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# Wolff-Denjoy theorems in non-smooth convex domains

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**ABSTRACT.** We give a short proof of Wolff-Denjoy theorem for (not necessarily smooth) strictly convex domains. With similar techniques we are also able to prove a Wolff-Denjoy theorem for weakly convex domains, again without any smoothness assumption on the boundary.

## 0. Introduction

Studying the dynamics of a holomorphic self-map  $f: \Delta \rightarrow \Delta$  of the unit disk  $\Delta \subset \mathbb{C}$  one is naturally led to consider two different cases. If  $f$  has a fixed point then Schwarz's lemma readily implies that either  $f$  is an elliptic automorphism, or the sequence  $\{f^k\}$  of iterates of  $f$  converges (uniformly on compact sets) to the fixed point. The classical Wolff-Denjoy theorem ([W], [D]) says what happens when  $f$  has no fixed points:

**Theorem 0.1:** (Wolff-Denjoy) *Let  $f: \Delta \rightarrow \Delta$  be a holomorphic self-map without fixed points. Then there exists a point  $\tau \in \partial\Delta$  such that the sequence  $\{f^k\}$  of iterates of  $f$  converges (uniformly on compact sets) to the constant map  $\tau$ .*

Since its discovery, a lot of work has been devoted to obtain similar statements in more general situations (surveys covering different aspects of this topic are [A3, RS, ES]). In one complex variable, there are results in multiply connected domains, multiply and infinitely connected Riemann surfaces, and even in the settings of one-parameter semigroups and of random dynamical systems (see, e.g., [H, L, B]). In several complex variables, the first Wolff-Denjoy theorems are due to Hervé [He1, 2]; in particular, in [He2] he proved a statement identical to the one above for fixed points free self-maps of the unit ball  $B^n \subset \mathbb{C}^n$ . Hervé's theorem has also been generalized in various ways to open unit balls of complex Hilbert and Banach spaces (see, e.g., [BKS, S] and references therein).

A breakthrough occurred in 1988, when the first author (see [A1]) showed how to prove a Wolff-Denjoy theorem for holomorphic self-maps of smoothly bounded strongly convex domains in  $\mathbb{C}^n$ . The techniques introduced there turned out to be quite effective in other contexts too (see, e.g., [A5, AR, Br1, Br2]); but in particular they led to Wolff-Denjoy theorems in smooth strongly pseudoconvex domains and smooth domains of finite type (see, e.g., [A4, Hu, RZ, Br3]).

Two natural questions were left open by the previous results : how much does the boundary smoothness matter? And, what happens in weakly (pseudo)convex domains? As already shown by the results obtained by Hervé [He1] in the bidisk, if we drop both boundary smoothness and strong convexity the situation becomes much more complicated; but most of Hervé's techniques were specific for the bidisk, and so not necessarily applicable to more general domains. On the other hand, for smooth weakly convex domains a Wolff-Denjoy theorem was already obtained in [A3] (but here we shall get a better result; see Corollary 3.2).

In 2012, Budzyńska [Bu2] (see also [BKR] and [Bu3] for infinite dimensional generalizations) finally proved a Wolff-Denjoy theorem for holomorphic fixed point free self-maps of a bounded strictly convex domain in  $\mathbb{C}^n$ , under no smoothness assumption on the boundary; but she did not deal with weakly convex domains.

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In Section 2 of this paper (Section 1 is devoted to recalling a few known preliminary facts) we shall give a simpler proof of Budzyńska's result, using only tools already introduced in [A1] and no additional machinery; it is worth mentioning that the final proof is also simpler than the proof presented in [A1] for the smooth case. In Section 3 we shall furthermore show how, combining our ideas with Budzyńska's new tools, one can obtain a Wolff-Denjoy theorem for weakly convex domains with no smoothness assumptions, thus addressing the second natural question mentioned above. Finally, in Section 4 we shall specialize our results to the polydisk, and we shall see that Hervé's results imply that our statements are essentially optimal.

## 1. Preliminaries

In this section we shall collect a few more or less known facts on bounded convex domains in  $\mathbb{C}^n$ .

### 1.1. Euclidean geometry.

Let us begin by recalling a few standard definitions and notations.

**Definition 1.1:** Given  $x, y \in \mathbb{C}^n$  let

$$[x, y] = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in [0, 1]\} \quad \text{and} \quad (x, y) = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in (0, 1)\}$$

denote the *closed*, respectively *open*, *segment* connecting  $x$  and  $y$ . A set  $D \subseteq \mathbb{C}^n$  is *convex* if  $[x, y] \subseteq D$  for all  $x, y \in D$ ; and *strictly convex* if  $(x, y) \subseteq D$  for all  $x, y \in \overline{D}$ .

An easy but useful observation is:

**Lemma 1.2:** *Let  $D \subset \mathbb{C}^n$  be a convex domain. Then:*

- (i)  $(z, w) \subset D$  for all  $z \in D$  and  $w \in \partial D$ ;
- (ii) if  $x, y \in \partial D$  then either  $(x, y) \subset \partial D$  or  $(x, y) \subset D$ .

This suggests the following

**Definition 1.3:** Let  $D \subset \mathbb{C}^n$  be a convex domain. Given  $x \in \partial D$ , we put

$$\text{ch}(x) = \{y \in \partial D \mid [x, y] \subset \partial D\};$$

we shall say that  $x$  is a *strictly convex point* if  $\text{ch}(x) = \{x\}$ . More generally, given  $F \subseteq \partial D$  we put

$$\text{ch}(F) = \bigcup_{x \in F} \text{ch}(x).$$

A similar construction having a more holomorphic character is the following:

**Definition 1.4:** Let  $D \subset \mathbb{C}^n$  be a convex domain. A *complex supporting functional* at  $x \in \partial D$  is a  $\mathbb{C}$ -linear map  $\sigma: \mathbb{C}^n \rightarrow \mathbb{C}$  such that  $\text{Re } \sigma(z) < \text{Re } \sigma(x)$  for all  $z \in D$ . A *complex supporting hyperplane* at  $x \in \partial D$  is an affine complex hyperplane  $L \subset \mathbb{C}^n$  of the form  $L = x + \ker \sigma$ , where  $\sigma$  is a complex supporting functional at  $x$  (the existence of complex supporting functionals and hyperplanes is guaranteed by the Hahn-Banach theorem). Given  $x \in \partial D$ , we shall denote by  $\text{Ch}(x)$  the intersection of  $\overline{D}$  with of all complex supporting hyperplanes at  $x$ . Clearly,  $\text{Ch}(x)$  is a closed convex set containing  $x$ ; in particular,  $\text{Ch}(x) \subseteq \text{ch}(x)$ . If  $\text{Ch}(x) = \{x\}$  we say that  $x$  is a *strictly  $\mathbb{C}$ -linearly convex point*; and we say that  $D$  is *strictly  $\mathbb{C}$ -linearly convex* if all points of  $\partial D$  are strictly  $\mathbb{C}$ -linearly convex. Finally, if  $F \subset \partial D$  we set

$$\text{Ch}(F) = \bigcup_{x \in F} \text{Ch}(x) \subseteq \text{ch}(F).$$

**Remark 1.5:** If  $\partial D$  is of class  $C^1$  then for each  $x \in \partial D$  there exists a unique complex supporting hyperplane at  $x$ , and thus  $\text{Ch}(x)$  coincides with the intersection of the complex supporting hyperplane with  $\partial D$ , which is smaller than the flat region introduced in [A3, p. 277] as the intersection of  $\partial D$  with the real supporting hyperplane. But non-smooth points can have more than one complex supporting hyperplanes; this happens for instance in the polydisk (see Section 4).

### 1.2. Intrinsic geometry

The intrinsic (complex) geometry of convex domains is conveniently described using the (intrinsic) Kobayashi distance. We refer to [A3], [JP] and [K] for details and much more on the Kobayashi (pseudo)distance in complex manifolds; here we shall just recall what is needed for our aims. Let  $k_\Delta$  denote the Poincaré distance on the unit disk  $\Delta \subset \mathbf{C}$ . If  $X$  is a complex manifold, the *Lempert function*  $\delta_X: X \times X \rightarrow \mathbf{R}^+$  of  $X$  is

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) \mid \text{there exists a holomorphic map } \phi: \Delta \rightarrow X \text{ with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\}$$

for all  $z, w \in X$ . In general, the *Kobayashi pseudodistance*  $k_X: X \times X \rightarrow \mathbf{R}^+$  of  $X$  is the largest pseudodistance on  $X$  bounded above by  $\delta_X$ ; when  $D \subset\subset \mathbb{C}^n$  is a bounded convex domain in  $\mathbb{C}^n$ , Lempert [Le] has proved that  $\delta_D$  is an actual distance, and thus it coincides with the Kobayashi distance  $k_D$  of  $D$ .

The main property of the Kobayashi (pseudo)distance is that it is contracted by holomorphic maps: if  $f: X \rightarrow Y$  is a holomorphic map then  $k_Y(f(z), f(w)) \leq k_X(z, w)$  for all  $z, w \in X$ . In particular, biholomorphisms are isometries, and holomorphic self-maps are  $k_X$ -nonexpansive.

The Kobayashi distance of convex domains enjoys several interesting properties; for instance, it coincides with the Carathéodory distance, and it is a complete distance (see, e.g., [A3] or [Le]); in particular,  $k_D$ -bounded subsets of  $D$  are relatively compact in  $D$ . We shall also need the following estimates:

**Lemma 1.6:** ([Le, KKR1, KS]) *Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain. Then:*

(i) *if  $z_1, z_2, w_1, w_2 \in D$  and  $s \in [0, 1]$  then*

$$k_D(sz_1 + (1-s)w_1, sz_2 + (1-s)w_2) \leq \max\{k_D(z_1, z_2), k_D(w_1, w_2)\};$$

(ii) *if  $z, w \in D$  and  $s, t \in [0, 1]$  then*

$$k_D(sz + (1-s)w, tz + (1-t)w) \leq k_D(z, w).$$

As a consequence we have:

**Lemma 1.7:** *Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain,  $x, y \in \partial D$ , and let  $\{z_\nu\}, \{w_\nu\} \subset D$  be two sequences converging to  $x$  and  $y$  respectively. If*

$$\sup_{\nu \in \mathbb{N}} k_D(z_\nu, w_\nu) = c < +\infty$$

*then  $[x, y] \subset \partial D$ . In particular, if  $x$  (or  $y$ ) is a strictly convex point then  $x = y$ .*

*Proof:* By Lemma 1.2 we know that either  $(x, y) \subset D$ , or  $(x, y) \subset \partial D$ . Assume by contradiction that  $(x, y) \subset D$ . Lemma 1.6 yields

$$k_D(sz_\nu + (1-s)w_\nu, tz_\nu + (1-t)w_\nu) \leq k_D(z_\nu, w_\nu) \leq c$$

for each  $\nu \in \mathbb{N}$  and for all  $s, t \in (0, 1)$ . Hence

$$k_D(sx + (1-s)y, tx + (1-t)y) = \lim_{\nu \rightarrow \infty} k_D(sz_\nu + (1-s)w_\nu, tz_\nu + (1-t)w_\nu) \leq c$$

for all  $s, t \in (0, 1)$ . But this implies that  $(x, y)$  is relatively compact in  $D$ , which is impossible because  $x, y \in \partial D$ .  $\square$

### 1.3. Dynamics

In this subsection we recall a few known facts about the dynamics of holomorphic (or more generally  $k_D$ -nonexpansive) self-maps of convex domains.

When  $D \subset\subset \mathbb{C}^n$  is a bounded domain, by Montel's theorem the space  $\text{Hol}(D, D)$  is relatively compact in  $\text{Hol}(D, \mathbb{C}^n)$ . In particular, if  $f \in \text{Hol}(D, D)$  then every sequence  $\{f^{k_j}\}$  of iterates contains a subsequence converging to a holomorphic map  $h \in \text{Hol}(D, \mathbb{C}^n)$ . Analogously, using this time Ascoli-Arzelà theorem, if  $f: D \rightarrow D$  is  $k_D$ -nonexpansive then every sequence  $\{f^{k_j}\}$  of iterates contains a subsequence converging to a continuous map  $h: D \rightarrow \bar{D} \subset \mathbb{C}^n$ .

**Definition 1.8:** Let  $D \subset\subset \mathbb{C}^n$  be a bounded domain, and  $f: D \rightarrow D$  a holomorphic or  $k_D$ -nonexpansive self-map. A map  $h: D \rightarrow \mathbb{C}^n$  is a *limit point* of the sequence  $\{f^k\}$  of iterates of  $f$  if there is a subsequence  $\{f^{k_j}\}$  of iterates converging (uniformly on compact subsets) to  $h$ ; we shall denote by  $\Gamma(f)$  the set of all limit points of  $\{f^k\}$ . The *target set*  $T(f)$  of  $f$  is then defined as the union of the images of limit points of the sequence of iterates:

$$T(f) = \bigcup_{h \in \Gamma(f)} h(D).$$

**Definition 1.9:** A sequence  $\{f_k\} \subset C(X, Y)$  of continuous maps between topological spaces is *compactly divergent* if for each pair of compact subsets  $H \subseteq X$  and  $K \subseteq Y$  there is  $k_0 \in \mathbb{N}$  such that  $f^k(H) \cap K = \emptyset$  for all  $k \geq k_0$ .

When  $D$  is a convex domain, the target set either is contained in  $D$  if  $f$  has a fixed point or is contained in  $\partial D$  if  $f$  has no fixed points. More precisely, we have the following statement (see [A1, A4, C, KKR1, KS, Bu1]):

**Theorem 1.10:** Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain, and  $f: D \rightarrow D$  a  $k_D$ -nonexpansive (e.g., holomorphic) self-map. Then the following assertions are equivalent:

- (i)  $f$  has a fixed point in  $D$ ;
- (ii) the sequence  $\{f^k\}$  is not compactly divergent;
- (iii) the sequence  $\{f^k\}$  has no compactly divergent subsequences;
- (iv)  $\{f^k(z)\}$  is relatively compact in  $D$  for all  $z \in D$ ;
- (v) there exists  $z_0 \in D$  such that  $\{f^k(z_0)\}$  is relatively compact in  $D$ ;
- (vi) there exists  $z_0 \in D$  such that  $\{f^k(z_0)\}$  admits a subsequence relatively compact in  $D$ .

**Remark 1.11:** For more general taut domains (and  $f$  holomorphic) the statements (ii)–(vi) are still equivalent. For some classes of domains, these statements are equivalent to  $f$  having a periodic point (see [A4] and [Hu]); however, there exist holomorphic self-maps of a taut topologically contractible smooth domain satisfying (ii)–(vi) but without fixed points (see [AH]).

When the sequence of iterates of  $f$  is not compactly divergent (i.e., when  $f$  has a fixed point if  $D$  is convex) then the target set of  $f$  has already been characterized ([Be], [A1, 5]). In particular, using Theorem 1.10 and repeating word by word the proof of [A3, Theorem 2.1.29] we obtain

**Theorem 1.12:** Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain, and  $f: D \rightarrow D$  a  $k_D$ -nonexpansive (e.g., holomorphic) self-map of  $D$ . Assume that  $f$  has a fixed point in  $D$ . Then  $T(f)$  is a  $k_D$ -nonexpansive (respectively, holomorphic) retract of  $D$ . More precisely, there exists a unique  $k_D$ -nonexpansive (respectively, holomorphic) retraction  $\rho: D \rightarrow T(f)$  which is a limit point of  $\{f^k\}$ , such that every limit point of  $\{f^k\}$  is of the form  $\gamma \circ \rho$ , where  $\gamma: T(f) \rightarrow T(f)$  is a (biholomorphic) invertible  $k_D$ -isometry, and  $f|_{T(f)}$  is a (biholomorphic) invertible  $k_D$ -isometry.

In this paper we want to describe the target set of fixed points free self-maps of bounded convex domains.

## 2. Strictly convex domains

Since Wolff's proof of the Wolff-Denjoy theorem [W], horospheres have been the main tool needed for the study of the dynamics of fixed points free holomorphic self-maps. Let us recall the general definitions introduced in [A1, 3].

**Definition 2.1:** Let  $D \subset\subset \mathbb{C}^n$  be a bounded domain,  $z_0 \in D$ ,  $x \in \partial D$  and  $R > 0$ . The *small horosphere*  $E_{z_0}(x, R)$  and the *large horosphere*  $F_{z_0}(x, R)$  of center  $x$ , pole  $z_0$  and radius  $R$  are defined by

$$E_{z_0}(x, R) = \left\{ z \in D \mid \limsup_{w \rightarrow x} [k_D(z, w) - k_D(z_0, w)] < \frac{1}{2} \log R \right\},$$

$$F_{z_0}(x, R) = \left\{ z \in D \mid \liminf_{w \rightarrow x} [k_D(z, w) - k_D(z_0, w)] < \frac{1}{2} \log R \right\}.$$

The following lemma contains some basic properties of horospheres, immediate consequence of the definition and of Lemma 1.6:

**Lemma 2.2:** Let  $D \subset \subset \mathbb{C}^n$  be a bounded domain,  $z_0 \in D$  and  $x \in \partial D$ . Then:

- (i) we have  $E_{z_0}(x, R) \subseteq F_{z_0}(x, R)$  for every  $R > 0$ ;
- (ii) we have  $\overline{E_{z_0}(x, R_1)} \cap D \subseteq E_{z_0}(x, R_2)$  and  $\overline{F_{z_0}(x, R_1)} \cap D \subseteq F_{z_0}(x, R_2)$  for every  $0 < R_1 < R_2$ ;
- (iii) we have  $B_D(z_0, \frac{1}{2} \log R) \subseteq E_{z_0}(x, R)$  for all  $R > 1$ , where  $B_D(z_0, r)$  denotes the Kobayashi ball of center  $z_0$  and radius  $r$ ;
- (iv) we have  $F_{z_0}(x, R) \cap B_D(z_0, -\frac{1}{2} \log R) = \emptyset$  for all  $0 < R < 1$ ;
- (v)  $\bigcup_{R>0} E_{z_0}(x, R) = \bigcup_{R>0} F_{z_0}(x, R) = D$  and  $\bigcap_{R>0} E_{z_0}(x, R) = \bigcap_{R>0} F_{z_0}(x, R) = \emptyset$ ;
- (vi) if moreover  $D$  is convex then  $E_{z_0}(x, R)$  is convex for every  $R > 0$ .

Large horospheres are not always convex, even if  $D$  is convex; an example is given by the horospheres in the polydisk (see Section 4). However, large horospheres in convex domains are always star-shaped with respect to the center:

**Lemma 2.3:** Let  $D \subset \subset \mathbb{C}^n$  be a bounded convex domain,  $z_0 \in D$ ,  $R > 0$  and  $x \in \partial D$ . Then we have  $[x, z] \subset F_{z_0}(x, R)$  for all  $z \in \overline{F_{z_0}(x, R)}$ . In particular,  $x$  always belongs to  $F_{z_0}(x, R)$ .

*Proof:* Given  $z \in F_{z_0}(x, R)$ , choose a sequence  $\{x_\nu\} \subset D$  converging to  $x$  and such that the limit of  $k_D(z, x_\nu) - k_D(z_0, x_\nu)$  exists and is less than  $\frac{1}{2} \log R$ . Given  $0 < s < 1$ , let  $h_\nu^s: D \rightarrow D$  be

$$\forall w \in D \quad h_\nu^s(w) = sw + (1-s)x_\nu;$$

then  $h_\nu^s(x_\nu) = x_\nu$ , and moreover

$$\forall z_1, z_2 \in D \quad k_D(h_\nu^s(z_1), h_\nu^s(z_2)) \leq k_D(z_1, z_2)$$

because  $h_\nu^s$  is a holomorphic self-map of  $D$ . In particular,

$$\limsup_{\nu \rightarrow +\infty} [k_D(h_\nu^s(z), x_\nu) - k_D(z_0, x_\nu)] \leq \lim_{\nu \rightarrow +\infty} [k_D(z, x_\nu) - k_D(z_0, x_\nu)] < \frac{1}{2} \log R.$$

Furthermore we have

$$|k_D(sz + (1-s)x, x_\nu) - k_D(h_\nu^s(z), x_\nu)| \leq k_D(sz + (1-s)x, sz + (1-s)x) \rightarrow 0$$

as  $\nu \rightarrow +\infty$ . Therefore

$$\begin{aligned} \liminf_{w \rightarrow x} [k_D(sz + (1-s)x, w) - k_D(z_0, w)] &\leq \limsup_{\nu \rightarrow +\infty} [k_D(sz + (1-s)x, x_\nu) - k_D(z_0, x_\nu)] \\ &\leq \limsup_{\nu \rightarrow +\infty} [k_D(h_\nu^s(z), x_\nu) - k_D(z_0, x_\nu)] + \lim_{\nu \rightarrow +\infty} [k_D(sz + (1-s)x, x_\nu) - k_D(h_\nu^s(z), x_\nu)] \\ &< \frac{1}{2} \log R, \end{aligned}$$

and thus  $sz + (1-s)x \in F_{z_0}(x, R)$ . Letting  $s \rightarrow 1$  we get  $x \in \overline{F_{z_0}(x, R)}$ , and we have proved the assertion for  $z \in F_{z_0}(x, R)$ . If  $z \in \partial F_{z_0}(x, R)$ , it suffices to apply the statement to a sequence in  $F_{z_0}(x, R)$  approaching  $z$ .  $\square$

One of the main points in the proof given in [A1] of the Wolff-Denjoy theorem for strongly convex  $C^2$  domains is the fact that in such domains the intersection between the closure of a large horosphere and the boundary of the domain reduces to the center of the horosphere. The following corollary will play the same rôle for not necessarily smooth convex domains:

**Corollary 2.4:** Let  $D \subset \subset \mathbb{C}^n$  be a bounded convex domain,  $z_0 \in D$ , and  $x \in \partial D$ . Then

$$\bigcap_{R>0} \overline{F_{z_0}(x, R)} \subseteq \text{ch}(x). \quad (2.1)$$

In particular, if  $x$  is a strictly convex point then  $\bigcap_{R>0} \overline{F_{z_0}(x, R)} = \{x\}$ .

*Proof:* First of all, Lemma 2.2 implies that the intersection in (2.1) is not empty and contained in  $\partial D$ . Take  $\tilde{x} \in \bigcap_{R>0} \overline{F_{z_0}(x, R)}$  different from  $x$ . Then Lemma 2.3 implies that the whole segment  $[x, \tilde{x}]$  is contained in the intersection, and thus in  $\partial D$ ; hence  $\tilde{x} \in \text{ch}(x)$ , and we are done.  $\square$

Let us turn now to the study of the target set. A first step in this direction is the following:

**Proposition 2.5:** *Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain. Then:*

- (i) *for every connected complex manifold  $X$  and every holomorphic map  $h: X \rightarrow \mathbb{C}^n$  such that  $h(X) \subset \overline{D}$  and  $h(X) \cap \partial D \neq \emptyset$  we have*

$$h(X) \subseteq \bigcap_{x \in X} \text{Ch}(h(x)) \subseteq \partial D .$$

*In particular, if  $h$  is a limit point of the sequence of iterates of a holomorphic self-map of  $D$  without fixed points we have*

$$h(D) \subseteq \bigcap_{z \in D} \text{Ch}(h(z)) .$$

- (ii) *Let  $f: D \rightarrow D$  be a  $k_D$ -nonexpansive self-map without fixed points, and  $h: D \rightarrow \mathbb{C}^n$  a limit point of  $\{f^k\}$ . Then*

$$h(D) \subseteq \bigcap_{z \in D} \text{ch}(h(z)) .$$

*Proof:* (i) The fact that  $h(X) \subseteq \partial D$  is an immediate consequence of the maximum principle (see, e.g., [AV, Lemma 2.1]).

Let now  $L = h(x_0) + \ker \sigma$  be a complex supporting hyperplane at  $h(x_0)$ . Then  $\text{Re}(\sigma \circ h) \leq \text{Re} \sigma(h(x_0))$  on  $X$ ; therefore, by the maximum principle,  $\sigma \circ h \equiv \sigma(h(x_0))$ , that is  $h(X) \subset L$ . Since this holds for all complex supporting hyperplanes at  $h(x_0)$  the assertion follows.

(ii) Let  $\{f^{k_j}\}$  be a subsequence of iterates converging to  $h$ . Since  $f$  has no fixed points, we know by Theorem 1.10 that  $h(D) \subseteq \partial D$ . Furthermore

$$\forall z, w \in D \quad k_D(f^{k_j}(z), f^{k_j}(w)) \leq k_D(z, w) < +\infty ;$$

therefore Lemma 1.7 implies  $[h(z), h(w)] \subset \partial D$ , and the assertion follows.  $\square$

The disadvantage of these statements is that the right-hand side still depends on the given limit point of the sequence of iterates; instead we would like to determine a subset of the boundary containing the whole target set. This can be accomplished as follows:

**Lemma 2.6:** *Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain, and  $f: D \rightarrow D$  a  $k_D$ -nonexpansive (respectively, holomorphic) self-map without fixed points. Assume there exist  $\emptyset \neq E \subseteq F \subset D$  such that  $f^k(E) \subset F$  for all  $k \in \mathbb{N}$ . Then we have*

$$T(f) \subseteq \text{ch}(\overline{F} \cap \partial D)$$

*if  $f$  is  $k_D$ -nonexpansive, or*

$$T(f) \subseteq \text{Ch}(\overline{F} \cap \partial D)$$

*if  $f$  is holomorphic.*

*Proof:* Let  $h$  be a limit point of the sequence of iterates of  $f$ . Since  $f$  has no fixed points, we know that  $h(D) \subseteq \partial D$ . Take  $z_0 \in E$ ; by assumption, the whole orbit of  $z_0$  is contained in  $F$ . Therefore  $h(z_0) \in \overline{F} \cap \partial D$ , and the assertion follows from Proposition 2.5.  $\square$

The Wolff lemma [A1, Theorem 2.3], whose proof can easily be adapted to the case of  $k_D$ -nonexpansive maps, is exactly what we need to apply Lemma 2.6:

**Lemma 2.7:** *Let  $D \subset\subset \mathbb{C}^n$  be a convex domain, and let  $f: D \rightarrow D$  be  $k_D$ -nonexpansive and without fixed points. Then there exists  $x \in \partial D$  such that for every  $z_0 \in D$ ,  $R > 0$  and  $k \in \mathbb{N}$  we have*

$$f^k(E_{z_0}(x, R)) \subseteq F_{z_0}(x, R) .$$

We can now give a proof of a Wolff-Denjoy theorem for  $k_D$ -nonexpansive self-maps of strictly convex domains in the same spirit as the proof given in [A1] of the Wolff-Denjoy theorem for holomorphic self-maps of  $C^2$  strongly convex domains, without requiring the machinery introduced in [Bu2] and [BKR]:

**Theorem 2.8:** *Let  $D \subset\subset \mathbb{C}^n$  be a bounded strictly convex domain, and  $f: D \rightarrow D$  a  $k_D$ -nonexpansive (e.g., holomorphic) self-map without fixed points. Then there exists a  $x_0 \in \partial D$  such that  $T(f) = \{x_0\}$ , that is the sequence of iterates  $\{f^k\}$  converges to the constant map  $x_0$ .*

*Proof:* Fix  $z_0 \in D$ . Lemmas 2.7 and 2.6 give  $x_0 \in \partial D$  such that

$$T(f) \subseteq \bigcap_{R>0} \text{ch}(\overline{F_{z_0}(x_0, R)} \cap \partial D).$$

But  $D$  is strictly convex; therefore  $\text{ch}(\overline{F_{z_0}(x_0, R)} \cap \partial D) = \overline{F_{z_0}(x_0, R)} \cap \partial D$ , and the assertion follows from Corollary 2.4.  $\square$

In [A2] the first author characterized converging one-parameter semigroups of holomorphic self-maps of smooth strongly convex domains. Theorem 2.8 allows us to extend that characterization to (not necessarily smooth) strictly convex domains:

**Corollary 2.9:** *Let  $D \subset\subset \mathbb{C}^n$  be a bounded strictly convex domain, and  $\Phi: \mathbb{R}^+ \rightarrow \text{Hol}(D, D)$  a one-parameter semigroup of holomorphic self-maps of  $D$ . Then  $\Phi$  converges if and only if*

- (i) *either  $\Phi$  has a fixed point  $z_0 \in D$  and the spectral generator at  $z_0$  of  $\Phi$  has no nonzero purely imaginary eigenvalues, or*
- (ii)  *$\Phi$  has no fixed points.*

*Proof:* It follows arguing as in [A2, Theorem 1.3], replacing the references to [A1] by Theorem 2.8.  $\square$

### 3. Weakly convex domains

As mentioned in the introduction, this approach works too when  $D$  is convex but not strictly convex. Simply applying the same argument used to prove Theorem 2.8 one obtains

$$T(f) \subseteq \bigcap_{R>0} \text{ch}(\overline{F_{z_0}(x_0, R)} \cap \partial D)$$

in the  $k_D$ -nonexpansive case, and

$$T(f) \subseteq \bigcap_{R>0} \text{Ch}(\overline{F_{z_0}(x_0, R)} \cap \partial D) \tag{3.1}$$

in the holomorphic case (and it is easy to see that these intersections do not depend on  $z_0 \in D$ ). This already can be used to strengthen the Wolff-Denjoy theorem obtained in [A3, Theorem 2.4.27] for weakly convex  $C^2$  domains. Indeed, we can prove the following:

**Proposition 3.1:** *Let  $D \subset\subset \mathbb{C}^n$  be a  $C^2$  bounded convex domain, and  $x \in \partial D$ . Then for every  $z_0 \in D$  and  $R > 0$  we have*

$$\overline{F_{z_0}(x, R)} \cap \partial D \subseteq \text{Ch}(x).$$

*In particular, if  $x$  is a strictly  $\mathbb{C}$ -linearly convex point then  $\overline{F_{z_0}(x, R)} \cap \partial D = \{x\}$ .*

*Proof:* For every  $x \in \partial D$  let  $\mathbf{n}_x$  denote the unit outer normal vector to  $\partial D$  in  $x$ , and put  $\sigma_x(z) = (z, \mathbf{n}_x)$ , where  $(\cdot, \cdot)$  is the canonical Hermitian product. Then  $\sigma_x$  is a complex supporting functional at  $x$  such that  $\sigma_x(y) = \sigma_x(x)$  for some  $y \in \partial D$  if and only if  $y \in \text{Ch}(x)$ .

We can now argue as in the proof of [A3, Proposition 2.4.26] replacing the  $P$ -function  $\Psi: \partial D \times \mathbb{C}^n \rightarrow \mathbb{C}$  given by  $\Psi(x, z) = \exp(\sigma_x(z) - \sigma_x(x))$ , with the  $P$ -function  $\widehat{\Psi}: \partial D \times \mathbb{C}^n \rightarrow \mathbb{C}$  given by

$$\widehat{\Psi}(x, z) = \frac{1}{1 - (\sigma_x(z) - \sigma_x(x))}.$$

$\square$



**Corollary 3.2:** *Let  $D \subset\subset \mathbb{C}^n$  be a  $C^2$  bounded convex domain, and  $f: D \rightarrow D$  a holomorphic self-map without fixed points. Then there exists  $x_0 \in \partial D$  such that  $T(f) \subseteq \text{Ch}(x_0)$ . In particular, if  $D$  is strictly  $\mathbb{C}$ -linearly convex then the sequence of iterates  $\{f^k\}$  converges to the constant map  $x_0$ .*

*Proof:* It follows from (3.1), Proposition 3.1, and the fact that in  $C^2$  convex domains each point in the boundary admits a unique complex supporting hyperplane.  $\square$

**Remark 3.3:** We conjecture that the final assertion of this corollary should also hold for not necessarily smooth strictly  $\mathbb{C}$ -linearly convex domains.

In weakly convex non-smooth domains large horospheres might be too large, and the right-hand side of (3.1) might coincide with the whole boundary of the domain (see Section 4 for an example in the polydisk); so to get an effective statement we need to replace them with smaller sets.

Small horospheres might be too small; as shown by Frosini [F], there are holomorphic self-maps of the polydisk with no invariant small horospheres. We thus need another kind of horospheres, defined by Kapeluszny, Kuczumow and Reich [KKR2], and studied in detail by Budzyńska [Bu2]. To introduce them we begin with a definition:

**Definition 3.4:** Let  $D \subset\subset \mathbb{C}^n$  be a bounded domain, and  $z_0 \in D$ . A sequence  $\mathbf{x} = \{x_\nu\} \subset D$  converging to  $x \in \partial D$  is a *horosphere sequence* at  $x$  if the limit of  $k_D(z, x_\nu) - k_D(z_0, x_\nu)$  as  $\nu \rightarrow +\infty$  exists for all  $z \in D$ .

**Remark 3.5:** It is easy to see that the notion of horosphere sequence does not depend on the point  $z_0$ .

**Remark 3.6:** In [BKR] it is shown that every sequence in  $D$  converging to  $x \in \partial D$  contains a subsequence which is a horosphere sequence at  $x$ . In strongly convex  $C^3$  domains all sequences converging to a boundary point are horosphere sequences (see [A3, Theorem 2.6.47] and [BPT]); in Section 4 we shall give an explicit example of horosphere sequence in the polydisk.

**Definition 3.7:** Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain. Given  $z_0 \in D$ , let  $\mathbf{x}$  be a horosphere sequence at  $x \in \partial D$ , and take  $R > 0$ . Then the *sequence horosphere*  $G_{z_0}(x, R, \mathbf{x})$  is defined as

$$G_{z_0}(x, R, \mathbf{x}) = \left\{ z \in D \mid \lim_{\nu \rightarrow +\infty} [k_D(z, x_\nu) - k_D(z_0, x_\nu)] < \frac{1}{2} \log R \right\}.$$

The basic properties of sequence horospheres are contained in the following:

**Proposition 3.8:** ([KKR2, Bu2, BKR]) *Let  $D \subset\subset \mathbb{C}^n$  be a bounded convex domain. Fix  $z_0 \in D$ , and let  $\mathbf{x} = \{x_\nu\} \subset D$  be a horosphere sequence at  $x \in \partial D$  for  $z_0$ . Then:*

- (i)  $E_{z_0}(x, R) \subseteq G_{z_0}(x, R, \mathbf{x}) \subseteq F_{z_0}(x, R)$  for all  $R > 0$ ;
- (ii)  $G_{z_0}(x, R, \mathbf{x})$  is nonempty and convex for all  $R > 0$ ;
- (iii)  $\overline{G_{z_0}(x, R_1, \mathbf{x})} \cap D \subset G_{z_0}(x, R_2, \mathbf{x})$  for all  $0 < R_1 < R_2$ ;
- (iv)  $B_D(z_0, \frac{1}{2} \log R) \subset G_{z_0}(x, R, \mathbf{x})$  for all  $R > 1$ ;
- (v)  $B_D(z_0, -\frac{1}{2} \log R) \cap G_{z_0}(x, R, \mathbf{x}) = \emptyset$  for all  $0 < R < 1$ ;
- (vi)  $\bigcup_{R>0} G_{z_0}(x, R, \mathbf{x}) = D$  and  $\bigcap_{R>0} G_{z_0}(x, R, \mathbf{x}) = \emptyset$ .

**Remark 3.9:** If  $\mathbf{x}$  is a horosphere sequence at  $x \in \partial D$  then it is not difficult to check that the family  $\{G_z(x, 1, \mathbf{x})\}_{z \in D}$  and the family  $\{G_{z_0}(x, R, \mathbf{x})\}_{R>0}$  with,  $z_0 \in D$  given, coincide.

It turns out that we can always find invariant sequence horospheres:

**Lemma 3.10:** *Let  $D \subset\subset \mathbb{C}^n$  be a convex domain, and let  $f: D \rightarrow D$  be  $k_D$ -nonexpansive and without fixed points. Then there exists  $x \in \partial D$  and a horosphere sequence  $\mathbf{x}$  at  $x$  such that*

$$f(G_{z_0}(x, R, \mathbf{x})) \subseteq G_{z_0}(x, R, \mathbf{x})$$

for every  $z_0 \in D$  and  $R > 0$ .

*Proof:* Arguing as in the proof of [A1, Theorem 2.3] we can find a sequence  $\{f_\nu\}$  of  $k_D$ -contractions with a unique fixed point  $x_\nu \in D$  such that  $f_\nu \rightarrow f$  and  $x_\nu \rightarrow x \in \partial D$  as  $\nu \rightarrow +\infty$ . Up to a subsequence, we can also assume (Remark 3.6) that  $\mathbf{x} = \{x_\nu\}$  is a horosphere sequence at  $x$ .

Now, for every  $z \in D$  we have

$$|k_D(f(z), x_\nu) - k_D(f_\nu(z), x_\nu)| \leq k_D(f_\nu(z), f(z)) \rightarrow 0$$

as  $\nu \rightarrow +\infty$ . Therefore if  $z \in G_{z_0}(x, R, \mathbf{x})$  we get

$$\begin{aligned} & \lim_{\nu \rightarrow +\infty} [k_D(f(z), x_\nu) - k_D(z_0, x_\nu)] \\ & \leq \limsup_{\nu \rightarrow +\infty} [k_D(f_\nu(z), x_\nu) - k_D(z_0, x_\nu)] + \limsup_{\nu \rightarrow +\infty} [k_D(f(z), x_\nu) - k_D(f_\nu(z), x_\nu)] \\ & \leq \lim_{\nu \rightarrow +\infty} [k_D(z, x_\nu) - k_D(z_0, x_\nu)] < \frac{1}{2} \log R \end{aligned}$$

because  $f_\nu(x_\nu) = x_\nu$  for all  $\nu \in \mathbb{N}$ , and we are done.  $\square$

Putting everything together we can prove the following Wolff-Denjoy theorem for (not necessarily strictly or smooth) convex domains:

**Theorem 3.11:** *Let  $D \subset \subset \mathbb{C}^n$  be a bounded convex domain, and  $f: D \rightarrow D$  a  $k_D$ -nonexpansive (respectively, holomorphic) self-map without fixed points. Then there exist  $x \in \partial D$  and a horosphere sequence  $\mathbf{x}$  at  $x$  such that for any  $z_0 \in D$  we have*

$$T(f) \subseteq \bigcap_{z \in D} \text{ch}(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D) = \bigcap_{R > 0} \text{ch}(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D)$$

if  $f$  is  $k_D$ -nonexpansive, or

$$T(f) \subseteq \bigcap_{z \in D} \text{Ch}(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D) = \bigcap_{R > 0} \text{Ch}(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D)$$

if  $f$  is holomorphic.

*Proof:* The equality of the intersections (and thus the independence of  $z_0$ ) is an immediate consequence of Remark 3.9. Then the assertion follows from Lemmas 3.10 and 2.6.  $\square$

**Corollary 3.12:** *Let  $D \subset \subset \mathbb{C}^n$  be a bounded strictly  $\mathbb{C}$ -linearly convex domain, and  $f: D \rightarrow D$  a holomorphic self-map of  $D$  without fixed points. Then there exist  $x \in \partial D$  and a horosphere sequence  $\mathbf{x}$  at  $x$  such that for any  $z_0 \in D$  we have*

$$T(f) \subseteq \bigcap_{z \in D} \overline{G_z(x, 1, \mathbf{x})} = \bigcap_{R > 0} \overline{G_{z_0}(x, R, \mathbf{x})}.$$

*Proof:* It follows immediately from Theorem 3.11 and the definition of strictly  $\mathbb{C}$ -linearly convex domain.  $\square$

#### 4. The polydisk

The polydisk  $\Delta^n \subset \mathbb{C}^n$  is the unit ball for the norm

$$\|z\| = \max\{|z_j| \mid j = 1, \dots, n\},$$

and therefore

$$\forall z, w \in \Delta^n \quad k_{\Delta^n}(z, w) = \frac{1}{2} \log \frac{1 + \|\gamma_z(w)\|}{1 - \|\gamma_z(w)\|},$$

where

$$\gamma_z(w) = \left( \frac{w_1 - z_1}{1 - \bar{z}_1 w_1}, \dots, \frac{w_n - z_n}{1 - \bar{z}_n w_n} \right)$$

is an automorphism of the polydisk with  $\gamma_z(z) = 0$ .

Thanks to the homogeneity of  $\Delta^n$ , we can restrict ourselves to consider only horospheres with pole  $z_0$  at the origin, and we have (see [A3, chapter 2.4.2] for detailed computations) the following description for horospheres with center  $\xi \in \partial\Delta^n$  and radius  $R > 0$ :

$$E_O(\xi, R) = \left\{ z \in \Delta^n \mid \max_j \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \mid |\xi_j| = 1 \right\} < R \right\}$$

and

$$F_O(\xi, R) = \left\{ z \in \Delta^n \mid \min_j \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \mid |\xi_j| = 1 \right\} < R \right\}.$$

Moreover,  $E_O(\xi, R) = F_O(\xi, R)$  if and only if  $\xi$  has only one component of modulus 1.

Given  $\xi \in \partial\Delta^n$ , a not difficult computation shows that

$$\text{ch}(\xi) = \bigcup_{|\xi_j|=1} \{\eta \in \partial\Delta^n \mid \eta_j = \xi_j\} \quad \text{and} \quad \text{Ch}(\xi) = \bigcap_{|\xi_j|=1} \{\eta \in \partial\Delta^n \mid \eta_j = \xi_j\}.$$

This implies that in the polydisk large horospheres are too large to give a sensible Wolff-Denjoy theorem. Indeed we have  $\text{ch}(\overline{F_O(\xi, R)} \cap \partial\Delta^n) = \text{Ch}(\overline{F_O(\xi, R)} \cap \partial\Delta^n) = \partial\Delta^n$  if  $\xi$  has at least two components of modulus 1, and

$$\text{ch}(\overline{F_O(\xi, R)} \cap \partial\Delta^n) = \text{Ch}(\overline{F_O(\xi, R)} \cap \partial\Delta^n) = \partial\Delta^n \setminus \{\eta \in \partial\Delta^n \mid \eta_{j_0} \neq \xi_{j_0}, |\eta_j| < 1 \text{ for } j \neq j_0\}$$

if  $|\xi_{j_0}| = 1$  and  $|\xi_j| < 1$  for  $j \neq j_0$ .

Let us then compute the sequence horospheres. Fix a horosphere sequence  $\mathbf{x} = \{x_\nu\}$  converging to  $\xi \in \partial\Delta^n$ . Arguing as in [A3, chapter 2.4.2], we arrive to the following

$$G_O(\xi, R, \mathbf{x}) = \left\{ z \in \Delta^n \mid \max_j \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \lim_{\nu \rightarrow +\infty} \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \mid |\xi_j| = 1 \right\} < R \right\}.$$

Since if  $|\xi_j| = 1$  we clearly have

$$\alpha_j := \lim_{\nu \rightarrow +\infty} \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \leq 1,$$

we get

$$G_O(\xi, R, \mathbf{x}) = \left\{ z \in \Delta^n \mid \max_j \left\{ \alpha_j \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \mid |\xi_j| = 1 \right\} < R \right\}.$$

In other words, we can write  $G_O(\xi, R, \mathbf{x})$  as a product

$$G_O(\xi, R, \mathbf{x}) = \prod_{j=1}^n E_j,$$

where, denoting by  $E^\Delta(\sigma, R) \subset \Delta$  the standard horocycle of center  $\sigma \in \partial\Delta$ , pole the origin and radius  $R > 0$ , we put

$$E_j = \begin{cases} \Delta & \text{if } |\xi_j| < 1, \\ E^\Delta(\xi_j, R/\alpha_j) & \text{if } |\xi_j| = 1. \end{cases}$$

As a consequence,

$$\text{ch}(\overline{G_O(\xi, R, \mathbf{x})} \cap \partial\Delta^n) = \text{Ch}(\overline{G_O(\xi, R, \mathbf{x})} \cap \partial\Delta^n) = \bigcup_{j=1}^n \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta},$$

where

$$C_j(\xi) = \begin{cases} \{\xi_j\} & \text{if } |\xi_j| = 1, \\ \partial\Delta & \text{if } |\xi_j| < 1. \end{cases}$$

Notice that the right-hand sides do not depend either on  $R$  or on the horosphere sequence  $\mathbf{x}$ , but only on  $\xi$ .

So Theorem 3.11 in the polydisk assumes the following form:

**Corollary 4.1:** *Let  $f: \Delta^n \rightarrow \Delta^n$  be a  $k_{\Delta^n}$ -nonexpansive (e.g., holomorphic) self-map without fixed points. Then there exists  $\xi \in \partial\Delta^n$  such that*

$$T(f) \subseteq \bigcup_{j=1}^n \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta}, \quad (4.1)$$

where

$$C_j(\xi) = \begin{cases} \{\xi_j\} & \text{if } |\xi_j| = 1, \\ \partial\Delta & \text{if } |\xi_j| < 1. \end{cases}$$

This is the best one can do, in the sense that while it is true that the image of a limit point of the sequence of iterates of  $f$  is always contained in just one of the set appearing in the right-hand side of (4.1), it is impossible to determine a priori in which one on the basis of the point  $\xi$  only; it is necessary to know something more about the map  $f$ . Indeed, Hervé has proved the following:

**Theorem 4.2:** ([He1]) *Let  $F = (f, g): \Delta^2 \rightarrow \Delta^2$  be a holomorphic self-map of the bidisk, and write  $f_w = f(\cdot, w)$  and  $g_z = g(z, \cdot)$ . Assume that  $\text{Fix}(F) = \emptyset$ . Then one and only one of the following cases occurs:*

- (0) *if  $g(z, w) \equiv w$  (respectively,  $f(z, w) \equiv z$ ) then the sequence of iterates of  $F$  converges uniformly on compact sets to  $h(z, w) = (\sigma, w)$ , where  $\sigma$  is the common Wolff point of the  $f_w$ 's (respectively, to  $h(z, w) = (z, \tau)$ , where  $\tau$  is the common Wolff point of the  $g_z$ 's);*
- (1) *if  $\text{Fix}(f_w) = \emptyset$  for all  $w \in \Delta$  and  $\text{Fix}(g_z) = \{y(z)\} \subset \Delta$  for all  $z \in \Delta$  (respectively, if  $\text{Fix}(f_w) = \{x(w)\}$  and  $\text{Fix}(g_z) = \emptyset$ ) then  $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$ , where  $\sigma \in \partial\Delta$  is the common Wolff point of the  $f_w$ 's (respectively,  $T(f) \subseteq \overline{\Delta} \times \{\tau\}$ , where  $\tau$  is the common Wolff point of the  $g_z$ 's);*
- (2) *if  $\text{Fix}(f_w) = \emptyset$  for all  $w \in \Delta$  and  $\text{Fix}(g_z) = \emptyset$  for all  $z \in \Delta$  then either  $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$  or  $T(f) \subseteq \overline{\Delta} \times \{\tau\}$ , where  $\sigma \in \partial\Delta$  is the common Wolff point of the  $f_w$ 's, and  $\tau \in \partial\Delta$  is the common Wolff point of the  $g_z$ ;*
- (3) *if  $\text{Fix}(f_w) = \{x(w)\} \subset \Delta$  for all  $w \in \Delta$  and  $\text{Fix}(g_z) = \{y(z)\} \subset \Delta$  for all  $z \in \Delta$  then there are  $\sigma, \tau \in \partial D$  such that the sequence of iterates converges to the constant map  $(\sigma, \tau)$ .*

We end this paper providing, as promised, an example of horosphere sequence. Given  $\xi \in \partial\Delta^n$ , put  $x_\nu = (1 - 1/\nu)^{1/2}\xi$ ; we claim that  $\mathbf{x} = \{x_\nu\}$  is a horosphere sequence. Indeed, arguing as in [A3, chapter 2.4.2] we see it suffices to show that

$$\max_{|\xi_j|=1} \left\{ \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \frac{|1 - \overline{z_j}x_{\nu,j}|^2}{1 - |z_j|^2} \right\}$$

converges as  $\nu \rightarrow +\infty$ . But indeed with this choice of  $x_\nu$  we have

$$\begin{aligned} \max_{|\xi_j|=1} \left\{ \min_h \left\{ \frac{1 - |x_{\nu,h}|^2}{1 - |x_{\nu,j}|^2} \right\} \frac{|1 - \overline{z_j}x_{\nu,j}|^2}{1 - |z_j|^2} \right\} &= \max_{|\xi_j|=1} \left\{ \min_h \{ \nu(1 - |\xi_h|^2) + |\xi_h|^2 \} \frac{|1 - \overline{z_j}x_{\nu,j}|^2}{1 - |z_j|^2} \right\} \\ &= \max_{|\xi_j|=1} \left\{ \frac{|1 - \overline{z_j}x_{\nu,j}|^2}{1 - |z_j|^2} \right\} \rightarrow \max_{|\xi_j|=1} \left\{ \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \right\}. \end{aligned}$$

In particular, using this horosphere sequence one obtains  $G_O(\xi, R, \mathbf{x}) = E_O(\xi, R)$ .

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