}.$$

The function  $\Xi$  is continuous and such that  $\Xi(1) = \delta_1$  and  $\Xi(0) = 0$ , so there exists  $s^* \in ]0, 1[$  such that  $\Xi(s^*) = \delta_0$ . Since, according to Lemma A.1, the function  $s \in [0, 1] \mapsto \ell(\Gamma_s)$  is increasing, we get:

$$\Phi_{\mathfrak{E}, \mathcal{K}}(\delta_0) \leq \ell(\Gamma_{s^*}) < \ell(\Gamma_1) = \Phi_{\mathfrak{E}, \mathcal{K}}(\delta_1),$$

which is in contradiction with (24). The proof is now completed. ■

### 3 Modeling

In this Section we aim to establish the dynamics governing the motion of low and high Reynolds numbers swimmers and show that they fall within the abstract framework introduced in Section 2.1.

Our purpose is to highlight that, although the properties of the fluid are different in both cases, the equations of motion turn out to have the exact same general form. The modeling is carried out in 3D, the 2D case being similar.

We assume that the swimmer is alone in the fluid and that the fluid-swimmer system fills the whole space. The buoyant force is not taken into account.

We consider a Galilean fixed frame  $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$  and an attached coordinate system  $\mathcal{R}_f := (O, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$  where  $O$  is a fixed point of  $\mathbb{R}^3$ . In  $\mathcal{R}_f$ , coordinates will be written with capital letters (as  $X := (X_1, X_2, X_3)$ ). Since we are interested in seeking optimal strokes, we will consider only shape changes that make the center of mass of the swimmer remains on the  $X_1$ -axis. Thus, the swimmer is compelled to swim along a straight line.

#### Kinematics

We denote by  $\mathbf{r}$  the center of mass of the swimmer (lying on the  $X_1$ -axis) and we consider the coordinate system  $\mathcal{R}_m := (\mathbf{r}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ , attached to the swimmer.

We assume that every possible shape of the swimmer, when described in  $\mathcal{R}_m$ , can be characterized by a so-called shape variable  $\mathbf{s}$  belonging to some connected analytic hypersurface  $\mathcal{S}$  of  $\mathbb{R}^{N+1}$  (for some integer  $N \geq 1$ ).

Thus, we denote by  $\mathcal{A}_m(\mathbf{s})$  the domain of  $\mathbb{R}^3$  occupied by the swimmer, in the coordinate system  $\mathcal{R}_m$ , and hence  $\mathcal{A}_f(\mathbf{s}) = \mathbf{r} + \mathcal{A}_m(\mathbf{s})$  is the same domain expressed in  $\mathcal{R}_f$ . For every  $\mathbf{s} \in \mathcal{S}$ , the set  $\mathcal{A}_m(\mathbf{s})$  is the image of the unit ball  $B$  by a  $C^1$  diffeomorphism  $\chi(\mathbf{s}, \cdot)$  depending on the parameter  $\mathbf{s}$ . Knowing every diffeomorphism  $\chi(\mathbf{s}, \cdot)$  for every shape variable  $\mathbf{s}$ , the shape changes over a time interval  $[0, T]$  can be merely described by means of a (shape) function:

$$t \in [0, T] \mapsto \mathbf{s}(t) \in \mathcal{S}.$$

The Eulerian velocity  $\mathbf{W}$  at any point  $X \in \mathcal{A}_f(\mathbf{s})$  of the swimmer is the sum of the rigid velocity  $\dot{\mathbf{r}} := \dot{r}\mathbf{E}_1$  ( $\dot{r} \in \mathbb{R}$ ) and the velocity of deformation

$$\mathbf{W}_d(\mathbf{s}, \dot{\mathbf{s}}, X) := \nabla_{\mathbf{s}}\chi(\mathbf{s}, \chi(\mathbf{s}, (X - \mathbf{r}))^{-1}) \cdot \dot{\mathbf{s}}.$$

Thus, we get:

$$\mathbf{W} = \dot{\mathbf{r}} + \mathbf{W}_d \quad \text{in } \mathcal{A}_f(\mathbf{s}).$$

In the coordinate system  $\mathcal{R}_m$ , this equality turns out to be:

$$\mathbf{w} = \dot{\mathbf{r}} + \mathbf{w}_d \quad \text{in } \mathcal{A}_m(\mathbf{s}),$$

where, for every  $x \in \mathcal{A}_m(\mathbf{s})$  we have set  $\mathbf{w}(x) := \mathbf{W}(x + \mathbf{r})$  and  $\mathbf{w}_d(\mathbf{s}, \dot{\mathbf{s}}, x) := \mathbf{W}_d(\mathbf{s}, \dot{\mathbf{s}}, x + \mathbf{r})$ .

## Dynamics

In  $\mathcal{R}_m$ , the density  $\varrho(\mathbf{s}, \cdot)$  of the body can be deduced from a given constant density  $\varrho_0 > 0$ , defined in  $B$ , according to the conservation of mass principle:

$$\varrho(\mathbf{s}, \chi(\mathbf{s}, x)) = \frac{\varrho_0}{|\det \nabla_x \chi(\mathbf{s}, x)|}, \quad x \in B.$$

The volume of the swimmer is  $\text{Vol}(\mathbf{s}) = \int_B |\det \nabla_x \chi(\mathbf{s}, x)| dx$  and its mass  $m = \varrho_0 \text{Vol}(B)$ .

Although prescribed, the deformations should be interpretable as produced by some internal forces. It means that in the absence of fluid, the swimmer is not able to modify its linear momentum, which reads:

$$\int_{\mathcal{A}_m(\mathbf{s})} \varrho(\mathbf{s}, x) \mathbf{w}_d(\mathbf{s}, \dot{\mathbf{s}}, x) dx = \varrho_0 \int_B \nabla_{\mathbf{s}} \chi(\mathbf{s}, x) \cdot \dot{\mathbf{s}} dx = \mathbf{0}. \quad (25)$$

We introduce the  $3 \times N$  matrix:

$$\mathbf{Q}^S(\mathbf{s}) := \varrho_0 \int_B \nabla_{\mathbf{s}} \chi(\mathbf{s}, x) dx, \quad (26)$$

and we rewrite (2) as:

$$\mathbf{Q}^S(\mathbf{s}) \dot{\mathbf{s}} = \mathbf{0}. \quad (27)$$

This equation has to be understood as a constraint on the shape variable and is referred to as the *self-propulsion hypothesis*.

The fluid obeys, in the whole generality, to the Navier-Stokes equations for incompressible fluid:

$$\varrho_f \frac{D}{Dt} \mathbf{U}(t, X) - \nabla_X \cdot \mathbb{T}_f(\mathbf{U}, P)(t, X) = 0 \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)); \quad (28a)$$

$$\nabla_X \cdot \mathbf{U}(t, X) = 0 \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)); \quad (28b)$$

where

1. For every  $\mathbf{s} \in \mathcal{S}$ ,  $\mathcal{F}_f(\mathbf{s}) := \mathbb{R}^3 \setminus \overline{\mathcal{A}_f(\mathbf{s})}$  is the domain of the fluid;
2.  $\varrho_f > 0$  is the fluid's density;
3.  $\mathbf{U}(t, X)$  is the Eulerian velocity of the fluid at the time  $t > 0$  and the point  $X \in \mathcal{F}_f(\mathbf{s}(t))$ ;
4.  $D/Dt := \partial/\partial t + (\mathbf{U}(t, X) \cdot \nabla_X)$  is the convective derivative;
5.  $\mathbb{T}_f(\mathbf{U}, P)(t, X) := \mu(\nabla_X \mathbf{U}(t, X) + \nabla_X \mathbf{U}^T(t, X)) - P(t, X)\mathbb{I}d$  is the stress tensor,  $\mu$  the dynamic viscosity and  $P$  the pressure.

The rigid displacement of the body is governed by Newton's laws for the linear momentum:

$$m\ddot{\mathbf{r}}(t) = - \int_{\partial\mathcal{A}_f(\mathbf{s})} \mathbf{E}_1 \cdot \mathbb{T}_f(\mathbf{U}, P)(t, X) \mathbf{n} \, d\sigma_X, \quad (t > 0),$$

where  $\mathbf{n}$  is the unit normal vector to  $\partial\mathcal{A}_f(\mathbf{s})$  directed towards the interior of  $\mathcal{A}_f(\mathbf{s})$ .

These equations have to be supplemented with boundary conditions on  $\partial\mathcal{A}_f(\mathbf{s})$ , which can be either

$$\mathbf{U} \cdot \mathbf{n} = \mathbf{W} \cdot \mathbf{n} \quad \text{on } \partial\mathcal{A}_f(\mathbf{s}),$$

known as *slip* or Navier boundary conditions or

$$\mathbf{U} = \mathbf{W} \quad \text{on } \partial\mathcal{A}_f(\mathbf{s}),$$

referred to as *no-slip* boundary conditions. Eventually, for the system to be well-posed, initial data are needed:

$$\mathbf{U}(0) = \mathbf{U}_0, \mathbf{r}(0) = \mathbf{r}_0 \text{ and } \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0.$$

As mentioned in the introduction, we focus on two limit problems connecting to the value of the Reynolds number  $\text{Re} := \varrho \bar{\mathbf{U}} L / \mu$  ( $\bar{\mathbf{U}}$  is the mean fluid velocity and  $L$  is a characteristic linear dimension). The first case  $\text{Re} \ll 1$  concerns low Reynolds swimmers like bacteria (or more generally so-called micro swimmers whose size is about  $1\mu m$ ). For the second  $\text{Re} \gg 1$ , we will restrain our study to irrotational flows and so it is relevant for large animals swimming quite slowly, a case where vorticity can be neglected.



### 3.1 Low Reynolds numbers swimmers

For micro-swimmers, scientists agree that inertia (for both the fluid and the body) can be neglected in the dynamics. It means that in the modeling, we can set  $\varrho_0 = \varrho_f = 0$ . In this case, the Navier-Stokes equations reduce to the steady Stokes equations

$$\begin{aligned} -\nabla_X \cdot \mathbb{T}_f(\mathbf{U}, P)(t, X) &= 0 & t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)); \\ \nabla_X \cdot \mathbf{U}(t, X) &= 0 & t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)); \end{aligned}$$

supplemented with no-slip boundary conditions

$$\mathbf{U} = \mathbf{W} \text{ on } \partial\mathcal{A}_f(\mathbf{s}).$$

Introducing, for all  $x \in \mathcal{F}_m(\mathbf{s}) := \mathbb{R}^3 \setminus \overline{\mathcal{A}_m(\mathbf{s})}$ ,

$$\mathbf{u}(t, x) := \mathbf{U}(t, x + \mathbf{r}(t)) \quad \text{and} \quad p(t, x) := P(t, x + \mathbf{r}(t)),$$

the equations keep the same form when expressed in the coordinate system  $\mathcal{R}_m$ , namely, with evident notation:

$$-\nabla_x \cdot \mathbb{T}_m(\mathbf{u}, p)(t, x) = 0 \quad t > 0, x \in \mathcal{F}_m(\mathbf{s}(t)); \quad (29a)$$

$$\nabla_x \cdot \mathbf{u}(t, x) = 0 \quad t > 0, x \in \mathcal{F}_m(\mathbf{s}(t)); \quad (29b)$$

$$\mathbf{u}(t, x) = \mathbf{w}(t, x) \quad t > 0, x \in \partial\mathcal{A}_m(\mathbf{s}(t)). \quad (29c)$$

From a mathematical point of view, the advantage is two folds:

1. The equations are now linear;
2. The fluid has no more proper degree of freedom. Indeed, the fluid equations simplify from an initial and boundary value problem into merely a boundary value problem. In particular, no more initial data is required.

Newton's law for linear momentum reads:

$$\int_{\partial\mathcal{A}_m(\mathbf{s})} \mathbf{E}_1 \cdot \mathbb{T}_m(\mathbf{u}, p)(t, x) \mathbf{n} d\sigma = 0.$$

The solution  $(\mathbf{u}, p)$  being linear with respect to the boundary data  $\mathbf{w}$  it can be decomposed as follows:

$$\begin{aligned} \mathbf{u}(t, x) &= \dot{r}(t) \mathbf{u}_r(\mathbf{s}(t), x) + \sum_{j=1}^N \dot{s}_j u_d^j(\mathbf{s}(t), x); \\ p(t, x) &= \dot{r}(t) p_r(\mathbf{s}(t), x) + \sum_{j=1}^N \dot{s}_j p_d^j(\mathbf{s}(t), x); \quad (t > 0, x \in \mathcal{F}_m(\mathbf{s})), \end{aligned}$$

where we are written  $\mathbf{s} = (s_1, \dots, s_N)$  in a local chart of  $\mathcal{S}$  and  $\dot{s} = (\dot{s}_1, \dots, \dot{s}_N)$  in the basis  $(\partial_{s_1}, \dots, \partial_{s_N})$  of the tangent space  $T_{\mathcal{S}}\mathcal{S}$ . It entails that the stress tensor:

$$\mathbb{T}_m(\mathbf{u}, p) := \mu(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T) - p \mathbb{I}d,$$

can also be decomposed as:

$$\mathbb{T}_m(\mathbf{u}, p) = \dot{r} \mathbb{T}_m(\mathbf{u}_r, p_r) + \sum_{j=1}^N \dot{s}_j \mathbb{T}_m(\mathbf{u}_d^j, p_d^j).$$

The *elementary solutions*  $(\mathbf{u}_r, p_r)$  and  $(\mathbf{u}_d^j, p_d^j)$  satisfy the Stokes system (29a-29b) with the boundary conditions:

$$\begin{aligned} \mathbf{u}_r(t, x) &= \mathbf{E}_1 & t > 0, x \in \partial \mathcal{A}_m(\mathbf{s}(t)); \\ \mathbf{u}_d^j(t, x) &= \mathbf{w}_d(\mathbf{s}(t), \partial_{s_j}, x) & t > 0, x \in \partial \mathcal{A}_m(\mathbf{s}(t)), j = 1, \dots, N. \end{aligned}$$

Notice that the elementary solutions  $(\mathbf{u}_r, p_r)$  and  $(\mathbf{u}_d^j, p_d^j)$  ( $j = 1, \dots, N$ ) depend on the time through the shape variable  $\mathbf{s}$  only. We next introduce the scalar:

$$M^r(\mathbf{s}) := \int_{\partial \mathcal{A}(\mathbf{s})} \mathbf{E}_1 \cdot (\mathbb{T}_m(\mathbf{u}_r, p_r)) \mathbf{n} d\sigma,$$

and the row vector  $\mathbf{N}(\mathbf{s})$  whose entries are:

$$N_j(\mathbf{s}) = \int_{\partial \mathcal{A}(\mathbf{s})} \mathbf{E}_1 \cdot (\mathbb{T}_m(\mathbf{u}_d^j, p_d^j)) \mathbf{n} d\sigma.$$

We can rewrite Newton's laws as

$$\dot{r} M^r(\mathbf{s}) + \mathbf{N}(\mathbf{s}) \dot{\mathbf{s}} = 0.$$

Upon an integration by parts, we get the equivalent definition  $M^r(\mathbf{s})$ :

$$M^r(\mathbf{s}) := \mu \int_{\mathcal{F}_m(\mathbf{s})} D(\mathbf{u}_r) : D(\mathbf{u}_r) dx, \quad (30a)$$

where  $D(\mathbf{u}_r) := (\nabla_x \mathbf{u}_r + \nabla_x \mathbf{u}_r^T)$ . We deduce that  $M^r(\mathbf{s})$  is always positive. The same arguments for  $\mathbf{N}(\mathbf{s})$  lead to the identity:

$$N_j(\mathbf{s}) = \mu \int_{\mathcal{F}_m(\mathbf{s})} D(\mathbf{u}^r) : D(\mathbf{u}_d^j) dx. \quad (30b)$$

Later on, we will also need the matrix:

$$\mathbb{M}^d(\mathbf{s}) = \left( \mu \int_{\mathcal{F}_m(\mathbf{s})} D(\mathbf{u}^i) : D(\mathbf{u}_d^j) dx \right)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}. \quad (30c)$$

We eventually obtain the Euler-Lagrange equation governing the rigid displacement with respect to the shape changes:

$$\dot{r} = \mathcal{L}_{\mathbf{s}} \dot{\mathbf{s}} \quad t > 0, \quad (31)$$

where we have set set, for every  $\mathbf{s} \in \mathcal{S}$ :

$$\mathcal{L}_{\mathbf{s}} = -M^r(\mathbf{s})^{-1} \mathbf{N}(\mathbf{s}).$$

Considering the expressions (30) and (31), we deduce:

**Proposition 3.1** *The dynamics of a micro-swimmer is independent of the viscosity of the fluid. Or, in other words, the same shape changes produce the same rigid displacement, whatever the viscosity of the fluid is.*

**Proof.** Let  $(\mathbf{u}, p)$  be an elementary solution (as defined in the modeling above) to the Stokes equations corresponding to a viscosity  $\mu > 0$ , then  $(\mathbf{u}, (\tilde{\mu}/\mu)p)$  is the solution corresponding to the viscosity  $\tilde{\mu} > 0$ . Since the Euler-Lagrange equation depends only on the Eulerian velocities  $\mathbf{u}$ , the proof is completed. ■

In the sequel, we will set  $\mu = 1$ .

The self-propelled constraint (2) does not make sense any longer for low Reynolds number swimmers because  $\varrho_0 = 0$ . However, since we still do not want the swimmer to be able to translate itself just by self-deforming, we require the shape function to satisfy (2) in which we define the matrix  $\mathbf{Q}^S(\mathbf{s})$  by:

$$\mathbf{Q}^S(\mathbf{s}) := \int_{\Sigma} \nabla_{\mathbf{s}} \chi(\mathbf{s}, x) dx, \quad (32)$$

where  $\Sigma = \partial B$ .

### 3.2 High Reynolds number swimmers

Assume now that the inertia is preponderant with respect to the viscous force (it is the case when  $\text{Re} \ll 1$ ). The Navier-Stokes equations (28) simplify into the Euler equations:

$$\varrho_f \frac{D}{Dt} \mathbf{U}(t, X) - \nabla_X \cdot \mathbb{T}_f(\mathbf{U}, P)(t, X) = 0 \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)); \quad (33a)$$

$$\nabla_X \cdot \mathbf{U}(t, X) = 0 \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)); \quad (33b)$$

$$\mathbf{U}(t, X) \cdot \mathbf{n} - \mathbf{W}(t, X) \cdot \mathbf{n} = 0 \quad t > 0, X \in \partial \mathcal{A}_f(\mathbf{s}(t)). \quad (33c)$$

where the stress tensor reads:

$$\mathbb{T}_f(\mathbf{U}, P)(t, X) = -P(t, X) \text{Id} \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)).$$

Like in the preceding Subsection, we will assume that According to Kelvin's circulation theorem, if the flow is irrotational at some moment (i.e.  $\nabla \times \mathbf{U} =$

0) then, it has always been and will always remain irrotational. Hence, we can suppose that  $\nabla \times \mathbf{U} = 0$  for all times and then, according to the Helmholtz decomposition, that there exists for all time  $t > 0$  a potential scalar function  $\Phi(t, \cdot)$  defined in  $\mathcal{F}_f(\mathbf{s})$ , such that

$$\mathbf{U}(t, X) = \nabla_X \Phi(t, X) \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)).$$

The divergence-free condition leads to

$$\Delta_X \Phi(t, X) = 0 \quad t > 0, X \in \mathcal{F}_f(\mathbf{s}(t)),$$

and the boundary condition reads:

$$\partial_{\mathbf{n}} \Phi(t, X) = \mathbf{W}(t, X) \cdot \mathbf{n} \quad t > 0, X \in \partial \mathcal{A}_f(\mathbf{s}(t)).$$

The function  $\varphi(t, \cdot)$  defined by:

$$\varphi(t, x) := \Phi(t, x - \mathbf{r}) \quad t > 0, x \in \mathcal{F}_m(\mathbf{s}(t)),$$

is harmonic in  $\mathcal{F}_m(\mathbf{s}(t))$  and satisfies the boundary condition:

$$\partial_{\mathbf{n}} \varphi(t, x) = \mathbf{w}(t, x) \cdot \mathbf{n} \quad t > 0, x \in \partial \mathcal{A}_m(\mathbf{s}(t)).$$

The potential  $\varphi$  is linear in  $\mathbf{w}$ , so it can be decomposed into

$$\varphi(t, x) = \dot{r} \varphi_r(t, x) + \sum_{j=1}^N \dot{s}_j \varphi_d(t, x) \quad t > 0, x \in \mathcal{F}_m(\mathbf{s}(t)),$$

where at every moment the elementary potentials  $\varphi_r(t, \cdot)$  and  $\varphi_d(t, \cdot)$  are harmonics in  $\mathcal{F}_m(\mathbf{s}(t))$  and satisfy the boundary conditions:

$$\begin{aligned} \partial_{\mathbf{n}} \varphi_r(t, x) &= \mathbf{E}_1 \cdot \mathbf{n}, \\ \partial_{\mathbf{n}} \varphi_d(t, x) &= \mathbf{w}_d(\mathbf{s}(t), \partial_{s_j}, x) \quad t > 0, x \in \partial \mathcal{A}_m(\mathbf{s}(t)). \end{aligned}$$

This process is usually referred to as Kirchhoff's law. At this point, we do not invoke Newton's laws to derive the Euler-Lagrange equation but rather use the formalism of Analytic Mechanics. Both approaches (Newton's laws of Classical Mechanics and the Least Action principle of Analytic Mechanics) are equivalent, as proved in [17], but the latter is simpler and shorter.

In the absence of buoyant force, the Lagrangian function  $L$  of the body-fluid system coincides with the kinetic energy:

$$L = m \frac{1}{2} |\dot{r}|^2 + \frac{1}{2} \int_{\mathcal{A}_m(\mathbf{s})} \varrho(\mathbf{s}, x) |\mathbf{w}_d(t, x)|^2 dx + \frac{1}{2} \int_{\mathcal{F}_m(\mathbf{s})} \varrho_f |\mathbf{u}(t, x)|^2 dx.$$

In this sum, one can identify, from the left to the right: the kinetic energy of the body connecting to the rigid motion, the kinetic energy resulting from

the deformations and the kinetic energy of the fluid. We can next compute that, upon a change of variables:

$$\int_{\mathcal{A}_m(\mathbf{s})} \varrho(\mathbf{s}, x) |\mathbf{w}_d(t, x)|^2 dx = \varrho_0 \int_B |\nabla_{\mathbf{s}} \chi(\mathbf{s}, x) \cdot \dot{\mathbf{s}}|^2 dx,$$

and

$$\int_{\mathcal{F}_m(\mathbf{s})} \varrho_f |\mathbf{u}(t, x)|^2 dx = \varrho_f \int_{\mathcal{F}_m(\mathbf{s})} |\nabla \varphi(t, x)|^2 dx.$$

It leads us to introduce the scalar:

$$M^r(\mathbf{s}) = m + \varrho_f \int_{\mathcal{F}_m(\mathbf{s})} |\nabla \varphi_r(\mathbf{s}, x)|^2 dx, \quad (34a)$$

the row vector  $\mathbf{N}(\mathbf{s})$  whose entries are:

$$N_j(\mathbf{s}) = \varrho_f \int_{\mathcal{F}_m(\mathbf{s})} \nabla \varphi_r(\mathbf{s}, x) \cdot \nabla \varphi_d^j(\mathbf{s}, x) dx, \quad j = 1, \dots, N, \quad (34b)$$

and the matrix  $\mathbb{M}^d(\mathbf{s})$ :

$$\begin{aligned} \mathbb{M}^d(\mathbf{s}) = \varrho_0 \int_B \nabla_{\mathbf{s}} \chi(\mathbf{s}, x) \otimes \nabla_{\mathbf{s}} \chi(\mathbf{s}, x) dx \\ + \varrho_f \left( \int_{\mathcal{F}_m(\mathbf{s})} \nabla \varphi_d^i(\mathbf{s}, x) \cdot \nabla \varphi_d^j(\mathbf{s}, x) dx \right)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}. \end{aligned}$$

Observe the similarity between relations (30) and (34). We can rewrite the Lagrangian function as:

$$L(\dot{r}, \mathbf{s}, \dot{\mathbf{s}}) = \frac{1}{2} M^r(\mathbf{s}) |\dot{r}|^2 + \dot{r} \mathbf{N}(\mathbf{s}) \dot{\mathbf{s}} + \dot{\mathbf{s}} \cdot \mathbb{M}^d(\mathbf{s}) \dot{\mathbf{s}}.$$

Invoking now the Least Action principle, we claim that the Euler-Lagrange equation is:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}}(\dot{r}, \mathbf{s}, \dot{\mathbf{s}}) - \frac{\partial L}{\partial r}(\dot{r}, \mathbf{s}, \dot{\mathbf{s}}) = 0,$$

which reduces to, since  $L$  does not depend on  $r$ :

$$\frac{d}{dt} (M^r(\mathbf{s}) \dot{r} + \mathbf{N}(\mathbf{s}) \dot{\mathbf{s}}) = 0.$$

Assuming that the impulse  $M^r(\mathbf{s}) \dot{r} + \mathbf{N}(\mathbf{s}) \dot{\mathbf{s}}$  is zero at the initial time, we get eventually for the dynamics the exact same expression as (31):

$$\dot{r} = \mathcal{L}_{\mathbf{s}} \dot{\mathbf{s}} \quad t > 0,$$

where, for every  $\mathbf{s} \in \mathcal{S}$ :

$$\mathcal{L}_{\mathbf{s}} = -M^r(\mathbf{s})^{-1} \mathbf{N}(\mathbf{s}). \quad (35)$$

It is easy to verify that the dynamics does not depend on  $\varrho_0$  and  $\varrho_f$  independently but only on the relative density  $\varrho_0/\varrho_f$ , which is assumed to be equal to 1 in the sequel.

### 3.3 Examples of cost functionals

For low Reynolds number swimmers, a classical notion of swimming efficiency (see [14] and [5]) is defined as the inverse of the ratio between the average power expended by the swimmer during a stroke starting and ending at the shape  $\mathbf{s}^\dagger$  and the power that an external force would spend to translate the system rigidly at the same average speed:

$$\text{Eff}^{-1} := \frac{\frac{1}{T} \int_0^T \left( \int_{\partial \mathcal{A}_m(\mathbf{s})} \mathbf{F}(\mathbf{s}, \dot{\mathbf{s}}, \dot{r}, x) \cdot \mathbf{u}(\mathbf{s}, \dot{\mathbf{s}}, \dot{r}, x) d\sigma_x \right) dt}{\bar{\mathbf{v}} \cdot \int_{\partial \mathcal{A}_m(\mathbf{s})} \mathbf{F}(\mathbf{s}^\dagger, \mathbf{0}, \bar{\mathbf{v}}, x) d\sigma_x},$$

where

$$\mathbf{F}(\mathbf{s}, \dot{\mathbf{s}}, \dot{r}, x) := \left( \dot{r} \mathbb{T}_m(\mathbf{u}_r, p_r)(\mathbf{s}, x) + \sum_{j=1}^N \dot{s}_j \mathbb{T}_m(\mathbf{u}_j^d, p_j^d)(\mathbf{s}, x) \right) \mathbf{n}$$

is the force in the normal direction exerted by the fluid at the point  $x$  of the surface of the swimmer, with shape  $\mathbf{s}$ , shape change rate  $\dot{\mathbf{s}}$  and rigid velocity  $\dot{r}$ . In the same way:

$$\mathbf{u}(\mathbf{s}, \dot{\mathbf{s}}, x) := \dot{r} \mathbf{E}_1 + \sum_{j=1}^N \dot{s}_j \mathbf{w}_d(\mathbf{s}, \partial_{s_j}, x),$$

is the velocity of the swimmer. Eventually  $\bar{\mathbf{v}}$  is the average speed:

$$\bar{\mathbf{v}} := \left( \frac{1}{T} \int_0^T \dot{r} dt \right) \mathbf{E}_1.$$

With the notation (30), the efficiency can be rewritten as:

$$\text{Eff}^{-1} := \frac{\frac{1}{T} \int_0^T (M^r(\mathbf{s}) |\dot{r}|^2 + \dot{r} \mathbf{N}(\mathbf{s}) \dot{\mathbf{s}} + \dot{\mathbf{s}} \cdot \mathbb{M}^d(\mathbf{s}) \dot{\mathbf{s}}) dt}{|\bar{\mathbf{v}}|^2 M^r(\mathbf{s}^\dagger)}. \quad (36)$$

It can easily be verified that:

$$M^r(\mathbf{s}) |\dot{r}|^2 + \dot{r} \mathbf{N}(\mathbf{s}) \dot{\mathbf{s}} + \dot{\mathbf{s}} \cdot \mathbb{M}^d(\mathbf{s}) \dot{\mathbf{s}} = \int_{\mathcal{F}_m(\mathbf{s})} D(\mathbf{u}, p) : D(\mathbf{u}, p) dx > 0,$$

where  $(\mathbf{u}, p)$  is the solution to the Stokes system (29).

For high Reynolds number swimmers, we can choose the same expression (36) for the efficiency, in which we use the definitions (34). In this case, the efficiency is the inverse of the ratio between the mean energy expended by the swimmer divided by the energy required to translate rigidly the swimmer at the same average speed.

Taking into account the dynamics and replacing  $\dot{r}$  by  $-M^r(\mathbf{s})^{-1}\mathbf{N}(\mathbf{s})\dot{\mathbf{s}}$  in (36), it leads us to consider on  $T_{\mathbf{s}}\mathcal{S}$  the following scalar product:

$$\mathbf{g}_{\mathbf{s}}(\dot{\mathbf{s}}_1, \dot{\mathbf{s}}_2) = \dot{\mathbf{s}}_1 \cdot \left( \mathbb{M}^d(\mathbf{s}) - \frac{\mathbf{N}(\mathbf{s}) \otimes \mathbf{N}(\mathbf{s})}{M^r(\mathbf{s})} \right) \dot{\mathbf{s}}_2, \quad (\dot{\mathbf{s}}_1, \dot{\mathbf{s}}_2 \in T_{\mathbf{s}}\mathcal{S}). \quad (37)$$

According to the abstract framework introduced in Section 2.1, the cost of an admissible shape change  $\mathbf{s} : [0, T] \mapsto \mathcal{S}$  will be:

$$\frac{1}{2} \int_0^T \mathbf{g}_{\mathbf{s}(t)}(\dot{\mathbf{s}}(t), \dot{\mathbf{s}}(t)) dt. \quad (38)$$

### 3.4 Regularity results

In Section 2.1, the manifold  $\mathcal{S}$  and the differential forms are all of them assumed to be analytic. The following Lemma ensures that, under a simple hypothesis, this regularity is ensured for swimmers in a perfect fluid and Stokesian swimmers.

We denote by  $M(N_1, N_2)$  the Euclidian space of the  $N_1 \times N_2$  matrices and we claim:

**Lemma 3.2** *Assume that the map  $\mathbf{s} \in \mathcal{S} \mapsto \chi(\mathbf{s}, \cdot) \in C^1(\bar{B}, \mathbb{R}^3)$  is analytic (we refer to [22] for the definitions and the properties of analytic functions valued in Banach spaces), then for both cases (low and high Reynolds number swimmers) the maps  $\mathbf{s} \in \mathcal{S} \mapsto \mathbf{Q}^{\mathcal{S}}(\mathbf{s}) \in M(3, N)$ ,  $\mathbf{s} \in \mathcal{S} \mapsto M^r(\mathbf{s}) \in \mathbb{R}$ ,  $\mathbf{s} \in \mathcal{S} \mapsto \mathbf{N}(\mathbf{s}) \in \mathbb{R}^N$  and  $\mathbf{s} \in \mathcal{S} \mapsto \mathbb{M}^d(\mathbf{s}) \in M(N, N)$  are analytic.*

The proofs can be found in [10] for swimmers in a perfect fluid and in [15] for Stokesian swimmers.

### 3.5 An example of swimmer in a potential flow

All along the paper, we will illustrate our purpose by applying our approach to a concrete example. We have chosen to deal with a simplified version of a 2D swimmer in a perfect fluid introduced in [9] and improved in [18].

#### Shape changes

Recall that in 2D, at every time,  $\mathcal{A}_m(\mathbf{s})$  is the image of the unit disk  $D$  by a diffeomorphism  $\chi(\mathbf{s}, \cdot)$  depending on the parameter  $\mathbf{s}$ , and whose form (with complex notation) is:

$$\chi(\mathbf{s}, z) = z + s_1 \bar{z} + s_2 \bar{z}^2 + s_3 \bar{z}^3, \quad (z \in \mathbb{C}, \mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3). \quad (39)$$

We define the following norm in  $\mathbb{R}^3$ :

$$\|\mathbf{s}\|_{\mathcal{S}} = \sup_{z \in \partial D} |s_1 + 2s_2 z + 3s_3 z^2|, \quad (\mathbf{s} \in \mathbb{R}^3).$$

and we claim (see [9] for details):

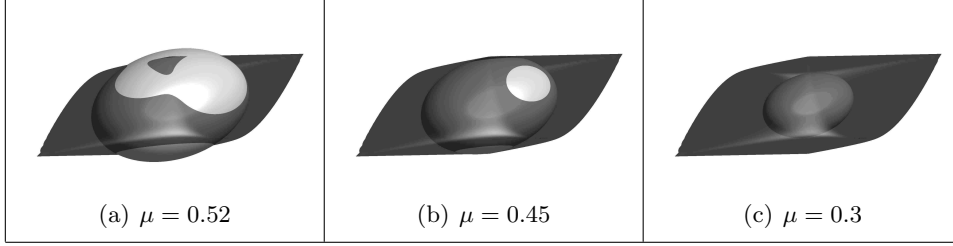


Figure 1: Examples of manifolds  $\mathcal{S}_\mu$  for different values of  $\mu$ . For small values of  $\mu$ ,  $\mathcal{S}_\mu$  turns out to be merely the surface of the ellipsoid since it is entirely included in the unit ball.

**Lemma 3.3** 1. The mapping  $\chi(\mathbf{s}, \cdot)$  is a  $C^\infty$  diffeomorphism from the unit ball  $D$  onto its image  $\mathcal{A}_m(\mathbf{s})$  if and only if  $\|\mathbf{s}\|_{\mathcal{S}} < 1$ .

2. The measure of the area of  $\mathcal{A}_m(\mathbf{s})$  is  $\pi(1 - s_1^2 + 2s_2^2 + 3s_3^2)$ .

Since we want both conditions (i) the mapping  $\chi(\mathbf{s}, \cdot)$  is a diffeomorphism and (ii) the area of  $\mathcal{A}_m(\mathbf{s}, \cdot)$  is of constant (and nonzero) measure, to be fulfilled, we introduce for every  $0 < \mu < 1$  the set (see Fig. 1):

$$\mathcal{S}_\mu = \{\mathbf{s} \in \mathbb{R}^3 : \|\mathbf{s}\|_{\mathcal{S}} < 1 \text{ and } s_1^2 + 2s_2^2 + 3s_3^2 = \mu^2\}.$$

For any  $0 < \mu < 1$ ,  $\mathcal{S}_\mu$  is a 2D analytic submanifold of  $\mathbb{R}^3$ . It consists in the parts the ellipsoid surface  $s_1^2 + 2s_2^2 + 3s_3^2 = \mu^2$  lying inside the unit ball  $\|\mathbf{s}\|_{\mathcal{S}} < 1$ .

To simplify, we will consider in the following that  $\mu$  is small enough. In this case, the ellipsoid  $s_1^2 + 2s_2^2 + 3s_3^2 \leq \mu^2$  is included in the unit ball  $\|\mathbf{s}\|_{\mathcal{S}} < 1$ , and hence  $\mathcal{S}_\mu$  reduces merely to the surface of the ellipsoid.

As a conclusion, once  $\mu$  (and therefore the measure of the swimmer) has been chosen and fixed, the shape changes over a time interval  $[0, T]$  are described by means of a function:

$$t \in [0, T] \mapsto \mathbf{s}(t) \in \mathcal{S}_\mu.$$

By direct computations, one verifies that the self-propelled constraints (2), ensuring that the swimmer can not modified its linear momentum by self-deforming, are automatically satisfied in this simplified case.

In Fig. 2, we have pictured some points of the the ellipsoid and the corresponding shapes for the swimmer.

Using the conformal mapping

$$\phi(\mathbf{s}, z) := z + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3}, \quad (z \in \mathbb{C} \setminus \bar{D}, \mathbf{s} = (s_1, s_2, s_3) \in \mathcal{S}_\mu),$$

which maps the exterior of the unit disk onto the fluid domain  $\mathcal{F}_m(\mathbf{s})$ , we can compute explicitly the elementary kirchhoff's potentials  $\varphi_r(\mathbf{s}, \cdot)$  and  $\varphi_d(\mathbf{s}, \cdot)$



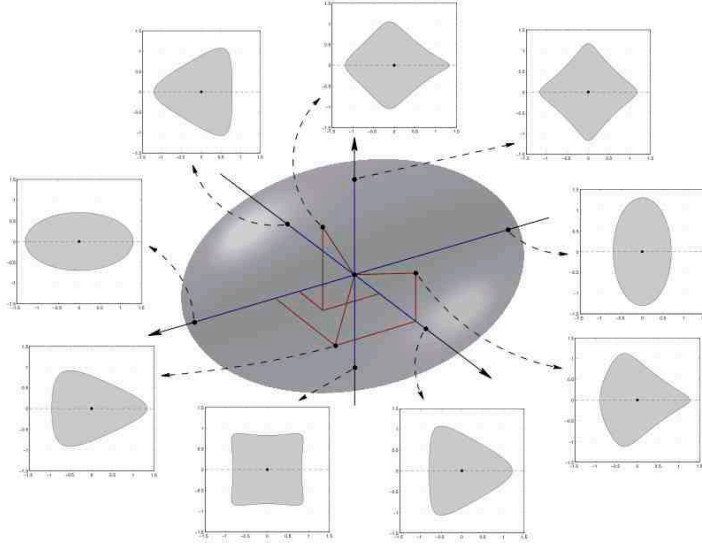


Figure 2: Some points on the ellipsoid  $\mathcal{S}_\mu$  ( $\mu = 0.3$ ) and the corresponding shapes of the swimmer.

(again, we refer to [9] or [18] for the details). We finally get the following expressions for the mass matrices introduced in (34):

$$M^r(\mathbf{s}) = 2 - 2s_1 \quad (40a)$$

$$\mathbf{N}(\mathbf{s}) = \begin{bmatrix} -3s_2 + 2s_2s_1 + 3s_2s_3 & -s_1 - 4s_3 + s_1^2 + 3s_1s_3 & -2s_2 + 3s_2s_1 \end{bmatrix} \quad (40b)$$

$$\mathbb{M}^d(\mathbf{s}) = \begin{bmatrix} 4s_2^2 - 3s_3 + \frac{9}{2}s_3^2 + 1 & 2s_1s_2 + 6s_2s_3 & 4s_2^2 - \frac{1}{2}s_1 + \frac{3}{2}s_1s_3 \\ 2s_1s_2 + 6s_2s_3 & s_1^2 + 6s_1s_3 + 9s_3^2 + \frac{2}{3} & 2s_1s_2 + 6s_2s_3 \\ 4s_2^2 - \frac{1}{2}s_1 + \frac{3}{2}s_1s_3 & 2s_1s_2 + 6s_2s_3 & 4s_2^2 + \frac{1}{2}s_1^2 + \frac{1}{2} \end{bmatrix} \quad (40c)$$

The 1-form  $\mathcal{L}$  defined in (35) as well as the scalar product  $\mathbf{g}$  defined in (37) and the cost functional (38) can now be explicitly computed. Instead of writing out their (complicated) expressions, we compute rather the Ricci-curvature induced by  $\mathbf{g}$  on the ellipsoid. The result is displayed on Fig 3.

### Controllability result for this swimmer

In this case, as already mentioned before, we have  $\mathbf{Q}^S = \mathbf{0}$  (the self-propelled constraints are always fulfilled) and hence  $\Delta^S = T\mathcal{S}_\mu$ . We define the following vector fields which are, for every  $\mathbf{s} \in \mathcal{S}_\mu$ , an analytic spanning set of  $T_{\mathbf{s}}\mathcal{S}_\mu$ :

$$\mathbf{X}_1(\mathbf{s}) := \begin{bmatrix} 3s_3(1 - s_1) \\ 0 \\ s_1(s_1 - 1) \end{bmatrix}, \quad \mathbf{X}_2(\mathbf{s}) := \begin{bmatrix} 2s_2(1 - s_1) \\ s_1(s_1 - 1) \\ 0 \end{bmatrix}, \quad \mathbf{X}_3(\mathbf{s}) := \begin{bmatrix} 0 \\ 3s_3(1 - s_1) \\ 2s_2(s_1 - 1) \end{bmatrix}.$$

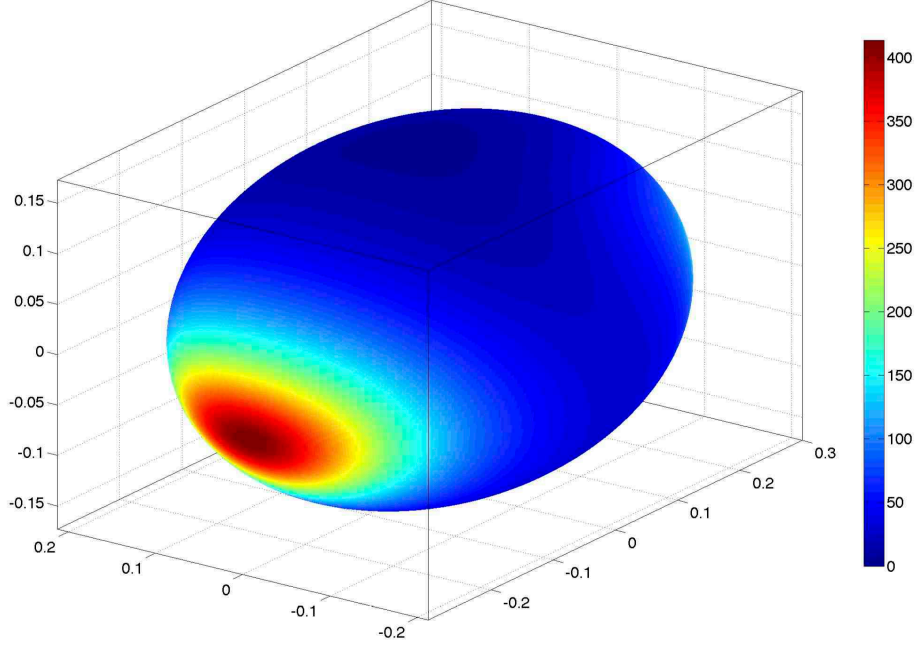


Figure 3: The Ricci curvature corresponding to the Riemannian metric  $\mathbf{g}$  defined in (37), on the ellipsoid.

Notice that, for all  $\mathbf{s} \in \mathcal{S}$ ,  $2s_2\mathbf{X}_1(\mathbf{s}) - 3s_3\mathbf{X}_2(\mathbf{s}) - s_1\mathbf{X}_3(\mathbf{s}) = \mathbf{0}$ . From this family of vectors, we build the vectors  $\mathbf{Z}_j$  ( $j = 1, 2, 3$ ) according to the definition (5) and the expressions (40). We get:

$$\begin{aligned} \mathbf{Z}_1(\boldsymbol{\xi}) &:= \begin{bmatrix} 3s_3(1-s_1) \\ 0 \\ s_1(s_1-1) \\ \frac{9}{2}s_2s_3 - 3s_1s_2s_3 - \frac{9}{2}s_2s_3^2 - s_1s_2 + \frac{3}{2}s_1^2s_2 \end{bmatrix} \\ \mathbf{Z}_2(\boldsymbol{\xi}) &:= \begin{bmatrix} 2s_2(1-s_1) \\ s_1(s_1-1) \\ 0 \\ 3s_2^2 - 2s_1s_2^2 - 3s_2^2s_3 - \frac{1}{2}s_1^2 - 2s_1s_3 + \frac{1}{2}s_1^3 + \frac{3}{2}s_1^2s_3 \end{bmatrix} \\ \mathbf{Z}_3(\boldsymbol{\xi}) &:= \begin{bmatrix} 0 \\ 3s_3(1-s_1) \\ 2s_2(s_1-1) \\ \frac{3}{2}s_1s_3 + 6s_3^3 - \frac{3}{2}s_1^2s_3 - \frac{9}{2}s_1s_3^2 - 2s_2^2 + 3s_1s_2^2 \end{bmatrix}. \end{aligned}$$

Obviously, we have again that  $2s_2\mathbf{Z}_1(\boldsymbol{\xi}) - 3s_3\mathbf{Z}_2(\boldsymbol{\xi}) - s_1\mathbf{Z}_3(\boldsymbol{\xi}) = \mathbf{0}$  for all  $\boldsymbol{\xi} = (s, r) \in \mathcal{M}$ . By direct calculation, one can check that for all  $\boldsymbol{\xi} \in \mathcal{M}$ :

$$[\mathbf{Z}_1(\boldsymbol{\xi}), \mathbf{Z}_2(\boldsymbol{\xi})] = \begin{bmatrix} 0 \\ 3s_3(2s_1 - 1)(s_1 - 1) \\ -2s_2(2s_1 - 1)(s_1 - 1) \\ -\frac{3}{2}s_1s_3 - 3s_1^2s_2^2 + \frac{3}{2}s_1^3s_3 + \frac{9}{2}s_1^2s_3^2 - \frac{21}{2}s_1s_3^2 - s_1^2 + 6s_3^2 + s_1^3 + 5s_1s_2^2 - 2s_2^2 \end{bmatrix}.$$

For  $\boldsymbol{\xi}^\dagger := (\mu, 0, 0, 0) \in \mathcal{M}$ , we have:

$$[\mathbf{Z}_1(\boldsymbol{\xi}^\dagger), \mathbf{Z}_2(\boldsymbol{\xi}^\dagger), [\mathbf{Z}_1(\boldsymbol{\xi}^\dagger), \mathbf{Z}_2(\boldsymbol{\xi}^\dagger)]] = \mu(\mu - 1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2}\mu^2 & \mu^2 \end{bmatrix},$$

and hence  $\dim \text{Lie}_{\boldsymbol{\xi}^\dagger} \{\mathbf{Z}_j, j = 1, 2, 3\} = 3$ . According to Theorem 2.3, we deduce that our example of swimmer in a perfect fluid is controllable.

## 4 The case $N = 2$

This section is devoted to the study of optimal strokes for controllable, trivialized, unconstrained swimmers, when the dimension  $N$  of the manifold  $\mathcal{S}$  is equal to 2.

### 4.1 Optimal Strokes and Isoperimetric Inequalities

In this subsection, we wish to give a hint of how the optimal stroke problem can be interpretable as an isoperimetric problem on the manifold  $\mathcal{S}$ .

Recall that a stroke is a closed (oriented) admissible curve  $\Gamma$ . Let  $\mathbf{s} : [0, T] \mapsto \mathcal{S}$  be a parameterization of a  $\Gamma$ . The traveled distance resulting from this stroke is:

$$\int_0^T \mathcal{L}_{\mathbf{s}(t)} \dot{\mathbf{s}}(t) dt = \int_\Gamma \mathcal{L}.$$

Denoting by  $\Omega$  the area enclosed by  $\Gamma$ , one gets from Stokes formula:

$$\int_\Gamma \mathcal{L} = \int_\Omega d\mathcal{L}.$$

Notice that these formula are metric independent but require  $\mathcal{S}$  to be orientable. The 2-form  $d\mathcal{L}$  defined on the 2 dimensional manifold  $\mathcal{S}$  can be seen as a signed measure on  $\mathcal{S}$ .

An example of cost functional considered in this paper is just the Riemannian length of  $\Gamma$ .

Roughly speaking, seeking optimal strokes consists in minimizing the Riemannian length of  $\Gamma$  while maximizing the measure of the area  $\Omega$  for the signed measure  $d\mathcal{L}$ .

In the case of the swimmer in the potential flow, we have drawn on Fig 4 the density function of the measure  $d\mathcal{L}$ . Notice in particular that, considering small strokes (and hence small closed curves), not only matter the shape of the curve but the position on the ellipsoid is also preponderant.

## 4.2 Pontryagin's maximum principle

We consider a trivialized, unconstrained, controllable swimmer  $\mathfrak{S} = (\mathcal{S}, \mathbf{g}, \mathbf{Q}^{\mathcal{S}}, \mathbf{s}^\dagger, \mathcal{L})$  (with  $\mathbf{Q}^{\mathcal{S}} = \mathbf{0}$  since it is *unconstrained*) and  $\mathcal{X} = \{\mathbf{X}_1, \mathbf{X}_2\}$  an analytic orthonormal basis of  $\Delta^{\mathcal{S}} = T\mathcal{S}$ . Then we define the analytic vector fields  $\mathcal{Z} = \{\mathbf{Z}_1, \mathbf{Z}_2\}$  on  $\mathcal{M}$ , according to formula (5).

In (6), the dynamics of the swimmer has been shown to be governed by the following differential equation

$$\dot{\boldsymbol{\xi}} = u_1 \mathbf{Z}_1(\boldsymbol{\xi}) + u_2 \mathbf{Z}_2(\boldsymbol{\xi}). \quad (41)$$

In the following, we need to strengthened Hypothesis 1 as follows.

**Hypothesis 4** *For every  $\boldsymbol{\xi} \in \mathcal{M}$ , we have:*

$$\dim \text{span}\{\mathbf{Z}_1(\boldsymbol{\xi}), \mathbf{Z}_2(\boldsymbol{\xi}), [\mathbf{Z}_1, \mathbf{Z}_2](\boldsymbol{\xi})\} = 3. \quad (42)$$

As already pointed out, this Hypothesis does not depend on the choice of  $\mathcal{Z}$  and hence is independent of the choice of the orthonormal basis  $\mathcal{X}$ . Actually, (42) can be rewritten as:

$$\dim \text{span}\{\Delta_{\boldsymbol{\xi}}^{\mathcal{M}}, [\Delta_{\boldsymbol{\xi}}^{\mathcal{M}}, \Delta_{\boldsymbol{\xi}}^{\mathcal{M}}]\} = 3.$$

Recall that according to Lemma 2.2, Hypothesis 1 leads to:

$$\dim \text{Lie}_{\boldsymbol{\xi}} \Delta^{\mathcal{M}} = \dim \mathcal{M}, \quad \forall \boldsymbol{\xi} \in \mathcal{M}.$$

In Hypothesis 4,  $\text{Lie}_{\boldsymbol{\xi}} \Delta^{\mathcal{M}}$  is required to be spanned by the Lie brackets of order no greater than 1.

In Theorem 2.6, it has been proved that Problems 1-3 are equivalent. Let us restate a version of this problem in the simplified 2 dimensional case we are considering in this Section:

**Problem 6** *For any given  $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1$  in  $\mathcal{M}$ , find  $T \geq 0$  and a measurable bounded function  $t \in [0, T] \mapsto (u_1(t), u_2(t)) \in \mathbb{R}^2$  which minimizes the cost*

$$\frac{1}{2} \int_0^T (u_1(t)^2 + u_2(t)^2) dt,$$

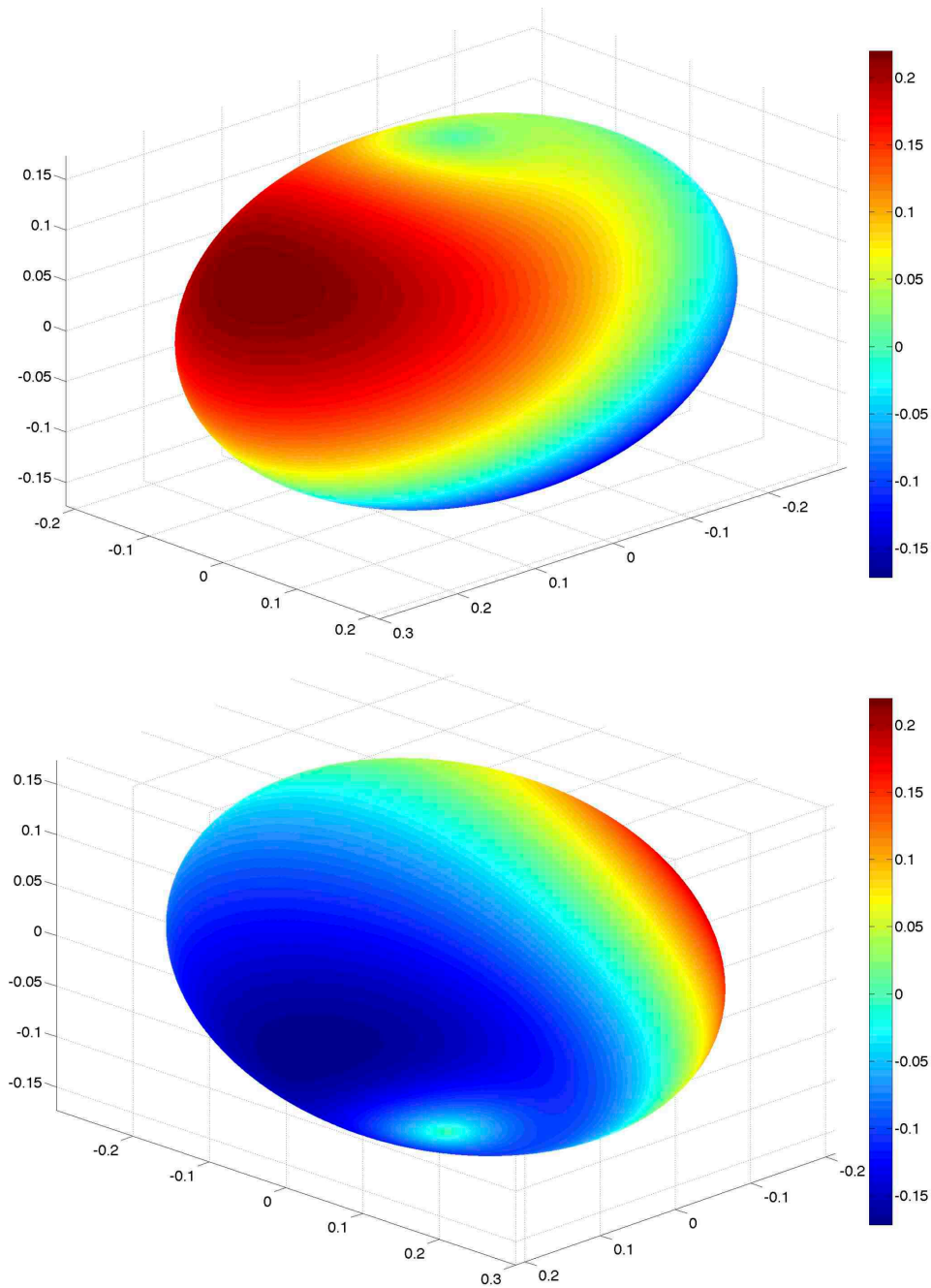


Figure 4: Density function of the signed measure defined by  $d\mathcal{L}$  on the ellipsoid. A stroke being a closed curve on the ellipsoid, the resulting travelled distance is obtained by measuring the area of the enclosed surface for the measure  $d\mathcal{L}$ . See also Fig 5 and 9.

such that there exists an absolutely continuous curve  $t \in [0, T] \mapsto \boldsymbol{\xi}(t) \in \mathcal{M}$  satisfying

$$\dot{\boldsymbol{\xi}}(t) = u_1(t)\mathbf{Z}_1(\boldsymbol{\xi}) + u_2(t)\mathbf{Z}_2(\boldsymbol{\xi}) \text{ for almost every } t \text{ in } [0, T], \quad (43a)$$

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \quad \boldsymbol{\xi}(T) = \boldsymbol{\xi}_1. \quad (43b)$$

and,

$$u_1(t)^2 + u_2(t)^2 = 1 \quad \forall t \in [0, T]. \quad (43c)$$

In order to apply Pontryagin's maximum principle (see [1]) to this problem, we denote by  $T^*\mathcal{M}$  the cotangent bundle of  $\mathcal{M}$  and we introduce the Hamiltonian (see [21] Chap.7.1)

$$\begin{aligned} H : \mathbb{R} \times \mathcal{M} \times T^*\mathcal{M} \times \mathbb{R} \times \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (t, \boldsymbol{\xi}, \mathbf{p}, p^0, (u_1, u_2)) &\mapsto H(t, \boldsymbol{\xi}, \mathbf{p}, p^0, (u_1, u_2)) \end{aligned} \quad (44a)$$

defined by:

$$\begin{aligned} H(t, \boldsymbol{\xi}, \mathbf{p}, p_0, \mathbf{U}) &= \langle \mathbf{p}, u_1(t)\mathbf{Z}_1(\boldsymbol{\xi}) + u_2(t)\mathbf{Z}_2(\boldsymbol{\xi}) \rangle \\ &\quad - p_0 (u_1^2(t) + u_2^2(t)) \end{aligned} \quad (44b)$$

where  $\langle \cdot, \cdot \rangle$  stands for the duality product  $T^*\mathcal{M} \times T\mathcal{M}$ . For every  $\boldsymbol{\xi}$  in  $\mathcal{M}$ , the quantity  $u_1(t)\mathbf{Z}_1(\boldsymbol{\xi}) + u_2(t)\mathbf{Z}_2(\boldsymbol{\xi})$  belongs to the tangent space  $T_{\boldsymbol{\xi}}\mathcal{M}$ ,  $\mathbf{p}$  is in the cotangent space  $T_{\boldsymbol{\xi}}^*\mathcal{M}$  and  $p_0$ , called the cost dual variable, is a time independent real constant.

In this setting and under Hypothesis 4, the Pontryagin's maximum principle reads

**Proposition 4.1 (Normal case of the PMP)** *If  $(u_1(\cdot), u_2(\cdot)) \in L^\infty([0, T], \mathbb{R}^2)$  is a solution to Problem 6 associated with curve  $t \in [0, T] \mapsto \boldsymbol{\xi}(t) \in \mathcal{M}$ , then, there exists a non trivial*

$$\mathbf{p} : t \in [0, T] \mapsto \mathbf{p}(t) \in T^*\mathcal{M}$$

solution of the system

$$\dot{\boldsymbol{\xi}} = u_1(t)\mathbf{Z}_1(\boldsymbol{\xi}) + u_2(t)\mathbf{Z}_2(\boldsymbol{\xi}), \quad (45a)$$

$$\dot{\mathbf{p}} = -u_1(t)\langle \mathbf{p}, D_{\boldsymbol{\xi}}\mathbf{Z}_1(\boldsymbol{\xi}) \rangle - u_2(t)\langle \mathbf{p}, D_{\boldsymbol{\xi}}\mathbf{Z}_2(\boldsymbol{\xi}) \rangle. \quad (45b)$$

Moreover, for all  $t \in [0, T]$ ,

$$u_i(t) = \langle \mathbf{p}(t), \mathbf{Z}_i(\boldsymbol{\xi}(t)) \rangle \text{ for } i = 1, 2.$$

**Definition 4.1** *The exponential map (from  $\boldsymbol{\xi}_0$ ), denoted by  $\varepsilon_{\boldsymbol{\xi}_0}$ , is the mapping which associates with every  $\mathbf{p}_0 \in T^*\mathcal{M}$ , the solution of System (45) with the initial condition  $\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0$  and  $\mathbf{p}(0) = \mathbf{p}_0$ . Such a trajectory is called an extremal.*

*We call  $\gamma_{\mathbf{p}_0}$  the curve on the manifolds  $\mathcal{M}$  of the solution  $\varepsilon_{\boldsymbol{\xi}_0}$ .*

**Remark 4.1** *The general statement of the Maximum Principle, as it can be found for instance in [1], is more intricate and involves so-called abnormal extremals. The study of such extremals is usually far from obvious, from both a theoretical and a numerical point of view. In our case, the distribution is 1-step generating, see Eq. (42). Hence, the Goh condition (see Chap. 10.2 of [7]) ensures that all optimal trajectories in  $\mathcal{M}$  can be lifted in  $T^*\mathcal{M}$  as normal extremals.*

An obvious consequence of Proposition 4.1 is

**Lemma 4.2** *For every solution of (45)  $(\boldsymbol{\xi}(\cdot), \mathbf{p}(\cdot)) : [0, T] \rightarrow T^*\mathcal{M}$ , there exists a continuous function  $\theta : [0, T] \rightarrow \mathbb{R}$  such that,*

$$\begin{aligned}\langle \mathbf{p}(\cdot), \mathbf{Z}_1(\boldsymbol{\xi}(\cdot)) \rangle &= \cos(\theta(\cdot)), \\ \langle \mathbf{p}(\cdot), \mathbf{Z}_2(\boldsymbol{\xi}(\cdot)) \rangle &= \sin(\theta(\cdot)).\end{aligned}$$

### 4.3 Swimmer in a potential flow: Numerics

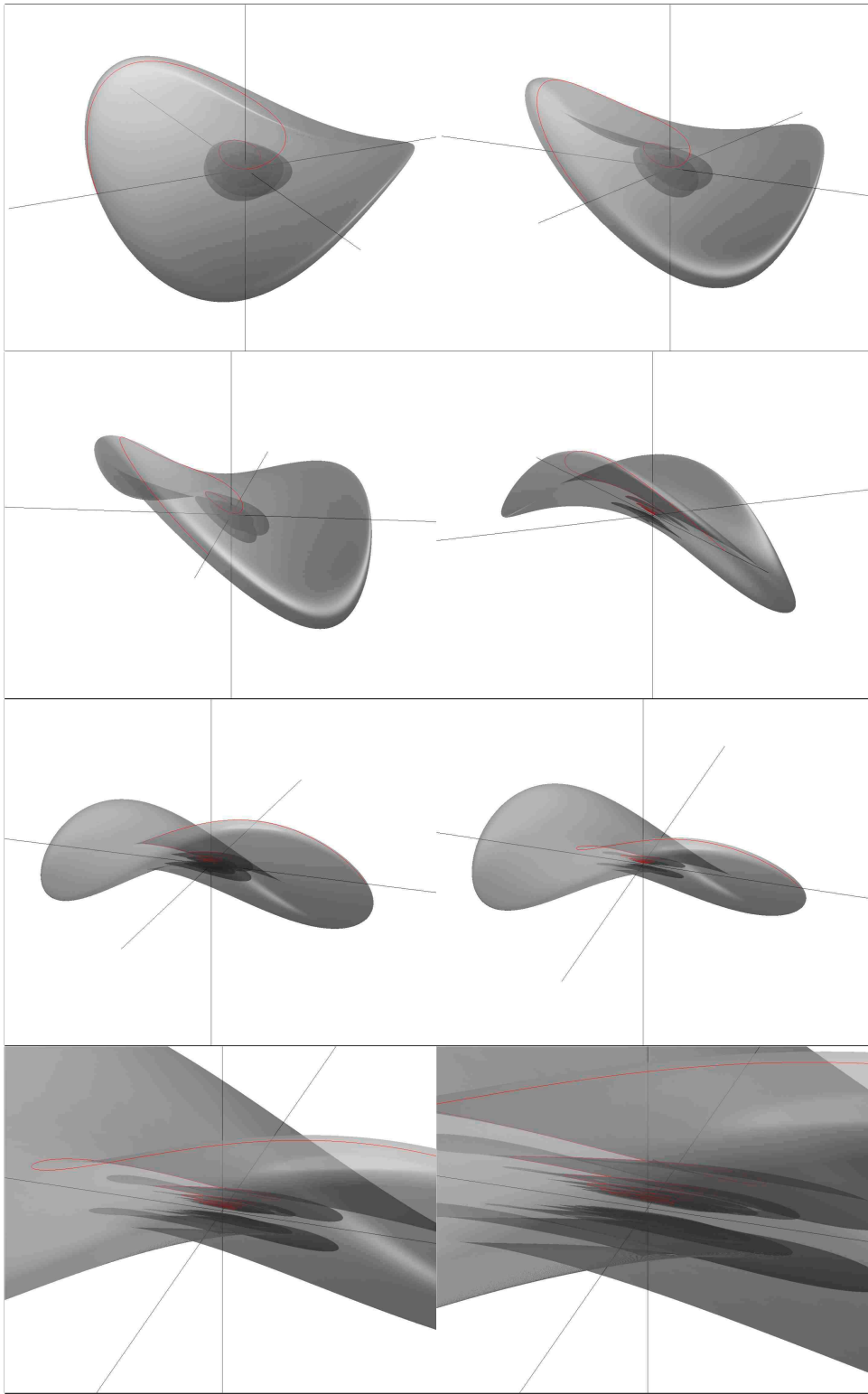
There are basically two methods to compute the optimal strokes corresponding to Problems 1-5 for the example of swimmer in a potential flow presented in Subsection (3.5).

The first one consists in integrating Pontryagin's maximum principle (45) over a time interval  $[0, T]$ , specifying some initial data  $(\boldsymbol{\xi}_0, \mathbf{p}_0)$ . Since we are mostly interested in the ending point  $\boldsymbol{\xi}(T)$ , we can implement a so-called shooting method. For instance, seeking strokes, we want the  $\mathcal{S}$  component of the ending point  $\pi_{\mathcal{S}}\boldsymbol{\xi}(T)$  to be the same as the  $\mathcal{S}$  component of the initial point  $\pi_{\mathcal{S}}\boldsymbol{\xi}_0$ . Let  $\boldsymbol{\xi}_0$  be given and for every  $\mathbf{p}_0$  denote by  $t \in [0, T] \mapsto \boldsymbol{\xi}(t, \mathbf{p}_0)$  the solution to (45) with Cauchy data  $(\boldsymbol{\xi}_0, \mathbf{p}_0)$ . We seek  $\mathbf{p}_0$  as a solution to the equation  $F(\mathbf{p}) = 0$  where  $F$  is for instance defined (in a chart) by:

$$\mathbf{p} \mapsto \|\pi_{\mathcal{S}}\boldsymbol{\xi}(T, \mathbf{p}) - \pi_{\mathcal{S}}\boldsymbol{\xi}_0\|_{\mathbb{R}^2}.$$

Notice however that not every solution to Pontryagin's maximum principle is an optimal curve. So, rather than determining optimal strokes, we integrate (45) a large number of times in order to draw sub-Riemannian wavefronts. Sub-Riemannian wavefronts are precisely the surfaces constituted by the ending points of all the extremal starting at a given point and having the same fixed sub-Riemannian length.

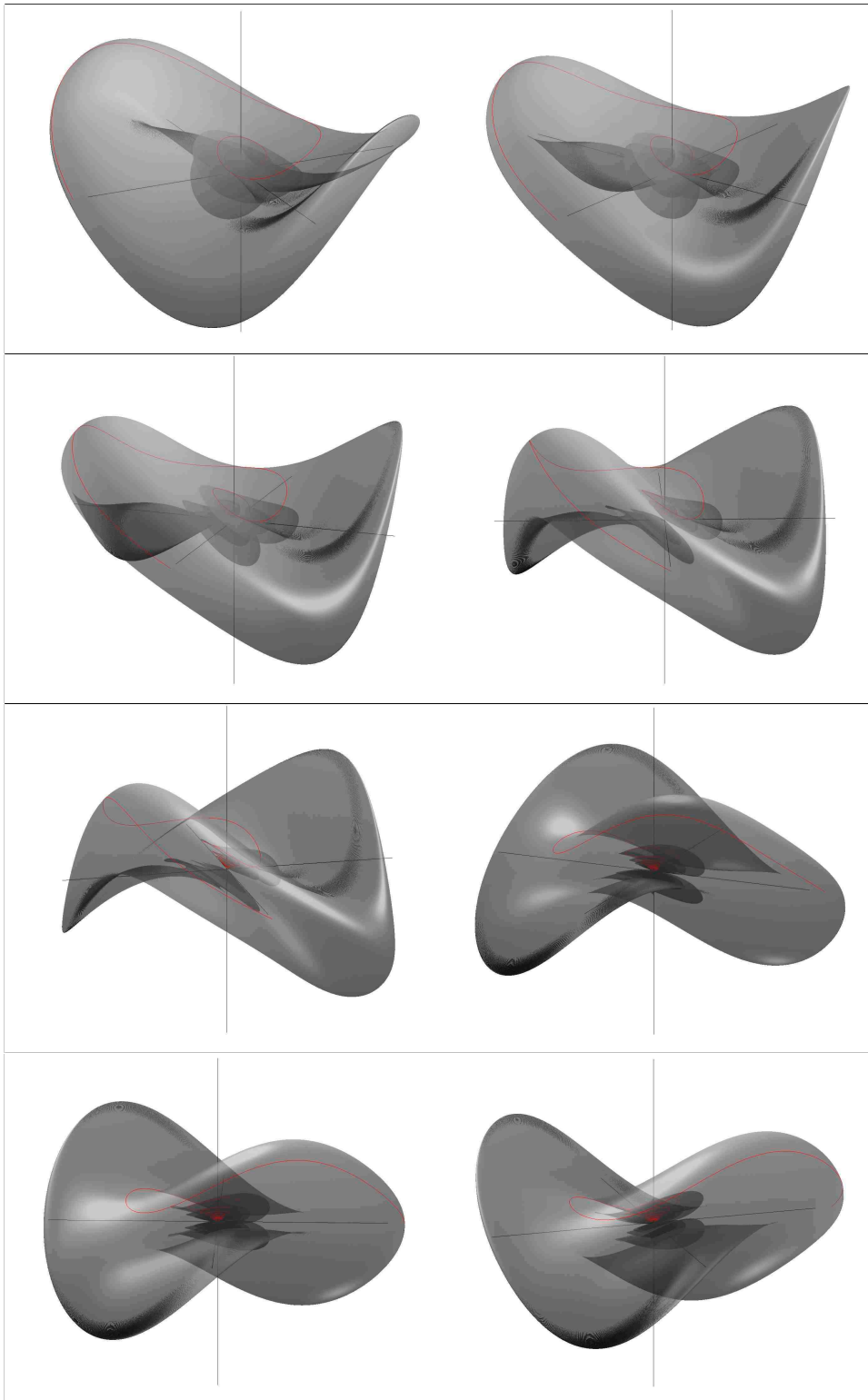
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Table 1: The sub-Riemannian wavefront of length 0.1 computed by integrating Pontryagin's maximum principle (45) with initial data satisfying  $\xi_0 = (\mathbf{s}_0, 0)$ ,  $\mathbf{s}_0 = (0.3, 0, 0)$ . The picture is draw in a chart, using spherical coordinates.



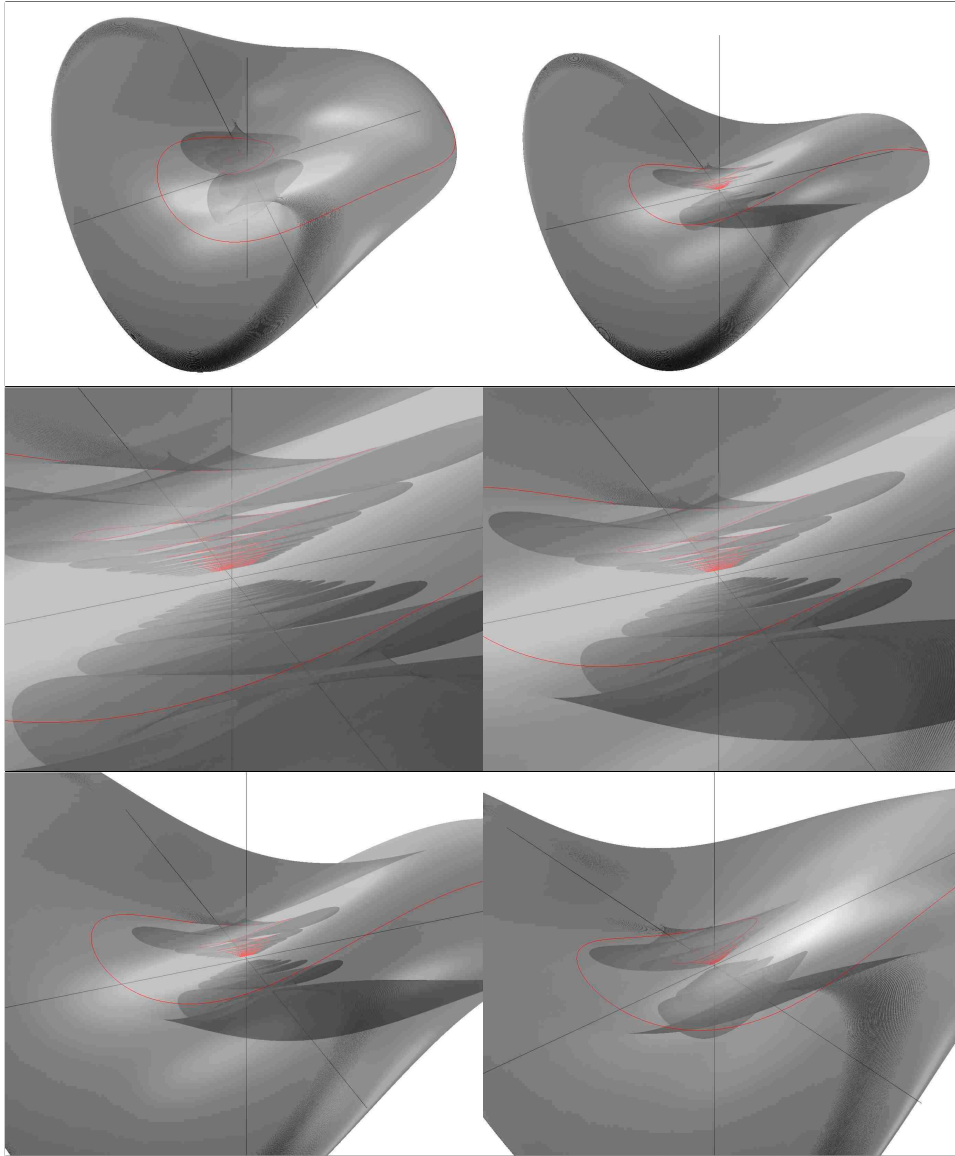


Table 2: The sub-Riemannian wavefront of length 0.2 computed by integrating Pontryagin's maximum principle (45) with initial data satisfying  $\xi_0 = (\mathbf{s}_0, 0)$ ,  $\mathbf{s}_0 = (0.3, 0, 0)$ . The picture is drawn in a chart, using spherical coordinates.

The second method consists in approximating any curve on  $\mathcal{S}$  by means of cubic splines. In the following examples, we use a basis of 20 cubic splines (actually, two bases since, working in spherical coordinates, we need 20 cubic splines for the polar angle and others 20 for the azimuth angle). So, we dispose of 40 parameters controlling the splines and the optimal

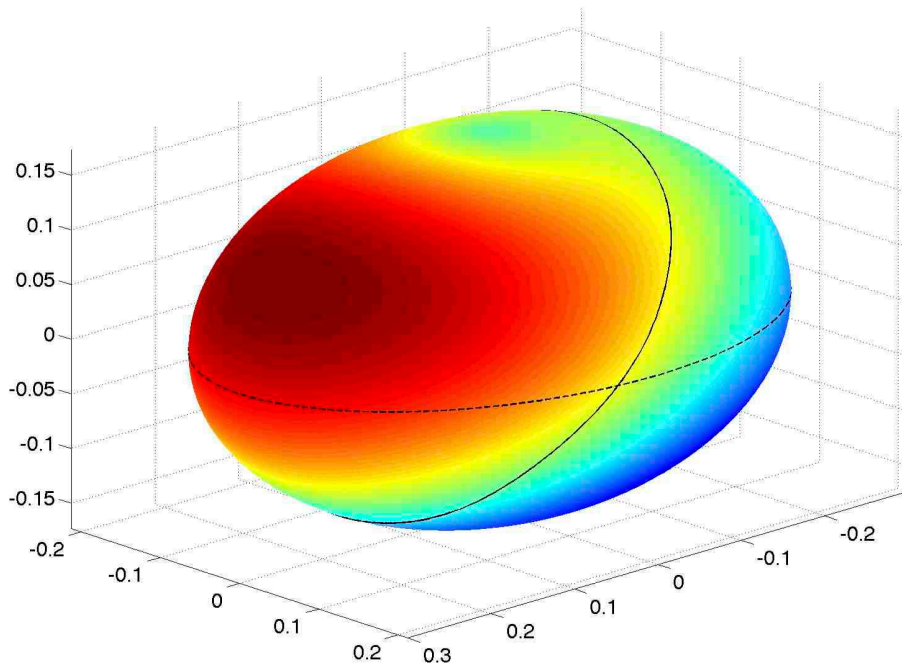


Figure 5: The initial closed curve is the equator of the ellipsoid (the dashed line). The optimized curve (the continuous line) is supposed to have the minimum cost for the same travelled distance. The colors are the same of in Fig 4.

problems under consideration turn into finite dimensional optimal problems. To every set of parameters, we can associate a travelled distance and a cost. To solve our optimal problems, we use the optimal toolbox of Matlab. The main difficulty is to manage the change of chart. Indeed, starting with the classical spherical coordinates, a curve cannot pass through the south or north pole of the ellipsoid. So we have to switch the axes, in such a way that the north pole becomes a regular point in spherical coordinates.

### First example: optimizing the cost

We consider a closed curve on the ellipsoid and compute the corresponding covered distance by the swimmer. Then, we try to minimize the cost among all the closed curves for which the travelled distance is the same. Notice that it is not exactly what is stated in Problem 1, because here there is no fixed starting point. The resulting curves on the ellipsoid are pictured on Fig 5, while the corresponding sequences of shapes are pictured on Fig 6 (initial guess) and 7 (optimized swimmer).

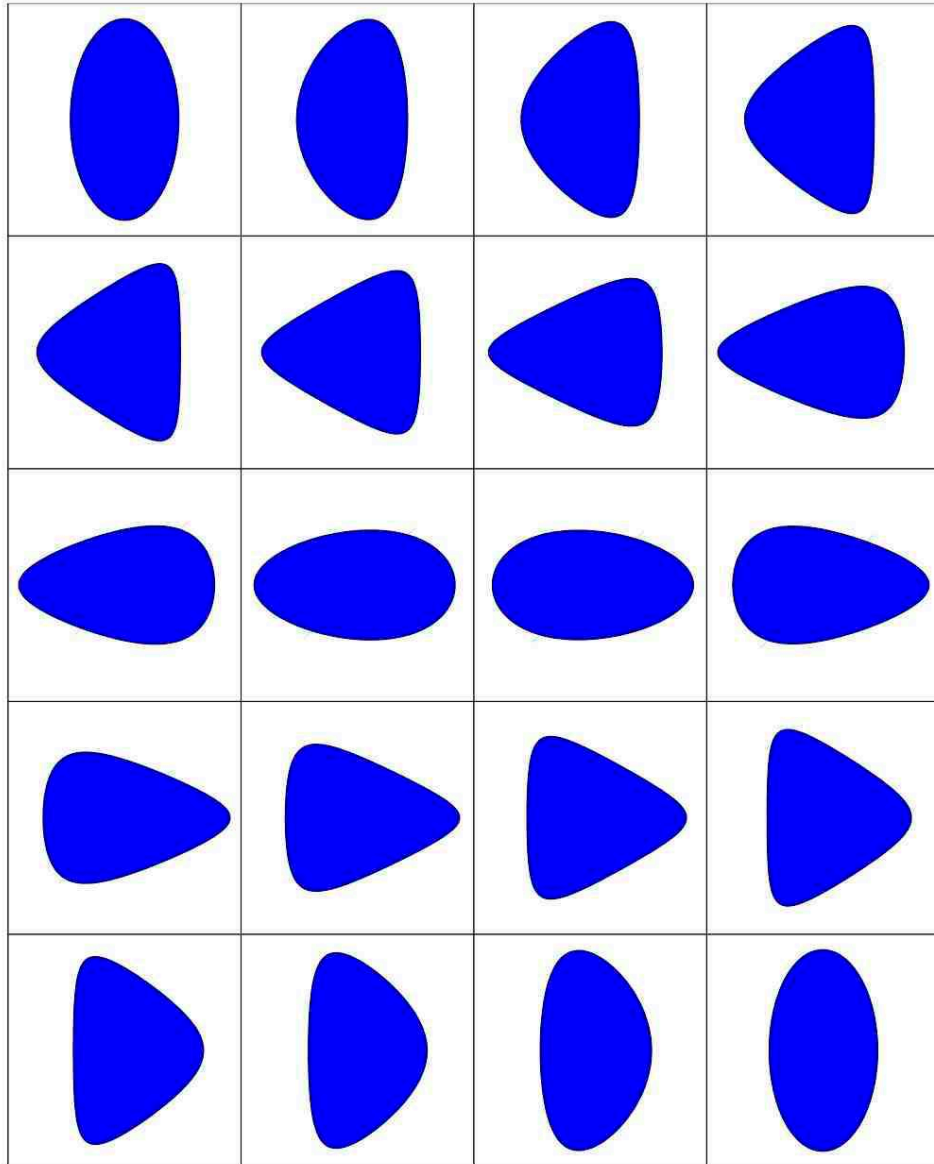


Figure 6: Sequence of 20 time equidistributed shapes for the initial guess.

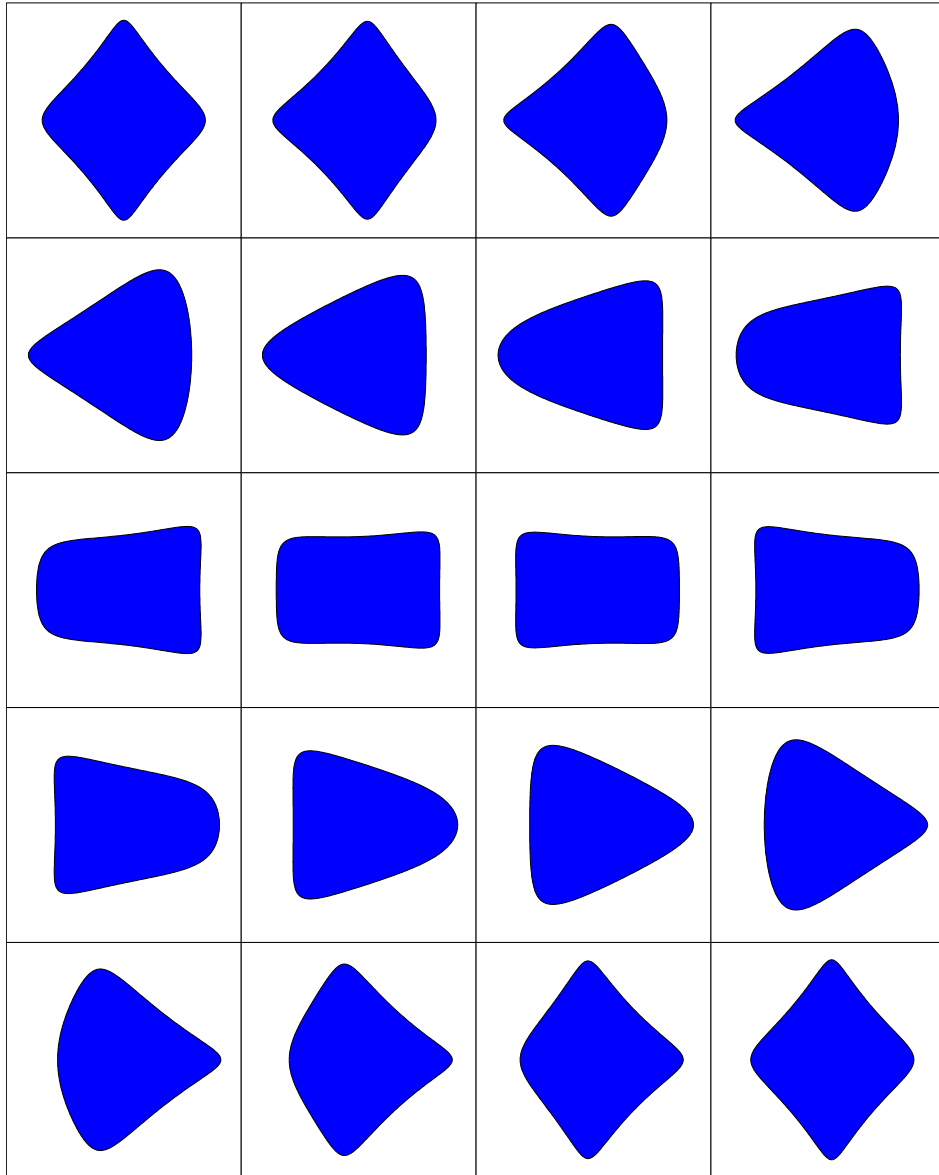


Figure 7: Sequence of 20 time equidistributed shapes after optimization.

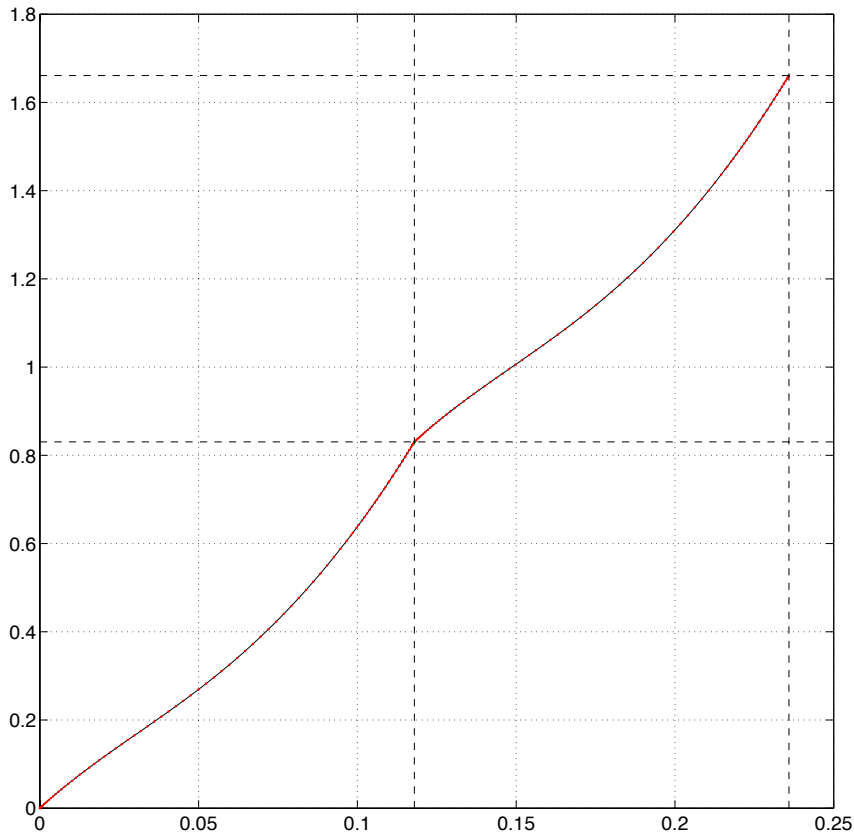


Figure 8: For every targeted distance (in abscissa) we compute the corresponding optimal cost (in ordinate). The curve is pseudo-periodic because, above a certain distance, the best stroke is made of two optimal smaller loops.

### Second example: The distance-cost function

Let us draw now the graph of the function defined (20c). Considering a shape *at rest* for the swimmer, we compute for every  $\delta \in \mathbb{R}_+$ , the optimal stroke (i.e. minimizing the cost) allowing the swimmer to cover the distance  $\delta$ . We choose as shape *at rest*, the converging point of all the curves on Fig 9. On the same picture, we draw all the curves corresponding to the optimal strokes for different values of  $\delta$ . A sequence of shapes corresponding to the longest curve is given in Fig 10. Finally, the graph of the distance-cost function is given in Fig 8.

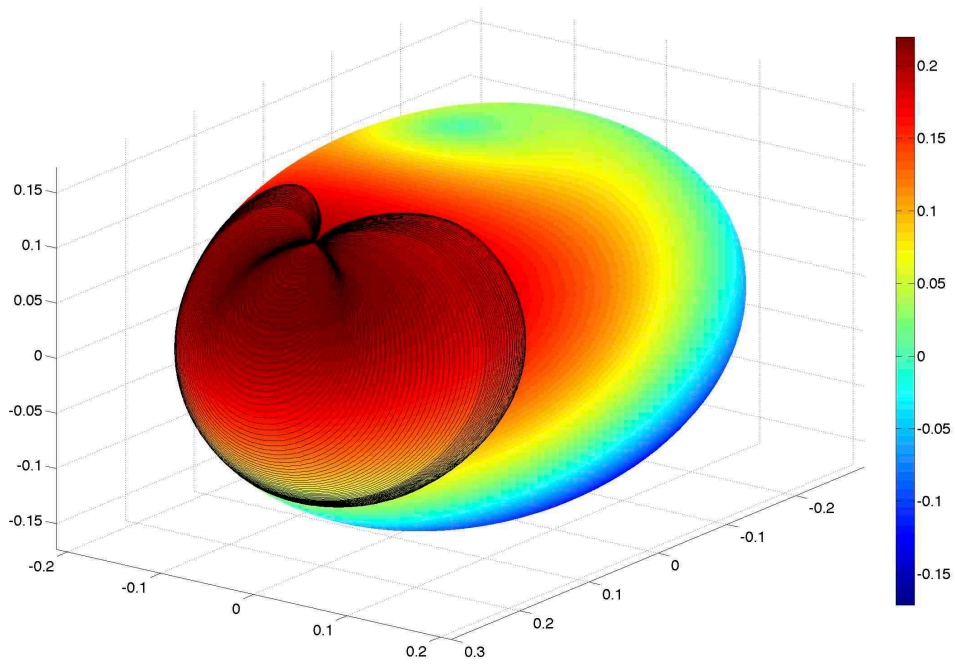


Figure 9: Every closed curve corresponds to an optimal stroke (minimizing the cost for a given distance).



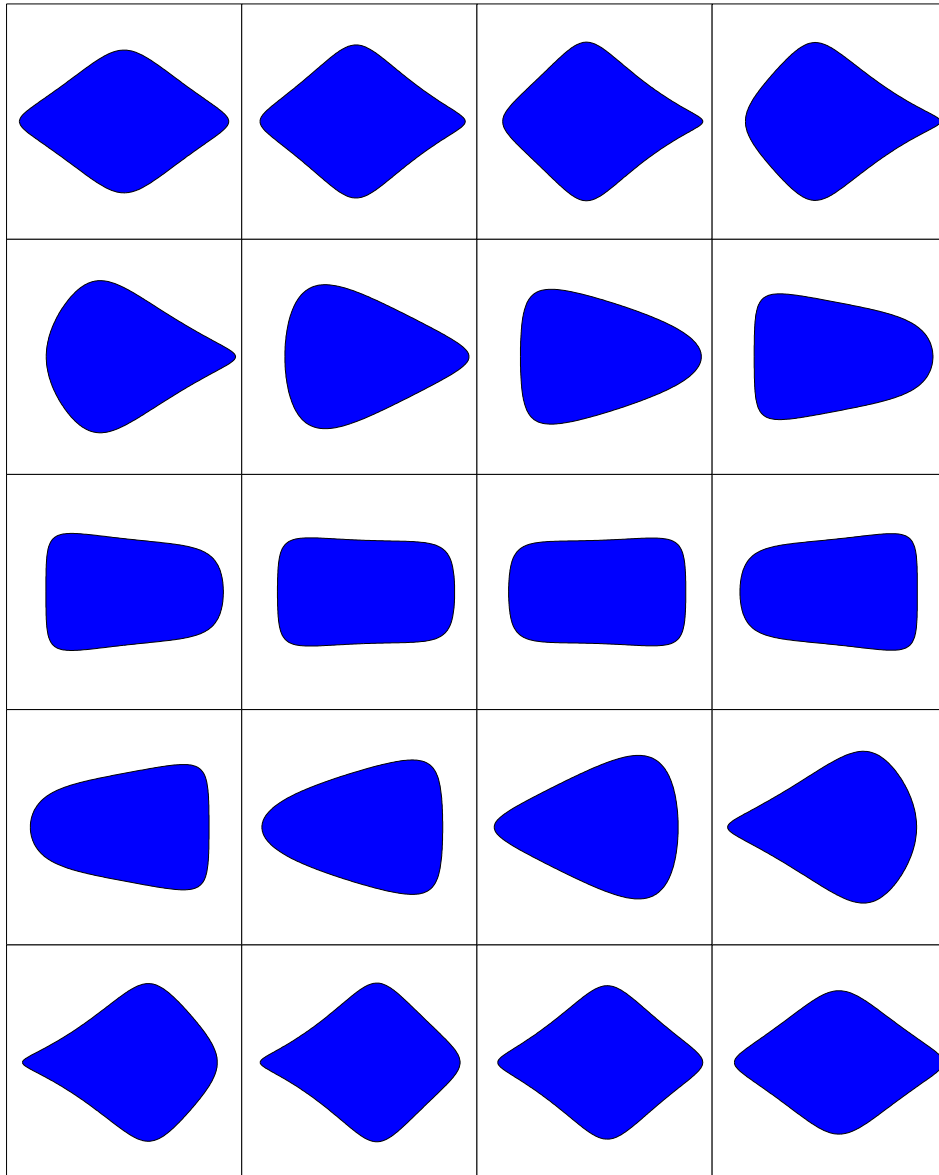


Figure 10: Sequence of 20 shapes time equi distributed corresponding to the longest curve on Fig 9.

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## A Riemannian Geometry

Let  $(M, g)$  be a Riemannian manifold. We denote  $\ell(\Gamma)$  the length of any rectifiable curve  $\Gamma \subset M$  and for every  $x \in M$  and  $r > 0$ , we denote by  $B(x, r)$  the Riemannian ball centered at  $x$  and of radius  $r$ . The following Lemma ensures the existence of a small monotone retract at any point of  $M$ .

**Lemma A.1** *Let  $x_0$  be a point of  $M$ . Then there exists  $r > 0$  such that, for every path  $\gamma : [0, T] \mapsto M$  absolutely continuous with essentially bounded first derivative, such that:*

1.  $\gamma(t) \in B(x_0, r)$  for every  $t \in [0, T]$ ;
2.  $\gamma(0) = \gamma(T) = x_0$ ;

there exists a continuous function  $\psi : [0, 1] \times [0, T] \mapsto M$  satisfying:

1. For every  $s \in [0, 1]$ ,  $t \in [0, T] \mapsto \psi(s, t)$  is continuous with essentially bounded first derivative.
2.  $\psi(1, \cdot) = \gamma$ ;
3.  $\psi(0, \cdot) = x_0$ ;
4. The function  $s \in [0, 1] \mapsto \ell(\Gamma_s)$  (where  $\Gamma_s$  is the curve parameterized by  $t \in [0, T] \mapsto \psi(s, t)$ ) is increasing.

**Proof.** Let  $\text{inj}(x_0)$  be the injectivity radius at  $x_0$  and denote  $\mathcal{V} := B(x_0, r)$ , for  $0 < r < \text{inj}(x_0)$  (the constant  $r$  will be fixed later on). Let  $\gamma$  be a path included in  $\mathcal{V}$  and satisfying the hypotheses of the lemma. Then define:

$$\zeta : t \in [0, T] \mapsto \zeta(t) := \exp_{x_0}^{-1}(\gamma(t)) \in T_{x_0}M.$$

This function has the same regularity as  $\gamma$  and  $\zeta(0) = \zeta(T) = 0$ . Define now, for every  $(s, t) \in [0, 1] \times [0, T]$ :

$$\psi(s, t) = \exp_{x_0}(s\zeta(t)).$$

This function has the required regularity and satisfies the equalities  $\psi(1, \cdot) = \gamma$  and  $\psi(0, \cdot) = x_0$ . So it remains only to prove that for  $r$  small enough, the length of  $\Gamma_s$  is increasing in  $s$ .

In the exponential map the metric  $g$  have the following Cartan local development:

$$g_{ij}(x) = \delta_i^j - \frac{1}{3} \sum_{k,l} R_{iklj}(0)x_k x_l + \mathcal{O}(\|x\|_E^3), \quad (46)$$

where  $\delta_i^j$  is the Kronecker symbol,  $R_{ijkl}$  are the coefficients of the Riemann curvature tensor and  $\|x\|_E$  stands for the Euclidean norm. The quantity we are interested in estimating is:

$$\ell(\Gamma_s) = \int_0^T \|\partial_t \psi(s, t)\|_{g(\psi(s,t))} dt, \quad (47)$$

and in the local chart, according to (46), we have:

$$\begin{aligned} \|\partial_t \psi(s, t)\|_{g(\psi(s,t))}^2 &= s^2 \|\dot{\zeta}(t)\|_E^2 \\ &- s^4 \left( \frac{1}{3} \sum_{i,j} \sum_{k,l} R_{iklj}(0) \zeta_k(t) \zeta_l(t) \dot{\zeta}^i(t) \dot{\zeta}^j(t) \right) \\ &+ s^3 \left( \|\dot{\zeta}(t)\|_E^2 \mathcal{O}(\|\zeta(t)\|_E^3) \right). \end{aligned}$$

So for  $r$  small enough,  $\zeta(t)$  is uniformly small and for every  $t \in [0, T]$ , the function  $s \in [0, 1] \mapsto \|\partial_t \psi(s, t)\|_{g(\psi(s,t))}^2$  is increasing. We draw the same conclusion for the quantity (47) and proof is completed. ■

## B A brief Survey of the Orbit Theorem

In this Appendix, we aim to recall the statement of the Orbit Theorem. The material presented below is now considered as a classical part of geometric control theory.

Throughout this section,  $M$  is a real analytic manifold, and  $\mathcal{G}$  a set of analytic vector fields on  $M$ . We do not assume in general that the fields from  $\mathcal{G}$  are complete.

### B.1 Attainable sets

Let  $f$  be an element of  $\mathcal{G}$  and  $q^*$  be an element of  $M$ . The Cauchy problem

$$\dot{q} = f(q), \quad q(0) = q^*, \quad (48)$$

admits a solution defined on the open interval  $I(f, q^*)$  containing 0. For any real  $t$  in  $I(f, q^*)$  we denote the value of the solution of (48) at time  $t$  by  $e^{tf}(q^*)$ . We denote by  $I(f, q^*)^+ = I(f, q^*) \cap ]0, +\infty[$  the positive elements of  $I(f, q^*)$ .

For any element  $q_0$  in  $M$  and any positive real number  $T$ , we define the *attainable set at time  $T$*  of  $\mathcal{G}$  from  $q_0$  by the set  $\mathcal{A}_{q_0}(T)$  of all points of  $M$  that can be attained with  $\mathcal{G}$  using piecewise constants controls in time  $T$

$$\mathcal{A}_{q_0}(T) = \left\{ e^{t_p f_p} \circ e^{t_{p-1} f_{p-1}} \circ \dots \circ e^{t_1 f_1}(q_0) : p \in \mathbf{N}, f_i \in \mathcal{G}, \right. \\ \left. t_i \in I(f_i, e^{t_{i-1} f_{i-1}} \circ \dots \circ e^{t_1 f_1}(q_0))^+, t_1 + \dots + t_p = T \right\},$$

the times  $t_i$  and the fields  $f_i$  being chosen in such a way that every written quantity exists. We define also the *orbit* of  $\mathcal{G}$  through  $q_0$  by the set  $\mathcal{O}_{q_0}$  of all points of  $M$  that can be attained with  $\mathcal{G}$  using piecewise constant controls, *at any positive or negative time*

$$\mathcal{O}_{q_0}(T) = \left\{ e^{t_p f_p} \circ e^{t_{p-1} f_{p-1}} \circ \dots \circ e^{t_1 f_1}(q_0) : p \in \mathbf{N}, f_i \in \mathcal{G}, \right. \\ \left. t_i \in I(f_i, e^{t_{i-1} f_{i-1}} \circ \dots \circ e^{t_1 f_1}(q_0)) \right\}.$$

Of course, if  $\mathcal{G}$  is a cone, that is if  $\lambda f \in \mathcal{G}$  for any positive  $\lambda$  as soon as  $f$  belongs to  $\mathcal{G}$ , the set  $\mathcal{A}_{q_0}(T)$  does not depend on the positive  $T$  but only on  $q_0$ . If  $\mathcal{G}$  is assumed to be symmetric, that is if  $-f$  belongs to  $\mathcal{G}$  as soon as  $f$  belongs to  $\mathcal{G}$ , then the orbit of  $\mathcal{G}$  through a point  $q_0$  is the union of all attainable sets at positive time of  $\mathcal{G}$  from  $q_0$ .

### B.2 Lie algebra of vector fields

If  $f_1$  and  $f_2$  are two vector fields on  $M$  and  $q$  is a point of  $M$ , the *Lie bracket*  $[f_1, f_2](q)$  of  $f_1$  and  $f_2$  at a point  $q$  is the derivative at  $t = 0$  of the curve

$t \mapsto \gamma(\sqrt{t})$  where  $\gamma$  is defined by  $\gamma(t) := e^{-tf_2}e^{-tf_1}e^{tf_2}e^{tf_1}(q)$  for  $t$  small enough. The Lie bracket of  $f_1$  and  $f_2$  at a point  $q$  is an element of the tangent space  $T_qM$  of  $M$  at the point  $q$ . The Lie bracket is bilinear and skew-symmetric in  $f_1$  and  $f_2$ , and measures the non-commutativity of the fields  $f_1$  and  $f_2$  (see [2, Prop 2.6]).

**Proposition B.1** *For any  $f_1, f_2$  in  $\mathcal{G}$ , we have the equivalence:*

$$e^{t_1f_1}e^{t_2f_2} = e^{t_2f_2}e^{t_1f_1} \Leftrightarrow [f_1, f_2] = 0$$

for all times  $t_1$  and  $t_2$  (if any) for which the expressions written in the left hand side of the above equivalence make sense.

Lie brackets of vectors fields are easy to compute with the following formulas (see [2, Prop 1.3] and [2, Exercise 2.2]).

**Proposition B.2** *For any  $f_1, f_2$  in  $\mathcal{G}$ , for any  $q$  in  $M$ ,*

$$[f_1, f_2](q) = \frac{df_2}{dq}f_1(q) - \frac{df_1}{dq}f_2(q).$$

Further, we have the useful property:

**Proposition B.3** *Let  $f_1$  and  $f_2$  be two smooth vector fields on  $M$ , and let  $a, b : M \rightarrow \mathbb{R}$  be two smooth functions. Then*

$$[aX, bY] = ab[X, Y] + \left(\frac{db}{dq}X\right)Y - \left(\frac{da}{dq}Y\right)X.$$

From the Lie brackets, we can define the Lie algebra:

**Definition B.1** *The Lie algebra of  $\mathcal{G}$  is the linear span of all Lie brackets, of any length, of the elements of  $\mathcal{G}$*

$$\text{Lie } \mathcal{G} = \text{span}\{[f_1, [\dots [f_{k-1}, f_k] \dots]], k \in \mathbf{N}, f_i \in \mathcal{G}\},$$

which is a subset of all the vector fields on  $M$ .

We denote by  $\text{Lie}_q\mathcal{G} := \{g(q), g \in \text{Lie } \mathcal{G}\}$  the evaluation  $\text{Lie}_q\mathcal{G}$  of the Lie algebra generated by  $\mathcal{G}$  at a point  $q$  of  $M$ .

### B.3 The Orbit Theorem

The Orbit Theorem describes the differential structure of the orbit through a point (see for instance [2, Th 5.1] for a proof).

**Theorem B.4 (Orbit Theorem)** *For any  $q$  and  $q_0$  in  $M$ :*

1.  $\mathcal{O}(q_0)$  is a connected immersed submanifold of  $M$ .

2. If  $q \in \mathcal{O}(q_0)$ , then  $T_q\mathcal{O}(q_0) = \text{Lie}_q\mathcal{G}$ .

**Remark B.1** *The conclusion (1) of the Orbit Theorem holds true even if  $M$  and  $\mathcal{G}$  are only assumed to be smooth (and not analytic). The conclusion (2) is false in general when  $\mathcal{G}$  is only assumed to be smooth.*

The Orbit Theorem has many consequences, among them the following useful properties (see [2, Th 5.2] for a proof and further discussion).

**Theorem B.5 (Rashevsky-Chow)** *If  $\text{Lie}_q\mathcal{G} = T_qM$  for every  $q$  in  $M$ , then the orbit of  $\mathcal{G}$  through  $q$  is equal to  $M$ .*

**Proposition B.6** *If  $\mathcal{G}$  is a symmetric cone such that  $\text{Lie}_q\mathcal{G} = T_qM$  for every  $q$  in  $M$ , then the attainable set at any positive time of any point of  $M$  is equal to  $M$ .*