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MARTINGALE PROBLEMS FOR SOME DEGENERATE KOLMOGOROV EQUATIONS

STÉPHANE MENOZZI

ABSTRACT. We obtain Calderón-Zygmund estimates for some degenerate equations of Kolmogorov type with inhomogeneous coefficients. We then derive the well-posedness of the martingale problem associated to related degenerate operators, and therefore uniqueness in law for the corresponding stochastic differential equations. Some density estimates are established as well.

1. INTRODUCTION

1.1. Statement of the problem. Consider the following system of Stochastic Differential Equations (SDEs in short)

$$(1.1) \quad \begin{aligned} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + \sigma(t, X_t^1, \dots, X_t^n)dW_t, \\ dX_t^2 &= F_2(t, X_t^1, \dots, X_t^n)dt, \\ dX_t^3 &= F_3(t, X_t^2, \dots, X_t^n)dt, \\ &\dots \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt, \end{aligned} \quad t \geq 0,$$

$(W_t)_{t \geq 0}$ standing for a d -dimensional Brownian motion, and each $(X_t^i)_{t \geq 0}$, $i \in \llbracket 1, n \rrbracket$, being \mathbb{R}^d -valued as well.

From the applicative viewpoint, systems of type (1.1) appear in many fields. Let us for instance mention for $n = 2$ stochastic Hamiltonian systems (see e.g. Soize [Soi94] for a general overview or Talay [Tal02] and Hérau and Nier [HN04] for convergence to equilibrium). Again for $n = 2$, the above dynamics is used in mathematical finance to price Asian options (see for example [BPV01]). For $n \geq 2$, it appears in heat conduction models (see e.g. Eckmann et al. [EPRB99] and Rey-Bellet and Thomas [RBT00] when the chain of differential equations is forced by two heat baths).

Under suitable assumptions on the coefficients, namely mild smoothness assumptions of the coefficients $(F_i)_{i \in \llbracket 1, n \rrbracket}$, σ , non degeneracy of the bounded diffusion matrix $a(t, \cdot) := \sigma \sigma^*(t, \cdot)$ and the gradients $(D_{x_{i-1}} F_i(t, \cdot))_{i \in \llbracket 2, n \rrbracket}$ (weak Hörmander condition), some multi-scale Gaussian Aronson like estimates have been proved in [DM10] for the density of (1.1) uniformly on the time set $(0, T]$, for fixed $T > 0$ (see Theorem 1.1 of that reference). For Hölder continuous diffusion coefficient $a(t, \cdot)$ and gradients $(D_{x_{i-1}} F_i(t, \cdot))_{i \in \llbracket 2, n \rrbracket}$ uniqueness in law has been investigated through the well posedness of the martingale problem in [Men11].

Anyhow, when studying the martingale problem, the natural framework is to consider non-degenerate continuous coefficients. In the special case $n = 1$ if $a(t, \cdot)$

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is bounded and uniformly elliptic, i.e. (1.1) corresponds to a non-degenerate SDE, it is well known that the martingale problem associated to the generator $(L_t)_{t \geq 0}$ of (1.1) is well posed as soon as the coefficient F_1 is bounded measurable and that $a(t, \cdot)$ is continuous in space, see e.g. Stroock and Varadhan [SV79]. The key ingredient consists in proving Calderón-Zygmund estimates, that write in that framework as controls in L^p norms, $p \in (1, +\infty)$, of the second order derivatives of a suitable Gaussian convolution solving the parabolic Cauchy problem with L^p source term for an operator with constant coefficients when $p := p(d)$ is large enough. These L^p controls then allow to derive a unique solution, for the parabolic Cauchy problem with L^p source term involving $(L_t)_{t \geq 0}$, by a perturbative argument (appropriate operator inversion) when the coefficients of $(L_t)_{t \geq 0}$ do not vary much. This procedure is recalled in Section 3 below. As a by product, this gives the uniqueness to the martingale problem, up to a suitable localization argument which is needed to extend the result for an operator whose coefficients are not *almost constant*. We refer to the monographs of Stein [Ste70] or Gilbarg and Trudinger [GT83] for a presentation of the Calderón-Zygmund theory for non-degenerate elliptic equations. In that framework a more probabilistic approach is proposed in Bass [Bas95]. We also mention the monograph of Coifman and Weiss [CW71], from which some Calderón-Zygmund estimates can be derived in some degenerate frameworks when there is an underlying *homogeneous space*.

In this work, for $n > 1$, under the previous assumptions of non-degeneracy and continuity on a and weak Hörmander condition of the $(D_{\mathbf{x}_{i-1}} F_i(t, \cdot))_{i \in \llbracket 2, n \rrbracket}$, we are interested in proving the well-posedness of the martingale problem for the generator $(L_t)_{t \geq 0}$ of (1.1).

To achieve this goal, we will establish some Calderón-Zygmund estimates for a suitable singular Gaussian kernel deriving from the degenerate system (1.1), already used in [DM10], [Men11], and reproduce the previously described arguments. This procedure will also lead to some density estimates for the unique weak solution of (1.1).

The main difficulty to derive the Calderón-Zygmund estimates consists in choosing a good parabolic metric associated to the system and in defining suitable metric balls that allow to average the singularities of the kernel. The parabolic metric will be chosen in order to reflect the multi-scale behavior of system (1.1) when a weak Hörmander like hypothesis is assumed to hold. Namely, if the $(D_{\mathbf{x}_{i-1}} F_i(t, \cdot))_{i \in \llbracket 2, n \rrbracket}$ are non-degenerate, the intrinsic time scale of the i^{th} component is $(2i - 1)/2$, $i \in \llbracket 1, n \rrbracket$ (characteristic time scale of the i^{th} iterated time integral of the Brownian motion). It will be seen in Section 3.1 that the geometry of the balls will be determined by the indicated multi-scale aspect and the transport of the spatial center by the deterministic differential system deriving from (1.1), that is the indicated equation with $\sigma = 0$.

Let us now mention some related works. In particular the papers by Bramanti *et al.* [BCLP10], [BCLP13]. They consider an operator

$$\mathcal{A} := \sum_{i,j=1}^{p_0} a_{ij} \partial_{x_i} \partial_{x_j} + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j},$$

where the matrices $(a_{ij})_{(i,j) \in \llbracket 1, p_0 \rrbracket^2}$ are symmetric positive definite, constant in [BCLP10], with continuous variable homogeneous coefficients in [BCLP13], and

the $(b_{ij})_{(i,j) \in [1,N]^2}$ are s.t. \mathcal{A} is hypoelliptic. The authors then establish global L^p estimates $p \in (1, +\infty)$ of the following type: $\exists c := c(a, b, p_0, N, p)$, $\forall u \in C_0^2(\mathbb{R}^N)$,

$$\|\partial_{x_i x_j} u\|_{L^p(\mathbb{R}^N)} \leq c\{\|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}\}, \quad (i, j) \in [1, p_0]^2.$$

Weak (1-1) estimates are also obtained. Observe that for $p_0 < N$, the operator \mathcal{A} is a particular case of generator associated to (1.1). The strategy in those works consists in estimating suitable singular integrals related to the fundamental solution of $L = \mathcal{A} - \partial_t$. To this end, a specific local quasi-distance related to a homogeneous norm associated to the “principal part of L ” is used; Here, by principal part we mean the operator formed taking only the drift coefficients of \mathcal{A} that guarantee the associated parabolic operator is homogeneous w.r.t. a family of dilations¹ and also left-invariant w.r.t. a suitable Lie group. The quasi-distance is *local* in the sense that it satisfies the quasi-triangle inequality on metric balls, i.e. compact sets, the associated constant depending on the radius of the ball (see Section 6.1). Such singular integrals involving that type of local quasi-distance have been investigated by Bramanti [Bra10]. The local quasi-distance in [BCLP10] (see Section 2 therein), associated to the particular structure of the degeneracy, is very close to the one we use for our analysis, see Section 4. Let us also mention that for variable homogeneous coefficients (namely in VMO_{loc} (resp. C^α) w.r.t. the distance induced by the vector fields), local L^p (resp. Schauder) estimates have been obtained by Bramanti and Zhu [BZ11] following the same lines. Concerning the link between the L^p estimates of [BCLP13] and the well-posedness of the martingale problem, we can refer to the recent work of Priola [Pri13] who introduces a rather general localization procedure that allows to extend the well posedness of the martingale problem from the case of *almost constant* coefficients to the natural one of continuous coefficients. The novelty of [Pri13], w.r.t. to the classic localization results of Stroock and Varadhan (see e.g. Chapter 6.6 in [SV79]), being the handling of unbounded coefficients.

The main novelty in our approach consists in considering inhomogeneous and non-linear coefficients for the degenerate part of equation (1.1). To this end we introduce a suitable kernel and do not exploit some underlying Lie group properties appearing in the quoted works. The key idea consists in viewing (1.1) as an ODE *perturbed* by a noise. This naturally yields to consider balls that are built around the characteristic lines of the ODE and reflect the multi-scale behavior of the process, where the various scales are once again those of the Brownian motion and its iterated integrals. This approach also allows, through a suitable localization procedure, to establish density estimates in L^q spaces.

The article is organized as follows. We state our assumptions and main results in Section 2. We then introduce in Section 3 the degenerate Gaussian kernel for which we establish Calderón-Zygmund estimates, recalling formally how uniqueness can be derived from these controls when the coefficients do not vary much. In Section 4 we specify the various steps that lead to the Calderón-Zygmund estimates of Theorem 3.1. We then perform in Section 5 a localization procedure and give some local and global controls on the density from the previous estimates. This requires some careful adaptations of the arguments of the non-degenerate framework, see

¹The time-homogeneity of the dilations is actually the same than the one associated to iterated integrals of the Brownian motion.

e.g. Sections 7.1, 7.2 and 9.1 in [SV79]). Section 6 is the technical core of the paper and is devoted to the proof of the technical results of Section 4.

2. ASSUMPTIONS AND MAIN RESULTS

2.1. Notations and Assumptions. In what follows, we denote a quantity in \mathbb{R}^{nd} by a bold letter: i.e. $\mathbf{0}$, stands for zero in \mathbb{R}^{nd} and the solution $(X_t^1, \dots, X_t^n)_{t \geq 0}$ to (1.1) is denoted by $(\mathbf{X}_t)_{t \geq 0}$. Introducing the embedding matrix B from \mathbb{R}^d into \mathbb{R}^{nd} , i.e. $B = (I_d, 0, \dots, 0)^*$, where “ $*$ ” stands for the transpose, we rewrite (1.1) in the shortened form

$$d\mathbf{X}_t = \mathbf{F}(t, \mathbf{X}_t)dt + B\sigma(t, \mathbf{X}_t)dW_t,$$

where $\mathbf{F} = (F_1, \dots, F_n)$ is an \mathbb{R}^{nd} -valued function.

With these notations the generator of (1.1) writes for all $t \geq 0$:

(2.1)

$$\forall \varphi \in C_0^2(\mathbb{R}^{nd}), \forall \mathbf{x} \in \mathbb{R}^{nd}, L_t \varphi(\mathbf{x}) = \langle \mathbf{F}(t, \mathbf{x}), \mathbf{D}_{\mathbf{x}} \varphi(\mathbf{x}) \rangle + \frac{1}{2} \text{tr}(a(t, \mathbf{x}) D_{\mathbf{x}_1}^2 \varphi(\mathbf{x})).$$

Also, for a point $\mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{R}^{nd}$, we will often denote for all $i \in \llbracket 1, n-1 \rrbracket$, $\mathbf{x}^{i,n} := (\mathbf{x}_i, \dots, \mathbf{x}_n)$.

Let us now introduce some assumptions concerning the coefficients of (1.1).

(C) The diffusion coefficient $(a(t, \cdot))_{t \geq 0}$ is bounded measurable and continuous in space, i.e.

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \sup_{0 \leq s \leq T} |a(s, \mathbf{y}) - a(s, \mathbf{x})| = 0$$

for all $T > 0$ and $\mathbf{x} \in \mathbb{R}^{nd}$.

(UE) There exists $\Lambda \geq 1$, $\forall t \geq 0, \mathbf{x} \in \mathbb{R}^{nd}, \xi \in \mathbb{R}^d, \Lambda^{-1}|\xi|^2 \leq \langle a(t, \mathbf{x})\xi, \xi \rangle \leq \Lambda|\xi|^2$.

(ND) The functions $(F_i)_{i \in \llbracket 1, n \rrbracket}$ are once continuously differentiable and uniformly Lipschitz continuous in space (with constant $\kappa > 0$), bounded measurable in time. Also, there exists a closed convex subset $\mathcal{E}_{i-1} \subset GL_d(\mathbb{R})$ (set of invertible $d \times d$ matrices) s.t., for all $t \geq 0$ and $(\mathbf{x}_{i-1}, \dots, \mathbf{x}_n) \in \mathbb{R}^{(n-i+2)d}$, the matrix $D_{\mathbf{x}_{i-1}} F_i(t, \mathbf{x}_{i-1}, \dots, \mathbf{x}_n)$ belongs to \mathcal{E}_{i-1} . For example, $\mathcal{E}_i, i \in \llbracket 1, n-1 \rrbracket$, may be a closed ball included in $GL_d(\mathbb{R})$, which is an open set.

Assumptions **(UE)**, **(ND)** can be seen as a kind of (weak) Hörmander condition. They allow to transmit the non degenerate noise of the first component to the other ones. Let us also recall that the last part of Assumption **(ND)** and the particular structure of $\mathbf{F}(t, \cdot) = (F_1(t, \cdot), \dots, F_n(t, \cdot))$ yield that the i^{th} component of the system (1.1) has intrinsic time scale $(2i-1)/2, i \in \llbracket 1, n \rrbracket$. This fact will be thoroughly used in our analysis (see Section 3 for details). We notice that the coefficients may be irregular in time. We say that assumption **(A_{SD})** is satisfied if $F_1(t, \mathbf{x}) := F_1(t, \mathbf{x}_1), \forall i \in \llbracket 2, n \rrbracket, F_i(t, \mathbf{x}) := F_i(t, \mathbf{x}_{i-1}, \mathbf{x}_i)$, assumptions **(C)**, **(UE)**, **(ND)** hold and

(S_{SD}) For each integer $i \in \llbracket 2, n \rrbracket, (t, \mathbf{x}_i) \in \mathbb{R}_+ \times \mathbb{R}^d$, the function $\mathbf{x}_{i-1} \in \mathbb{R}^d \mapsto D_{\mathbf{x}_{i-1}} F_i(t, \mathbf{x}_{i-1}, \mathbf{x}_i)$ is uniformly η -Hölder continuous with constant κ ($\eta \in (0, 1]$).

Precisely, under **(A_{SD})** the dynamics of the deterministic differential system associated to (1.1) is *subdiagonal*. Under this assumption, the local quasi-distance we introduce below is actually a usual quasi-distance in the sense of Coifman and Weiss, see Chapter 3 in [CW71]. This could allow to derive the Calderón-Zygmund estimates rather directly. The problem is that we unfortunately did not succeed in establishing the required L^2 estimate on our kernel without some additional

smoothness on the drift term \mathbf{F} . In the special case where \mathbf{F} is inhomogeneous and linear, we obtained the required estimate thanks to Theorem 2.4 in [CW71] (see Sections 4 and 6 for details).

In whole generality, we say that (\mathbf{A}_G) holds if (\mathbf{C}) , (\mathbf{UE}) , (\mathbf{ND}) are satisfied and

(\mathbf{S}_G) The $(F_i)_{i \in \llbracket 1, n \rrbracket}$ and the $(D_{\mathbf{x}_{i-1}} F_i)_{i \in \llbracket 1, n \rrbracket}$ are twice continuously differentiable with bounded derivatives.

This allows to consider a *general* non linear dynamics of the form $F_i(t, \mathbf{x}) := F_i(t, \mathbf{x}^{i-1, n})$ in (1.1). In such case, we need to localize the singularities of the kernel and to adapt the techniques of [BCLP10], which requires some additional smoothness on the coefficient F since the symmetrized kernel is involved. We say that (\mathbf{A}) is in force whenever (\mathbf{A}_{SD}) or (\mathbf{A}_G) holds true. Under (\mathbf{A}_G) , the variables η appearing in the results below should be set to 1.

2.2. Main Results. Our main result is the following theorem.

Theorem 2.1. Under (\mathbf{A}_G) the martingale problem associated to $(L_t)_{t \geq 0}$ in (2.1) is well-posed. That is, for every $\mathbf{x} \in \mathbb{R}^{nd}$, there exists a unique probability measure \mathbb{P} on $C(\mathbb{R}^+, \mathbb{R}^{nd})$ s.t. denoting by $(\mathbf{X}_t)_{t \geq 0}$ the canonical process, $\mathbb{P}[\mathbf{X}_0 = \mathbf{x}] = 1$ and for all $\varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^{nd}, \mathbb{R})$, $\varphi(t, \mathbf{X}_t) - \varphi(0, \mathbf{x}) - \int_0^t (\partial_s + L_s)\varphi(s, \mathbf{X}_s) ds$ is a \mathbb{P} -martingale. In particular, weak uniqueness in law holds for the SDE (1.1).

Also, if the diffusion coefficient a is uniformly continuous, the unique weak solution of (1.1) admits a density in the following sense. Letting $P(s, t, \mathbf{x}, \cdot)$ be the transition probability determined by $(L_t)_{t \geq 0}$, then for a given $T > 0$, almost all $t \in (s, T]$ and all $\Gamma \in \mathcal{B}(\mathbb{R}^{nd})$, $P(s, t, \mathbf{x}, \Gamma) = \int_{\Gamma} p(s, t, \mathbf{x}, \mathbf{y}) d\mathbf{y}$.

More specifically, for any $f \in L^p([0, T] \times \mathbb{R}^{nd})$, $p > \frac{(n^2 d + 2)}{2}$, there exists $C_{2.2} := C_{2.2}(T, p, (\mathbf{A}_G))$ s.t. for all $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$:

$$(2.2) \quad |\mathbb{E}_{s, \mathbf{x}}^{\mathbb{P}}[\int_s^T f(t, X_t) dt]| \leq C_{2.2}(1 + |\mathbf{x}|) \|f\|_{L^p([0, T] \times \mathbb{R}^{nd})}.$$

Remark 2.1. By duality, the previous control gives a bound for the density in $L^q([0, T] \times \mathbb{R}^{nd})$ where $q^{-1} + p^{-1} = 1$.

3. “FROZEN” KERNEL AND FORMAL DERIVATION OF UNIQUENESS FROM CALDERÓN-ZYGMUND ESTIMATES

Assume (\mathbf{A}) is in force. One of the main differences between the uniform Hölder continuity assumed in [BP09] in the non degenerate case or in [Men11] for the current framework and the continuity statement of (\mathbf{C}) is that in the first two cases no localization is needed. Indeed, the global Hölder continuity allows to remove globally the time singularities coming from the second order spatial derivatives of suitable Gaussian kernels arising in a parametrix like expansion of the density. In the current framework we first focus on the “local case”. As in the non-degenerate case, we assume the diffusion coefficient $a(t, \cdot) := \sigma \sigma^*(t, \cdot)$ of (1.1) “does not vary much” (see e.g. Chapter 7 of [SV79]). For technical reasons (see the proofs of Lemmas 4.2 and 6.1), we also suppose that the gradients $(D_{\mathbf{x}_{i-1}} F_i(t, \cdot))_{i \in \llbracket 2, n \rrbracket}$ do not vary much as well, that is the degenerate components are “almost affine” w.r.t the components that transmit the noise. Precisely, we first assume that there exists

$\mathbf{x}_0 \in \mathbb{R}^{nd}$ such that:

$$\begin{aligned} \varepsilon_a &:= \sup_{0 \leq t \leq T} \sup_{\mathbf{x} \in \mathbb{R}^{nd}} |a(t, \mathbf{x}) - a(t, \mathbf{x}_0)|, \\ \varepsilon_{\mathbf{F}} &:= \sup_{i \in \llbracket 2, n \rrbracket} \sup_{0 \leq t \leq T} \sup_{(\mathbf{x}, \mathbf{z}) \in (\mathbb{R}^{nd})^2} |D_{\mathbf{x}_{i-1}} F_i(t, \mathbf{x}_{i-1}, \mathbf{z}^{i,n}) - D_{\mathbf{x}_{i-1}} F_i(t, (\mathbf{x}_0)_{i-1}, \mathbf{z}^{i,n})| \end{aligned} \quad (3.1)$$

are small, without supposing *a priori* continuity of a .

To define the Gaussian kernel needed for the analysis we first introduce the backward deterministic differential system associated to (1.1). For fixed $T > 0$, $\mathbf{y} \in \mathbb{R}^{nd}$ we define:

$$(3.2) \quad \dot{\boldsymbol{\theta}}_{t,T}(\mathbf{y}) = \mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})), \quad \boldsymbol{\theta}_{T,T}(\mathbf{y}) = \mathbf{y}.$$

Consider now the deterministic ODE

$$(3.3) \quad \frac{d}{dt} \tilde{\boldsymbol{\phi}}_t = \mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})) + D\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))[\tilde{\boldsymbol{\phi}}_t - \boldsymbol{\theta}_{t,T}(\mathbf{y})], \quad t \geq 0,$$

where for all $\mathbf{x} \in \mathbb{R}^{nd}$,

$$D\mathbf{F}(t, \mathbf{x}) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ D_{\mathbf{x}_1} \mathbf{F}_2(t, \mathbf{x}) & 0 & \cdots & \cdots & 0 \\ 0 & D_{\mathbf{x}_2} \mathbf{F}_3(t, \mathbf{x}) & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots & \\ 0 & \cdots & 0 & D_{\mathbf{x}_{n-1}} \mathbf{F}_n(t, \mathbf{x}) & 0 \end{pmatrix}$$

denotes the subdiagonal of the Jacobian matrix $D_{\mathbf{x}} \mathbf{F}$ at point \mathbf{x} .

We denote by $(\tilde{\boldsymbol{\theta}}_{t,s}^{T,\mathbf{y}})_{s,t \geq 0}$ the associated flow, i.e. $\tilde{\boldsymbol{\theta}}_{t,s}^{T,\mathbf{y}}(\mathbf{x})$ is the value of $\tilde{\boldsymbol{\phi}}_t$ when $\tilde{\boldsymbol{\phi}}_s = \mathbf{x}$. It is affine:

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_{t,s}^{T,\mathbf{y}}(\mathbf{x}) &= \tilde{\mathbf{R}}^{T,\mathbf{y}}(t, s) \mathbf{x} \\ &+ \int_s^t \tilde{\mathbf{R}}^{T,\mathbf{y}}(t, u) (\mathbf{F}(u, \boldsymbol{\theta}_{u,T}(\mathbf{y})) - D\mathbf{F}(u, \boldsymbol{\theta}_{u,T}(\mathbf{y})) \boldsymbol{\theta}_{u,T}(\mathbf{y})) du \\ &:= \tilde{\mathbf{R}}^{T,\mathbf{y}}(t, s) \mathbf{x} + \mathbf{m}^{T,\mathbf{y}}(s, t). \end{aligned} \quad (3.4)$$

Above, $(\tilde{\mathbf{R}}^{T,\mathbf{y}}(t, s))_{s,t \geq 0}$ stands for the resolvent associated with the subdiagonal matrices $(D\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})))_{t \geq 0}$.

We now introduce for all $0 \leq s < t$, $(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{nd})^2$ the kernel:

$$\begin{aligned} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) &:= \frac{1}{(2\pi)^{nd/2} (\tilde{\mathbf{K}}^{\mathbf{y}}(s, t))^{1/2}} \\ &\times \exp \left(-\frac{1}{2} \langle \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right), \end{aligned} \quad (3.5)$$

where $\tilde{\mathbf{K}}^{\mathbf{y}}(s, t) := \int_s^t \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, u) B a(u, \mathbf{x}_0) B^* \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, u)^* du$. In other words, under (\mathbf{A}) , $\tilde{q}(s, t, \mathbf{x}, \mathbf{y})$ is the density at time t and point \mathbf{y} of the diffusion $(\tilde{\mathbf{X}}_u^{t,\mathbf{y}})_{u \in [s,t]}$ with dynamics:

$$\begin{aligned} d\tilde{\mathbf{X}}_u^{t,\mathbf{y}} &= [\mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y})) + D\mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}))(\tilde{\mathbf{X}}_u^{t,\mathbf{y}} - \boldsymbol{\theta}_{u,t}(\mathbf{y}))] du + B\sigma(u, \mathbf{x}_0) dW_u, \\ \forall u \in [s, t], \quad \tilde{\mathbf{X}}_s^{t,\mathbf{y}} &= \mathbf{x}. \end{aligned} \quad (3.6)$$

Assumption **(A)** also guarantees that the covariance matrix $(\tilde{\mathbf{K}}^{\mathbf{y}}(s, t))_{0 \leq s < t}$ satisfies *uniformly* in $\mathbf{y} \in \mathbb{R}^{nd}$ a *good scaling property* in the sense of Definition 3.2 in [DM10] (see also Proposition 3.4 of that reference). That is: for all fixed $T > 0$, there exists $C_{3.7} := C_{3.7}(T, (\mathbf{A})) \geq 1$ s.t. for all $0 \leq s < t \leq T$, for all $\mathbf{y} \in \mathbb{R}^{nd}$:

$$(3.7) \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{nd}, C_{3.7}^{-1}(t-s)^{-1} |\mathbb{T}_{t-s} \boldsymbol{\xi}|^2 \leq \langle \tilde{\mathbf{K}}^{\mathbf{y}}(s, t) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \leq C_{3.7}(t-s)^{-1} |\mathbb{T}_{t-s} \boldsymbol{\xi}|^2,$$

where for all $t > 0$, $\mathbb{T}_t = \text{diag}((t^i I_d)_{i \in \llbracket 1, n \rrbracket})$ is a scale matrix. As pointed out in the previous section, equation (3.7) indicates that the i^{th} component of (3.6) has characteristic time scale of order $(2i - 1)/2$.

From (3.5) and (3.7), we directly derive that for all $T > 0$ there exists $C_{3.8} := C_{3.8}(T, (\mathbf{A})) \geq 1$ s.t. for all $0 \leq s < t \leq T$, $(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{nd})^2$:

$$(3.8) \quad \begin{aligned} C_{3.8}^{-1}(t-s)^{-n^2 d/2} \exp(-C_{3.8}(t-s) |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2) &\leq \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) \\ &\leq C_{3.8}(t-s)^{-n^2 d/2} \exp(-C_{3.8}^{-1}(t-s) |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2). \end{aligned}$$

Also, Lemma 5.3 and Equation (5.11) from the proof of Lemma 5.5 in [DM10] give that there exists $C := C(T, (\mathbf{A}))$ s.t.:

$$(3.9) \quad \begin{aligned} C^{-1} |\mathbb{T}_{t-s}^{-1}(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))| &\leq |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| \leq C |\mathbb{T}_{t-s}^{-1}(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))|, \\ |D_{\mathbf{x}_j} \tilde{q}(s, t, \mathbf{x}, \mathbf{y})| &\leq C(t-s)^{-j+1} |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| \tilde{q}(s, t, \mathbf{x}, \mathbf{y}), \quad j \in \llbracket 1, n \rrbracket. \end{aligned}$$

On the other hand it is crucial to observe that $\tilde{q}(s, t, \mathbf{x}, \mathbf{y})$ satisfies the following Backward Kolmogorov equation for all $(t, \mathbf{y}) \in \mathbb{R}^{+*} \times \mathbb{R}^{nd}$:

$$(3.10) \quad \left(\partial_s + \tilde{L}_s^{t,\mathbf{y}} \right) \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) = 0, \quad (s, \mathbf{x}) \in [0, t) \times \mathbb{R}^{nd}, \quad \tilde{q}(s, t, \cdot, \mathbf{y}) \xrightarrow{s \uparrow t} \delta_{\mathbf{y}}(\cdot).$$

In the above equation we wrote:

$$\begin{aligned} \tilde{L}_s^{t,\mathbf{y}} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) &:= \langle \mathbf{F}(s, \boldsymbol{\theta}_{s,t}(\mathbf{y})) + D\mathbf{F}(s, \boldsymbol{\theta}_{s,t}(\mathbf{y}))(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y})), D_{\mathbf{x}} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) \rangle \\ &\quad + \frac{1}{2} \text{tr}(a(s, \mathbf{x}_0) D_{\mathbf{x}_1}^2 \tilde{q}(s, t, \mathbf{x}, \mathbf{y})). \end{aligned}$$

For $0 \leq s < T$ and a function $f \in C_0^\infty([0, T) \times \mathbb{R}^{nd})$ we now define for all $\mathbf{x} \in \mathbb{R}^{nd}$:

$$(3.11) \quad \tilde{G}f(s, \mathbf{x}) := \int_s^T dt \int_{\mathbb{R}^{nd}} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y}) d\mathbf{y}.$$

From (3.10) one easily gets that

$$\partial_s \tilde{G}f(s, \mathbf{x}) + \tilde{M}f(s, \mathbf{x}) = -f(s, \mathbf{x}), \quad (s, \mathbf{x}) \in [0, T) \times \mathbb{R}^{nd}, \quad \tilde{G}f(s, \cdot) \xrightarrow{s \uparrow T} 0,$$

with $\tilde{M}f(s, \mathbf{x}) := \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} \tilde{L}_s^{t,\mathbf{y}} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y})$.

Hence,

$$\partial_s \tilde{G}f(s, \mathbf{x}) + L_s \tilde{G}f(s, \mathbf{x}) = (-f + Rf)(s, \mathbf{x}), \quad (s, \mathbf{x}) \in [0, T) \times \mathbb{R}^{nd},$$

where $Rf(s, \mathbf{x}) := (L_s \tilde{G}f - \tilde{M}f)(s, \mathbf{x}) = \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} (L_s - \tilde{L}_s^{t,\mathbf{y}}) \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y})$.

Now, the local condition (3.1) yields:

$$(3.12) \quad |Rf(s, \mathbf{x})| \leq |Nf(s, \mathbf{x})| + \sum_{i=2}^n |D_{\mathbf{x}_i} R_i f(s, \mathbf{x})| + \frac{\varepsilon_a}{2} |D_{\mathbf{x}_1}^2 \tilde{G}f(s, \mathbf{x})|,$$

where setting

$$\begin{aligned} \mathbf{F}^{t,\mathbf{y}}(s, \mathbf{x}) &:= (\mathbf{F}_1(s, \boldsymbol{\theta}_{s,t}(\mathbf{y})), \mathbf{F}_2(s, \mathbf{x}_1, (\boldsymbol{\theta}_{s,t}(\mathbf{y}))^{2,n}), \mathbf{F}_3(s, \mathbf{x}_2, (\boldsymbol{\theta}_{s,t}(\mathbf{y}))^{3,n}), \dots, \\ &\quad \mathbf{F}_n(s, \mathbf{x}_{n-1}, (\boldsymbol{\theta}_{s,t}(\mathbf{y}))_n)), \\ Nf(s, \mathbf{x}) &:= \sum_{i=1}^n \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} \langle ((\mathbf{F} - \mathbf{F}^{t,\mathbf{y}})(s, \mathbf{x}))_i, \mathbf{D}_{\mathbf{x}_i} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) \rangle f(t, \mathbf{y}), \end{aligned} \quad (3.13)$$

and for all $i \in \llbracket 2, n \rrbracket$,

$$\begin{aligned} R_i f(s, \mathbf{x}) &:= \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y}) \{ \mathbf{F}_i^{t,\mathbf{y}}(s, \mathbf{x}) \\ &\quad - [\mathbf{F}_i(s, \boldsymbol{\theta}_{s,t}(\mathbf{y})) + D_{\mathbf{x}_{i-1}} \mathbf{F}_i(s, \boldsymbol{\theta}_{s,t}(\mathbf{y}))(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_{i-1}] \}. \end{aligned} \quad (3.14)$$

Remark 3.1. Observe from the above equation that if, for all $i \in \llbracket 2, n \rrbracket$, the function \mathbf{F}_i is linear w.r.t. to the $(i-1)^{\text{th}}$ variable (component that transmits the noise), then for all $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$, $R_i f(s, \mathbf{x}) = 0$.

The terms Nf in (3.13) and $(D_{\mathbf{x}_i} R_i f)_{i \in \llbracket 2, n \rrbracket}$ in (3.14) do not have time singularities. Let us justify this point. Write first:

$$D_{\mathbf{x}_j} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) = -[(\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s))^* (\tilde{\mathbf{K}}^{\mathbf{y}}(s, t))^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})]_j \tilde{q}(s, t, \mathbf{x}, \mathbf{y}), \quad j \in \llbracket 1, n \rrbracket.$$

Using (3.8), (3.9), we then derive from (3.13) that:

$$\begin{aligned} |Nf(s, \mathbf{x})| &\leq C \sum_{i=1}^n \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} |(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))^{i,n}| |D_{\mathbf{x}_i} \tilde{q}(s, t, \mathbf{x}, \mathbf{y})| |f(t, \mathbf{y})| \\ &\leq C \sum_{i=1}^n \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} \left\{ \frac{|(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))^{i,n}|}{(t-s)^{(2i-1)/2}} \right\} (t-s)^{1/2} |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) |f(t, \mathbf{y})| \\ &\leq C \int_s^T dt \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(t-s)^{n^2 d/2}} \exp(-C^{-1}(t-s) |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2) |f(t, \mathbf{y})|, \quad C := C((\mathbf{A})). \end{aligned}$$

Now, as a consequence of Hölder's inequality we derive that for all $p > 1$, $p^{-1} + q^{-1} = 1$:

$$\begin{aligned} |Nf(s, \mathbf{x})|^p &\leq C(p, (\mathbf{A})) T^{p/q} \int_s^T dt \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(t-s)^{n^2 d/2}} \\ &\quad \exp(-C^{-1}(t-s) |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2) |f(t, \mathbf{y})|^p. \end{aligned}$$

The Fubini Theorem and (3.9) then yields:

$$(3.15) \quad \|Nf\|_{L^p([0, T] \times \mathbb{R}^{nd})} \leq C(p, (\mathbf{A})) T \|f\|_{L^p([0, T] \times \mathbb{R}^{nd})}.$$

From (3.8), (3.9) we also derive for all $i \in \llbracket 2, n \rrbracket$:

$$\begin{aligned} |D_{\mathbf{x}_i} R_i f(s, \mathbf{x})| &\leq C \int_s^T dt \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(t-s)^{n^2 d/2}} (t-s)^{-i+1/2} |(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_{i-1}|^{1+\eta} \\ &\quad \times \exp\left(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2\right) |f(t, \mathbf{y})| \\ &\leq C \int_s^T dt (t-s)^{-1+(i-\frac{3}{2})\eta} \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(t-s)^{n^2 d/2}} |f(t, \mathbf{y})| \\ &\quad \times \exp\left(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2\right). \end{aligned}$$

Hence, by Hölder's inequality and for $p > 2/\eta$, $p^{-1} + q^{-1} = 1$:

$$\begin{aligned} |D_{\mathbf{x}_i} R_i f(s, \mathbf{x})|^p &\leq C(p, (\mathbf{A})) \left(\int_s^T dt (t-s)^{(-1+(i-\frac{3}{2})\eta)q} \right)^{p/q} \\ &\times \left(\int_s^T dt \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(t-s)^{n^2 d/2}} \exp\left(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2\right) |f(t, \mathbf{y})|^p \right), \\ (3.16) \quad \|D_{\mathbf{x}_i} R_i f\|_{L^p([0,T] \times \mathbb{R}^{nd})} &\leq C(p, (\mathbf{A})) (T^{\eta/2} \vee T^{(n-3/2)\eta}) \|f\|_{L^p([0,T] \times \mathbb{R}^{nd})}, \end{aligned}$$

where the last control again follows from Fubini's theorem.

Now, the key tool to prove uniqueness for the martingale problem derives from the following Calderón and Zygmund type estimate for the Green function $\tilde{G}f$. Namely, we have the following theorem.

Theorem 3.1. Assume that Assumption (\mathbf{A}_G) is in force. Suppose also that the local condition (3.1) holds for \mathbf{F} , i.e. $\varepsilon_{\mathbf{F}} \leq \varepsilon_0(n, d)$ for a small enough $\varepsilon_0(n, d)$, and that $T \in (0, T_0]$, $T_0 := T_0((\mathbf{A}_G)) \leq 1$. Then, for all $p \in (1, +\infty)$ there exists $C_{3.17} := C_{3.17}(T_0, p, (\mathbf{A}_G))$ s.t. for all $f \in L^p([0, T] \times \mathbb{R}^{nd})$,

$$(3.17) \quad \|D_{\mathbf{x}_1}^2 \tilde{G}f\|_{L^p([0,T] \times \mathbb{R}^{nd})} \leq C_{3.17} \|f\|_{L^p([0,T] \times \mathbb{R}^{nd})},$$

where the Green function $\tilde{G}f$ is defined in (3.11) with the kernel \tilde{q} introduced in (3.5).

Remark 3.2. Let us specify here that the *small time* condition appearing here is due to the fact that we are led to compare the flow $\boldsymbol{\theta}$ and its *linearization*. It is clear that this procedure can be a good approximation in small time only.

Hence, plugging (3.17), (3.16) and (3.15) into (3.12) we derive that under $(\mathbf{A}_G)^2$, for $p > 2$,

$$(3.18) \quad \|Rf\|_{L^p([0,T] \times \mathbb{R}^{nd})} \leq \left(\frac{\varepsilon_a}{2} C_{3.17} + C(p, (\mathbf{A})) T^{\frac{1}{2}}\right) \|f\|_{L^p([0,T] \times \mathbb{R}^{nd})}.$$

Thus, for $\varepsilon_a < C_{3.17}^{-1}$ and $T < (4C(p, (\mathbf{A})))^{-2}$, the operator $I - R$ admits a bounded inverse on $L^p([0, T] \times \mathbb{R}^{nd})$, and formally $Gf(s, \mathbf{x}) := \tilde{G} \circ (I - R)^{-1} f(s, \mathbf{x})$, $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$ solves the Cauchy problem:

$$\begin{cases} (\partial_t + L_t)u(t, \mathbf{x}) = -f(t, \mathbf{x}), & (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}, \\ u(T, \mathbf{x}) = 0, \end{cases}$$

²recall indeed that under (\mathbf{A}_G) the parameter η must be set to 1.

for $f \in L^p([0, T] \times \mathbb{R}^{nd})$, $p > (n^2d+2)/2 \vee 2/\eta$. This last condition on p is needed to give a pointwise sense to Gf . Observe indeed from (3.8) that, for all $p > n^2d+2/2$, $f \in L^p([0, T] \times \mathbb{R}^{nd})$,

$$(3.19) \quad |\tilde{G}f(s, \mathbf{x})| \leq CT^{1-(2+n^2d)/(2p)} \|f\|_{L^p([s, T] \times \mathbb{R}^{nd})}.$$

From the probabilistic viewpoint we will prove that there is only one probability \mathbb{P} on $C([0, T], \mathbb{R}^{nd})$ solving the martingale problem and therefore derive $Gf(s, \mathbf{x}) = \mathbb{E}^{\mathbb{P}}[\int_s^T f(t, \mathbf{X}_t) dt]$, $(\mathbf{X}_t)_{t \in [0, T]}$ being the canonical process. A localization argument similar to the one in Priola [Pri13] then allows to extend the well posedness of the martingale problem under the sole continuity assumption **(C)** in **(A)**, i.e. without the local condition (3.1), see Section 5.

4. DERIVATION OF THE CALDERÓN-ZYGMUND ESTIMATES

We assume **(A)** is in force and that $T \leq T_0(\mathbf{A}) \leq 1$. We also suppose for the whole section that the local condition (3.1) on **F** holds.

4.1. Quasi Metric Structure and Covering. To derive Theorem 3.1, a crucial step consists in considering a “good” parabolic metric and in taking into account the unbounded transport term in (1.1). In order to take into consideration our various time-scales, associated to the propagation of the noise into the system, we introduce the following metric:

$$(4.1) \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^{nd}, \quad \rho(t, \mathbf{x}) := \left(t^{2n-1} + \sum_{i=1}^n |\mathbf{x}_i|^{\frac{2(2n-1)}{(2i-1)}} \right)^{\frac{1}{2(2n-1)}}.$$

Remark 4.1. Recalling the definition of the scale matrix $\mathbb{T}_t := \text{diag}((t^i I_d)_{i \in \llbracket 1, n \rrbracket})$, we can now observe that $\mathbf{x} \in \mathbb{R}^{nd} \mapsto \rho(t, t^{-1/2} \mathbb{T}_t \mathbf{x})$ is $1/2$ homogeneous in the time variable, i.e. $\rho(t, t^{-1/2} \mathbb{T}_t \mathbf{x}) = t^{1/2} \rho(1, \mathbf{x})$.

The metric introduced in (4.1) is equivalent to the one that appears in all the quoted works related to equation (1.1), in Priola, see [Pri09] to study regularity properties in Hölder spaces for semi-groups associated to special cases of equations of the form (1.1), and in [BCLP10], [BCLP13] for L^p regularity.

Setting now $S = [-T, T] \times \mathbb{R}^{nd}$, we then define for $(s, \mathbf{x}), (t, \mathbf{y}) \in S^2$ the distances:

$$(4.2) \quad \begin{aligned} d((s, \mathbf{x}), (t, \mathbf{y})) &:= \rho(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}), \\ d^*((s, \mathbf{x}), (t, \mathbf{y})) &:= d((t, \mathbf{y}), (s, \mathbf{x})) := \rho(|t-s|, \mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y})), \end{aligned}$$

with ρ as in (4.1). We now define the metric balls in the following way:

$$(4.3) \quad \begin{aligned} \forall (s, \mathbf{x}) \in S, \quad \forall \delta > 0, \\ B((s, \mathbf{x}), \delta) &:= \{(t, \mathbf{y}) \in S : d((s, \mathbf{x}), (t, \mathbf{y})) \leq \delta\}. \end{aligned}$$

We mention that the natural extension of the balls considered in [BCLP10], [BCLP13] would have been to consider d^* in the above definition. For this choice, in the linear, homogeneous case $\boldsymbol{\theta}_{s,t}(\mathbf{y}) := \mathbf{R}_{s,t} \mathbf{y}$, \mathbf{R} standing for the resolvent of the linear differential system deriving from (1.1), which can indeed be seen as a group action. We choose here to follow the characteristic associated with the center of the ball, considering a *metric tube* around it. Anyhow those choices are very close and locally equivalent, see Proposition 4.1.

There is now, as in the previously mentioned works, a double difficulty, first the “distance” used to define the balls satisfies the triangle inequality only locally. A natural choice would then consist in considering singular integrals for the “homogeneous space” associated to the balls of the above form, but in such case it is not clear that such balls enjoy the doubling property, which is however satisfied on the whole strip $S = [-T, T] \times \mathbb{R}^{nd}$.

The first key-point is the following result.

Proposition 4.1. Let $S := [-T, T] \times \mathbb{R}^{nd}$. The space $(S, d, dt d\mathbf{x})$ is a locally invariant quasi-metric space in the following sense: there exists a constant $C_{4.1} := C_{4.1}(\mathbf{A}, T) > 0$ s.t.

a) For all $(s, \mathbf{x}), (t, \mathbf{y}) \in S$, if $d((s, \mathbf{x}), (t, \mathbf{y})) \leq 1$ then

$$d((t, \mathbf{y}), (s, \mathbf{x})) \leq C_{4.1} d((s, \mathbf{x}), (t, \mathbf{y})) = \rho(|t - s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}),$$

and for $(\sigma, \boldsymbol{\xi}) \in S$ s.t. $d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi})) \leq 1$ and $d((t, \mathbf{y}), (\sigma, \boldsymbol{\xi})) \leq 1$ then

$$d((s, \mathbf{x}), (t, \mathbf{y})) \leq C_{4.1} (d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi})) + d((\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))).$$

b) Every d -ball in the sense of (4.3) has positive and finite measure and every non-empty intersection of two balls has positive measure.

c) There exists $R > 0$ s.t. for $0 < R_1 < R_2 \leq R$ there exists $C := C(R_1, R_2)$ s.t. for all $(s, \mathbf{x}) \in S$,

$$|B((s, \mathbf{x}), R_2)| \leq C |B((s, \mathbf{x}), R_1)|,$$

where $|\cdot|$ stands here for the Lebesgue measure of the balls.

Remark 4.2 (General and Subdiagonal structure). Let us stress, as it will appear from the proof in Section 6.1, that for the general form of \mathbf{F} in the dynamics of $\boldsymbol{\theta}$ (Assumption (\mathbf{A}_G)), the constant $C_{4.1}$ appearing here depends on the specific radius, 1, chosen for the balls.

However, the proof also emphasizes that when the function \mathbf{F} has the following structure, $\mathbf{F}_1(t, \mathbf{x}) = \mathbf{F}_1(t, \mathbf{x}_1)$, $\forall i \in \llbracket 2, n \rrbracket$, $\mathbf{F}_i(t, \mathbf{x}^{i-1, n}) = \mathbf{F}_i(t, \mathbf{x}_{i-1}, \mathbf{x}_i)$ (subdiagonal case of Assumption (\mathbf{A}_{SD})), then the constant $C_{4.1}$ does not depend on the radius (see Remark 6.1). Hence, in this latter case, point a) of the proposition gives that the distances d and d^* involving respectively the forward and backward transport are actually equivalent. In such a case d is a usual quasi-distance in the sense of Coifman and Weiss [CW71] and the strip S can be seen as a homogeneous space.

From Proposition 4.1 we can use Theorem 25 in [BCLP10] that we now state in our specific case.

Theorem 4.1 (Covering Theorem). For every $\delta_0 > 0$ and $K > 1$ there exists $\delta \in (0, \delta_0)$, a positive integer M and a countable set $((s_i, \mathbf{x}_i))_{i \in A} \subset S$ s.t.

1. $S = \bigcup_{i \in A} B((s_i, \mathbf{x}_i), \delta)$.
2. $\sum_{i \in A} \mathbb{I}_{B((s_i, \mathbf{x}_i), K\delta)} \leq M^2$.

4.2. Singular kernel and associated estimates. Fix $T > 0$ and introduce for $\epsilon \in (0, 1)$ and $(s, t, \mathbf{x}, \mathbf{y}) \in (\mathbb{R}^+)^2 \times (\mathbb{R}^{nd})^2$, $k^\epsilon(s, t, \mathbf{x}, \mathbf{y}) := \mathbb{I}_{\mathbb{R}^+}(s) \mathbb{I}_{[\epsilon, T-s]}(t - s) D_{\mathbf{x}_1}^2 \tilde{q}(s, t, \mathbf{x}, \mathbf{y})$. From (3.4) and (3.5) a direct computation yields (see also the

proof of Lemma 5.5 in [DM10]):

$$(4.4) \quad \begin{aligned} k^\epsilon(s, t, \mathbf{x}, \mathbf{y}) &= \mathbb{I}_{\mathbb{R}^+}(s) \mathbb{I}_{[\epsilon, T-s]}(t-s) \left(-[\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s)]_{11} \right. \\ &\quad \left. + [\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})]_1^{\otimes 2} \right) \tilde{q}(s, t, \mathbf{x}, \mathbf{y}). \end{aligned}$$

In the above equation, for a matrix $\mathbf{M} \in \mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ (resp. a vector $\mathbf{z} \in \mathbb{R}^{nd}$), the notation $[\mathbf{M}]_{11}$ stands for the $d \times d$ submatrix $(\mathbf{M}_{ij})_{(i,j) \in \llbracket 1, d \rrbracket}$ (resp. $[\mathbf{z}]_1$ stands for the subvector of \mathbb{R}^d , $(\mathbf{z}_i)_{i \in \llbracket 1, d \rrbracket}$).

From (3.5), (3.7) and the scaling Lemma B.1 (see also equations (5.10), (5.11) in [DM10]), we have that there exists $C := C(T, (\mathbf{A}))$ s.t.:

$$\begin{aligned} |[\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})]_i| &\leq C \left((t-s)^{-i+1} |\mathbb{T}_{t-s}^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| \right), \\ |[\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s)]_{11} + [\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})]_1^{\otimes 2}| \\ &\leq C((t-s)^{-1} |I_d| + |\mathbb{T}_{t-s}^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2), \end{aligned}$$

so that (3.7), (4.4) yield that $\exists(c_{4.5}, C_{4.5}) := (c_{4.5}, C_{4.5})(T, (\mathbf{A}))$ s.t.

$$(4.5) \quad |k^\epsilon(s, t, \mathbf{x}, \mathbf{y})| \leq C_{4.5} \mathbb{I}_{[\epsilon, T-s]}(t-s) (t-s)^{-1} q_{c_{4.5}}(s, t, \mathbf{x}, \mathbf{y}),$$

where for all $c > 0$,

$$q_c(s, t, \mathbf{x}, \mathbf{y}) := \frac{c^{nd/2}}{(2\pi)^{nd/2} (t-s)^{n^2 d/2}} \exp\left(-c(t-s) |\mathbb{T}_{t-s}^{-1} (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2\right).$$

Observe that this is the same order of singularity than in the non-degenerate case. This is anyhow expectable since we are considering the derivatives w.r.t. the non-degenerate variables. From equation (4.5) and the cut-off in time, we have $\int dt dy |k^\epsilon(s, t, \mathbf{x}, \mathbf{y})| < \infty$. Hence, for all $\epsilon \in (0, 1)$, $\forall (i, j) \in \llbracket 1, d \rrbracket^2$, $f \in L^\infty(\mathbb{R} \times \mathbb{R}^{nd})$ the kernel

$$(4.6) \quad K_{ij}^\epsilon f(s, \mathbf{x}) := \int dt dy k_{ij}^\epsilon(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y}), \quad \forall (i, j) \in \llbracket 1, d \rrbracket^2,$$

is well defined.

4.3. Proof of Theorem 3.1.

4.3.1. *A priori controls.* We have the following results.

Lemma 4.1 (L^2 -control of the kernel). Assume (\mathbf{A}) is in force and that the drift is linear. There exists $C_{4.1} := C_{4.1}(T, (\mathbf{A}))$ s.t. for all $\epsilon \in (0, 1)$, $(i, j) \in \llbracket 1, d \rrbracket^2$, for all $f \in L^2([0, T] \times \mathbb{R}^{nd}) \cap L^\infty([0, T] \times \mathbb{R}^{nd})$,

$$\|K_{ij}^\epsilon f\|_{L^2([0, T] \times \mathbb{R}^{nd})} \leq C_{4.1} \|f\|_{L^2([0, T] \times \mathbb{R}^{nd})}, \quad \forall (i, j) \in \llbracket 1, d \rrbracket^2.$$

Lemma 4.2 (L^∞ -control of the kernel). Assume (\mathbf{A}) is in force. There exist constants $c_\infty := c_\infty(n, d) > 1$, $\varepsilon_0(n, d) < 1$, $C_\infty := C_\infty(T, (\mathbf{A}), \varepsilon_0) > 0$ such that $\forall (s, \mathbf{x}), (\sigma, \boldsymbol{\xi}) \in [0, T] \times \mathbb{R}^{nd}$ and $\varepsilon_{\mathbf{F}} \leq \varepsilon_0 := \varepsilon_0(n, d)$ in (3.1),

$$(4.7) \quad \int_{\rho(t-s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \geq c_\infty \rho(|\sigma-s|, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))} |k_{ij}^\epsilon(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}^\epsilon(\sigma, t, \boldsymbol{\xi}, \mathbf{y})| dt dy \leq C_\infty.$$

If (\mathbf{A}_{SD}) holds, from Remark 4.2, the local quasi-distance distance is actually global and we can take the strip as homogeneous space. Theorem 3.1, as well as associated weak (1,1) estimates, then follow *when the drift is linear* from Lemmas 4.1,4.2 using Theorem 2.4 Chapter 3 in [CW71]. The only reason why we require here the drift to be linear is because we failed to prove Lemma 4.1 under (\mathbf{A}_{SD}) with a non-linear drift. Even with the subdiagonal structure we need the same smoothness of the drift as in the general case to obtain the required L^2 estimate that could be for instance derived from Theorem 11 in [Bra10] which exploits some properties of the symmetrized kernel (whereas we do not in order to establish Lemma 4.1). In the general case we need some additional controls.

4.3.2. Additional Properties in the General Case. Under (\mathbf{A}_G) we need to be more precise on the estimates. The first thing to do consists in splitting the kernel into a singular and a non singular part observing that the singularity is *diagonal*. Specifically, for a given fixed $\delta > 0$ there exist $(c, C) := (c, C)(T, (\mathbf{A}), \delta)$ s.t. if for $(s, \mathbf{x}), (t, \mathbf{y}) \in S^2$, $d((s, \mathbf{x}), (t, \mathbf{y})) = \rho(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \geq \delta$, then from (4.5) we have:

$$(4.8) \quad |k^\epsilon(s, t, \mathbf{x}, \mathbf{y})| \mathbb{1}_{\rho(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \geq \delta} \leq Cq_c(s, t, \mathbf{x}, \mathbf{y}).$$

Let us now split $K_{ij}^\epsilon f$ into:

$$\begin{aligned} K_{ij}^\epsilon f(s, \mathbf{x}) &= \int_S dt d\mathbf{y} k_{ij}^\epsilon(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y}) \eta_\delta(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \\ &\quad + \int_S dt d\mathbf{y} k_{ij}^\epsilon(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y}) (1 - \eta_\delta)(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \\ &:= K_{ij}^{\epsilon,d} f(s, \mathbf{x}) + K_{ij}^{\epsilon,\infty} f(s, \mathbf{x}), \end{aligned}$$

where η_δ is a smooth cut-off function s.t. for all $(u, z) \in \mathbb{R}^+ \times \mathbb{R}^{nd}$, $\eta_\delta(u, z) = 1$ if $\rho(u, z) \leq \delta$ and $\eta_\delta(u, z) = 0$ if $\rho(u, z) \geq 2\delta$. It is then easily seen from (4.8) that for all $f \in L^p(S, \mathbb{R})$, $p \in [1, +\infty]$,

$$(4.9) \quad \|K_{ij}^{\epsilon,\infty} f\|_{L^p(S, \mathbb{R})} \leq C_{4.9} \|f\|_{L^p(S, \mathbb{R})}, \quad C_{4.9} := C_{4.9}(T, (\mathbf{A}), \delta, p).$$

The singular part of the kernel requires a much more subtle handling. Setting $k_{ij}^{\epsilon,d}(s, t, \mathbf{x}, \mathbf{y}) = \eta_\delta(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) k_{ij}^\epsilon(s, t, \mathbf{x}, \mathbf{y})$ we will prove the following proposition.

Proposition 4.2 (Calderón-Zygmund Kernel).

- i) $\exists C_{4.2} := C_{4.2}(T, (\mathbf{A}_G), \delta)$, $\forall (s, \mathbf{x}), (t, \mathbf{y}) \in S^2$, $|k_{ij}^{\epsilon,d}(s, t, \mathbf{x}, \mathbf{y})| \leq \frac{C_{4.2}}{d((s, \mathbf{x}), (t, \mathbf{y}))^{n^2 d + 2}}$.
ii) There exists a constant $c_{4.2}$ s.t.

$$|k_{ij}^{\epsilon,d}(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}^{\epsilon,d}(\sigma, t, \boldsymbol{\xi}, \mathbf{y})| \leq C_{4.2} \frac{d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}))}{d((s, \mathbf{x}), (t, \mathbf{y}))^{n^2 d + 3}},$$

$\forall (s, \mathbf{x}), (\sigma, \boldsymbol{\xi}) \in S^2$, $c_{4.2} d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi})) \leq d((s, \mathbf{x}), (t, \mathbf{y})) \leq 1$.

- iii) The two previous “standard estimates” hold for the symmetric kernel

$$k_{ij}^{\epsilon,d,*}(s, t, \mathbf{x}, \mathbf{y}) := k_{ij}^{\epsilon,d}(t, s, \mathbf{y}, \mathbf{x}).$$

iv) Cancellation Property:

$$\begin{aligned} & \sup_{\epsilon > 0} \left| \int_{d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y} \right| \\ & + \sup_{\epsilon > 0} \left| \int_{d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,*}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y} \right| < +\infty. \end{aligned}$$

Also the limits:

$$\lim_{\epsilon \rightarrow 0} \int_{d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y}, \quad \lim_{\epsilon \rightarrow 0} \int_{d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,*}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y}$$

exist and are finite for almost every $\mathbf{x} \in S$.

The strategy is now to exploit those estimates to derive L^p controls on the covering of S with our metric balls introduced in the Theorem 4.1. But to do so, we have to carefully check that the cancellation property appearing in Proposition 4.2 for the whole strip still holds on the metric balls. This property can be conserved thanks to a Hölder continuous cut-off as in Proposition 18 from [BCLP10]. Namely, from Proposition 4.2 it can be derived similarly to the previous reference that:

Proposition 4.3 (Localized Cancellation). There exists a constant $R_0 > 0$ s.t. for $(s_0, \mathbf{x}_0) \in S$, $R \leq R_0$, if a, b stand for two cut-off functions belonging to $C^\alpha(\mathbb{R}^{n+1}, \mathbb{R})$, $\alpha > 0$ and with support in $B((s_0, \mathbf{x}_0), R)$, then defining

$$k_{ij}^{\epsilon,d,\text{loc}}(s,t,\mathbf{x},\mathbf{y}) := a(s,\mathbf{x}) k_{ij}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) b(t,\mathbf{y}),$$

we have that:

- $k_{ij}^{\epsilon,d,*,\text{loc}}$ satisfies the first three points of Proposition (4.2) and for all $(s, \mathbf{x}) \in B((s_0, \mathbf{x}_0), R)$:

$$\begin{aligned} & \sup_{\epsilon > 0} \left| \int_{(t,\mathbf{y}) \in B((s_0,\mathbf{x}_0),R), d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,\text{loc}}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y} \right| \\ & + \sup_{\epsilon > 0} \left| \int_{(t,\mathbf{y}) \in B((s_0,\mathbf{x}_0),R), d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,*,\text{loc}}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y} \right| < +\infty. \end{aligned}$$

- For almost all $(s, \mathbf{x}) \in B((s_0, \mathbf{x}_0), R)$ the limits

$$\lim_{\epsilon \rightarrow 0} \int_{d((s_0,\mathbf{x}_0),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,\text{loc}}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y}, \quad \lim_{\epsilon \rightarrow 0} \int_{d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,*,\text{loc}}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y}$$

exist and are finite.

Now from Propositions 4.2 and 4.3 we derive from Theorem 3 in [Bra10], that for every $(s, \mathbf{x}) \in S$, $R \leq R_0$, $p \in (1, +\infty)$, there exists a constant $C_{p,T,(\mathbf{A}_G)}$ independent of (s, \mathbf{x}) s.t. setting

$$\begin{aligned} T_{ij} f & := \lim_{\epsilon \rightarrow 0} \int_{(t,\mathbf{y}) \in B((s,\mathbf{x}),R), d((s,\mathbf{x}),(t,\mathbf{y})) > \epsilon} k_{ij}^{\epsilon,d,\text{loc}}(s,t,\mathbf{x},\mathbf{y}) f(t,\mathbf{y}) dt d\mathbf{y}, \\ \|T_{ij} f\|_{L^p(B((s,\mathbf{x}),R))} & \leq C_{p,T,(\mathbf{A}_G)} \|f\|_{L^p(B((s,\mathbf{x}),R))}. \end{aligned}$$

The covering Theorem 4.1 then gives, similarly to the proof of Theorem 22 in [BCLP10], that the result can be extended to the strip for the initial operator without cut-off, i.e. for every $p \in (1, +\infty)$, there exists a constant $C_{p,T,(\mathbf{A}_G)}$ s.t.

$$\|D_{\mathbf{x}_1^i, \mathbf{x}_1^j}^2 \tilde{G} f\|_{L^p(S)} \leq C_{p,T,(\mathbf{A}_G)} \|f\|_{L^p(S)},$$

which concludes the proof of Theorem 3.1 under (\mathbf{A}_G) .

5. DERIVATION OF THEOREM 2.1 FROM THE CALDERÓN-ZYGMUND ESTIMATES

5.1. Well posedness of the martingale problem. Existence can be obtained by usual compactness arguments, see e.g. Theorem 6.1.7 in [SV79] that can be adapted to the current framework. On the other hand, from the definition in (3.11) and (3.8), observe that for $f \in C_0^\infty([0, T] \times \mathbb{R}^{nd})$ and $p > (n^2d + 2)/2$,

$$(5.1) \quad \begin{aligned} & \exists C_{5.1} := C_{5.1}(p, T, (\mathbf{A})), \quad \forall (s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}, \\ & |\tilde{G}f(s, \mathbf{x})| \leq C_{5.1} T^{1-(n^2d+2)/(2p)} \|f\|_{L^p([s, T] \times \mathbb{R}^{nd})}. \end{aligned}$$

Assume now that $\varepsilon_{\mathbf{F}} \leq \varepsilon_0(n, d)$ as in Lemma 4.2, $\varepsilon_a \leq C_{3.17}^{-1}$ in the local condition (3.1), and $T < (4C(p, (\mathbf{A})))^{-2}$. Exploiting (5.1), (3.18), the uniqueness to the martingale problem can be derived following the steps of Section 7.1 in [SV79].

Now, without the local condition, the continuity assumed in (\mathbf{A}) allows to localize. Precisely, for all $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$ we can find $\delta > 0$ s.t. if $\mathcal{G} := [(s - \delta) \vee 0, s + \delta] \times B(\mathbf{x}, \delta)$ then

$$\begin{aligned} & \sup_{(t, \mathbf{y}) \in \mathcal{G}} |a(t, \mathbf{y}) - a(t, \mathbf{x})| \leq \varepsilon_a, \\ & \sup_{i \in \llbracket 2, n \rrbracket} \sup_{(t, \mathbf{y}) \in \mathcal{G}, \mathbf{z} \in \mathbb{R}^{nd}} |D_{\mathbf{x}_{i-1}} F_i(t, \mathbf{y}_{i-1}, \mathbf{z}^{i,n}) - D_{\mathbf{x}_{i-1}} F_i(t, \mathbf{x}_{i-1}, \mathbf{z}^{i,n})| \leq \varepsilon_{\mathbf{F}}. \end{aligned}$$

Define then $\forall (t, \mathbf{y}) \in [0, T] \times \mathbb{R}^{nd}$, $\tilde{a}(t, \mathbf{y}) = a(t, \mathbf{y}) \mathbb{I}_{(t, \mathbf{y}) \in \mathcal{G}} + (1 - \mathbb{I}_{(t, \mathbf{y}) \in \mathcal{G}}) a(t, \mathbf{x})$ and \tilde{F} s.t. $D_{\mathbf{x}_{i-1}} \tilde{F}_i(t, \mathbf{y}_{i-1}, \mathbf{z}^{i,n}) := D_{\mathbf{x}_{i-1}} F_i(t, (\Pi_{\mathbf{x}_{i-1}, \delta}(\mathbf{y}_{i-1}), \mathbf{z}^{i,n}))$ where:

$$\Pi_{\mathbf{x}_{i-1}, \delta}(\mathbf{y}_{i-1}) = \mathbf{y}_{i-1} \mathbb{I}_{|\mathbf{y}_{i-1} - \mathbf{x}_{i-1}| \leq \delta} + \left(\mathbf{x}_{i-1} + \frac{\mathbf{y}_{i-1} - \mathbf{x}_{i-1}}{|\mathbf{y}_{i-1} - \mathbf{x}_{i-1}|} \delta \right) \mathbb{I}_{|\mathbf{y}_{i-1} - \mathbf{x}_{i-1}| > \delta}.$$

Then the coefficients $\tilde{a}, \tilde{\mathbf{F}}$ satisfy (\mathbf{A}) and the local condition (3.1). We can then invoke Theorem 27 in Priola [Pri13] to derive global uniqueness in *small time*. Indeed, it is not hard to directly construct a covering of S with sets of the previous type. An alternative consists in exploiting the covering result of Theorem 4.1. Let us emphasize that we cannot here directly apply Theorem 6.6.1 in [SV79], which being valid for bounded coefficients would have required a truncation of the drift. It is not clear to derive Calderón-Zygmund estimates of the previous type that are uniform w.r.t. to the truncation.

The well posedness of the martingale for an arbitrary given $T > 0$ then follows from the Markov property (see Chapter 6.2 of [SV79] for details).

5.2. Existence of the Density and Associated Estimates. The goal of this section is to prove equation (2.2) of Theorem 2.1. To this end, we will need the following result which extends to our current degenerate setting Theorem 9.1.9 in [SV79] in *small time*.

Theorem 5.1 (Local existence of the density and associated estimates). Assume $T \leq T_0((\mathbf{A})) \leq 1$ as in Theorem 3.1.

- If the diffusion coefficient is uniformly continuous, then, for $0 \leq s < T$, and $q \in [1, 2)$, the density p satisfies:

$$\left(\int_s^T dt (t-s)^\alpha \int_{\mathbb{R}^{nd}} dy |p(s, t, \mathbf{x}, \mathbf{y})|^q \right)^{1/q} \leq C_{5.1}^1 (1 + |\mathbf{x}|),$$

where $\alpha = \left(\frac{n^2d+2}{2}\right)(q-1)$ and $C_{5.1}^1 := C_{5.1}^1(T, q, (\mathbf{A}), \delta_T)$, denoting for all $\varepsilon > 0$, $\delta_T(\varepsilon) := \operatorname{argmax}_{\zeta \in \mathbb{R}^+} \{\sup_{s \in [0, T], |\mathbf{x} - \mathbf{y}| \leq \zeta} |a(s, \mathbf{x}) - a(s, \mathbf{y})| < \varepsilon\}$ the modulus of continuity of a .

- For all $0 \leq s < T$ and $q \in [1, 2)$, $\delta > 0$,

$$\left(\int_s^T dt \int_{\mathbb{R}^{nd} \setminus B^E(\boldsymbol{\theta}_{t,s}(\mathbf{x}), \delta)} d\mathbf{y} |p(s, t, \mathbf{x}, \mathbf{y})|^q \right)^{1/q} \leq C_{5.1}^2(1 + |\mathbf{x}|),$$

with $C_{5.1}^2 := C_{5.1}^2(T, q, (\mathbf{A}), \delta_T, \delta)$, $B^E(\boldsymbol{\theta}_{t,s}(\mathbf{x}), \delta)$ standing for the Euclidean ball of \mathbb{R}^{nd} with radius δ and center $\boldsymbol{\theta}_{t,s}(\mathbf{x})$, recalling $\boldsymbol{\theta}_{t,s}(\mathbf{x}) = \mathbf{x} + \int_s^t \mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) du$ (i.e. $\boldsymbol{\theta}_{t,s}(\mathbf{x})$ is the solution at time t of the deterministic differential system associated to (1.1) starting from \mathbf{x} at time s).

Remark 5.1. There are three differences w.r.t. to the indicated theorem. First, the norm of the initial point in the r.h.s. of the above controls is due to the transport by unbounded coefficients. Second, the small time constraint follows from our linearization strategy employed to derive the Calderon-Zygmund estimates of Theorem 3.1. At last, the upper bound on q comes from the control on the remainders in (3.16). When the system is linear w.r.t. the components that transmit the noise (see Remark 3.1) this constraint disappears and the result of Theorem 5.1 hold for $q \in [1, +\infty)$.

We provide below the principal lines needed to adapt the proof of that theorem, stressing which specific modifications are needed in the degenerate case and mainly concern the *localization* arguments. Once again the key idea is to localize along the characteristic lines associated to the deterministic differential system instead of using spatial balls only as in [SV79]. Observe anyhow that, when the drift is bounded, the product of the time interval and the spatial ball can be seen as a *characteristic line*. Indeed, if the drift is 0 then the deterministic differential system does not leave its initial condition; if it is bounded, the image of a spatial ball by the deterministic system will stay uniformly in time in a ball whose radius only depend on the bound of the drift, the final time and the initial radius, **but not** on the points of the initial ball.

5.2.1. *Controls for Slowly Varying Coefficients.* We use here freely the notations of Section 3 for the operators \tilde{G}, R (see equations (3.11)-(3.14)). Also, in order to keep notations close to those in [SV79], we introduce for $r > 2$, the class $\mathcal{A}(r, T)$ of measurable coefficients $a : \mathbb{R}^+ \times \mathbb{R}^{nd}$ satisfying **(UE)** and \mathbf{F} satisfying **(ND)**, **(SG)**, for which there exists $\mathbf{x}_0 \in \mathbb{R}^{nd}$ s.t. with the notations of (3.1), $\varepsilon_{\mathbf{F}} \leq \varepsilon_0$ and for all $\rho \in [r, \frac{n^2d+4}{2} \vee r]$, $(\frac{\varepsilon_a}{2} C_{3.17}(T, \rho, (\mathbf{A})) + C(\rho, (\mathbf{A}))T^{\frac{1}{2}}) < 3/4$ so that from (3.18), we have that $(I - R)^{-1}$ is consistent as bounded operator from $L^\rho([0, T] \times \mathbb{R}^{nd})$ into itself. In particular this imposes that $T \leq 1$ is sufficiently small. Setting then $K := \tilde{G} \circ R^{-1}$, we thus derive that it is consistent as bounded operator from $L^\rho([0, T] \times \mathbb{R}^{nd})$ into $L^\sigma([0, T] \times \mathbb{R}^{nd})$ provided that

$$0 \leq \frac{1}{\rho} - \frac{1}{\sigma} < \frac{2}{n^2d+2}.$$

Here comes the first Lemma emphasizing some regularizing effects of K which can be derived similarly to Lemma 9.1.2 in [SV79].

Lemma 5.1. Let $a, \mathbf{F} \in \mathcal{A}(r, T)$. Then, for $N = \lceil \frac{n^2 d}{2r} \rceil$, K^{N+1} maps $L^r([0, T] \times \mathbb{R}^{nd})$ into $C_b([0, T] \times \mathbb{R}^{nd})$ (space of real valued bounded continuous functions). Precisely, for all $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$:

$$|K^{N+1}f(s, \mathbf{x})| \leq C_{5.1} \|f\|_{L^r([0, T] \times \mathbb{R}^{nd})},$$

where $C_{5.1} := C_{5.1}(T, r, (\mathbf{A}))$.

From Lemma 5.1, Lemma 9.1.3 in [SV79] and the well posedness of the martingale problem, denoting by $P(s, x, t, \cdot)$ the associated transition function, one then gets: $Pf(s, x) = \int_s^T dt \int_{\mathbb{R}^{nd}} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) = Kf(s, x)$ if $f \in L^\rho([0, T] \times \mathbb{R}^{nd}) \cap L^\infty([0, T] \times \mathbb{R}^{nd})$. Therefore, for $N = \lceil (n^2 d + 2)/2r \rceil$, $f \in C_0([0, T] \times \mathbb{R}^{nd})$ (functions with compact support),

$$(5.2) \quad |P^{N+1}f(s, \mathbf{x})| \leq C_{5.1} \|f\|_{L^r([0, T] \times \mathbb{R}^{nd})}.$$

This observation then yields the following result.

Lemma 5.2. If $a, \mathbf{F} \in \mathcal{A}(r, T)$ denoting by P the transition function associated to $(L_t)_{t \in [0, T]}$ then for $r \leq \rho \leq +\infty$,

$$\int_s^T dt (t-s)^N \int_{\mathbb{R}^{nd}} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \leq C_{5.2}^1 \|f\|_{L^\rho([0, T] \times \mathbb{R}^{nd})},$$

with $C_{5.2}^1 := C_{5.2}^1(T, r, (\mathbf{A}))$. Also, for each $\delta > 0$, $r < \rho \leq \infty$,

$$\int_s^T dt \int_{\mathbb{R}^{nd} \setminus B^E(\boldsymbol{\theta}_{t,s}(\mathbf{x}), \delta)} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \leq C_{5.2}^2 \|f\|_{L^\rho([0, T] \times \mathbb{R}^{nd})},$$

where $C_{5.2}^2 := C_{5.2}^2(T, r, \rho, (\mathbf{A}))$.

Remark 5.2. This is the first statement that differs from [SV79]. Indeed, the unbounded transport contribution appears here for the first time. To fully justify this aspect we give below the full proof of this result.

Proof. The first statement of the Lemma still follows from Lemma 9.1.3 in [SV79] and (5.2) from an interpolation argument. For the second one, we can assume w.l.o.g. that $T \geq s + c$, $c > 0$. In that case:

$$\begin{aligned} & \left| \int_s^T dt \int_{\mathbb{R}^{nd} \setminus B^E(\boldsymbol{\theta}_{t,s}(\mathbf{x}), \delta)} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \right| \\ & \leq \sum_{n \geq 1} \int_{s+c/(n+1)}^{s+c/n} dt \int_{\mathbb{R}^{nd} \setminus B^E(\boldsymbol{\theta}_{t,s}(\mathbf{x}), \delta)} P(s, t, \mathbf{x}, d\mathbf{y}) |f(t, \mathbf{y})| \\ & \quad + c^{-N} \int_{s+c}^T (t-s)^N \int_{\mathbb{R}^{nd}} P(s, t, \mathbf{x}, d\mathbf{y}) |f(t, \mathbf{y})|. \end{aligned}$$

The last contribution can be bounded directly by the first statement of the Lemma. To control the sum, we see that introducing

$$\Lambda_{s, \mathbf{x}}^n \varphi := \int_{s+c/(n+1)}^{s+c/n} dt \int_{\mathbb{R}^{nd} \setminus B^E(\boldsymbol{\theta}_{t,s}(\mathbf{x}), \delta)} P(s, t, \mathbf{x}, d\mathbf{y}) \varphi(t, \mathbf{y}),$$

we indeed get, from Lemma 9.1.3 in [SV79] and the first part of the lemma, that as a linear operator on $L^r([0, T] \times \mathbb{R}^{nd})$, $\Lambda_{s, \mathbf{x}}^n$ is bounded by $N!((n+1)/c)^N$. Now

for $\varphi \in L^\infty([0, T] \times \mathbb{R}^d)$,

$$|\Lambda_{s, \mathbf{x}}^n \varphi| \leq \frac{C}{n} |\varphi|_\infty \mathbb{P}_{s, \mathbf{x}} \left[\sup_{t \in [s, s+c/n]} |\mathbf{X}_t - \boldsymbol{\theta}_{t,s}(\mathbf{x})| \geq \delta \right].$$

Let us emphasize that it is precisely because we consider the deviations of the process from the deterministic differential system, that we can control the previous term with Bernstein like inequalities. Precisely, from Gronwall's lemma:

$$|\mathbf{X}_t - \boldsymbol{\theta}_{t,s}(\mathbf{x})| \leq \exp(CT) \left| \int_s^t \sigma(u, \mathbf{X}_u) dW_u \right| \leq \exp(C) \left| \int_s^t \sigma(u, \mathbf{X}_u) dW_u \right|,$$

with $C := C((\mathbf{A}))$, so that, from Bernstein's inequality:

$$\mathbb{P} \left[\sup_{t \in [s, s+c/n]} |\mathbf{X}_t - \boldsymbol{\theta}_{t,s}(\mathbf{x})| \geq \delta \right] \leq C \exp(-C^{-1} n \delta^2 / c), \quad C := C((\mathbf{A})),$$

up to a modification of C . The result then once again follows from standard interpolation. \square

5.2.2. Localization arguments. Now we adapt more significantly the arguments in [SV79] to get our results. The leading idea is the same as in the proof of Lemma 5.2: to exploit the Bernstein-like deviations of the process from the deterministic system. We now want to localize carefully to get rid off the *quasi-constant* coefficients of the previous section. We have the following *tubular* localization.

Lemma 5.3 (Tubular estimate). For $s_0 \in [0, T)$, $\mathbf{x}_1 \in \mathbb{R}^{nd}$ let $\mathbb{P}_{s_0, \mathbf{x}_1}$ denote the solution to the martingale problem associated to $(L_t)_{t \in [s_0, T]}$. For $0 < R_1 < R_2$, $\mathbf{x}_0 \in \mathbb{R}^{nd}$ defining $\tau_{-1} = s_0$ and for all $n \in \mathbb{N}$,

$$\begin{aligned} \tau_{2n} &:= \inf \{ t \geq \tau_{2n-1} : |\mathbf{X}_t - \boldsymbol{\theta}_{t,s}(\mathbf{x}_0)| = R_2 \}, \\ \tau_{2n+1} &:= \inf \{ t \geq \tau_{2n} : |\mathbf{X}_t - \boldsymbol{\theta}_{t,s}(\mathbf{x}_0)| = R_1 \}, \end{aligned}$$

then

$$\mathbb{E}^{\mathbb{P}_{s_0, \mathbf{x}_1}} \left[\sum_{n \geq 0} \mathbb{I}_{\tau_{2n} \in [0, T]} \right] \leq C_{5.3} := C_{5.3}(T, (\mathbf{A}), R_2 - R_1).$$

The proof can be performed as in Lemma 9.1.6 in [SV79]. The previous definitions of the stopping times allows to apply the required Bernstein like arguments similarly to the proof of Lemma 5.2.

The following result differs once again in the localization argument from Lemma 9.1.7 in [SV79], even though it can be proved rather similarly from Lemma 5.3. We emphasize here that the localization has to be performed in time and space. Roughly speaking this is needed in order to *partition* in time the characteristic tubes in subtubes for which the local condition 3.1 is valid. This is the key of the proof.

Lemma 5.4 (First Localization Lemma). Let P solve the martingale problem for $(L_t)_{t \in [0, T]}$ starting from $(s_0, \mathbf{x}_1) \in [0, T] \times \mathbb{R}^{nd}$. Suppose now that the martingale problem associated to the operator

$$\tilde{L}_t = \tilde{\mathbf{F}}(t, \cdot) \cdot \nabla + \frac{1}{2} \text{Tr}(\tilde{a}(t, \cdot) D_{\mathbf{x}_1}^2)$$

is well posed and that $\tilde{\mathbf{F}} = \mathbf{F}$, $\tilde{a} = a$ on $\mathcal{C}_{\underline{t}, \bar{t}, R}(s_0, \mathbf{x}_0) := \{(t, \mathbf{y}) \in [0, T] \times \mathbb{R}^{nd} : t \in [\underline{t}, \bar{t}], \boldsymbol{\theta}_{s_0, t}(\mathbf{y}) \in B^E(\mathbf{x}_0, R)\}$ for some $0 \leq \underline{t} < \bar{t} \leq T$, $R > 0$. Then for each

$\delta \in (0, R)$:

$$\begin{aligned} |\mathbb{E}^{\mathbb{P}}[\int_{s_0}^T f(t, \mathbf{X}_t) dt]| &\leq \mathbb{E}^{\tilde{\mathbb{P}}_{s_0, \mathbf{x}_1}}[\int_{s_0}^T |f(t, \mathbf{X}_t)| dt] \\ &+ C_{5.4} \sup_{(s, \mathbf{x}) \in \partial \mathcal{C}_{\underline{t}, \bar{t}, R-\delta}(s, \mathbf{x}_0)} \mathbb{E}^{\tilde{\mathbb{P}}_{s, \mathbf{x}}}[\int_s^T |f(t, \mathbf{X}_t)| dt], \end{aligned}$$

for all $f \in C_0(\mathcal{C}_{\underline{t}, \bar{t}, R-\delta}(s_0, \mathbf{x}_0))$ and $C_{5.4} := C_{5.4}(T, (\mathbf{A}), R, \delta)$.

We now specify how this Lemma needs to be used. As a direct corollary of Lemmas 5.4 and 5.2 we derive:

Lemma 5.5 (Second Localization Lemma). Let $\tilde{a}(s, \mathbf{x}) := a(s, \mathbf{x})$, $\tilde{\mathbf{F}}(s, \mathbf{x}) := \mathbf{F}(s, \mathbf{x})$ in $\mathcal{C}_{\underline{t}, \bar{t}, R}(s_0, \mathbf{x}_0)$ and $\tilde{a}(s, \mathbf{x}) := a(s, \mathbf{x}_0)$, $\tilde{\mathbf{F}}(s, \mathbf{x}) := \mathbf{F}(s, \mathbf{x}_0)$ elsewhere. Assume that the functions $\tilde{a}, \tilde{\mathbf{F}} \in \mathcal{A}(r, T)$ for some $r \in (2, +\infty)$. Let P solve the martingale problem for L starting at $s_0, \mathbf{x}_1 \in [0, T] \times \mathbb{R}^{nd}$. Then for each $0 < \alpha < R$ and $r < \rho \leq +\infty$.

$$|\mathbb{E}^{\mathbb{P}}[\int_{s_0}^T (t - s_0)^N f(t, \mathbf{X}_t) dt]| \leq C_{5.5}^1 \|f\|_{L^\rho([s_0, T] \times \mathbb{R}^{nd})},$$

for all $f \in C_0(\mathcal{C}_{\underline{t}, \bar{t}, \alpha}(s_0, \mathbf{x}_0))$, where $N = \lceil (n^2 d + 2)/2r \rceil$ and $C_{5.5}^1 := C_{5.5}^1(T, (\mathbf{A}), r, \rho, R - \alpha)$. If additionally, $|\mathbf{x}_1 - \mathbf{x}_0| > \alpha$, then

$$|\mathbb{E}^{\mathbb{P}}[\int_{s_0}^T f(t, \mathbf{X}_t) dt]| \leq C_{5.5}^2 \|f\|_{L^\rho([s_0, T] \times \mathbb{R}^{nd})},$$

where $C_{5.5}^2 := C_{5.5}^2(T, (\mathbf{A}), r, \rho, R - \alpha, |\mathbf{x}_1 - \mathbf{x}_0| - \alpha)$.

5.2.3. Proof of Theorem 5.1. From the previous localization Lemmas, the idea is now to specifically *partition* the space in order to have crowns of the previous type, $\mathcal{C}_{\underline{t}, \bar{t}, R}(s_0, \mathbf{x}_0)$ on which the local condition (3.1) holds. Let q' denote the conjugate of $q \in (1, 2)$ and choose $r \in (2, q')$ s.t. $N := \lceil \frac{n^2 d + 2}{2} \frac{1}{r} \rceil = \lceil \frac{n^2 d + 2}{2} \frac{1}{q} \rceil$. Choose $T \leq (4(\sup_{\rho \in [r, (\frac{n^2 d + 2}{2} \vee r) + 1]} C(\rho, (\mathbf{A}))))^{-2}$ and set $\varepsilon^{-1} = C_{(3.17)}(T, r, (\mathbf{A})) \vee C_{(3.17)}(T, (\frac{n^2 d + 2}{2} \vee r) + 1, (\mathbf{A})) \vee \varepsilon_0^{-1}$ in (3.18). Let us introduce for a fixed starting point (s, \mathbf{x}) of the martingale problem, the spatial balls

$$Q_{\mathbf{k}} := \{\mathbf{x} + \mathbf{y} : |\mathbf{y}_j - \mathbf{k}_j \gamma / (nd)^{1/2}| \leq \gamma / (nd)^{1/2}, j \in \llbracket 1, nd \rrbracket\}, \mathbf{k} \in \mathbb{Z}^{nd},$$

where $\gamma := \frac{\delta_T(\varepsilon)}{C_1}$, and recalling that δ_T stands for the modulus of continuity of a , the constant C_1 is then chosen large enough so that for all $\mathbf{k} \in \mathbb{Z}^{nd}$, $\mathbf{y}_0, \mathbf{y}_1 \in Q_{\mathbf{k}}$, $t \in [s, T]$,

$$\begin{aligned} &|a(t, \boldsymbol{\theta}_{t,s}(\mathbf{y}_0) - a(t, \boldsymbol{\theta}_{t,s}(\mathbf{y}_1))| \\ &+ \sup_{i \in \llbracket 2, n \rrbracket, 0 \leq t \leq T, (\mathbf{x}, \mathbf{z}) \in (\mathbb{R}^{nd})^2} |D_{\mathbf{x}_{i-1}} F_i(t, (\boldsymbol{\theta}_{t,s}(\mathbf{y}_0))_{i-1}, \mathbf{z}^{i,n}) \\ &- D_{\mathbf{x}_{i-1}} F_i(t, ((\boldsymbol{\theta}_{t,s}(\mathbf{y}_1))_{i-1})_{i-1}, \mathbf{z}^{i,n})| \leq \varepsilon. \end{aligned} \tag{5.3}$$

This means that the local condition is satisfied on the time section of the transport of $Q_{\mathbf{k}}$ by the flow. In order to apply the previous results, we also need to handle the time contribution. Define now $h^{\mathbf{k}} := \frac{T-s}{\lceil C_2(1+|\mathbf{k}\gamma/(nd)^{1/2}+\mathbf{x}|) \rceil}$ where the constant $C_2 := C_2(\varepsilon)$ is chosen large enough, so that for all $i \in \llbracket 0, \lceil C_2(1+|\mathbf{k}\gamma/(nd)^{1/2}+\mathbf{x}|) \rceil \rrbracket$,

setting $t_i^{\mathbf{k}} := s + ih^{\mathbf{k}}$, the coefficients a, \mathbf{F} restricted to $\mathcal{C}_{t_i^{\mathbf{k}}, t_{i+1}^{\mathbf{k}}, \gamma}(s, \mathbf{x} + \mathbf{k}\gamma/(nd)^{1/2})$ coincide with some $\tilde{a}, \tilde{\mathbf{F}}$ satisfying $\mathcal{A}(r, T)$.

This choice simply means that the length of the time intervals for which we partition the set $\boldsymbol{\theta}_{T,s}(Q_{\mathbf{k}}) := \{(t, \mathbf{z}) \in [s, T] \times \mathbb{R}^{nd} : \boldsymbol{\theta}_{s,t}(\mathbf{z}) \in Q_{\mathbf{k}}\}$ (image of $Q_{\mathbf{k}}$ by the flow) highly depend on the norm of the starting point. This is specifically due to the unbounded drift. Precisely we can write:

$$(5.4) \quad \boldsymbol{\theta}_{T,s}(Q_{\mathbf{k}}) := \bigcup_{i=0}^{\lceil C_2(1+|\mathbf{k}\gamma/(nd)^{1/2}+\mathbf{x}|)-1 \rceil} \mathcal{C}_{t_i^{\mathbf{k}}, t_{i+1}^{\mathbf{k}}, \gamma}(s, \mathbf{x} + \mathbf{k}\gamma/(nd)^{1/2}).$$

Now from Lemma 5.5 we get that for $\rho = (r + q')/2$, for all $i \in \llbracket 0, \lceil C_2(1 + |\mathbf{k}\gamma/(nd)^{1/2} + \mathbf{x}|) - 1 \rceil \rrbracket$

$$(5.5) \quad \left| \int_s^T dt (t-s)^N \int_{\mathcal{C}_{t_i^{\mathbf{k}}, t_{i+1}^{\mathbf{k}}, \gamma}(s, \mathbf{x} + \mathbf{k}\gamma/(nd)^{1/2})} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \right| \leq C_{5.5} \|f\|_{L^\rho([0, T] \times \mathbb{R}^{nd})},$$

where $C_{5.5} := C_{5.5}(T, (\mathbf{A}), r, \rho)$. On the other hand, comparing deviations along the characteristics allows once again to use Bernstein inequalities, similarly to the proof of Lemma 5.2. Namely,

$$\begin{aligned} P(s, t, \mathbf{x}, Q_{\mathbf{k}}) &\leq \mathbb{P}_{s, \mathbf{x}}[\exists u \in [s, t], |\mathbf{X}_u - \boldsymbol{\theta}_{u,s}(\mathbf{x})| > |\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s}(\mathbf{x} + \mathbf{k}\gamma/(nd)^{1/2})|] \\ &\leq \mathbb{P}_{s, \mathbf{x}}\left[\sup_{u \in [s, T]} \left| \int_s^u \sigma(v, \mathbf{X}_v) dW_v \right| \geq C|\mathbf{k}\gamma|\right] \\ &\leq 2d \exp(-\bar{C}^{-1} \frac{|\mathbf{k}|^2 \gamma^2}{Td^2}), \quad \bar{C} := \bar{C}((\mathbf{A})) \geq 1, \end{aligned}$$

using Gronwall's Lemma (see proof of Lemma 5.2) and the bi-Lipschitz property of the flow for the last but one inequality. We therefore obtain:

$$(5.6) \quad \left| \int_s^T dt (t-s)^N \int_{\mathcal{C}_{t_i^{\mathbf{k}}, t_{i+1}^{\mathbf{k}}, \gamma}(s, \mathbf{x} + \mathbf{k}\gamma/(nd)^{1/2})} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \right| \leq C_{5.6} \exp(-\bar{C}^{-1} \frac{|\mathbf{k}|^2 \gamma^2}{Td^2}) \|f\|_\infty,$$

where $C_{5.6} := C_{5.6}(T, n, d, N)$. We thus get by interpolation that for $\vartheta = 1 - \frac{d}{q'} \in (0, 1)$:

$$\begin{aligned} &\left| \int_s^T dt (t-s)^N \int_{\mathcal{C}_{t_i^{\mathbf{k}}, t_{i+1}^{\mathbf{k}}, \gamma}(s, \mathbf{x} + \mathbf{k}\gamma/(nd)^{1/2})} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \right| \\ &\leq C_{5.5}^{1-\vartheta} C_{5.6}^\vartheta \exp(-\bar{C}^{-1} \frac{\vartheta |\mathbf{k}|^2 \gamma^2}{Td^2}) \|f\|_{L^{q'}([0, T] \times \mathbb{R}^{nd})}. \end{aligned}$$

Summing for a given $\mathbf{k} \in \mathbb{Z}^d$ first over $i \in \llbracket 0, \lceil C_2(1 + |\mathbf{k}\gamma/(nd)^{1/2} + \mathbf{x}|) - 1 \rceil \rrbracket$ (that is according to (5.4) on $\boldsymbol{\theta}_{T,s}(Q_{\mathbf{k}})$) we obtain

$$\begin{aligned} &\left| \int_s^T dt (t-s)^N \int_{\boldsymbol{\theta}_{T,s}(Q_{\mathbf{k}})} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \right| \\ &\leq \tilde{C}(T^{1/2} + |\mathbf{x}|) C_{5.5}^{1-\vartheta} C_{5.6}^\vartheta \exp(-\bar{C}^{-1} \frac{\vartheta |\mathbf{k}|^2 \gamma^2}{Td^2}) \|f\|_{L^{q'}([0, T] \times \mathbb{R}^{nd})}, \quad \tilde{C} := \tilde{C}((\mathbf{A}), \vartheta). \end{aligned}$$

Summing now over $\mathbf{k} \in \mathbb{Z}^{nd}$ yields:

$$(5.7) \quad \begin{aligned} & \left| \int_s^T dt (t-s)^N \int_{\mathbb{R}^{nd}} P(s, t, \mathbf{x}, d\mathbf{y}) f(t, \mathbf{y}) \right| \\ & \leq C_{5.7}(1 + |\mathbf{x}|) \|f\|_{L^{q'}([0, T] \times \mathbb{R}^{nd})}, C_{5.7} := C_{5.7}(T, q, (\mathbf{A}), \gamma). \end{aligned}$$

This contribution already emphasizes the main difference w.r.t. the non degenerate case: the estimate depends on the initial point. The proof can then be completed similarly to Theorem 9.1.9 in [SV79]. This achieves the proof of the existence of the density and the associated estimates when T is small enough.

The existence of the density in Theorem 2.1 then follows from a chaining argument.

5.2.4. *Derivation of equation (2.2) in Theorem 2.1.* To prove (2.2) we will inductively apply the results of the previous section along a time grid which time-steps are lower than T_0 in Theorem 5.1. We can assume w.l.o.g. that $T := T_0 N$, $N \in \mathbb{N}$. Setting now $t_i := iT_0$, $i \in \llbracket 0, N \rrbracket$, write from the strong Markov property:

$$(5.8) \quad \begin{aligned} & \mathbb{E}_{\mathbf{x}} \left[\int_0^T f(t, \mathbf{X}_t) dt \right] = \sum_{i=0}^{N-1} \mathbb{E}_{\mathbf{x}} [\mathbb{E}_{t_i, \mathbf{X}_{t_i}} \left[\int_{t_i}^{t_{i+1}} f(t, \mathbf{X}_t) dt \right]] \\ & = \left\{ \sum_{i=0}^{N-1} \mathbb{E}_{\mathbf{x}} [\mathbb{E}_{t_i, \mathbf{X}_{t_i}} \left[\int_{t_i}^{t_{i+1}} f(t, \mathbf{X}_t) \mathbb{1}_{\mathbf{X}_t \in B^E(\boldsymbol{\theta}_{t_i, t_i}(\mathbf{X}_{t_i}), \delta)} dt \right]] \right\} \\ & \quad + \left\{ \sum_{i=0}^{N-1} \mathbb{E}_{\mathbf{x}} [\mathbb{E}_{t_i, \mathbf{X}_{t_i}} \left[\int_{t_i}^{t_{i+1}} f(t, \mathbf{X}_t) \mathbb{1}_{\mathbf{X}_t \notin B^E(\boldsymbol{\theta}_{t_i, t_i}(\mathbf{X}_{t_i}), \delta)} dt \right]] \right\} := T_C + T_F, \end{aligned}$$

for some $\delta > 0$ to be specified later on. From the second part of Theorem 5.1 we obtain that for $p > n^2 d/2 + 1$,

$$(5.9) \quad |T_F| \leq C_{5.1}^2 \sum_{i=0}^{N-1} \mathbb{E}_{\mathbf{x}} [(1 + |\mathbf{X}_{t_i}|)] \|f\|_{L^p([0, T] \times \mathbb{R}^{nd})} \leq N C_{5.1}^2 C_F (1 + |\mathbf{x}|),$$

where $C_F := C_F(T_0, (\mathbf{A}))$. On the other hand, we can follow the localization procedure of the previous proof (see equation (5.3)), and find $\delta > 0$ s.t. setting $h := \frac{t_{i+1} - t_i}{\lceil C_2(1 + |\mathbf{X}_{t_i}|) \rceil}$ and $t_i^j := t_i + hj$, $j \in \llbracket 0, \lceil C_2(1 + |\mathbf{X}_{t_i}|) \rceil \rrbracket$ then the coefficients $\tilde{a}, \tilde{\mathbf{F}}$ equal to a, \mathbf{F} on $\mathcal{C}_{t_i^j, t_i^{j+1}, 2\delta}(t_i, \mathbf{X}_{t_i})$ and to $a(t_i, \mathbf{X}_{t_i}), \mathbf{F}(t_i, \mathbf{X}_{t_i})$ elsewhere, belong to the class $\mathcal{A}(r, T)$ for some $r \in (2, n^2 d/2 + 1)$. We then derive from Lemma 5.4:

$$(5.10) \quad \begin{aligned} |T_C| & \leq \sum_{i=0}^{N-1} \mathbb{E}_{\mathbf{x}} \left[\sum_{j=0}^{\lceil C_2(1 + |\mathbf{X}_{t_i}|) \rceil - 1} \mathbb{E}_{t_i, \mathbf{X}_{t_i}} \left[\mathbb{E}_{\tilde{\mathbb{P}}_{t_i^j, \mathbf{X}_{t_i^j}}} \left[\int_{t_i^j}^{t_i^{j+1}} |f(t, \mathbf{X}_t)| \mathbb{1}_{\mathbf{X}_t \in B^E(\boldsymbol{\theta}_{t_i, t_i}(\mathbf{X}_{t_i}), \delta)} dt \right] \right. \right. \\ & \quad \left. \left. + \sup_{(s, \mathbf{y}) \in \partial \mathcal{C}_{t_i^j, t_i^{j+1}, \delta}(t_i, \mathbf{X}_{t_i})} \mathbb{E}_{\tilde{\mathbb{P}}^{s, \mathbf{y}}} \left[\int_s^{t_i^{j+1}} |f(t, \mathbf{X}_t)| \mathbb{1}_{\mathbf{X}_t \in B^E(\boldsymbol{\theta}_{t_i, t_i}(\mathbf{X}_{t_i}), \delta)} dt \right] \right] \right]. \end{aligned}$$

Now, from equation (3.19), we derive

$$(5.10) \quad |T_C| \leq C \|f\|_{L^p([0, T] \times \mathbb{R}^{nd})} \sum_{i=0}^{N-1} \mathbb{E} [\lceil C_2(1 + |\mathbf{X}_{t_i}|) \rceil] \leq CN(1 + |\mathbf{x}|) \|f\|_{L^p([0, T] \times \mathbb{R}^{nd})},$$

up to a modification of C . The result follows from (5.10), (5.9), (5.8).

6. PROOFS OF THE TECHNICAL RESULTS

6.1. **Proofs concerning the quasi-metric structure (Proposition (4.1)).** Let us first observe from the definition of the balls, see equations (4.1), (4.2), that there exists $C_1 := C_1((\mathbf{A})) > 0$ s.t. for all $\delta > 0$, $(s, \mathbf{x}) \in S$,

$$|B((s, \mathbf{x}), \delta)| \leq C_1 \delta^{2+d \sum_{i=1}^n (2i-1)} = C_1 \delta^{2+n^2 d}.$$

On the other hand, introducing

$$\begin{aligned} \bar{B}((s, \mathbf{x}), \delta) &:= \left\{ (t, \mathbf{y}) \in S : |t-s| \leq \frac{\delta^2}{4}, \rho_{\text{Sp}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \leq \frac{\delta}{2} \right\} \\ &= \bigcup_{\substack{|t-s| \leq \frac{\delta^2}{4}, \\ s \in [-T, T]}} \left\{ \mathbf{y} \in \mathbb{R}^{nd} : \rho_{\text{Sp}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \leq \frac{\delta}{2} \right\}, \\ \forall \mathbf{z} \in \mathbb{R}^{nd}, \rho_{\text{Sp}}(\mathbf{z}) &:= \left(\sum_{i=1}^n |\mathbf{z}_i|^{\frac{2(2n-1)}{2i-1}} \right)^{\frac{1}{2(2n-1)}}, \end{aligned}$$

i.e. ρ_{Sp} corresponds to the spatial contribution in the metric (4.1), we have that $\bar{B}((s, \mathbf{x}), \delta) \subset B((s, \mathbf{x}), \delta)$. Indeed, for all $(t, \mathbf{y}) \in \bar{B}((s, \mathbf{x}), \delta)$, $d((s, \mathbf{x}), (t, \mathbf{y})) := \rho(|t-s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \leq |t-s|^{1/2} + \rho_{\text{Sp}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \leq \delta$. Since we also have, up to a modification of C_1 that for all $(s, \mathbf{x}) \in S$, $\delta > 0$, $|\bar{B}((s, \mathbf{x}), \delta)| \geq C_1^{-1} \delta$ we therefore derive:

$$C_1^{-1} \delta^{2+n^2 d} \leq |B((s, \mathbf{x}), \delta)| \leq C_1 \delta^{2+n^2 d},$$

which gives b) and c). To derive a), we need to exploit the specific structure of the dynamics. Let us first recall how to relate the forward and backward dynamics. Precisely, one has for all $v \in I(t, s) := ([s, t] \mathbb{I}_{s < t}) \cup ([t, s] \mathbb{I}_{s \geq t})$,

$$(6.1) \quad \boldsymbol{\theta}_{v,s}(\mathbf{x}) - \boldsymbol{\theta}_{v,t}(\mathbf{y}) = \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y} - \int_v^t (\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) - \mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}))) du,$$

which for $v = s$ yields:

$$\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}) = \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y} - \int_s^t (\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) - \mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}))) du.$$

Starting from the last components, and assuming w.l.o.g. that $t > s$, we have:

$$\begin{aligned} |(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_n| &\leq |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_n| \\ &+ C_2 \int_s^t (|(\boldsymbol{\theta}_{v,s}(\mathbf{x}) - \boldsymbol{\theta}_{v,t}(\mathbf{y}))_{n-1}| + |(\boldsymbol{\theta}_{v,s}(\mathbf{x}) - \boldsymbol{\theta}_{v,t}(\mathbf{y}))_n|) dv \\ &\leq \exp(C_2(t-s)) \left(|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_n| + C_2 \int_s^t |(\boldsymbol{\theta}_{v,s}(\mathbf{x}) - \boldsymbol{\theta}_{v,t}(\mathbf{y}))_{n-1}| dv \right), \end{aligned}$$

where $C_2 := C_2((\mathbf{A}))$ and using Gronwall's Lemma for the last inequality. Using iteratively (6.1) and Gronwall's Lemma we derive that there exists $C_3 := C_3(T, (\mathbf{A}))$ s.t.

$$(6.2) \quad |(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_n| \leq C_3 \sum_{j=1}^n |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j| |t-s|^{n-j}.$$

Using Young's inequality with $p_j = \frac{2n-1}{2j-1}$, $q_j = \frac{2n-1}{2(n-j)}$ in order to make the homogeneous exponent of coordinate $j \in \llbracket 1, n-1 \rrbracket$ appear, we get:

$$(6.3) \quad |(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_n|^{1/(2n-1)} \leq C_3^{1/(2n-1)} \left[|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_n|^{1/(2n-1)} + \sum_{j=1}^{n-1} \left(\frac{|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j|^{1/(2j-1)}}{p_j} + \frac{|t-s|^{1/2}}{q_j} \right) \right] \leq C_4 d((s, \mathbf{x}), (t, \mathbf{y})),$$

for $C_4 := C_4(T, (\mathbf{A}))$.

The above estimate does not exploit the fact that $d((s, \mathbf{x}), (t, \mathbf{y})) \leq 1$. This last assumption is actually needed for the components $i \in \llbracket 1, n-1 \rrbracket$ whose differential dynamics potentially involves coordinates $j > i$ with higher characteristic time-scales in small times but that are not neglectable in the ‘‘homogeneous’’ norm we consider. Namely, similarly to (6.2) we derive for all $i \in \llbracket 1, n-1 \rrbracket$ up to a modification of C_3 :

$$(6.4) \quad |(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_i| \leq C_3 \left(\sum_{j=1}^i |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j| |t-s|^{i-j} + \sum_{j=i+1}^n |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j| |t-s| \right).$$

Thus,

$$|(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_i|^{1/(2i-1)} \leq C_3^{1/(2i-1)} \left(\sum_{j=1}^i (|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j| |t-s|^{i-j})^{1/(2i-1)} + \sum_{j=i+1}^n (|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j| |t-s|)^{1/(2i-1)} \right).$$

For the first contribution of the r.h.s. we can use again Young's inequality with $p_j = \frac{2i-1}{2j-1}$, $q_j = \frac{2i-1}{2(i-j)}$. For the second contribution we exploit that since $d((s, \mathbf{x}), (t, \mathbf{y})) \leq 1$ then for all $j \in \llbracket 1, n \rrbracket$, $|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j| \leq 1$ which for $j \geq i+1$ yields $|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j|^{1/(2i-1)} \leq |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_j|^{1/(2j-1)}$. We therefore get up to a modification of C_4 that:

$$|(\mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y}))_i|^{1/(2i-1)} \leq C_4 d((s, \mathbf{x}), (t, \mathbf{y})),$$

which together with (6.3) indeed gives that there exists $C_{4.1} := C_{4.1}((\mathbf{A}), T)$ s.t. $d((t, \mathbf{y}), (s, \mathbf{x})) \leq C_{4.1} d((s, \mathbf{x}), (t, \mathbf{y}))$ for $(s, \mathbf{x}), (t, \mathbf{y}) \in S$, $d((s, \mathbf{x}), (t, \mathbf{y})) \leq 1$ which is the first part of a). It remains to prove the quasi-triangle inequality. Recalling that ρ defined in (4.1) satisfies the quasi-triangle inequality, let us write:

$$(6.5) \quad \begin{aligned} d((s, \mathbf{x}), (t, \mathbf{y})) &= \rho(|t - \sigma + \sigma - s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}) + \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}) - \mathbf{y}) \\ &\leq K(\rho(|\sigma - s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi})) + \rho(|t - \sigma|, \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}) - \mathbf{y})) \\ &:= K(\rho(|\sigma - s|, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi})) + d((\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))). \end{aligned}$$

On the other hand, using the specific form of \mathbf{F} in the dynamics of $\boldsymbol{\theta}$, we can derive similarly to (6.2), (6.4) using the direct forward dynamics that for all $i \in \llbracket 1, n \rrbracket$:

$$\begin{aligned} & |(\boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x})) - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}))_i| \\ & \leq C_3 \left(\sum_{j=1}^i |(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})_j| |t - \sigma|^{i-j} + \sum_{j=i+1}^n |(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})_j| |t - \sigma| \right). \end{aligned}$$

Thus, using as above Young inequalities and the fact that $d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi})) \leq 1$ we get for all $i \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} & |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}))_i|^{1/(2i-1)} \\ & \leq C_3^{1/(2i-1)} \left(\sum_{j=1}^i \left[\frac{|(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})_j|^{1/(2j-1)}}{p_j} + \frac{|t - \sigma|^{1/2} + |\sigma - s|^{1/2}}{q_j} \right] \right) \\ & + \sum_{j=i+1}^n |(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})_j|^{1/(2j-1)} |t - \sigma|^{1/(2i-1)} \leq C_4 (d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi})) + d((\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))), \end{aligned}$$

with $p_j = \frac{2i-1}{2j-1}$, $q_j = \frac{2i-1}{2(i-j)}$ in the last but one equality. Hence $\rho(\sigma - s, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi})) \leq C_5 (d((s, \mathbf{x}), (\sigma, \boldsymbol{\xi})) + d((\sigma, \boldsymbol{\xi}), (t, \mathbf{y})))$, $C_5 := C_5((\mathbf{A}), T)$, which plugged into (6.5) concludes the proof up to a modification of $C_{4.1}$. \square

Remark 6.1 (Subdiagonal structure). Observe from the previous proof that when the function \mathbf{F} in the dynamics of $\boldsymbol{\theta}$ has the following structure, $\mathbf{F}_1(t, \mathbf{x}) = \mathbf{F}_1(t, \mathbf{x}_1)$, and for all $i \in \llbracket 2, n \rrbracket$, $\mathbf{F}_i(t, \mathbf{x}^{i-1, n}) = \mathbf{F}_i(t, \mathbf{x}_{i-1}, \mathbf{x}_i)$, then the terms in $j \in \llbracket i+1, n \rrbracket$ do not appear in equation (6.4). Hence, the distances respectively associated to the forward and backward transport are actually equivalent.

6.2. Proofs of the results concerning the kernel.

6.2.1. *Proof of Lemma 4.1.* From now on for $\varphi \in L^1(\mathbb{R} \times \mathbb{R}^{nd})$, we denote by $\hat{\varphi}^{\mathbf{x}}(s, \boldsymbol{\xi}) = \int_{\mathbb{R}^{nd}} \exp(i\langle \boldsymbol{\xi}, x \rangle) f(s, \mathbf{x}) d\mathbf{x}$ its Fourier transform in space. For $f \in C_0^\infty([0, T] \times \mathbb{R}^{nd})$ we easily derive from (4.4), (4.5) that $K_{ij}^\varepsilon f$ is continuous and rapidly decreasing on $\mathbb{R} \times \mathbb{R}^{nd}$. Hence, for all $s \in [0, T - \varepsilon]$, $\boldsymbol{\xi} \in \mathbb{R}^{nd}$,

$$\begin{aligned} \widehat{K_{ij}^\varepsilon f}^{\mathbf{x}}(s, \boldsymbol{\xi}) &= \int_{\mathbb{R}^{nd}} \exp(i\langle \boldsymbol{\xi}, \mathbf{x} \rangle) \left(\int_{\mathbb{R}^{nd}} \int_{s+\varepsilon}^T \partial_{\mathbf{x}_i^i \mathbf{x}_1^j} \tilde{q}(s, t, \mathbf{x}, \mathbf{y}) f(t, \mathbf{y}) dt d\mathbf{y} \right) d\mathbf{x} \\ &\stackrel{(3.4)}{=} -\boldsymbol{\xi}_1^i \boldsymbol{\xi}_1^j \int_{\mathbb{R}^{nd}} d\mathbf{y} \int_{s+\varepsilon}^T dt \int_{\mathbb{R}^{nd}} d\mathbf{x} \exp(i\langle \boldsymbol{\xi}, \mathbf{x} \rangle) f(t, \mathbf{y}) \\ &\frac{\exp\left(-\frac{1}{2}\langle \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1}(\tilde{\mathbf{R}}^{t, \mathbf{y}}(t, s)\mathbf{x} + \tilde{\mathbf{m}}^{t, \mathbf{y}}(s, t) - \mathbf{y}), \tilde{\mathbf{R}}^{t, \mathbf{y}}(t, s)\mathbf{x} - \tilde{\mathbf{m}}^{t, \mathbf{y}}(s, t) - \mathbf{y} \rangle\right)}{(2\pi)^{nd/2} \det(\tilde{\mathbf{K}}^{\mathbf{y}}(s, t))^{1/2}} \\ &= -\boldsymbol{\xi}_1^i \boldsymbol{\xi}_1^j \int_{\mathbb{R}^{nd}} d\mathbf{y} \int_{s+\varepsilon}^T dt \int_{\mathbb{R}^{nd}} d\mathbf{x} \exp(i\langle \boldsymbol{\xi}, \mathbf{x} \rangle) f(t, \mathbf{y}) \\ &\frac{\exp\left(-\frac{1}{2}\langle \tilde{\mathbf{H}}^{t, \mathbf{y}}(s, t)^{-1}(\mathbf{x} - \tilde{\mathbf{R}}^{t, \mathbf{y}}(s, t)(\mathbf{y} - \tilde{\mathbf{m}}^{t, \mathbf{y}}(s, t)), \mathbf{x} - \tilde{\mathbf{R}}^{t, \mathbf{y}}(s, t)(\mathbf{y} - \tilde{\mathbf{m}}^{t, \mathbf{y}}(s, t))) \rangle\right)}{(2\pi)^{nd/2} \det(\tilde{\mathbf{K}}^{\mathbf{y}}(s, t))^{1/2}}, \end{aligned}$$

setting

$$(6.6) \quad \tilde{\mathbf{H}}^{t, \mathbf{y}}(s, t) := \tilde{\mathbf{R}}^{t, \mathbf{y}}(s, t) \tilde{\mathbf{K}}^{\mathbf{y}}(s, t) \tilde{\mathbf{R}}^{t, \mathbf{y}}(s, t)^*.$$

Observe now from equation (3.4) that $\tilde{\mathbf{R}}^{t,\mathbf{y}}(s,t)(\mathbf{y} - \tilde{\mathbf{m}}^{t,\mathbf{y}}(s,t)) = \boldsymbol{\theta}_{s,t}(\mathbf{y})$ (pull-back by the deterministic system of the final point \mathbf{y} from t to s). Thus, recalling from the subdiagonal structure in (3.4) that $\det(\tilde{\mathbf{R}}^{t,\mathbf{y}}(s,t)) = 1$:

$$\begin{aligned} \widehat{K_{ij}^\epsilon f^{\mathbf{x}}}(s, \boldsymbol{\xi}) &= -\boldsymbol{\xi}_1^i \boldsymbol{\xi}_1^j \int_{s+\epsilon}^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} f(t, \mathbf{y}) \exp(i\langle \boldsymbol{\xi}, \boldsymbol{\theta}_{s,t}(\mathbf{y}) \rangle) \\ &\quad \times \left(\int_{\mathbb{R}^{nd}} d\mathbf{z} \frac{\exp\left(-\frac{1}{2}\langle \tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t)^{-1} \mathbf{z}, \mathbf{z} \rangle\right)}{(2\pi)^{nd/2} \det(\tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t))^{1/2}} \exp(i\langle \boldsymbol{\xi}, \mathbf{z} \rangle) \right) \\ &= -\boldsymbol{\xi}_1^i \boldsymbol{\xi}_1^j \int_{s+\epsilon}^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} f(t, \mathbf{y}) \exp(i\langle \boldsymbol{\xi}, \boldsymbol{\theta}_{s,t}(\mathbf{y}) \rangle) \exp\left(-\frac{1}{2}\langle \tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle\right). \end{aligned} \quad (6.7)$$

Remark 6.2. Observe that up to now we have not exploited the specific linear structure of the drift assumed in Lemma 4.1. From (6.7) this is indeed a very useful way to get rid-off the transport by simply changing the Fourier argument, writing $\exp(i\langle \boldsymbol{\xi}, \boldsymbol{\theta}_{s,t}(\mathbf{y}) \rangle) = \exp(i\langle \boldsymbol{\xi}, \mathbf{R}(s,t)\mathbf{y} \rangle) = \exp(i\langle \mathbf{R}(s,t)^* \boldsymbol{\xi}, \mathbf{y} \rangle)$, the flow writing as the transport by the resolvent³. In the non-linear case, even under the local condition (3.1), that expresses that the deterministic differential system is *almost linear*, it does not seem easy to control the remainders in a Fourier inversion starting from (6.7). This is mainly due to the dependence in the integration variable \mathbf{y} of the covariance matrix $\tilde{\mathbf{H}}^{t,\mathbf{y}}$.

When the drift is linear, we manage to adapt the proofs of Fabes, Stroock and Varadhan (see Appendix in [SV79]). In that case, equation (6.7) rewrites:

$$\begin{aligned} \widehat{K_{ij}^\epsilon f^{\mathbf{x}}}(s, \boldsymbol{\xi}) &= -\boldsymbol{\xi}_1^i \boldsymbol{\xi}_1^j \int_{s+\epsilon}^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y} f(t, \mathbf{y}) \exp(i\langle \mathbf{R}(s,t)^* \boldsymbol{\xi}, \mathbf{y} \rangle) \exp\left(-\frac{1}{2}\langle \tilde{\mathbf{H}}(s,t) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle\right) \\ &= -\boldsymbol{\xi}_1^i \boldsymbol{\xi}_1^j \int_{s+\epsilon}^T dt \hat{f}(t, \mathbf{R}(s,t)^* \boldsymbol{\xi}) \exp\left(-\frac{1}{2}\langle \tilde{\mathbf{H}}(s,t) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle\right). \end{aligned}$$

Observe now from (6.6), Lemma B.1 and (3.7) (good scaling property) that there exists $c := c(T, (\mathbf{A}))$ s.t.:

$$|\widehat{K_{ij}^\epsilon f^{\mathbf{x}}}(s, \boldsymbol{\xi})| \leq |\boldsymbol{\xi}_1|^2 \int_s^T dt |\hat{f}(t, \mathbf{R}(s,t)^* \boldsymbol{\xi})| \exp(-c(t-s)^{-1} |\mathbb{T}_{t-s} \boldsymbol{\xi}|^2).$$

Thus, the Cauchy-Schwarz inequality yields:

$$\begin{aligned} \|\widehat{K_{ij}^\epsilon f^{\mathbf{x}}}\|_{L^2([0,T] \times \mathbb{R}^{nd})}^2 &\leq \int_0^T ds \int_{\mathbb{R}^{nd}} d\boldsymbol{\xi} |\boldsymbol{\xi}_1|^4 \int_s^T dt |\hat{f}(t, \mathbf{R}(s,t)^* \boldsymbol{\xi})|^2 \\ &\quad \times \exp(-c(t-s)^{-1} |\mathbb{T}_{t-s} \boldsymbol{\xi}|^2) \int_s^T dt \exp(-c(t-s)^{-1} |\mathbb{T}_{t-s} \boldsymbol{\xi}|^2) \\ &\leq c^{-1} \int_0^T ds \int_{\mathbb{R}^{nd}} d\boldsymbol{\xi} |\boldsymbol{\xi}_1|^2 \int_s^T dt |\hat{f}(t, \mathbf{R}(s,t)^* \boldsymbol{\xi})|^2 \times \exp\left(-\frac{c}{C^2}(t-s)^{-1} |\mathbb{T}_{t-s} \mathbf{R}(s,t)^* \boldsymbol{\xi}|^2\right) \end{aligned}$$

recalling from the Scaling Lemma B.1 that $|\mathbb{T}_{t-s} \mathbf{R}(s,t)^* \boldsymbol{\xi}| \leq C |\mathbb{T}_{t-s} \boldsymbol{\xi}|$, $C(T, (\mathbf{A}))$ for the last inequality. The key idea is now to set $\tilde{\boldsymbol{\xi}} := \mathbf{R}(s,t)^* \boldsymbol{\xi}$ and to exploit the

³We stress that even in the linear case $\mathbf{R} \neq \tilde{\mathbf{R}}$, the latter corresponding to the resolvent associated to the subdiagonal part of the system. In that case, the matrices $\tilde{\mathbf{K}}, \tilde{\mathbf{H}}$ are defined from $\tilde{\mathbf{R}}$ as before and do not depend on t nor \mathbf{y} .

structure of the resolvent. Still from the Scaling Lemma B.1, one indeed derives that there exists $\bar{C} := \bar{C}(T, (\mathbf{A}))$, $\forall i \in \llbracket 1, n \rrbracket$, $|(\mathbf{R}(t, s)^*)_{1,i}| \leq \bar{C}|t-s|^{i-1}$. We therefore get up to a modification of the previous C :

$$\begin{aligned} \|\widehat{K_{ij}^\epsilon} f^{\mathbf{x}}\|_{L^2([0,T] \times \mathbb{R}^{nd})}^2 &\leq C \int_0^T ds \int_{\mathbb{R}^{nd}} d\bar{\xi} |(\mathbf{R}(t, s)^* \bar{\xi})_1|^2 \int_s^T dt |\hat{f}(t, \bar{\xi})|^2 \\ &\quad \times \exp(-C^{-1}(t-s)^{-1} |\mathbb{T}_{t-s} \bar{\xi}|^2) \\ &\leq C \bar{C} \int_0^T ds \int_{\mathbb{R}^{nd}} d\bar{\xi} \sum_{j=1}^n (t-s)^{2(j-1)} |\bar{\xi}_j|^2 \int_s^T dt |\hat{f}(t, \bar{\xi})|^2 \\ &\quad \times \exp(-C^{-1}(t-s)^{-1} |\mathbb{T}_{t-s} \bar{\xi}|^2) \\ &\leq C \bar{C} \int_0^T dt \int_{\mathbb{R}^{nd}} d\bar{\xi} |\hat{f}(t, \bar{\xi})|^2 \int_0^t ds \sum_{j=1}^n (t-s)^{2(j-1)} |\bar{\xi}_j|^2 \exp(-C^{-1} \sum_{j=1}^n (t-s)^{2j-1} |\bar{\xi}_j|^2) \\ &\leq \tilde{C} \int_0^T dt \int_{\mathbb{R}^{nd}} d\bar{\xi} |\hat{f}(t, \bar{\xi})|^2, \quad \tilde{C} := \tilde{C}(T, (\mathbf{A})). \end{aligned}$$

The statement follows from Parseval's identity.

6.2.2. *Proof of Lemma 4.2.* We have:

$$\begin{aligned} &\int_{\rho(t-s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \geq c_\infty \rho(|\sigma-s|, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))} |k_{ij}^\epsilon(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}^\epsilon(\sigma, t, \boldsymbol{\xi}, \mathbf{y})| dt d\mathbf{y} \\ &\leq \int_{\rho(t-s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \geq c_\infty \rho(|\sigma-s|, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))} |k_{ij}(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}(\sigma, t, \boldsymbol{\xi}, \mathbf{y})| dt d\mathbf{y} \\ &+ \int_{\rho(t-s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \geq c_\infty \rho(|\sigma-s|, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))} |\mathbb{I}_{t \in [s+\epsilon, T]} - \mathbb{I}_{t \in [\sigma+\epsilon, T]}| |k_{ij}(s, t, \mathbf{x}, \mathbf{y})| dt d\mathbf{y} \\ (6.8) \qquad \qquad \qquad &:= T_1 + T_2^\epsilon, \end{aligned}$$

with $k_{ij}(s, t, \mathbf{x}, \mathbf{y}) = \mathbb{I}_{0 \leq s < t \leq T} D_{\mathbf{x}_1^i \mathbf{x}_1^j}^2 \tilde{q}(s, t, \mathbf{x}, \mathbf{y})$, $\forall (s, t) \times (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times (\mathbb{R}^{nd})^2$.

Set $\alpha := \rho(|\sigma-s|, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))$. Similarly to (4.5), we derive that $|k_{ij}(s, t, \mathbf{x}, \mathbf{y})| \leq C_{4.5} \mathbb{I}_{0 \leq s < t \leq T} (t-s)^{-1} q_{c_{4.5}}(s, t, \mathbf{x}, \mathbf{y})$, $\forall (s, t) \times (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times (\mathbb{R}^{nd})^2$. Hence:

$$\begin{aligned} T_2^\epsilon &\leq C_{4.5} \int_{\rho(t-s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \geq c_\infty \alpha} |\mathbb{I}_{t \in [s+\epsilon, T]} - \mathbb{I}_{t \in [\sigma+\epsilon, T]}| \frac{1}{t-s} q_{c_{4.5}}(s, t, \mathbf{x}, \mathbf{y}) dt d\mathbf{y} \\ &\leq C \int_{\rho(t, \mathbf{y} - \boldsymbol{\theta}_{t+s,s}(\mathbf{x})) \geq c_\infty \alpha} |\mathbb{I}_{t \in [\epsilon, T-s]} - \mathbb{I}_{t-(\sigma-s) \in [\epsilon, T-\sigma]}| \frac{\mathbb{I}_{t>0}}{t^{n^2 d/2+1}} \\ &\quad \exp(-c_{4.5} t |\mathbb{T}_t^{-1}(\boldsymbol{\theta}_{t+s,s}^{t+s, \mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2) dt d\mathbf{y}, \quad C := C(T, (\mathbf{A})). \end{aligned}$$

Now from (3.9) and the bi-Lipschitz property of the flow we have:

$$\begin{aligned} T_2^\epsilon &\leq C \int_{\rho(t, \mathbf{y} - \boldsymbol{\theta}_{t+s,s}(\mathbf{x})) \geq c_\infty \alpha} |\mathbb{I}_{t \in [\epsilon, T-s]} - \mathbb{I}_{t-(\sigma-s) \in [\epsilon, T-\sigma]}| \frac{\mathbb{I}_{t>0}}{t^{n^2 d/2+1}} \\ &\quad \exp(-ct |\mathbb{T}_t^{-1}(\boldsymbol{\theta}_{t+s,s}(\mathbf{x}) - \mathbf{y})|^2) dt d\mathbf{y}, \quad c := c(T, (\mathbf{A})). \end{aligned}$$

Setting $\mathbf{Y} := t^{1/2} \mathbb{T}_t^{-1}(\boldsymbol{\theta}_{t+s,s}(\mathbf{x}) - \mathbf{y})$ we get

$$T_2^\epsilon \leq C \int_{\rho(t, t^{-1/2} \mathbb{T}_t \mathbf{Y}) \geq c_\infty \alpha, t>0} |\mathbb{I}_{t \in [\epsilon, T-s]} - \mathbb{I}_{t-(\sigma-s) \in [\epsilon, T-\sigma]}| \frac{\exp(-c|\mathbf{Y}|^2)}{t} dt d\mathbf{Y}.$$

From the homogeneity properties of the metric ρ defined in (4.1) (see Remark 4.1) we derive:

$$T_2^\epsilon \leq C \int_{\rho(1, \mathbf{Y}) \geq c_\infty \alpha t^{-1/2}, t > 0} |\mathbb{I}_{t \in [\epsilon, T-s]} - \mathbb{I}_{t - (\sigma-s) \in [\epsilon, T-\sigma]}| \frac{\exp(-c|\mathbf{Y}|^2)}{t} dt d\mathbf{Y}.$$

Recalling that $|\sigma - s| \leq \alpha^2$ we obtain:

$$\begin{aligned} T_2^\epsilon &\leq C \sup_{S \geq 0} \int_S^{S+\alpha^2} \frac{dt}{t} \int_{\rho(1, \mathbf{Y}) \geq c_\infty \alpha t^{-1/2}} \exp(-c|\mathbf{Y}|^2) d\mathbf{Y} \\ &\leq C \sup_{S \geq 0} \int_{S/\alpha^2}^{S/\alpha^2+1} \frac{dt}{t} \int_{\rho(1, \mathbf{Y}) \geq c_\infty t^{-1/2}} \exp(-c|\mathbf{Y}|^2) d\mathbf{Y} \\ &\leq C \sup_{S \geq 0} \int_S^{S+1} \frac{dt}{t} \int_{\rho(1, \mathbf{Y}) \geq c_\infty t^{-1/2}} \exp(-c|\mathbf{Y}|^2) d\mathbf{Y}. \end{aligned}$$

We now perform a dichotomy on the values of t considering the cases:

- $t > \left(\frac{c_\infty^{4n-2}}{2}\right)^{\frac{1}{2n-1}}$. There is then no time singularity in the above integral.
- $t \leq \left(\frac{c_\infty^{4n-2}}{2}\right)^{\frac{1}{2n-1}}$. In this case

$$\begin{aligned} &\{\mathbf{Y} \in \mathbb{R}^{nd} : \rho(1, \mathbf{Y}) := (1 + \sum_{j=1}^n |\mathbf{Y}_j|^{\frac{2(2n-1)}{2j-1}})^{\frac{1}{2(2n-1)}} \geq c_\infty t^{-1/2}\} \\ &\subset \{\mathbf{Y} \in \mathbb{R}^{nd} : \rho(1, \mathbf{Y}) := (1 + \sum_{j=1}^n |\mathbf{Y}_j|^{\frac{2(2n-1)}{2j-1}}) \geq \frac{c_\infty^{2(2n-1)}}{2t^{2n-1}} + 1\} \\ &\subset \cup_{j=1}^n \{\mathbf{Y}_j \in \mathbb{R}^d : |\mathbf{Y}_j|^2 \geq t^{-(2j-1)} c(n, j), c(n, j) := \left(\frac{c_\infty^2}{(2n)^{1/(2n-1)}}\right)^{2j-1}\}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\rho(1, \mathbf{Y}) \geq c_\infty t^{-1/2}} \exp(-c|\mathbf{Y}|^2) d\mathbf{Y} &\leq C \sum_{j=1}^n \exp\left(-\frac{c \times c(n, j)}{2t^{2j-1}}\right) \int \exp\left(-\frac{c}{2} |\mathbf{Y}_j|^2\right) d\mathbf{Y}_j \\ &\leq C \sum_{j=1}^n \exp\left(-\frac{c \times c(n, j)}{2t^{2j-1}}\right). \end{aligned}$$

These controls yield

$$(6.9) \quad T_2^\epsilon \leq C \sup_{S \geq 0} \int_S^{S+1} dt \left\{ \left(\frac{2}{c_\infty^{4n-2}}\right)^{\frac{1}{2n-1}} + t^{-1} \sum_{i=1}^n \exp\left(-\frac{c \times c(n, j)}{2t^{2j-1}}\right) \right\} \leq C.$$

Let us now turn to the control of T_1 in (6.8). The integrand writes:

$$\begin{aligned} |k_{ij}(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}(\sigma, t, \boldsymbol{\xi}, \mathbf{y})| &\leq |k_{ij}(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}(\sigma, t, \tilde{\boldsymbol{\theta}}_{\sigma, s}^{t, \mathbf{y}}(\mathbf{x}), \mathbf{y})| \\ &\quad + |k_{ij}(\sigma, t, \tilde{\boldsymbol{\theta}}_{\sigma, s}^{t, \mathbf{y}}(\mathbf{x}), \mathbf{y}) - k_{ij}(\sigma, t, \boldsymbol{\xi}, \mathbf{y})| \\ (6.10) \quad &:= I_1(s, \sigma, t, \mathbf{x}, \mathbf{y}) + I_2(\sigma, t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{y}). \end{aligned}$$

Remark 6.3. The previous splitting of $|k_{ij}(s, t, \mathbf{x}, \mathbf{y}) - k_{ij}(\sigma, t, \boldsymbol{\xi}, \mathbf{y})|$ has been done to separate the time and space sensitivities. In I_1 the space variable is frozen and from (4.4) and the flow property of $\tilde{\boldsymbol{\theta}}^{t, \mathbf{y}}$ its value is equal to $\tilde{\boldsymbol{\theta}}_{t, s}^{t, \mathbf{y}}(\mathbf{x}) - \mathbf{y}$. In I_2

the time variables are equal to $t - \sigma$. Also, the intermediate spatial point $\tilde{\boldsymbol{\theta}}_{\sigma,s}^{t,\mathbf{y}}(\mathbf{x})$ yields from I_2 a difference of the form $|\tilde{\boldsymbol{\theta}}_{\sigma,s}^{t,\mathbf{y}}(\mathbf{x}) - \boldsymbol{\xi}|$ which up to a linearization error has the same order as $|\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \mathbf{y}|$, norm of the spatial point appearing in the integration set in (4.7).

Setting for all $0 \leq s < t \leq T$, $(\mathbf{z}, \mathbf{y}) \in (\mathbb{R}^{nd})^2$:

$$\begin{aligned} \bar{k}_{ij}(s, t, \mathbf{z}, \mathbf{y}) &:= \mathbb{I}_{0 \leq s < t \leq T} \left(-[\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s)]_{11} \right. \\ &\quad \left. + [\tilde{\mathbf{R}}^{t,\mathbf{y}}(t, s) * \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} (\mathbf{z} - \mathbf{y})]_1^{\otimes 2} \right) \times \left(\frac{1}{(2\pi)^{nd} \det(\tilde{\mathbf{K}}^{\mathbf{y}}(s, t))^{1/2}} \right. \\ &\quad \left. \times \exp\left(-\frac{1}{2} \langle \tilde{\mathbf{K}}^{\mathbf{y}}(s, t)^{-1} (\mathbf{z} - \mathbf{y}), \mathbf{z} - \mathbf{y} \rangle \right) \right), \end{aligned}$$

we can rewrite $I_1(s, \sigma, t, \mathbf{x}, \mathbf{y}) = |\bar{k}_{ij}(s, t, \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}), \mathbf{y}) - \bar{k}_{ij}(\sigma, t, \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}), \mathbf{y})|$. Thus, from (4.4), we derive similarly to (4.5) (see also the proof of Proposition 3.7 in [DM10] for a thorough discussion on the time sensitivities of the covariance matrix) that $\exists(c, C) := (c, C)(T, \mathbf{A}) > 0$ s.t.

$$\begin{aligned} I_1(s, \sigma, t, \mathbf{x}, \mathbf{y}) &\leq |s - \sigma| \sup_{\tau \in [s \wedge \sigma, (s \vee \sigma) \vee t]} |\partial_\tau \bar{k}_{ij}(\tau, t, \mathbf{z}, \mathbf{y})|_{\mathbf{z} = \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x})} \\ (6.11) \quad &\leq C |s - \sigma| \times \sup_{\tau \in [s \wedge \sigma, (s \vee \sigma) \vee t]} \frac{\bar{q}_c(\tau, t, \mathbf{z}, \mathbf{y})}{(t - \tau)^2} \Big|_{\mathbf{z} = \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x})}, \end{aligned}$$

where $\bar{q}_c(\tau, t, \mathbf{z}, \mathbf{y}) := \frac{c^{nd/2}}{(2\pi)^{nd/2} (t - \tau)^{n^2 d/2}} \exp(-\frac{c}{2} (t - \tau) |\mathbb{T}_{t-\tau}^{-1}(\mathbf{z} - \mathbf{y})|^2)$.

Introduce now $\Sigma_1 := \{(u, \mathbf{z}) \in [0, T] \times \mathbb{R}^{nd} : \rho(u, \mathbf{z}) = 1\}$ with ρ defined in (4.1), i.e. Σ_1 is the level curve at 1 of the parabolic metric. Observe also that with the definition of ρ , Σ_1 is a C^1 hypersurface. With this definition we can introduce the mappings:

$$\begin{aligned} J_1 : (t, \mathbf{y}) \in [0, T] \times \mathbb{R}^{nd} &\mapsto (s + \rho^2 \bar{s}, \boldsymbol{\theta}_{t,s}(\mathbf{x}) + \rho^{-1} \mathbb{T}_{\rho^2} \bar{\mathbf{x}}), \quad \rho := \rho(t - s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \\ (6.12) \quad &(\bar{s}, \bar{\mathbf{x}}) := ((t - s)\rho^{-2}, \rho \mathbb{T}_{\rho^{-2}}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))) \in \Sigma_1. \end{aligned}$$

$$\begin{aligned} J_2 : (\sigma, \boldsymbol{\xi}) \in [0, T] \times \mathbb{R}^{nd} &\mapsto (s + \alpha^2 \bar{s}, \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) + \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}}), \quad \alpha := \rho(\sigma - s, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x})) \\ (6.13) \quad &(\bar{s}, \bar{\mathbf{x}}) := ((\sigma - s)\alpha^{-2}, \alpha \mathbb{T}_{\alpha^{-2}}(\boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))) \in \Sigma_1. \end{aligned}$$

From the definition of T_1 in (6.8), we have to integrate the terms I_1, I_2 on the set $\{(t, \mathbf{y}) \in [0, T] \times \mathbb{R}^{nd} : t \geq s \vee \sigma, \rho := \rho(t - s, \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \geq c_\infty \alpha := c_\infty \rho(\sigma - s, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))\}$.

From (6.12) we get that for all $\tau \in [s \wedge \sigma, (s \vee \sigma) \vee t]$:

$$\begin{aligned} \frac{\bar{q}_c(\tau, t, \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}), \mathbf{y})}{(t - \tau)^2} &\leq \frac{C}{\rho^{4+n^2 d}} \frac{1}{\left(\bar{s} - \frac{\tau - s}{\rho^2}\right)^{2+n^2 d/2}} \\ (6.14) \quad &\times \exp\left(-c(t - \tau) |\mathbb{T}_{t-\tau}^{-1}(\rho^{-1} \mathbb{T}_{\rho^2} \{\bar{\mathbf{x}} + \mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y})\})|^2\right), \end{aligned}$$

where $\mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}) := \rho \mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}))$ is a term that takes into account the approximation of the non-linear flow by the linear one.

Now, if $|\bar{s} - \frac{\tau - s}{\rho^2}|^{2n-1} \geq \frac{2^{2n-1}}{c_\infty^2} := \tilde{c}$, for $c_\infty \geq 1$ to be specified later on, we have from the above expression and (6.11) that $I_1(s, \sigma, t, \mathbf{x}, \mathbf{y}) \leq \frac{C|s-\sigma|}{\rho^{4+n^2 d}} \leq \frac{C\alpha^2}{\rho^{4+n^2 d}}$

using (6.13) for the last inequality. On the other hand, recalling from (6.8) that on the integration set in (6.8) $\rho \geq c_\infty \alpha$, we get $|\tau - s| \leq |\sigma - s| \leq \alpha^2 \leq \frac{\rho^2}{c_\infty^2}$. Since $(\tilde{s}, \tilde{\mathbf{x}}) \in \Sigma_1$, we thus derive:

$$\begin{aligned} \sum_{j=1}^n |\tilde{\mathbf{x}}_j|^{\frac{2(2n-1)}{2j-1}} + \left(\tilde{s} - \frac{\tau-s}{\rho^2} \right)^{2n-1} &= 1 + \sum_{i=0}^{2n-2} C_{2n-1}^i(\tilde{s})^i \left(-\frac{(\tau-s)}{\rho^2} \right)^{2n-1-i} \\ &\geq 1 - \frac{\sum_{i=0}^{2n-1} C_{2n-1}^i}{c_\infty^2} := 1 - \tilde{c}. \end{aligned}$$

Hence, for $|\tilde{s} - \frac{\tau-s}{\rho^2}|^{2n-1} \leq \tilde{c}$, we obtain

$$\begin{aligned} \sum_{j=1}^n |\tilde{\mathbf{x}}_j|^{\frac{2(2n-1)}{2j-1}} &\geq 1 - 2\tilde{c} \text{ and } \exists j_0 \in \llbracket 1, n \rrbracket, |\tilde{\mathbf{x}}_{j_0}|^{\frac{2(2n-1)}{2j_0-1}} \geq \frac{1-2\tilde{c}}{n} > 0, \text{ for } c_\infty > 2^n. \\ (6.15) \end{aligned}$$

Write now:

$$\begin{aligned} &(t-\tau) |\mathbb{T}_{t-\tau}^{-1}(\rho^{-1} \mathbb{T}_{\rho^2} \{ \tilde{\mathbf{x}} + \mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}) \})|^2 \\ &:= \sum_{j=1}^n \left(\tilde{s} - \frac{\tau-s}{\rho^2} \right)^{-(2j-1)} |\{ \tilde{\mathbf{x}} + \mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}) \}_j|^2. \end{aligned}$$

From a stability analysis similar to the one of equations (A-8), (A-10) in [Men11], we derive the following lemma whose proof is for the sake of completeness proposed in Appendix A.

Lemma 6.1. Assume that (\mathbf{A}_{SD}) holds or that $0 < \rho \leq \Lambda$. Then, under the local condition (3.1), there exists $C_{6.1} := C_{6.1}((\mathbf{A}), \Lambda)$ s.t.:

$$|\mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y})| \leq C_{6.1}(\varepsilon_{\mathbf{F}} + (t-s)) |\tilde{\mathbf{x}}|.$$

Also, if $\rho := \rho(t-s, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}) \geq c_\infty \alpha := \rho(|\sigma-s|, \boldsymbol{\xi} - \boldsymbol{\theta}_{\sigma,s}(\mathbf{x}))$, then

$$|\mathcal{R}_{t,\sigma}^\rho(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y})| \leq C_{6.1}(\varepsilon_{\mathbf{F}} + t - \sigma) |\tilde{\mathbf{x}}|.$$

This implies that taking $j_0 \in \llbracket 1, n \rrbracket$ s.t. $|\tilde{\mathbf{x}}_{j_0}| := \sup_{i \in \llbracket 1, n \rrbracket} |\tilde{\mathbf{x}}_i|$ one has for all $j \in \llbracket 1, n \rrbracket$:

$$\begin{aligned} &|\{ \mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}) \}_j| \leq C_{6.1} n (\varepsilon_{\mathbf{F}} + (t-s)) |\tilde{\mathbf{x}}_{j_0}|, \\ (6.16) \quad &|\{ \mathcal{R}_{t,\sigma}^\rho(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y}) \}_j| \leq C_{6.1} n (\varepsilon_{\mathbf{F}} + (t-\sigma)) |\tilde{\mathbf{x}}_{j_0}|. \end{aligned}$$

Remark 6.4. The structure or boundedness is here really required to control the remainders, see Remark A.1 in the proof of Lemma 6.1 for details. Anyhow, under the general assumption (\mathbf{A}_G) a truncation similar to the one mentioned in Lemma 4.3 can be performed in order to consider bounded arguments ρ . We thus consider for simplicity until the end of the proof that (\mathbf{A}_{SD}) holds. All the computations below would anyhow remain valid up to the previous truncation on ρ .

Thus, for T and $\varepsilon_{\mathbf{F}}$ s.t. $C_{6.1}n(\varepsilon_{\mathbf{F}} + (t - s)) \leq 1/2$:

$$\begin{aligned} & (t - \tau) |\mathbb{T}_{t-\tau}^{-1}(\rho^{-1} \mathbb{T}_{\rho^2} \{\tilde{\mathbf{x}} + \mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y})\})|^2 \\ & \geq \left(\tilde{s} - \frac{\tau - s}{\rho^2} \right)^{-(2j_0-1)} |\tilde{\mathbf{x}}_{j_0}|^2 \left(\frac{1}{2} - (C_{6.1}n(\varepsilon_{\mathbf{F}} + (t - s)))^2 \right) \\ & \geq \frac{1}{4} \left(\tilde{s} - \frac{\tau - s}{\rho^2} \right)^{-(2j_0-1)} \left(\frac{1 - 2\tilde{c}}{n} \right)^{\frac{2j_0-1}{2n-1}}, \end{aligned}$$

using (6.15) for the last inequality. Plugging the above control into (6.14) yields:

$$\begin{aligned} & \frac{\bar{q}_c(\tau, t, \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}), \mathbf{y})}{(t - \tau)^2} \leq \frac{C}{\rho^{4+n^2d}} \frac{1}{\left(\tilde{s} - \frac{\tau - s}{\rho^2} \right)^{2+n^2d/2}} \\ & \times \exp \left(-\frac{c}{4} \left(\tilde{s} - \frac{\tau - s}{\rho^2} \right)^{-(2j_0-1)} \left(\frac{1 - 2\tilde{c}}{n} \right)^{\frac{2j_0-1}{2n-1}} \right). \end{aligned}$$

From the above computations, we finally get the global bound:

$$(6.17) \quad \exists C_1 := C_1(T, (\mathbf{A})) > 0, \quad I_1(s, \sigma, t, \mathbf{x}, \mathbf{y}) \leq \frac{C_1 \alpha^2}{\rho^{4+n^2d}}.$$

Let us now turn to $I_2(\sigma, t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{y})$. From Proposition B.1 and (4.4), we get similarly to (6.11) that $\exists(c, C) := (c, C)(T, (\mathbf{A})) > 0$ s.t.:

$$\begin{aligned} I_2(\sigma, t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{y}) & \leq C(t - \sigma)^{1/2} |\mathbb{T}_{t-\sigma}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\xi}))| \\ & \quad \times \frac{1}{(t - \sigma)} \sup_{\gamma \in [0,1]} \bar{q}_c(\sigma, t, \gamma \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) + (1 - \gamma) \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\xi}), \mathbf{y}) \\ & \leq C(t - \sigma)^{1/2} |\mathbb{T}_{t-\sigma}^{-1}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})| \\ & \quad \times \frac{1}{(t - \sigma)} \sup_{\gamma \in [0,1]} \bar{q}_c(\sigma, t, \gamma \tilde{\boldsymbol{\theta}}_{\sigma,s}^{t,\mathbf{y}}(\mathbf{x}) + (1 - \gamma) \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\xi}), \mathbf{y}), \end{aligned}$$

using the bi-Lipschitz property of the flows $\tilde{\boldsymbol{\theta}}^{t,\mathbf{y}}$, $\boldsymbol{\theta}$ and (3.9) for the last inequality. Hence:

$$\begin{aligned} I_2(\sigma, t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{y}) & \leq C \left\{ \sum_{k=1}^n \frac{|(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})_k|}{\rho^{2+(2k-1)+n^2d}} \frac{1}{\left(\tilde{s} - \frac{\sigma - s}{\rho^2} \right)^{\frac{2+(2k-1)+n^2d}{2}}} \right\} \\ & \quad \times \sup_{\gamma \in [0,1]} \exp \left(-c(t - \sigma) \left| \mathbb{T}_{t-\sigma}^{-1} \left(\gamma \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) + (1 - \gamma) \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\xi}) - \mathbf{y} \right) \right|^2 \right). \end{aligned}$$

Thus, if $|\tilde{s} - \frac{\sigma - s}{\rho^2}|^{2n-1} \geq \tilde{c}$, we get

$$I_2(\sigma, t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{y}) \leq C \sum_{k=1}^n \frac{|(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \boldsymbol{\xi})_k|}{\rho^{2+(2k-1)+n^2d}} \leq C \sum_{k=1}^n \frac{\alpha^{2k-1}}{\rho^{2+(2k-1)+n^2d}},$$

using (6.13) for the last inequality. Recall now from (6.12), (6.13) that

$$\begin{aligned}
 & (\gamma \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) + (1-\gamma) \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\xi})) - \mathbf{y} \\
 &= \gamma \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) + (1-\gamma) \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) + \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}}) - \mathbf{y} \\
 &\stackrel{(3.4)}{=} \gamma \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) + (1-\gamma) \left\{ \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x})) + \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, \sigma) \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}} \right\} - \mathbf{y} \\
 &= \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y} + \gamma (\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x})) + (1-\gamma) (\tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x})) - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \\
 &\quad + (1-\gamma) \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, \sigma) \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}} \\
 &:= -\rho^{-1} \mathbb{T}_{\rho^2} \{ \tilde{\mathbf{x}} + \gamma \mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}) + (1-\gamma) \mathcal{R}_{t,\sigma}^\rho(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y}) \} + (1-\gamma) \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, \sigma) \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}}, \\
 &\quad (6.18)
 \end{aligned}$$

where $(\bar{s}, \bar{\mathbf{x}}) \in \Sigma_1$. Observe that, from (6.16) we have for all $j \in \llbracket 1, n \rrbracket$,

$$\begin{aligned}
 & |(\mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}))_j| + |(\mathcal{R}_{t,\sigma}^\rho(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y}))_j| \leq C_{6.1} n (2\varepsilon_{\mathbf{F}} + (t-\sigma) + (t-s)) |\tilde{\mathbf{x}}_{j_0}|. \\
 &\quad (6.19)
 \end{aligned}$$

On the other hand, from the scaling Lemma B.1 we obtain that

$$(t-\sigma)^{1/2} \mathbb{T}_{t-\sigma}^{-1} \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, \sigma) \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}} = \hat{\mathbf{R}}^{t,\mathbf{y},t-\sigma}(1, 0) (t-\sigma)^{1/2} \mathbb{T}_{t-\sigma}^{-1} \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}}$$

with $|\hat{\mathbf{R}}^{t,\mathbf{y},t-\sigma}(1, 0)| \leq \hat{C} := \hat{C}(T, (\mathbf{A}))$. Thus, recalling that from the structure of the linearized system the resolvent is subdiagonal (see (3.3), (3.4)), we derive for all $j \in \llbracket 1, n \rrbracket$:

$$\begin{aligned}
 & (t-\sigma)^{1/2} |(\mathbb{T}_{t-\sigma}^{-1} \tilde{\mathbf{R}}^{t,\mathbf{y}}(t, \sigma) \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}})_j| = (t-\sigma)^{1/2} |(\hat{\mathbf{R}}^{t,\mathbf{y},t-\sigma}(1, 0) \mathbb{T}_{t-\sigma}^{-1} \alpha^{-1} \mathbb{T}_{\alpha^2} \bar{\mathbf{x}})_j| \\
 &\quad \leq \hat{C} \sum_{i=1}^j \left(\frac{\alpha^2}{t-\sigma} \right)^{i-1/2} |\bar{\mathbf{x}}_i| \leq \hat{C} \sum_{i=1}^j \left(\frac{\alpha^2}{\rho^2 \{ \tilde{s} - \frac{\sigma-s}{\rho^2} \}} \right)^{i-1/2} |\bar{\mathbf{x}}| \\
 &\quad (6.20) \quad \leq \hat{C} \left(\tilde{s} - \frac{\sigma-s}{\rho^2} \right)^{-(j-1/2)} \sum_{i=1}^j \left(\frac{\alpha^2}{\rho^2} \right)^{i-1/2} |\bar{\mathbf{x}}|
 \end{aligned}$$

as soon as $\tilde{c} \leq 1$ and $|\tilde{s} - \frac{\sigma-s}{\rho^2}|^{2n-1} \leq \tilde{c}$. In that case, using (6.15), (6.18), (6.19), (6.20) we then derive that:

$$\begin{aligned}
 & (t-\sigma) |\mathbb{T}_{t-\sigma}^{-1} (\gamma \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) + (1-\gamma) \tilde{\boldsymbol{\theta}}_{t,\sigma}^{t,\mathbf{y}}(\boldsymbol{\xi}) - \mathbf{y})|^2 \\
 &\geq \left(\tilde{s} - \frac{\sigma-s}{\rho^2} \right)^{-(2j_0-1)} \left(\frac{1}{2} |\tilde{\mathbf{x}}_{j_0}|^2 - 2 \left\{ (|\mathcal{R}_{t,s}^\rho(\mathbf{x}, \mathbf{y}))_{j_0}| + |(\mathcal{R}_{t,\sigma}^\rho(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y}))_{j_0}| \right\}^2 \right. \\
 &\quad \left. + \hat{C}^2 \left\{ \sum_{i=1}^{j_0} \left(\frac{\alpha^2}{\rho^2} \right)^{i-1/2} \right\}^2 |\bar{\mathbf{x}}|^2 \right) \\
 &\geq \left(\tilde{s} - \frac{\sigma-s}{\rho^2} \right)^{-(2j_0-1)} \left(\frac{1}{2} |\tilde{\mathbf{x}}_{j_0}|^2 - 2n^2 [C_{6.1}^2 (2\varepsilon_{\mathbf{F}} + 2(t-\sigma \wedge s))^2 + \hat{C}^2 c_\infty^{-2}] \right) \\
 &\quad \geq \left(\tilde{s} - \frac{\sigma-s}{\rho^2} \right)^{-(2j_0-1)} \tilde{c},
 \end{aligned}$$

where $\bar{c} > 0$ for $T, \varepsilon_{\mathbf{F}}$ small enough and a sufficiently large c_{∞} . We thus obtain the global bound:

$$(6.21) \quad \exists C_2 := C_2(T, (\mathbf{A})) > 0, \quad I_2(\sigma, t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{y}) \leq C_2 \left(\sum_{k=1}^n \frac{\alpha^{2k-1}}{\rho^{2+(2k-1)+n^2d}} \right).$$

Plugging (6.17), (6.21) into (6.10), (6.8) we get:

$$T_1 \leq C \int_{\rho \geq c_{\infty} \alpha} \frac{dtdy}{\rho^{n^2d+1}} \left(\frac{\alpha^2}{\rho^3} + \sum_{k=1}^n \frac{\alpha^{2k-1}}{\rho^{(2k-1)+1}} \right), \quad C := C(T, (\mathbf{A})).$$

The idea is now to use the ‘‘polar’’ coordinates parametrized by (6.12), i.e. we consider the surface integral on the smooth hypersurface Σ_1 which is the boundary of the convex set $\{(u, \mathbf{z}) \in [0, T] \times \mathbb{R}^{nd} : \rho(u, \mathbf{z}) \leq 1\}$. Changing coordinates according to (6.12) we have that $\text{Jac}(J_1) = \rho^{n^2d+1} J_{n,d}(\tilde{s}, \tilde{\mathbf{x}})$, where $J_{n,d}$ is a smooth function on Σ_1 s.t. $\exists C_{n,d} \in [1, \infty)$, $\forall \boldsymbol{\eta} \in \Sigma_1$, $J_{n,d}(\boldsymbol{\eta}) \in [C_{n,d}^{-1}, C_{n,d}]$. We therefore derive:

$$T_1 \leq C \int_{\rho \geq c_{\infty} \alpha} d\rho \left(\frac{\alpha^2}{\rho^3} + \sum_{k=1}^n \frac{\alpha^{2k-1}}{\rho^{(2k-1)+1}} \right) \int_{\Sigma_1} J_{n,d}(\boldsymbol{\eta}) d\boldsymbol{\eta} \leq C \left(\frac{1}{c_{\infty}^2} + \sum_{k=1}^n \frac{1}{c_{\infty}^{2k-1}} \right) \leq C,$$

up to modifications of C , which plugged into (6.8) together with (6.9) concludes the proof. \square

6.3. Proof of the auxiliary propositions in the general case (\mathbf{A}_G) . This section is devoted to the proof of Propositions 4.2 and 4.3 which provide the key estimates to derive Theorem 3.1 under the general assumption (\mathbf{A}_G) .

6.3.1. Proof of Proposition 4.2. Point *i*) can be derived for both kernels $k_{ij}^{\varepsilon,d}$, $k_{ij}^{\varepsilon,d,*}$ recalling from (4.5) and (3.9) that there exists $C := C((\mathbf{A}), T) \geq 1$ s.t. $\forall((s, \mathbf{x}), (t, \mathbf{y})) \in S^2$,

$$(6.22) \quad \begin{aligned} & |k_{i,j}^{\varepsilon,d}(s, t, \mathbf{x}, \mathbf{y})| + |k_{i,j}^{\varepsilon,d,*}(s, t, \mathbf{x}, \mathbf{y})| \\ & \leq \frac{C}{(t-s)^{1+n^2d/2}} \exp(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})^2|). \end{aligned}$$

Now for a given $c_1 > 2$, if $c_1|t-s|^{1/2} \geq d((s, \mathbf{x}), (t, \mathbf{y}))$ then the r.h.s of (6.22) can directly be upper bounded by $Cc_1^{2+n^2d}/d((s, \mathbf{x}), (t, \mathbf{y}))^{2+n^2d}$. On the other hand, if $c_1|t-s|^{1/2} \leq d((s, \mathbf{x}), (t, \mathbf{y}))$ then, by definition of d in (4.2) we derive that $\exists i \in \llbracket 1, n \rrbracket$ s.t.

$$|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_i|^{1/(2i-1)} \geq \frac{1}{n^{1/(2(2n-1))}} \left(1 - \frac{1}{c_1}\right) d((s, \mathbf{x}), (t, \mathbf{y})).$$

This property yields:

$$\begin{aligned} |k_{i,j}^{\varepsilon,d}(s, t, \mathbf{x}, \mathbf{y})| + |k_{i,j}^{\varepsilon,d,*}(s, t, \mathbf{x}, \mathbf{y})| & \leq \frac{C}{|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_i|^{\frac{n^2d+2}{2i-1}}} \left(\frac{|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})_i|}{|t-s|^{1/2(2i-1)}} \right)^{\frac{n^2d+2}{2i-1}} \\ & \quad \times \exp(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})^2|) \\ & \leq \frac{\tilde{C}}{d((s, \mathbf{x}), (t, \mathbf{y}))^{n^2d+2}} \exp(-\tilde{C}^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})^2|), \end{aligned}$$

where $\tilde{C} := \tilde{C}((\mathbf{A}), T, c_1)$, $\bar{C} := \bar{C}((\mathbf{A}), T)$. This gives the first claim.

Now, point *ii*) can be derived for the kernel $k_{i,j}^{\epsilon,d}$ similarly to (6.17) and (6.21). For the kernel $k_{i,j}^{d,\epsilon,*}$, when investigating the difference

$$\begin{aligned} D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}), (t, \mathbf{y})) &:= k_{i,j}^{d,\epsilon,*}(s, t, \mathbf{x}, \mathbf{y}) - k_{i,j}^{d,\epsilon,*}(\sigma, t, \boldsymbol{\xi}, \mathbf{y}) \\ &= k_{i,j}^{d,\epsilon}(t, s, \mathbf{y}, \mathbf{x}) - k_{i,j}^{d,\epsilon}(t, \sigma, \mathbf{y}, \boldsymbol{\xi}), \end{aligned}$$

we are led to consider the linearized systems $\tilde{\boldsymbol{\theta}}_{s,t}^{s,\mathbf{x}}(\mathbf{y}), \tilde{\boldsymbol{\theta}}_{\sigma,t}^{\sigma,\boldsymbol{\xi}}(\mathbf{y})$. Recall now from definitions (3.5) and (6.6) that the exponential bounds write:

$$\begin{aligned} &\langle \tilde{\mathbf{K}}^{\mathbf{x}}(t, s)^{-1}(\tilde{\boldsymbol{\theta}}_{s,t}^{s,\mathbf{x}}(\mathbf{y}) - \mathbf{x}), \tilde{\boldsymbol{\theta}}_{s,t}^{s,\mathbf{x}}(\mathbf{y}) - \mathbf{x} \rangle \\ &= \langle \tilde{\mathbf{R}}^{s,\mathbf{x}}(s, t)^* \tilde{\mathbf{K}}^{\mathbf{x}}(t, s)^{-1} \tilde{\mathbf{R}}^{s,\mathbf{x}}(s, t)(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})), \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}) \rangle \\ &= \langle \tilde{\mathbf{H}}^{s,\mathbf{x}}(t, s)^{-1}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})), \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}) \rangle, \\ \langle \tilde{\mathbf{K}}^{\boldsymbol{\xi}}(t, \sigma)^{-1}(\tilde{\boldsymbol{\theta}}_{\sigma,t}^{\sigma,\boldsymbol{\xi}}(\mathbf{y}) - \boldsymbol{\xi}), \tilde{\boldsymbol{\theta}}_{\sigma,t}^{\sigma,\boldsymbol{\xi}}(\mathbf{y}) - \boldsymbol{\xi} \rangle &= \langle \tilde{\mathbf{H}}^{\sigma,\boldsymbol{\xi}}(t, \sigma)^{-1}(\mathbf{y} - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi})), \mathbf{y} - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}) \rangle. \end{aligned}$$

Introducing, for all $(s, t, u, \mathbf{x}) \in [0, T]^3 \times \mathbb{R}^{nd}$, $\mathbf{z} \in \mathbb{R}^{nd}$,

$$(6.23) \quad \check{k}^{s,\mathbf{x},u,t,\epsilon}(\mathbf{z}) := \mathbb{I}_{|s-t| \geq \epsilon} \left\{ -[\tilde{\mathbf{H}}^{s,\mathbf{x}}(u, t)^{-1}]_{1,1} + [\tilde{\mathbf{H}}^{s,\mathbf{x}}(u, t)^{-1} \mathbf{z}]_1^{\otimes 2} \right\} \\ \frac{1}{(2\pi)^{nd/2} \det(\tilde{\mathbf{K}}^{s,\mathbf{x}}(u, t))^{1/2}} \times \exp\left(-\frac{1}{2} \langle \tilde{\mathbf{H}}^{s,\mathbf{x}}(u, t)^{-1} \mathbf{z}, \mathbf{z} \rangle\right),$$

we can rewrite:

$$\begin{aligned} |D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))| &= |\check{k}_{i,j}^{s,\mathbf{x},s,t}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) - \check{k}_{i,j}^{\sigma,\boldsymbol{\xi},\sigma,t}(\mathbf{y} - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}))| \\ &\leq |\check{k}_{i,j}^{s,\mathbf{x},s,t}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) - \check{k}_{i,j}^{s,\mathbf{x},\sigma,t}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))| \\ &\quad + |\check{k}_{i,j}^{s,\mathbf{x},\sigma,t}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) - \check{k}_{i,j}^{\sigma,\boldsymbol{\xi},\sigma,t}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))| \\ + |\check{k}_{i,j}^{\sigma,\boldsymbol{\xi},\sigma,t}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) - \check{k}_{i,j}^{\sigma,\boldsymbol{\xi},\sigma,t}(\mathbf{y} - \boldsymbol{\theta}_{t,\sigma}(\boldsymbol{\xi}))| &:= \sum_{l=1}^3 |\{D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))\}_l|. \end{aligned}$$

Now the terms $|\{D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))\}_1|$ and $|\{D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))\}_3|$ respectively involve time and space sensitivities when the freezing parameters in the covariance matrix are fixed. Those contributions can therefore be investigated as terms I_1 and I_2 in (6.17), (6.21). The term $|\{D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \boldsymbol{\xi}), (t, \mathbf{y}))\}_2|$ involves two different covariance matrices observed at the same time but that are respectively associated to the freezing points (s, \mathbf{x}) and $(\sigma, \boldsymbol{\xi})$ in the linearization of (3.2). To analyze this difference we proceed as in the proof of Lemma 2.4 in [Men11]. Namely, using the scaling lemma B.1, we rewrite:

$$\tilde{\mathbf{H}}^{s,\mathbf{x}}(\sigma, t) = |t - \sigma|^{-1} \mathbb{T}_{|t-\sigma|} \check{\tilde{\mathbf{H}}}_1^{s,\mathbf{x},\sigma,t} \mathbb{T}_{|t-\sigma|}, \quad \tilde{\mathbf{H}}^{\sigma,\boldsymbol{\xi}}(\sigma, t) = |t - \sigma|^{-1} \mathbb{T}_{|t-\sigma|} \check{\tilde{\mathbf{H}}}_1^{\sigma,\boldsymbol{\xi},\sigma,t} \mathbb{T}_{|t-\sigma|},$$

where $\check{\tilde{\mathbf{H}}}_1^{s,\mathbf{x},\sigma,t}, \check{\tilde{\mathbf{H}}}_1^{\sigma,\boldsymbol{\xi},\sigma,t}$ are uniformly elliptic bounded matrices on $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$. Thus,

$$\begin{aligned} &\langle (\tilde{\mathbf{H}}^{s,\mathbf{x}}(\sigma, t) - \tilde{\mathbf{H}}^{\sigma,\boldsymbol{\xi}}(\sigma, t))(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})), \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}) \rangle \\ &= \langle (\check{\tilde{\mathbf{H}}}_1^{s,\mathbf{x},\sigma,t} - \check{\tilde{\mathbf{H}}}_1^{\sigma,\boldsymbol{\xi},\sigma,t})(|t - \sigma|^{-1/2} \mathbb{T}_{|t-\sigma|}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))), |t - \sigma|^{-1/2} \mathbb{T}_{|t-\sigma|}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \rangle. \end{aligned} \quad (6.24)$$

We now want to control the difference $\langle \check{\mathbf{H}}_1^{s,\mathbf{x},\sigma,t} - \check{\mathbf{H}}_1^{\sigma,\xi,\sigma,t} \rangle$ in (6.24). Let us first write:

$$\begin{aligned} & |\check{\mathbf{R}}^{s,\mathbf{x}}(\sigma,t) - \check{\mathbf{R}}^{\sigma,\xi}(\sigma,t)| = \\ & \left| \int_t^\sigma \left\{ D\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) \check{\mathbf{R}}^{s,\mathbf{x}}(u,t) - D\mathbf{F}(u, \boldsymbol{\theta}_{u,\sigma}(\xi)) \check{\mathbf{R}}^{\sigma,\xi}(u,t) \right\} du \right| \\ & \leq \left| \int_t^\sigma \left\{ D\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) (\check{\mathbf{R}}^{s,\mathbf{x}}(u,t) - \check{\mathbf{R}}^{\sigma,\xi}(u,t)) \right. \right. \\ & \left. \left. + (D\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) - D\mathbf{F}(u, \boldsymbol{\theta}_{u,\sigma}(\xi))) \check{\mathbf{R}}^{\sigma,\xi}(u,t) \right\} du \right| \leq C|t - \sigma| |\boldsymbol{\theta}_{\sigma,s}(\xi) - \xi|, \end{aligned}$$

where $C := C((\mathbf{A}))$, using the smoothness conditions assumed in (\mathbf{S}_G) , the bi-Lipschitz property of the flow and Gronwall's Lemma for the last inequality. The scaling Lemma B.1 then transfers this control to $|\check{\mathbf{H}}_1^{s,\mathbf{x},\sigma,t} - \check{\mathbf{H}}_1^{\sigma,\xi,\sigma,t}|$ to give:

$$(6.25) \quad \begin{aligned} & \langle (\check{\mathbf{H}}^{s,\mathbf{x}}(\sigma,t) - \check{\mathbf{H}}^{\sigma,\xi}(\sigma,t))(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})), \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}) \rangle \\ & \leq C |\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \xi| |t - \sigma|^{-1/2} \mathbb{T}_{|t-\sigma|}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))|^2, \quad C := C((\mathbf{A})). \end{aligned}$$

Because of the non-degeneracy of a , the inverse matrices $(\check{\mathbf{H}}^{s,\mathbf{x}}(\sigma,t))^{-1}$, $(\check{\mathbf{H}}^{\sigma,\xi}(\sigma,t))^{-1}$ have the same Hölder regularity. Indeed, up to a change of coordinates one can assume that one of the two matrices is diagonal at the considered point and that the other has dominant diagonal if $|\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \xi|$ is small enough (depending on the ellipticity bounds in (\mathbf{A}) and the dimension). This reduces to the scalar case. Hence,

$$\begin{aligned} & \langle (\check{\mathbf{H}}^{s,\mathbf{x}}(\sigma,t))^{-1} - (\check{\mathbf{H}}^{\sigma,\xi}(\sigma,t))^{-1}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})), \mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}) \rangle \\ & = \langle ((\check{\mathbf{H}}_1^{s,\mathbf{x},\sigma,t})^{-1} - (\check{\mathbf{H}}_1^{\sigma,\xi,\sigma,t})^{-1})(|t - \sigma|^{1/2} \mathbb{T}_{|t-\sigma|}^{-1}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))), |t - \sigma|^{1/2} \mathbb{T}_{|t-\sigma|}^{-1}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x})) \rangle \\ & \leq C |\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}) - \xi| |t - \sigma|^{1/2} \mathbb{T}_{|t-\sigma|}^{-1}(\mathbf{y} - \boldsymbol{\theta}_{t,s}(\mathbf{x}))|^2. \end{aligned}$$

We therefore derive:

$$|\{D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \xi), (t, \mathbf{y}))\}_2| \leq C \frac{d((s, \mathbf{x}), (\sigma, \xi))}{|t - \sigma|} \bar{q}_c(\sigma, t, \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y}),$$

which can be analyzed similarly to I_2 (see equation (6.10) and page 31) in the previous proof and yields:

$$(6.26) \quad |\{D_{i,j}^{d,\epsilon}((s, \mathbf{x}), (\sigma, \xi), (t, \mathbf{y}))\}_2| \leq C \frac{d((s, \mathbf{x}), (\sigma, \xi))}{d((s, \mathbf{x}), (t, \mathbf{y}))^{2+n^2d}} \leq C \frac{d((s, \mathbf{x}), (\sigma, \xi))}{d((s, \mathbf{x}), (t, \mathbf{y}))^{3+n^2d}},$$

recalling that $d((s, \mathbf{x}), (t, \mathbf{y})) \leq 1$. This gives points *ii*) and *iii*).

The cancellation property *iv*) is the more subtle to derive. From the definitions of the kernels it suffices to prove:

$$\sup_{\epsilon > 0} \left| \int_{d((t, \mathbf{y}), (s, \mathbf{x})) > \epsilon} k_{i,j}^{\epsilon,d}(s, t, \mathbf{x}, \mathbf{y}) dt d\mathbf{y} \right| + \sup_{\epsilon > 0} \left| \int_{d((t, \mathbf{y}), (s, \mathbf{x})) > \epsilon} k_{i,j}^{\epsilon,d}(s, t, \mathbf{x}, \mathbf{y}) ds d\mathbf{x} \right| < +\infty.$$

Let us begin with the second statement which is easier. From the definitions in (6.23), (6.6) and recalling that $\det(\check{\mathbf{R}}^{t,\mathbf{y}}(s,t)) = 1$ (Since $D\mathbf{F}$ is subdiagonal, see

(3.3)), we have to investigate:

$$\begin{aligned} & \int_{d((t,\mathbf{y}),(s,\mathbf{x}))>\epsilon} k_{i,j}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) ds d\mathbf{x} = \int_{\rho(|t-s|,\mathbf{x}-\boldsymbol{\theta}_{s,t}(\mathbf{y}))\geq\epsilon} \check{k}^{t,\mathbf{y},s,t,\epsilon}(\mathbf{x}-\boldsymbol{\theta}_{s,t}(\mathbf{y})) ds d\mathbf{x} \\ & = \int_{\rho(|t-s|,\mathbf{z})\geq\epsilon} \partial_{\mathbf{z}_1^i, \mathbf{z}_1^j} \left(\frac{1}{(2\pi)^{nd/2} \det(\tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t))^{1/2}} \exp\left(-\frac{1}{2} \langle \tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t) \mathbf{z}, \mathbf{z} \rangle\right) \right) ds d\mathbf{z}. \end{aligned} \quad (6.27)$$

By the divergence theorem:

$$\begin{aligned} \left| \int_{d((t,\mathbf{y}),(s,\mathbf{x}))>\epsilon} k_{i,j}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) ds d\mathbf{x} \right| & \leq \left| \int_{\rho(|t-s|,\mathbf{z})=\epsilon} \partial_{\mathbf{z}_1^i} \left(\frac{1}{(2\pi)^{nd/2} \det(\tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t))^{1/2}} \right. \right. \\ & \quad \left. \left. \times \exp\left(-\frac{1}{2} \langle (\tilde{\mathbf{H}}^{t,\mathbf{y}}(s,t))^{-1} \mathbf{z}, \mathbf{z} \rangle\right) \right) d\nu_\epsilon((t-s), \mathbf{z}) \right|, \end{aligned}$$

where ν_ϵ stands for the surface element of $\rho(|t-s|, \mathbf{z}) = \epsilon$. Now, from the metric homogeneity (see Remark 4.1), the good scaling property (3.7) that is valid for $\tilde{\mathbf{H}}^{t,\mathbf{y}}$, it can be shown, changing variables similarly to (6.12), that for ϵ small enough:

$$\left| \int_{d((t,\mathbf{y}),(s,\mathbf{x}))>\epsilon} k_{i,j}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) ds d\mathbf{x} \right| \leq C \int_{\rho(|\bar{s}|,\bar{\mathbf{z}})=1} \frac{d\nu_1(\bar{s}, \bar{\mathbf{z}})}{\bar{s}^{n^2 d/2+1/2}} \exp\left(-C|\bar{s}| |\mathbb{T}_{|\bar{s}|}^{-1} \bar{\mathbf{z}}|^2\right) < +\infty, \quad (6.28)$$

and that $\int_{d((t,\mathbf{y}),(s,\mathbf{x}))>\epsilon} k_{i,j}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) ds d\mathbf{x}$ admits a limit when $\epsilon \rightarrow 0$. From (6.28), and (6.27) we thus get point *iii*) for the kernel $k_{i,j}^{\epsilon,d,*}$. To derive the same property for $k_{i,j}^{\epsilon,d}$ there is an additional step. We first write as before:

$$\int_{d((t,\mathbf{y}),(s,\mathbf{x}))>\epsilon} k_{i,j}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y} = \int_{\rho(|t-s|,\mathbf{x}-\boldsymbol{\theta}_{s,t}(\mathbf{y}))\geq\epsilon} \check{k}^{t,\mathbf{y},s,t,\epsilon}(\mathbf{x}-\boldsymbol{\theta}_{s,t}(\mathbf{y})) dt d\mathbf{y}.$$

We still set $\mathbf{z} := \mathbf{x} - \boldsymbol{\theta}_{s,t}(\mathbf{y})$ that yields $d\mathbf{z} = \det(\text{Jac}_{\boldsymbol{\theta}_{s,t}}(\mathbf{y})) d\mathbf{y}$ where for T small enough $\det(\text{Jac}_{\boldsymbol{\theta}_{s,t}}(\mathbf{y})) = 1 + O(t-s)$. Denote now for $(\mathbf{A}, \mathbf{z}) \in \mathbb{R}^{nd} \otimes \mathbb{R}^{nd} \times \mathbb{R}^{nd}$, $P_{i,j}(\mathbf{A}, \mathbf{z}) := \{[\mathbf{A}]_{1,1} + [\mathbf{A}\mathbf{z}]_1^{\otimes 2}\}_{ij}$. Since

$$\begin{aligned} & \int_{\rho(|t-s|,\mathbf{z})\geq\epsilon} |t-s| \exp\left(-\frac{1}{2} \langle (\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1} \mathbf{z}, \mathbf{z} \rangle\right) \\ & \quad \times P_{i,j}(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1}, \mathbf{z}) \frac{1}{(2\pi)^{nd} \det(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{1/2}} dt d\mathbf{z} \\ & \leq C \int_{\rho(|t-s|,\mathbf{z})\geq\epsilon} \exp(-C^{-1}|t-s| |\mathbb{T}_{|t-s|}^{-1} \mathbf{z}|^2) \frac{1}{|t-s|^{n^2 d/2}} dt d\mathbf{z} = O(1), \end{aligned}$$

we derive:

$$\begin{aligned} \int_{d((t,\mathbf{y}),(s,\mathbf{x}))>\epsilon} k_{i,j}^{\epsilon,d}(s,t,\mathbf{x},\mathbf{y}) dt d\mathbf{y} & = \int_{\rho(|t-s|,\mathbf{z})\geq\epsilon} \exp\left(-\frac{1}{2} \langle (\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1} \mathbf{z}, \mathbf{z} \rangle\right) \\ & \quad \times P_{i,j}(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1}, \mathbf{z}) \frac{1}{(2\pi)^{nd} \det(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{1/2}} dt d\mathbf{z} + O(1), \end{aligned}$$

Now:

$$\begin{aligned}
& \exp\left(-\frac{1}{2}\langle(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1}\mathbf{z},\mathbf{z}\rangle\right) \\
& \times P_{i,j}(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1},\mathbf{z}) \frac{1}{(2\pi)^{nd}\det(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{1/2}} \\
& = \partial_{\mathbf{z}_i\mathbf{z}_j} \left\{ \exp\left(-\frac{1}{2}\langle(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{-1}\mathbf{z},\mathbf{z}\rangle\right) \frac{1}{(2\pi)^{nd}\det(\tilde{\mathbf{H}}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t))^{1/2}} \right\} \\
& \quad + \mathcal{R}_{i,j}(s,t,\mathbf{x},\mathbf{z}).
\end{aligned}$$

Reproducing the arguments that lead to (6.25) from (6.24), we observe that the sensitivities of the covariance matrices w.r.t. the freezing parameters do not induce additional singularities. Hence, it can be checked that:

$$|\mathcal{R}_{i,j}(s,t,\mathbf{x},\mathbf{z})| \leq \frac{C}{|t-s|^{n^2d/2+1/2}} \exp(-|t-s|\|\mathbb{T}_{|t-s|}^{-1}\mathbf{z}\|^2), \quad C := C((\mathbf{A}),T).$$

The smoothness on the coefficient \mathbf{F} , which is assumed under (\mathbf{A}_G) to be C^2 in space and s.t. the $D_{\mathbf{x}_{i-1}}\mathbf{F}_i$ is C^2 in space as well, is actually just required here to differentiate twice the dynamics of the resolvent which already corresponds to a first order linearization of the initial system. Such terms appear in $\mathcal{R}_{i,j}$ and derive from the sensitivities of $\mathbf{H}^{t,\boldsymbol{\theta}_{t,s}(\mathbf{x}-\mathbf{z})}(s,t)^{-1}$ w.r.t. $\mathbf{z}_1^i, \mathbf{z}_1^j$. The proof of the cancellation property can then be obtained as previously. \square

APPENDIX A. PROOF OF LEMMA 6.1

Let us prove the first claim of the Lemma. Recalling equations (3.2), (3.3) and (3.13), (3.14), we write:

$$\begin{aligned}
& \mathcal{R}_{t,s}^\rho(\mathbf{x},\mathbf{y}) := \rho\mathbb{T}_{\rho^{-2}} \left\{ \boldsymbol{\theta}_{t,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) \right\} = \\
& \rho\mathbb{T}_{\rho^{-2}} \left\{ \int_s^t du \left[\left(\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) - \mathbf{F}^{t,\mathbf{y}}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) \right) \right. \right. \\
& \quad \left. \left. + \left(D\mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}))(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{u,s}^{t,\mathbf{y}}(\mathbf{x})) \right) \right] \right. \\
& \quad \left. + \left(\int_0^1 d\delta(D\mathbf{F}^{t,\mathbf{y}}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}) + \delta(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y}))) \right. \right. \\
& \quad \left. \left. - D\mathbf{F}^{t,\mathbf{y}}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}))(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y})) \right) \right] \right\} \\
& (A.1) \quad \quad \quad := (\mathcal{R}_{t,s}^{\rho,1} + \mathcal{R}_{t,s}^{\rho,2} + \mathcal{R}_{t,s}^{\rho,3})(\mathbf{x},\mathbf{y}),
\end{aligned}$$

where for $(u, \mathbf{z}) \in [s, t] \times \mathbb{R}^{nd}$, $D\mathbf{F}^{t,\mathbf{y}}(u, \mathbf{z})$ is the $(nd) \times (nd)$ matrix with only non zero $d \times d$ matrix entries $(D\mathbf{F}^{t,\mathbf{y}}(u, \mathbf{z}))_{j,j-1} := D_{\mathbf{x}_{j-1}}\mathbf{F}_j(u, \mathbf{z}_{j-1}, \boldsymbol{\theta}_{u,t}(\mathbf{y})^{j,n})$, $j \in \llbracket 2, n \rrbracket$, so that in particular $D\mathbf{F}^{t,\mathbf{y}}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y})) = D\mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y}))$.

The structure of the “partial gradient” $D\mathbf{F}^{t,\mathbf{y}}$ and the small variation condition (3.1) yield that there exists $C_3 := C_3(T, (\mathbf{A}))$ s.t. for all $j \in \llbracket 2, d \rrbracket$:

$$\begin{aligned}
 |(\mathcal{R}_{t,s}^{\rho,3}(\mathbf{x}, \mathbf{y}))_j| &\leq C_3 \varepsilon_{\mathbf{F}} \rho^{1-2j} \int_s^t du |(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y}))_{j-1}| \\
 &\leq C_3 \varepsilon_{\mathbf{F}} \rho^{-2} \int_s^t du \left(\sum_{k=2}^n \rho^{1-2(k-1)} |(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y}))_{k-1}| \right) \\
 &\leq C_3 \varepsilon_{\mathbf{F}} \rho^{-2} \int_s^t du (\rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y}))|) \\
 &\leq C_3 \varepsilon_{\mathbf{F}} \rho^{-2} \int_s^t du (\rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})|) \\
 \text{(A.2)} \quad &\leq C_3 \varepsilon_{\mathbf{F}} \rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})|,
 \end{aligned}$$

up to a modification of C_3 , using the bi-Lipschitz property of the flow $\boldsymbol{\theta}$ for the last but one inequality (see the end of the proof of Proposition 5.1 in [DM10] for details) and recalling from (6.12) that $|t-s|/\rho^2 \leq 1$ for the last one.

On the other hand, the term $\mathcal{R}_{t,s}^{\rho,1}(\mathbf{x}, \mathbf{y})$ can be seen as a remainder w.r.t. the characteristic time scales. Precisely, there exists $C_1 := C_1(T, (\mathbf{A}))$ (possibly changing from line to line) s.t. for all $j \in \llbracket 1, n \rrbracket$:

$$\begin{aligned}
 |(\mathcal{R}_{t,s}^{\rho,1}(\mathbf{x}, \mathbf{y}))_j| &\leq C_1 \rho^{1-2j} \int_s^t du \sum_{k=j}^n |(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y}))_k| \\
 &\leq C_1 \int_s^t du \rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,t}(\mathbf{y}))| \\
 \text{(A.3)} \quad &\leq C_1 (t-s) \rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})|
 \end{aligned}$$

using once again the bi-Lipschitz property of the flow $\boldsymbol{\theta}$ for the last inequality.

Remark A.1. Let us observe carefully that the second inequality is valid in whole generality when ρ is bounded. On the other hand, under (\mathbf{A}_{SD}) there is only one entry corresponding to $k=j$ in the first inequality and the second one is always true. These are exactly the assumptions required in the statement of the Lemma.

Recall now that $\mathcal{R}_{t,s}^{\rho,2}(\mathbf{x}, \mathbf{y})$ is the linear part of equation (A.1), i.e. it can be rewritten

$$\begin{aligned}
 \mathcal{R}_{t,s}^{\rho,2}(\mathbf{x}, \mathbf{y}) &= \int_s^t du \{ \rho \mathbb{T}_{\rho^{-2}} D\mathbf{F}(u, \boldsymbol{\theta}_{u,t}(\mathbf{y})) \rho^{-1} \mathbb{T}_{\rho^2} \} \\
 &\quad \times \left(\rho \mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{u,s}^{t,\mathbf{y}}(\mathbf{x})) \right) \\
 &:= \int_s^t du \alpha_{t,u}^{\rho}(\mathbf{y}) \left(\rho \mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{u,s}^{t,\mathbf{y}}(\mathbf{x})) \right) = \int_s^t du \alpha_{t,u}^{\rho}(\mathbf{y}) \mathcal{R}_{u,s}^{\rho}(\mathbf{x}, \mathbf{y}),
 \end{aligned}$$

where there exists a constant $C_2 := C_2(T, (\mathbf{A}))$ s.t. $\int_s^t du |\alpha_{t,u}^{\rho}(\mathbf{y})| \leq C_2$. From (A.3), (A.2), (A.1) and Gronwall’s Lemma we derive

$$\begin{aligned}
 \exists C_4 := C_4(T, (\mathbf{A})), \quad |\mathcal{R}_{t,s}^{\rho}(\mathbf{y})| &\leq C_4 (\varepsilon_{\mathbf{F}} + (t-s)) \rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})| \\
 &\leq C_4 (\varepsilon_{\mathbf{F}} + (t-s)) |\tilde{\mathbf{x}}|
 \end{aligned}$$

recalling (6.12) for the last inequality. This gives the first part of the Lemma. For the second one, the previous proof can be adapted with obvious modifications

using thoroughly that $\boldsymbol{\theta}_{u,\sigma}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x})) := \boldsymbol{\theta}_{u,s}(\mathbf{x})$ and the bi-Lipschitz property of the flow. The main differences are that the time integrals are taken between σ and t . Following the computations leading to (A.2), the contribution $\mathcal{R}_{t,\sigma}^{\rho,3}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y})$ would be bounded by $C_3 \varepsilon_{\mathbf{F}} \rho^{-2} |t - \sigma| \{\rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})|\} \leq C_3 \varepsilon_{\mathbf{F}} \rho^{-2} (|t - s| + |s - \sigma|) \{\rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})|\}$. Recalling also that $|t - s| + |\sigma - s| \leq \rho^2 + \alpha^2 \leq (1 + c_\infty) \rho^2$ on the considered set, we get that (A.2) still holds in that case. We would similarly have $|\mathcal{R}_{t,\sigma}^{\rho,1}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y})| \leq C_1 (t - \sigma) \rho |\mathbb{T}_{\rho^{-2}}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{y})|$ giving (A.3) in that case. Eventually, the same previous triangle inequality would give that, on the considered set $\int_\sigma^t du |\alpha_{t,y}^\rho(\mathbf{y})| \leq C$, so that $\mathcal{R}_{t,\sigma}^{\rho,2}(\boldsymbol{\theta}_{\sigma,s}(\mathbf{x}), \mathbf{y})$ can still be viewed as the well controlled linear part of the inequality. The proof then again follows from Gronwall's lemma. \square

APPENDIX B. AUXILIARY TECHNICAL ESTIMATES

For the analysis we need the following lemma whose proof can be found in [DM10]. Let us introduce for $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, $\mathbf{L}_t \mathbf{x} = (0, \alpha_t^1 x_1, \dots, \alpha_t^{n-1} x_{n-1})^* + \mathbf{U}_t \mathbf{x}$, where $\mathbf{U}_t \in \mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ is an ‘‘upper triangular’’ block matrix, i.e. the sub-diagonal blocks of \mathbf{U}_t are zero. Let us now consider a linear equation of the form

$$(B.1) \quad d\mathbf{G}_t = \mathbf{L}_t \mathbf{G}_t dt + B \Sigma_t dW_t,$$

where Σ_t is a bounded uniformly elliptic matrix of $\mathbb{R}^d \otimes \mathbb{R}^d$.

Lemma B.1 (Scaling Lemma). Let $T > 0$ and Assumption **(A)** hold for the system (B.1). Fix $t \in (0, T]$ and set $\hat{\mathbf{G}}_s^t = t^{1/2} \mathbb{T}_t^{-1} \mathbf{G}_{st}$, $0 \leq s \leq 1$. Then, $(\hat{\mathbf{G}}_s^t)_{0 \leq s \leq 1}$ satisfies (B.1) with respect to $(\hat{\mathbf{L}}_s^t = t \mathbb{T}_t^{-1} \mathbf{L}_{st} \mathbb{T}_t)_{0 \leq s \leq 1}$, $(\hat{\Sigma}_s^t = \Sigma_{ts})_{0 \leq s \leq 1}$ and $(\hat{W}_s^t = t^{-1/2} W_{st})_{0 \leq s \leq 1}$. The resolvent $[\hat{\mathbf{R}}^t(s_1, s_0)]_{0 \leq s_0, s_1 \leq 1}$ associated with $(\hat{\mathbf{L}}_s^t)_{0 \leq s \leq 1}$ has the form $\hat{\mathbf{R}}^t(s_1, s_0) = \mathbb{T}_t^{-1} \mathbf{R}(s_1 t, s_0 t) \mathbb{T}_t$, $s_0, s_1 \in [0, 1]$ and the covariance matrix of $\hat{\mathbf{G}}_s^t$, $0 \leq s \leq 1$, is given by $\hat{\mathbf{K}}_s^t := \text{Cov}(\hat{\mathbf{G}}_s^t) = t \mathbb{T}_t^{-1} \mathbf{K}_{st} \mathbb{T}_t^{-1}$. The matrices $(\hat{\mathbf{L}}_s^t)_{0 \leq s \leq 1}$ and $(\hat{\Sigma}_s^t)_{0 \leq s \leq 1}$ satisfy for all $0 \leq s \leq 1$:

$$|\hat{\mathbf{L}}_s^t| \leq (1 \vee T^n) \kappa, \quad [\hat{\mathbf{L}}_s^t]_{i,i-1} \in \mathcal{E}_{i-1}, \quad 2 \leq i \leq n, \quad ; \quad [\hat{\mathbf{L}}_s^t]_{i,j} = 0, \quad 1 \leq j \leq j+2 \leq i \leq n, \\ \text{Spectrum}(\hat{\Sigma}_s^t (\hat{\Sigma}_s^t)^*) \subset [\Lambda^{-1}, \Lambda].$$

Also, there exists $C := C(T, \mathbf{(A)})$ s.t. for all $(s_1, s_0) \in [0, T]^2$, $|\hat{\mathbf{R}}^t(s_1, s_0)| \leq C$.

We will also need some controls on the various derivatives of the density (3.5).

Proposition B.1. There exist constants $C_{B.1} := C_{B.1}(T, \mathbf{(A)})$, $c_{B.1} := c_{B.1}(T, \mathbf{(A)})$ s.t. for all multi index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \sum_{i=1}^n \alpha_i \leq 3$ we have $\forall 0 \leq s < t \leq T, \forall (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^{nd})^2$,

$$|\partial_{\mathbf{x}}^\alpha \tilde{q}(s, t, \mathbf{x}, \mathbf{y})| \leq \frac{C_{B.1}}{(t-s)^{\sum_{i=1}^n \frac{(2i-1)\alpha_i}{2}}} q_{c_{B.1}}(s, t, \mathbf{x}, \mathbf{y}), \\ \forall c > 0, \quad q_c(s, t, \mathbf{x}, \mathbf{y}) := \frac{c^{nd/2}}{(t-s)^{n^2 d/2}} \exp\left(-c(t-s) |\mathbb{T}_{t-s}^{-1}(\tilde{\boldsymbol{\theta}}_{t,s}^{t,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2\right).$$

This proposition can be proved using the explicit expression for the density and the previous scaling Lemma (see also the arguments in Section 5 of [DM10]).

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