Isogeny graphs with maximal real multiplication

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Abstract. An isogeny graph is a graph whose vertices are principally polarizable abelian varieties and whose edges are isogenies between these varieties. In his thesis, Kohel describes the structure of isogeny graphs for elliptic curves and shows that one may compute the endomorphism ring of an elliptic curve defined over a finite field by using a depth-first search (DFS) algorithm in the graph. In dimension 2, the structure of isogeny graphs is less understood and existing algorithms for computing endomorphism rings are very expensive. In this article, we show that, under certain circumstances, the problem of determining the endomorphism ring can also be solved in genus 2 with a DFS-based algorithm. We consider the case of genus-2 Jacobians with complex multiplication, with the assumptions that the real multiplication subring is maximal and has class number one. We describe the isogeny graphs in that case, locally at prime numbers which split in the real multiplication subfield. The resulting algorithm is implemented over finite fields, and examples are provided. To the best of our knowledge, this is the first DFS-based algorithm in genus 2.

1 Introduction

Isogeny graphs are non-oriented graphs whose vertices are principally polarizable simple abelian varieties and whose edges are isogenies between these varieties. Isogeny graphs were first studied by Kohel [15], who proves that in the case of elliptic curves, we may use these structures to compute the endomorphism ring of an elliptic curve. Kohel identifies two types of ℓ-isogenies (i.e. of degree ℓ) in the graph: ascending-descending and horizontal. The first type corresponds to the case of an isogeny between two elliptic curves, such that the endomorphism ring of one curve is contained in the endomorphism ring of the other. The second type is that of an isogeny between two genus 1 curves with isomorphic endomorphism ring. As a consequence, computing the ℓ-adic valuation of the conductor of the endomorphism ring can be done by a depth-first search algorithm in the isogeny graph [15]. In the case of genus-2 Jacobians, designing a similar algorithm for endomorphism ring computation requires a good understanding of the isogeny graph structure.

Let K be a primitive quartic CM field and K_0 its totally real subfield. In this paper, we study subgraphs of isogenies whose vertices are all genus-2 Jacobians with endomorphism ring isomorphic to an order of K which contains the maximal order $\mathcal{O}_{K_0}$. Furthermore, we assume that $\mathcal{O}_{K_0}$ is principal and that ℓ splits in $\mathcal{O}_{K_0}$.

We show that the lattice of orders meeting these conditions has a simple 2-dimensional grid structure locally at ℓ. This results into a classification of isogenies in the isogeny graph: ascending-descending and horizontal, where these qualificatives apply separately to the two "dimensions" of the lattice of orders. Moreover, we show that any ℓ-isogeny which is such that the two endomorphism rings contain $\mathcal{O}_{K_0}$ is a composition of two isogenies of degree ℓ which preserve real multiplication. As a consequence, we design a depth-first search algorithm for computing endomorphism rings in the ℓ-isogeny graph, based on Cosset and Robert's algorithm for constructing ℓ-isogenies over finite fields. To the best of our knowledge, this is the first depth-first search algorithm for computing locally at small prime numbers ℓ the endomorphism ring of an ordinary genus-2 Jacobian. With our method, as well as with the Eisenträger-Lauter algorithm [7], the dominant part of the complexity is given by the computation of a subgroup of the ℓ-torsion. Our

analysis shows that our algorithm performs faster, since a smaller torsion subgroup is computed, defined over a smaller field.

This paper is organized as follows. Section 2 provides background material concerning isogeny graphs, $\mathcal{O}_{K_0}$-orders of quartic CM fields, as well as the definition and some properties of the Tate pairing. In Section 3 we give formulae for cyclic isogenies between principally polarizable complex tori, with maximal real multiplication. The structure of the graph given by reductions over finite fields of these isogenies is proved in Section 4. In Section 5 we show that the computation of the Tate pairing allows to orient ourselves in the isogeny graph. Finally, in Section 6 we give our algorithm for endomorphism ring computation when the real multiplication is maximal, compare its performance to the one of Eisenträger and Lauter's algorithm, and report on practical experiments over finite fields.

2 Background and notations

It is well known that in the case of elliptic curves with complex multiplication by an imaginary quadratic field K, the lattice of orders of K has the structure of a tower. This results into a easy way to classify isogenies and navigate in isogeny graphs [15,8,14].

Throughout this paper, we are concerned with the genus 2 case. Let then K be a primitive quartic CM field, with totally real subfield K_0. In this paper, we assume that principally polarized abelian surfaces are *simple*, i.e. not isogenous to a product of elliptic curves. The quartic CM field K is primitive, i.e. it does not contain a totally imaginary subfield. Let $K = \mathbb{Q}(\gamma)$, with $\gamma = i\sqrt{a + b\sqrt{d}}$ if $d \equiv 2, 3 \mod 4$ or $\gamma = i\sqrt{a + b\left(\frac{-1+\sqrt{d}}{2}\right)}$ if $d \equiv 1 \mod 4$. A CM-type Φ is a pair of non-complex conjugate embeddings of K in $\mathbb{C}$

$$\Phi(z) = \{\phi_1(z), \phi_2(z)\}.$$

We assume that K_0 has class number one. This implies in particular that the maximal order $\mathcal{O}_K$ is a module over the principal ideal ring $\mathcal{O}_{K_0}$, whence we may define η such that

$$\mathcal{O}_K = \mathcal{O}_{K_0} + \mathcal{O}_{K_0}\eta.$$

The notation η will be retained throughout the paper.

Several results of the article will involve a prime number ℓ and also the finite field $\mathbb{F}_p$ or its extensions. We always implicitly assume that ℓ is coprime to p. Furthermore, the case which matters for our point of view is when ℓ splits as two distinct degree-one prime ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ in $\mathcal{O}_{K_0}$. How the ideals $\mathfrak{l}_{1,2}$ split in $\mathcal{O}_K$ is not determined a priori, however.

2.1 Isogeny graphs: definitions and terminology

In this paper, we are interested in isogeny graphs whose nodes are principally polarizable abelian surfaces (i.e. Jacobians of hyperelliptic genus-2 curves) and whose edges are isogenies between them.

Definition 1. *Let $I : A_1 \to A_2$ be an isogeny between polarized abelian varieties and let E be a fixed polarization on A_2. The induced polarization on A_1, that we denote by I^*E, is defined by*

$$I^*E(x, y) = E(I(x), I(y)),$$

for all $x, y \in A$.

We recall here the definition of an ℓ-isogeny.

Definition 2. *Let $I : (A_1, E_1) \to (A_2, E_2)$ be an isogeny between principally polarized abelian varieties. We will say that I is an ℓ-isogeny if $I^*E_2 = \ell E_1$.*

One can easily see that these isogenies have degree ℓ^2 and that they are a generalization of genus 1 ℓ-isogenies. Hence a natural idea would be to consider the graph given by *ℓ-isogenies* between principally polarized abelian surfaces. Recent developments on the construction of ℓ-isogenies [17,5] allowed to compute examples of isogeny graphs over finite fields, whose edges are rational ℓ-isogenies [2]. It was noticed in this way that the corresponding lattice of orders has a much more complicated structure when compared to its genus-1 equivalent. Figure 1 displays an example of an ℓ-isogeny graph. The corresponding lattice of orders contains two orders of index 3 (in the maximal order), which are not contained one in the other. The existence of rational isogenies between Jacobians corresponding to these two orders shows that we cannot classify isogenies into ascending/descending and horizontal ones. This is a major obstacle to designing a depth-first search algorithm for computing the endomorphism ring.

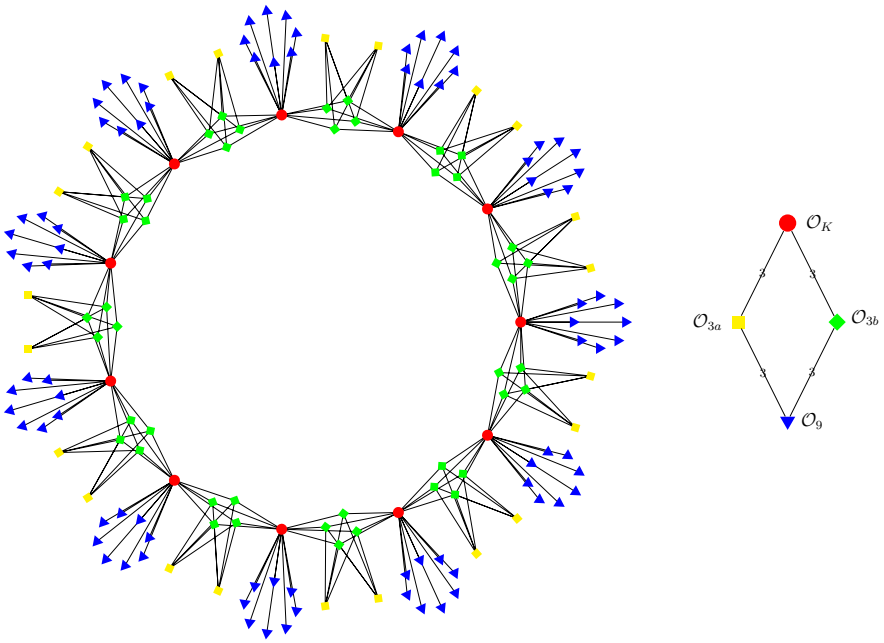


Fig. 1. Example of an ℓ-isogeny graph for $\ell = 3$ defined over a finite field $\mathbb{F}_p$, with $p = 211$ and K defined by $\alpha^4 + 81\alpha^2 + 1181$.

Note that here and all through the paper, we shall only distinguish isogenies up to isomorphism, regarding isogenies I_1 and I_2 as equivalent if $I_1 = i_1 \circ I_2 \circ i_2$ for any automorphisms i_1 and i_2. Also, note that we consider isogenies between abelian surfaces admitting principal polarizations, but without fixing polarizations (i.e. if $I : A \to B$ is an isogeny and E a polarization on B, we do not fix the polarization on A to be the one induced by E.) The approach we will take here is to consider the graph of *all* (equivalence classes of) isogenies between principally polarizable abelian surfaces and decompose it into subgraphs whose vertices are abelian surfaces with real multiplication by a fixed order $\mathcal{O}$ of K_0.

Definition 3. *Let A be a (principally polarized) abelian variety with complex multiplication by a quartic field K. Let K_0 be the real multiplication subfield of K and $\mathcal{O}$ and order in K_0. We say that A has real multiplication by $\mathcal{O}$ if there is a ring embedding $i_A : \mathcal{O} \hookrightarrow \mathrm{End}(A)$.*

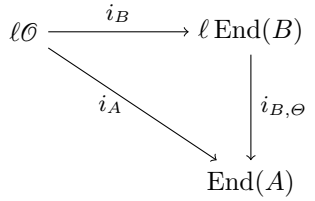
Let A and B two principally polarized abelian varieties with real multiplication by an order $\mathcal{O}$. Let $I : A \to B$ be an isogeny and I' such that $II' = [e_I]$, where e_I is the exponent of I (i.e. the exponent of the finite group $\mathrm{Ker}(I)$). We define the following map:

$$\Theta_B : \mathrm{End}(B) \to \mathrm{End}(A)$$
$$\phi \to I' \circ \phi \circ I.$$

Assume that e_I equals ℓ. This implies that there is an embedding $i_{B,\Theta} : \ell\,\mathrm{End}(B) \hookrightarrow \mathrm{End}(A)$. In particular, if B has real multiplication by an order $\mathcal{O}$ and $i_B : \mathcal{O} \to \mathrm{End}(B)$, this gives an embedding of $\ell\mathcal{O}$ in $\mathrm{End}(A)$. In a symmetric way, we obtain $i_{A,\Theta} : \ell\,\mathrm{End}(A) \hookrightarrow \mathrm{End}(B)$.

Definition 4. *With the notation above, let A and B two abelian varieties and $I : A \to B$ an isogeny between them. We say that I preserves the real multiplication by an order $\mathcal{O}$ in K_0 if the following conditions are satisfied:*

1. *A and B have real multiplication by $\mathcal{O}$, and no larger order.*
2. *The following diagram and its symmetric by inter-changing the roles of A and B are commutative:*



With this definition, we call layer in the graph the subgraph whose edges are isogenies preserving real multiplication by an order in $\mathcal{O}_{K_0}$. Understanding the structure of the graph then comes down to explaining the structure of each layer and in a later step classifying isogenies beween two vertices lying at different layers of the graph.

In this paper, we fully describe the structure of the maximal real multiplication layer. Working towards this goal, we first identify cyclic isogenies of degree ℓ between principally polarizable abelian varieties with maximal real multiplication. We show in Proposition 11 that a sufficient condition to guarantee the existence of isogenies of degree ℓ between principally polarizable abelian varieties is that $\ell\mathcal{O}_{K_0}$ decomposes as a product of principal ideals of degree 1, whose generators are totally positive.

As a consequence, we will assume that ℓ splits in $\mathcal{O}_{K_0}$ into two principal ideals. Under these restrictions, we describe the simple and interesting structure of the graph of cyclic isogenies, which fits into the ascending/descending and horizontal framework. Using this graph structure, we characterize all isogenies between principally polarizable abelian surfaces which preserve maximal real multiplication. This leads in particular to viewing Figure 1 as derived from a more structured graph, whose characteristics are well explained.

The case when ℓ is ramified the graph structure is similar, as explained in Section 4 (Remark 22). In the case of ℓ inert, we explain in Section 3 that there are no degree ℓ isogenies between principally polarizable abelian varieties with CM by K, preserving real multiplication. We chose not to treat this case in this work.

2.2 The lattice of $\mathcal{O}_{K_0}$-orders in a quartic CM field K

A major obstacle to explaining the structure of genus 2 isogeny graphs is that the lattice of orders of K lacks a concise description. Given an isogeny $I : J_1 \to J_2$ between two abelian surfaces with degree ℓ, the corresponding endomorphism rings are such that $\ell\mathcal{O}_{J_1} \subset \mathcal{O}_{J_2}$ and $\ell\mathcal{O}_{J_2} \subset \mathcal{O}_{J_1}$. Hence, even if a inclusion relation is guaranteed $\mathcal{O}_{J_2} \subset \mathcal{O}_{J_1}$, the index of one order in the other is bounded by ℓ^3. Since the $\mathbb{Z}$-rank of orders is 4, there could be several suborders of $\mathcal{O}_{J_1}$ with the same index.

In this paper, we study the structure of the isogeny graph between abelian varieties with maximal real multiplication. The first step in this direction is to describe the structure of the

lattice of orders of K which contain $\mathcal{O}_{K_0}$. Following [10], we call such an order an $\mathcal{O}_{K_0}$-order. We study the conductors of such orders. We recall that the conductor of an order $\mathcal{O}$ is the ideal

$$\mathfrak{f}_{\mathcal{O}} = \{x \in \mathcal{O}_K \mid x\mathcal{O}_K \subset \mathcal{O}\}.$$

The following lemma was given by Goren and Lauter [10].

Lemma 5. *Let K be a quartic CM-field and K_0 its real multiplication subfield. Assume that the class number of K_0 is 1. Then the following hold:*

1. *For any $\mathcal{O}_{K_0}$-order $\mathcal{O}$ of K there is $\alpha \in \mathcal{O}_{K_0}$, $\alpha \neq 0$ such that $\mathcal{O} = \mathcal{O}_{K_0}[\alpha\eta]$. The element α is unique up to units of $\mathcal{O}_{K_0}$. The conductor of $\mathcal{O}$ is the principal $\mathcal{O}_K$-ideal $\alpha\mathcal{O}_K$.*
2. *For any element $\alpha \in \mathcal{O}_{K_0}$, $\mathcal{O}_{K_0}[\alpha\eta]$ is an order of conductor $\alpha\mathcal{O}_K$.*

A first consequence of Lemma 5 is that there is a bijection between $\mathcal{O}_{K_0}$-orders and principal ideals in $\mathcal{O}_{K_0}$, which associates to every order the ideal $\mathfrak{f} \cap \mathcal{O}_{K_0}$, which for brevity we still call the conductor and denote by $\mathfrak{f}$.

Using the particular shape of $\mathcal{O}_K$ as a monogenic $\mathcal{O}_{K_0}$-module, we may rewrite the conductor differently. For a fixed element $\omega \in \mathcal{O}_K$, we define the conductor of $\mathcal{O}$ with respect to ω to be the ideal

$$\mathfrak{f}_{\omega,\mathcal{O}} = \{x \in \mathcal{O}_K \mid x\omega \in \mathcal{O}\}.$$

The following statement is an immediate consequence of Lemma 5.

Lemma 6. *For any $\mathcal{O}_{K_0}$-order $\mathcal{O}$ and any η such that $\mathcal{O}_K = \mathcal{O}_{K_0}[\eta]$, we have $\mathfrak{f}_{\mathcal{O}} = \mathfrak{f}_{\eta,\mathcal{O}}$.*

Let now $\mathcal{O}$ be an $\mathcal{O}_{K_0}$-order whose index is divisible by a power of ℓ. Assume that ℓ splits in $\mathcal{O}_{K_0}$ and let $\ell = \mathfrak{l}_1\mathfrak{l}_2$. Then by Lemma 5 the conductor $\mathfrak{f}$ has a unique factorization into prime ideals containing $\mathfrak{l}_1^{e_1}\mathfrak{l}_2^{e_2}$. Locally at ℓ, the lattice of orders of index divisible by ℓ has the form given in Figure 2. This is equivalent to the following statement.

Lemma 7. *Let $\mathcal{O}$ be an $\mathcal{O}_{K_0}$-order in K. Locally at ℓ, the position of $\mathcal{O}$ within the lattice of $\mathcal{O}_{K_0}$-orders is given by the valuations $\nu_{\mathfrak{l}_i}(\mathfrak{f}_{\mathcal{O}})$, for $i = 1, 2$.*

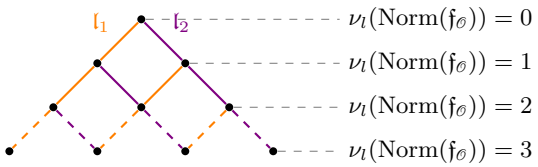


Fig. 2. The lattice of orders

We call level in the lattice of orders the set of all orders having the same ℓ-adic valuation of the norm of the conductor. For example, level 2 in Figure 2 is formed by three orders with conductors $\mathfrak{l}_1, \mathfrak{l}_1\mathfrak{l}_2$ and $\mathfrak{l}_2^2$, respectively. This distribution of orders on levels leads to a classification of isogenies into descending and ascending ones, which is the key point to a DFS algorithm for navigating in the isogeny graph, just like in the elliptic curve case. This will be furthered detailed in Section 4.

2.3 The Tate pairing

Let A be a polarized abelian variety, defined over a field L. We denote by $J[m]$ the m-torsion subgroup. We denote by μ_m the group of m-th roots of unity. Let

$$W_m : A[m] \times \hat{A}[m] \to \mu_m$$

be the m-Weil pairing.

In this paper, we are only interested in the Tate pairing over finite fields. We give a specialized definition of the pairing in this case, following [21,12]. More precisely, suppose we have $m \mid \#A(\mathbb{F}_q)$ and denote by k the *embedding degree with respect to* m, i.e. the smallest integer $k \geq 0$ such that $m \mid q^k - 1$. Moreover, we assume that $A[m]$ is defined over $\mathbb{F}_{q^k}$. We define the Tate pairing as

$$t_m(\cdot,\cdot) : \begin{cases} A(\mathbb{F}_{q^k})/mA(\mathbb{F}_{q^k}) \times \hat{A}[m](\mathbb{F}_{q^k}) \to \mu_m \\ (P,Q) \mapsto W_m(\pi_k(\bar{P}) - \bar{P}, Q), \end{cases}$$

where π_k is k-th power of the Frobenius endomorphism of A and $\bar{P}$ is any point such that $m\bar{P} = P$. Note that since $A[m] \subseteq A(\mathbb{F}_{q^k})$, this definition is independent of the choice of $\bar{P}$. Indeed, if $\bar{P}_1$ is a second point such that $m\bar{P}_1 = P$, then $\bar{P}_1 = \bar{P} + T$, where T is a m-torsion point, and $\pi_k(\bar{P}_1) - \bar{P}_1 = \pi_k(\bar{P}) - \bar{P}$.
For a fixed polarization $\lambda : A \to \hat{A}$ we define a pairing on A itself

$$t_m^\lambda(\cdot,\cdot) : \begin{cases} A(\mathbb{F}_{q^k})/mA(\mathbb{F}_{q^k}) \times A[m](\mathbb{F}_{q^k}) \to \mu_m \\ (P,Q) \mapsto t_m(P, \lambda(Q)). \end{cases}$$

If A has a distinguished principal polarization and there is no risk of confusion, we write simply $t_m(\cdot,\cdot)$ instead of $t_m^\lambda(\cdot,\cdot)$.

Lichtenbaum [16] describes a version of the Tate pairing on Jacobian varieties. Since we use Lichtenbaum's formula for computations, we briefly recall it here. Let $D_1 \in A(\mathbb{F}_{q^k})$ and $D_2 \in A[m](\mathbb{F}_{q^k})$ be two divisor classes, represented by two divisors such that $\mathrm{supp}(D_1) \cap \mathrm{supp}(D_2) = \emptyset$. Since D_2 has order m, there is a function f_{m,D_2} such that $\mathrm{div}(f_{m,D_2}) = mD_2$. The Lichtenbaum pairing of the divisor classes D_1 and D_2 is computed as

$$T_m(D_1, D_2) = f_{m,D_2}(D_1).$$

The output of this pairing is defined up to a coset of $(\mathbb{F}_{q^k}^*)^m$. Given that $\mathbb{F}_{q^k}^*/(\mathbb{F}_{q^k}^*)^m \simeq \mu_m$, we obtain a pairing defined as

$$\begin{aligned} t_m(\cdot,\cdot) : A(\mathbb{F}_{q^k})/mA(\mathbb{F}_{q^k}) \times A[m](\mathbb{F}_{q^k}) &\to \mu_m \\ (P,Q) &\to T_m(P,Q)^{(q^k-1)/m}. \end{aligned}$$

The function $f_{m,D_2}(D_1)$ is computed using Miller's algorithm [18] in $O(\log m)$ operations in $\mathbb{F}_{q^k}$.

3 Isogenies preserving real multiplication

An abelian surface over $\mathbb{C}$ with complex multiplication by an order $\mathcal{O} \subset K$ is of the form $A = \mathbb{C}^2/\Phi(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of $\mathcal{O}$ and $\Phi = (\phi_1, \phi_2)$ is a CM-type. This variety is said to be of CM-type $(\mathcal{O}, \Phi)$. Recall that we focus on the case where $\mathcal{O}_{K_0} \subset \mathcal{O}$. Since $\mathcal{O}_{K_0}$ is a Dedekind domain and the ideal $\mathfrak{a}$ is an $\mathcal{O}_{K_0}$-module, we may then write it as $\mathfrak{a} = \Lambda_1\alpha + \Lambda_2\beta$, with $\alpha, \beta \in K$, and $\Lambda_{1,2}$ two $\mathcal{O}_{K_0}$-ideals. Hence we have $A \cong \mathbb{C}^2/\Phi(\Lambda)$ and $\Lambda = \alpha^{-1}\mathfrak{a} = \Lambda_1 + \Lambda_2\tau$, with Λ_1 and Λ_2 lattices in K_0 and $\tau = \frac{\beta}{\alpha} \in K$. We may arrange so that $(\tau^{\phi_1}, \tau^{\phi_2}) \in \mathbb{H}_1^2$, where $\mathbb{H}_1$ is the upper-half plane. Note that in the more restrictive setting we have selected, K_0 has class number one, which entails that we can choose $\Lambda_1 = \Lambda_2 = \mathcal{O}_{K_0}$.

Every Riemann form on $\mathbb{C}^2/\Phi(\Lambda)$ is of the form

$$H_\xi(z,w) = \sum_{r=1}^{2} \frac{\xi^{\phi_r} z^{\phi_r} \bar{w}^{\phi_r}}{\Im(\tau^{\phi_r})},$$

for $\xi \in K_0$ totally positive and all $z, w \in \mathbb{C}^2$. The imaginary part E_ξ satisfies

$$E_\xi(z,w) = \sum_{r=1}^{2} \xi^{\phi_r}(x'^{\phi_r} y^{\phi_r} - x^{\phi_r} y'^{\phi_r}),$$

with $z = x + y\tau, w = x' + y'\tau$, where $x, y, x', y' \in \mathbb{R}$.

The isogenies discussed by the following proposition were brought to our attention by John Boxall.

Proposition 8. *Let K and K_0 be as previously stated. Let ℓ be a prime, and $\mathfrak{l} \subset \mathcal{O}_{K_0}$ a prime $\mathcal{O}_{K_0}$-ideal of norm ℓ. Let $A = \mathbb{C}^2/\Phi(\Lambda)$ be an abelian surface over $\mathbb{C}$ with complex multiplication by an $\mathcal{O}_{K_0}$-order $\mathcal{O} \subset K$, with $\Lambda = \Lambda_1 + \Lambda_2\tau$. A set of representatives of the cyclic subgroups of $(\Lambda/\mathfrak{l})/\Lambda$, and more precisely of the isogenies on A having these subgroup as kernels is given by $\{I_\infty\} \cup \{I_\rho,\ \rho \in \Lambda_1\Lambda_2^{-1}/\mathfrak{l}\Lambda_1\Lambda_2^{-1}\}$, where:*

$$I_\infty : \begin{cases} A \to \mathbb{C}^2/\Phi(\frac{\Lambda_1}{\mathfrak{l}} + \Lambda_2\tau), \\ z \mapsto z, \end{cases} \quad I_\rho : \begin{cases} A \to \mathbb{C}^2/\Phi(\Lambda_1 + \frac{\Lambda_2}{\mathfrak{l}}(\tau + \rho)), \\ z \mapsto z. \end{cases} \tag{1}$$

Proof. Our hypotheses imply that Λ is an $\mathcal{O}_{K_0}$-module of rank two, from which it follows that $(\Lambda/\mathfrak{l})/\Lambda$ is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^2$. The $\ell + 1$ cyclic subgroups of $(\Lambda/\mathfrak{l})/\Lambda$ are the kernels of the isogenies given in the Proposition. □

The isogenies above are central to this article, and are denoted by $\mathfrak{l}$-isogenies, defined as follows.

Definition 9. *Let $\mathfrak{l}$ be an ideal of $\mathcal{O}_{K_0}$. Then the $\mathfrak{l}$-torsion of an abelian variety A with real multiplication by $\mathcal{O}_{K_0}$ is given by*

$$A[\mathfrak{l}] = \{x \in A \ s.t.\ \alpha x = 0,\ \forall \alpha \in \mathfrak{l}\}.$$

Isogenies with kernel a cyclic subgroup of $A[\mathfrak{l}]$ of order ℓ are called $\mathfrak{l}$-isogenies.

For the commonly encountered case where $\mathfrak{l} = \alpha\mathcal{O}_{K_0}$ for some generator $\alpha \in \mathcal{O}_{K_0}$ (which occurs in our setting since $\mathcal{O}_{K_0}$ is assumed principal), the notation $A[\mathfrak{l}]$ above matches with the notation $A[\alpha]$ representing the kernel of the endomorphism represented by α. In this situation, the term α-isogeny may also be used.

Given definition 9, Proposition 8 can be regarded as giving formulae for a set of representatives for isomorphism classes of $\mathfrak{l}$-isogenies over the complex numbers.

The following trivial observation that $\mathfrak{l}$-isogenies preserve the maximal real multiplication follows directly from $\mathrm{End}(\frac{\Lambda_i}{\mathfrak{l}}) = \mathrm{End}(\Lambda_i)$. We shall investigate a converse to this statement later in this article.

Proposition 10. *Let A be an abelian surface with $\mathrm{End}(A)$ an $\mathcal{O}_{K_0}$-order. Let $I : A \to B$ be an $\mathfrak{l}$-isogeny. Then $\mathrm{End}(B)$ is also an $\mathcal{O}_{K_0}$-order.*

The following proposition shows how polarizations can be transported through $\mathfrak{l}$-isogenies. We use here the fact that $\mathcal{O}_{K_0}$ is assumed to have class number one.

Proposition 11. *Let A be an abelian surface with $\mathrm{End}(A)$ an $\mathcal{O}_{K_0}$-order. Let $I : A \to B$ be an $\mathfrak{l}$-isogeny (following the notations of Proposition 8). Let E_ξ define a principal polarization of A. We have:*

1. *If $\mathfrak{l} = (\alpha)$ with $\alpha \in K_0$ totally positive, $E_{\alpha\xi}$ defines a principal polarization on B.*
2. *$I^*E_{\alpha\xi} = \alpha E_\xi$.*

Proof. We follow notations of Proposition 8 and take $I = I_\infty$ as an example (the other cases are similar). We can write

$$\begin{aligned} E_\xi(x + y\tau, x' + y'\tau) &= E_{\alpha\xi}(\frac{x}{\alpha} + y\tau, \frac{x'}{\alpha} + y'\tau), \\ &= E_{\alpha\xi}(x + \frac{y}{\alpha}(\tau + \rho), x' + \frac{y'}{\alpha}(\tau + \rho)). \end{aligned}$$

Hence if H_ξ defines a principal polarization on $\mathbb{C}^2/\Phi(\Lambda_1+\Lambda_2\tau)$ and α is totally positive then $H_{\alpha\xi}$ defines principal polarizations on the variety $\mathbb{C}^2/\Phi(\frac{\Lambda_1}{\alpha}+\Lambda_2\tau)$ (we just showed that the matrices of the corresponding Riemann forms are equal).

The fact that $I^*E_{\alpha\xi}=\alpha E_\xi$ follows from the definition of I^*, and of the Riemann forms E_ξ and $E_{\alpha\xi}$. □

Lemma 12. *Let A be a principally polarized abelian variety under the assumptions in Proposition 11. The dual of an α-isogeny starting from A is an α-isogeny.*

Proof. This follows trivially from $I^*E_{\alpha\xi}=\alpha E_\xi$, since this implies that $\alpha\lambda_\xi=\hat{I}\circ\lambda_{\alpha\xi}\circ I$, where λ_ξ and and $\lambda_{\alpha\xi}$ are the isogenies corresponding to polarizations E_ξ and $E_{\alpha\xi}$.

Robert [20] shows that if $I:(A,E_1)\to(B,E_2)$ is an isogeny between principally polarizable abelian varieties, then the homomorphism corresponding to the induced polarization writes as $\lambda_{I^*E_2}=\lambda_{E_1}\circ\phi$, where ϕ is a real endomorphism. If moreover, I is of degree ℓ, the endomorphism ϕ corresponds to a totally positive element α of norm ℓ in $\mathcal{O}_{K_0}$. As a consequence, in the remainder of this paper, we assume that such an α exists. This implies that we restrict to the cases where ℓ is either split or ramified. In this paper, we deliberately chose to focus on the split case. This restriction allows us to further design an algorithm for endomorphism ring computation, as we will explain in Section 6.

Remark 13. If ℓ is a prime number such that $\ell\mathcal{O}_{K_0}=\mathfrak{l}_1\mathfrak{l}_2$, we denote by α_i, $i=\{1,2\}$, elements of $\mathcal{O}_{K_0}$ such that $\mathfrak{l}_i=\alpha_i\mathcal{O}_{K_0}$. Let $I:A\to B$ be an $\mathfrak{l}_1$-isogeny. Proposition 11 implies that for a given polarization ξ on A, $\ell\lambda_\xi=\hat{I}\circ(\alpha_2\lambda_{\alpha_1\xi})\circ I$.

Note that if ℓ is such that $\ell\mathcal{O}_{K_0}=\mathfrak{l}_1\mathfrak{l}_2$, with $\mathfrak{l}_1+\mathfrak{l}_2=(1)$, then the factorization of ℓ yields a symplectic basis for the ℓ-torsion. Indeed, we have $J[\ell]=J[\mathfrak{l}_1]+J[\mathfrak{l}_2]$, and the following proposition establishes the symplectic property.

Proposition 14. *Let J be a principally polarized abelian surface defined over a number field L. With the notations above, we have $W_\ell(P_1,P_2)=1$ for any $P_1\in J[\mathfrak{l}_1]$ and $P_2\in J[\mathfrak{l}_2]$.*

Proof. This can be easily checked on the complex torus $\mathbb{C}^2/\Phi(\Lambda_1+\Lambda_2\tau)$. Let $P_1=\frac{x_1}{\alpha_1}+\frac{x_2}{\alpha_1}\tau\in J[\alpha_1]$ and $P_2=\frac{y_1}{\alpha_2}+\frac{y_2}{\alpha_2}\tau\in J[\alpha_2]$, where $x_1,y_1\in\Lambda_1$ and $x_2,y_2\in\Lambda_2$. Then $W_\ell(P_1,P_2)=\exp(-2\pi i\ell\frac{E_\xi(x_1+x_2\tau,y_1+y_2\tau)}{\ell}))=1$. □

4 The structure of the real multiplication isogeny graph over finite fields

In this Section, we study the structure of the graph given by rational isogenies between principally polarizable abelian surfaces defined over a finite field, such that the corresponding endomorphism rings are $\mathcal{O}_{K_0}$-orders. The endomorphism ring of an ordinary Jacobian J over a finite field $\mathbb{F}_q$ ($q=p^n$) is an order in the quartic CM field K such that

$$\mathbb{Z}[\pi,\bar{\pi}]\subset\mathrm{End}(J)\subset\mathcal{O}_K,$$

where $\mathbb{Z}[\pi,\bar{\pi}]$ denotes the order generated by π, the Frobenius endomorphism and by $\bar{\pi}$, the Verschiebung. Moreover, the assumption that $\mathrm{End}(J)$ is an $\mathcal{O}_{K_0}$-order implies that it contains $\mathcal{O}_{K_0}[\pi-\bar{\pi}]\subset\mathcal{O}_{K_0}\mathbb{Z}[\pi,\bar{\pi}]$, where the two latter orders coincide locally at all primes except 2.

The notion of $\mathfrak{l}$-isogeny defined in Definition 9 has been used so far for abelian surfaces defined over $\mathbb{C}$. We remark than whenever an abelian variety defined over a finite field $\mathbb{F}_q$ has endomorphism ring some $\mathcal{O}_{K_0}$-order in the quartic field K, we may define the notion of $\mathfrak{l}$-isogeny exactly the same way.

Proposition 15. *Let J be a principally polarized ordinary abelian variety defined over a finite field $\mathbb{F}_q$ of characteristic p and having real multiplication by $\mathcal{O}_{K_0}$. Let $\mathfrak{l}$ be a prime ideal of degree 1 over ℓ in $\mathcal{O}_{K_0}$, with $\ell \neq p$. Then a $\mathfrak{l}$-isogeny $I : J \to J'$ preserves real multiplication and the target variety J' is principally polarizable.*

Proof. By the theory of canonical lifts, we may choose abelian surfaces $\tilde{J}$ defined over a number field L, such that J is isomorphic to the reduction of $\tilde{J}$ modulo a ideal $\mathfrak{P}$ lying over p in L. We have that $J[\mathfrak{l}] \simeq \tilde{J}[\mathfrak{l}]$ and the reductions of $\mathfrak{l}$-isogenies starting from $\tilde{J}$ give $\ell + 1$ (equivalence classes of) isogenies starting from J. Hence there is an isogeny $\tilde{I} : \tilde{J} \to J_1$ such that J_1 has good reduction (mod $\mathfrak{P}$) and its reduction is isomorphic to J'. Since the reduction map $\mathrm{End}(J_1) \to \mathrm{End}(J')$ is injective, it follows that $\mathrm{End}(J')$ is an $\mathcal{O}_{K_0}$-order. By reducing polarizations given in Proposition 11, we deduce that if J is principally polarizable, then J' is also principally polarizable.

We are now interested in determining the field of definition of $\mathfrak{l}$-isogenies starting from J. For that, we need several definitions.

Let $\mathfrak{l}$ be an ideal in $\mathcal{O}_{K_0}$ and α a generator of this ideal. Let $\mathcal{O}$ be an order of K and let $\theta \in \mathcal{O}$. We define the $\mathfrak{l}$-adic valuation of θ in $\mathcal{O}$ as

$$\nu_{\mathfrak{l},\mathcal{O}}(\theta) := \max_{m \geq 0}\{m : \theta \in \mathcal{O}_{K_0} + \mathfrak{l}^m\mathcal{O}\}.$$

Recall that for a Jacobian J with maximal real multiplication, we are interested (by Lemma 7) in computing the $\mathfrak{l}$-adic valuation of the conductor of the endomorphism ring $\mathcal{O}_J$. We remark that it suffices to determine $\nu_{\mathfrak{l},\mathcal{O}_J}(\pi)$. Indeed, we have $\mathcal{O}_J = \mathcal{O}_{K_0} + \mathcal{O}_{K_0}\mathfrak{f}_{\eta,\mathcal{O}_J}\eta$ and

$$\nu_{\mathfrak{l}}(\mathfrak{f}_{\eta,\mathcal{O}_J}) = \nu_{\mathfrak{l},\mathcal{O}_K}(\pi) - \nu_{\mathfrak{l},\mathcal{O}_J}(\pi). \tag{2}$$

In the remainder of this paper, we denote by $\nu_{\mathfrak{l}_i,J}(\pi) := \nu_{\mathfrak{l},\mathcal{O}_J}(\pi)$.

Proposition 16. *Let ℓ be an odd prime number, such that $(\ell) = \mathfrak{l}_1\mathfrak{l}_2$ in $\mathcal{O}_{K_0}$. Then the largest integer n such that the Frobenius matrix on $J[\mathfrak{l}_i^n]$ is of the form*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \bmod \ell^n \tag{3}$$

is $\nu_{\mathfrak{l}_i,J}(\pi)$.

Proof. First we assume that $\nu_{\mathfrak{l}_i,J}(\pi) = n$ and we show that the matrix of the Frobenius has the form given by equation 3. Let D be an element of $J[\mathfrak{l}_i^n]$. Then π acts on D as an element of $\mathcal{O}_{K_0}/\mathfrak{l}_i^n \simeq \mathbb{Z}/\ell^n\mathbb{Z}$. Hence $\pi(D) = \lambda D$ for some $\lambda \in \mathbb{Z}$.

Conversely, suppose that the matrix of the Frobenius on $J[\mathfrak{l}_i]$ is of the form 3 and take α the real multiplication endomorphism such that $\alpha(D) = \lambda D$, for all $D \in J[\mathfrak{l}_i^n]$. (Since any real multiplication endomorphism acts on $J[\mathfrak{l}_i^n]$ as λI_2, it is easy to see that such an α exists.) Then $\pi - \alpha$ is zero on $J[\mathfrak{l}_i^n]$, which implies that is an element of $\mathfrak{l}_i^n\mathcal{O}$.

Remark 17. A natural consequence of Proposition 16 is that the cyclic subgroups of $J[\mathfrak{l}_i^n]$ are rational if and only if $\nu_{\mathfrak{l}_i,J}(\pi) \geq n$. In particular, the $\ell + 1$ isogenies whose kernel is a cyclic subgroup of $J[\mathfrak{l}_i]$ are rational if and only if $\nu_{\mathfrak{l}_i}(\pi) > 0$.

Example 18. Let H be the genus-2 curve given by the equation

$$y^2 = 31x^6 + 79x^5 + 109x^4 + 130x^3 + 62x^2 + 164x + 56$$

defined over $\mathbb{F}_{211}$. The Jacobian J has complex multiplication by a quartic CM field K with defining equation $X^4 + 81X^2 + 1181$. The real subfield is $K_0 = \mathbb{Q}(\sqrt{1837})$, and has class number 1. The endomorphism ring of J contains the real maximal order $\mathcal{O}_{K_0}$. In the real subfield K_0, we have $3 = \alpha_1\alpha_2$, with $\alpha_1 = \frac{43+\sqrt{1837}}{2}$ and α_2 its conjugate. The 3-torsion is defined over an extension field of degree 6, but $J[\alpha_1] \subset J(\mathbb{F}_{q^6})$ and $J[\alpha_2] \subset J(\mathbb{F}_{q^2})$. We have that $\nu_{\alpha_i}(\mathfrak{f}_{\mathcal{O}_{K_0}[\pi,\bar{\pi}]}) = 1$, for $i = 1, 2$, where π has relative norm 211 in $\mathcal{O}_K$.

In particular, Remark 17 implies that if an $\mathfrak{l}$-isogeny $I : J_1 \to J_2$ is such that $\mathcal{O}_{K_0}[\pi, \bar{\pi}] \subset \mathrm{End}(J_1)$ and $\mathcal{O}_{K_0}[\pi, \bar{\pi}] \subset \mathrm{End}(J_2)$, then I is an isogeny in the graph of rational isogenies preserving the real multiplication. We will show that the $\{\mathfrak{l}_1, \mathfrak{l}_2\}$-isogeny graph is in fact the subgraph of rational isogenies preserving the maximal real multiplication.

Lemma 19. *Let A and B be two abelian varieties defined and isogenous over $\mathbb{F}_q$ and denote by $\mathcal{O}_A$ and $\mathcal{O}_B$ the corresponding endomorphism rings. Let $\mathfrak{l}$ be an ideal of norm ℓ in $\mathcal{O}_{K_0}$. Assume that the $\mathfrak{l}$-adic valuations of the conductors of $\mathcal{O}_A$ and $\mathcal{O}_B$ are different. Then for any isogeny $I : A \to B$ defined over $\mathbb{F}_q$ we have* $\mathrm{Ker}\, I \cap A[\mathfrak{l}] \neq \{0\}$.

Proof. We prove the contrapositive statement. Assume that there is an isogeny $I : A \to B$ defined over $\mathbb{F}_q$ with $\mathrm{Ker}\, I \cap A[\mathfrak{l}] = \{0\}$. We then have that $I(A[\mathfrak{l}^n]) = B[\mathfrak{l}^n]$, for all $n \geq 1$. Since $\pi_B \circ I = I \circ \pi_A$, it follows that the $\mathfrak{l}$-adic valuations $\nu_{\mathfrak{l},\mathcal{O}_A}(\pi_A)$ and $\nu_{\mathfrak{l},\mathcal{O}_B}(\pi_B)$ are equal. By equation (2), it follows that the $\mathfrak{l}$-adic valuations of the conductors of endomorphism rings of A and B are equal. □

The converse of Lemma 19 does not hold, as it is possible for an $\mathfrak{l}$-isogeny to have a kernel within $A[\mathfrak{l}]$, and yet leave the $\mathfrak{l}$-valuation of the conductor of the endomorphism ring unchanged.

The following statement is a converse to Proposition 10.

Proposition 20. *Let ℓ be an odd prime number, split in K_0. All cyclic isogenies of degree ℓ between principally polarizable abelian varieties defined over $\mathbb{F}_q$ having maximal real multiplication are $\mathfrak{l}$-isogenies, for some degree 1 ideal $\mathfrak{l}$ in $\mathcal{O}_{K_0}$.*

Proof. Let $\ell\mathcal{O}_{K_0} = \mathfrak{l}_1\mathfrak{l}_2$. Let $I : A \to B$ be a rational degree-ℓ isogeny which preserves the real multiplication $\mathcal{O}_{K_0}$. The endomorphism rings $\mathcal{O}_A$ and $\mathcal{O}_B$ are orders in the lattice of orders described by Figure 2. First, by [4, Section 8], we have that either $\ell\mathcal{O}_A \subset \mathcal{O}_B$, and $\ell\mathcal{O}_B \subset \mathcal{O}_A$. Hence the two orders lie either on the same level, either on consecutive levels in the lattice of orders. If $\mathcal{O}_A$ and $\mathcal{O}_B$ lie on consecutive levels, then there is an ideal $\mathfrak{l}$ of norm ℓ in $\mathcal{O}_{K_0}$ such that the $\mathfrak{l}$-adic valuation of the conductors is different. By Lemma 19, it follows that the kernel of any cyclic ℓ-isogeny between A and B is a cyclic subgroup of $A[\mathfrak{l}]$.

Assume now that $\mathcal{O}_A$ and $\mathcal{O}_B$ lie at the same level in the lattice of orders. If the two endomorphism rings are isomorphic as $\mathcal{O}_{K_0}$-algebras, then an isogeny between them is a horizontal isogeny coresponding to an invertible ideal $\mathfrak{u}$ of $\mathcal{O}_A$ such that $\mathfrak{u}\bar{\mathfrak{u}} = \mathfrak{l}$, with $\mathfrak{l}$ an ideal of norm ℓ in $\mathcal{O}_{K_0}$ [23]. Hence it is an $\mathfrak{l}$-isogeny, for some ideal $\mathfrak{l}$.

If the two orders lie at the same level and are not isomorphic, then both the $\mathfrak{l}_1$-adic and $\mathfrak{l}_2$-adic valuations of the corresponding conductors are different. It then follows that the kernel of any isogeny from A to B contains a subgroup of $A[\mathfrak{l}_1]$ and $A[\mathfrak{l}_2]$. This is not possible if the isogeny is cyclic. □

Associated to an ordinary principally polarizable abelian surface defined over $\mathbb{F}_q$ and whose endomorphism ring is an $\mathcal{O}_{K_0}$-order, we define the $\{\mathfrak{l}_1, \mathfrak{l}_2\}$-*isogeny graph* whose edges are either equivalence classes of $\mathfrak{l}_1$- or $\mathfrak{l}_2$-isogenies, and whose vertices are isomorphism classes of principally polarizable abelian surfaces over $\mathbb{F}_q$ reached (transitively) by such isogenies.

A natural consequence of Proposition 20 is that over finite fields, the $\{\mathfrak{l}_1, \mathfrak{l}_2\}$-isogeny graph is the graph of all isogenies of degree ℓ between principally polarizable abelian surfaces having maximal real multiplication. In this graph, we may classify cyclic isogenies preserving real multiplication (therefore, $\mathfrak{l}$-isogenies) in three categories. Let $\mathfrak{l}$ be such that the isogeny $I : A \to B$ being considered is an $\mathfrak{l}$-isogeny. If $\mathcal{O}_A \simeq \mathcal{O}_B$, we say that the isogeny is *horizontal*. If not, then the two orders lie on consecutive levels of the lattice given by Figure 2. If $\mathcal{O}_B$ is properly contained in $\mathcal{O}_A$, we say that the isogeny is *descending*. In the opposite situation, we say the isogeny is *ascending*.

Proposition 21. *Let A be a principally polarizable abelian surface defined over a field k which is either $\mathbb{C}$ or a finite field. Assume that the endomorphism ring $\mathcal{O}$ of A is an $\mathcal{O}_{K_0}$-order in a CM quartic field different from $\mathbb{Q}(\zeta_5)$. Let $\mathfrak{l}$ be an ideal of prime norm ℓ in $\mathcal{O}_{K_0}$.*

1. *Assume that $\mathfrak{l}\mathcal{O}_K$ is prime with the conductor of $\mathcal{O}$, that we denote by $\mathfrak{f}$. Then we have:*
   (a) *If $\mathfrak{l}$ splits into two ideals in $\mathcal{O}_K$, then there are, up to an isomorphism, exactly two horizontal $\mathfrak{l}$-isogenies starting from A and all the others are descending.*
   (b) *If $\mathfrak{l}$ ramifies in $\mathcal{O}_K$, up to an isomorphism, there is exactly one horizontal $\mathfrak{l}$-isogeny starting from A and all the others are descending.*
   (c) *If $\mathfrak{l}$ is inert in K, all $\ell+1$ $\mathfrak{l}$-isogenies starting from A are descending.*
2. *If $\mathfrak{l}$ is not coprime to $\mathfrak{f}$, then up to an isomorphism, there is exactly one ascending $\mathfrak{l}$-isogeny and ℓ descending ones starting from A.*

Proof. The number of horizontal isogenies is given by the number of projective ideals lying over $\mathfrak{l}$. In the case of abelian varieties defined over a number field, see [23, Chapter 17]. If A and B are defined over a finite field and have CM by the ring of integers $\mathcal{O}_K$, this was proven by Waterhouse [25]. Assume now that $I : A \to B$ is a horizontal isogeny between abelian varieties defined over a finite field having endomorphism ring $\mathcal{O}$ and assume the conductor $\mathfrak{f}$ is prime to $\mathfrak{l}$. Following [22], there are $\mathfrak{f}$-transforms towards $\lambda_{A,\mathfrak{l}} : A \to A_\mathfrak{l}$ and $\lambda_{B,\mathfrak{l}} : B \to B_\mathfrak{l}$, where $A_\mathfrak{l}$ and $B_\mathfrak{l}$ have CM by $\mathcal{O}_K$. Then there is an isogeny $I' : A \to B$ such that the following diagram is commutative:

$$\begin{array}{ccc} A_\mathfrak{l} & \xrightarrow{\quad I' \quad} & B_\mathfrak{l} \\ {\scriptstyle \lambda_{A,\mathfrak{l}}}\uparrow & & \uparrow{\scriptstyle \lambda_{B,\mathfrak{l}}} \\ A & \xrightarrow{\quad I \quad} & B \end{array}$$

Then I' is an isogeny corresponding to a projective ideal $\mathfrak{l}_1$, lying over $\mathfrak{l}$ in $\mathcal{O}_K$. Since $\mathfrak{l}$ is prime to the conductor $\mathfrak{f}$, it follows that I corresponds to the ideal $\mathfrak{l}_1 \cap \mathcal{O}$ in $\mathcal{O}$. In order to count descending isogenies, we count the abelian surfaces lying at a given level in the graph (up to isomorphism), by applying class number relations. More precisely, we have the exact sequence

$$1 \to \mathcal{O}^\times \to \mathcal{O}_K^\times \to (\mathcal{O}_K/\mathfrak{f}\mathcal{O}_K)^\times/(\mathcal{O}/\mathfrak{f}\mathcal{O}_K)^\times \to \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}(\mathcal{O}_K) \to 1. \tag{4}$$

Hence we have the formula for the class number

$$\#\,\mathrm{Cl}(\mathcal{O}) = \frac{\#\,\mathrm{Cl}(\mathcal{O}_K)}{[\mathcal{O}_K^\times : \mathcal{O}^\times]} \frac{\#(\mathcal{O}_K/\mathfrak{f}\mathcal{O}_K)^\times}{\#(\mathcal{O}/\mathfrak{f}\mathcal{O}_K)^\times}.$$

We have that $\mathcal{O}_K^\times = \mathcal{O}_{K_0}^\times$ (see [24, Lemma 4.15]). Since $\mathcal{O}_{K_0} \subset \mathcal{O}$, it follows that $[\mathcal{O}_K^\times : \mathcal{O}^\times] = 1$.

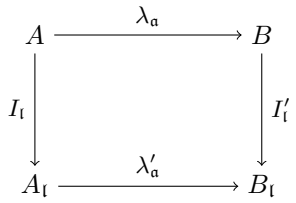
We note that $\mathcal{O}/\mathfrak{f}\mathcal{O}_K \simeq \mathbb{Z}/f\mathbb{Z}$, where $f = N(\mathfrak{f})$. Hence we have that $\#(\mathcal{O}/\mathfrak{f}\mathcal{O}_K)^\times = f\prod_{p|f}(1-\frac{1}{p})$. Moreover, we have

$$\#(\mathcal{O}_K/\mathfrak{f}O_K)^\times = N(\mathfrak{f})\prod_{\mathfrak{p}|\mathfrak{f}}(1-\frac{1}{N(\mathfrak{p})}),$$

where the ideals in the product are all prime ideals of $\mathcal{O}_K$, dividing the conductor. Let $\mathcal{O}_\mathfrak{l}$ be the $\mathcal{O}_{K_0}$-order of conductor $\mathfrak{l}\mathfrak{f}$. By writing the exact sequence (4) for the order $\mathcal{O}_\mathfrak{l}$, we obtain that

$$\begin{aligned} \#\,\mathrm{Cl}(\mathcal{O}_\mathfrak{l}) &= \#\,\mathrm{Cl}(\mathcal{O})\frac{\#(\mathcal{O}/\mathfrak{f}\mathcal{O}_K)^\times}{\#(\mathcal{O}_\mathfrak{l}/\mathfrak{f}\mathfrak{l}\mathcal{O}_K)^\times}N(\mathfrak{l})\prod_{\mathfrak{p}|\mathfrak{l}}(1-\frac{1}{N(\mathfrak{p})}), \\ &= \#\,\mathrm{Cl}(\mathcal{O})\frac{1}{\ell-1}N(\mathfrak{l})\prod_{\mathfrak{p}|\mathfrak{l}}(1-\frac{1}{N(\mathfrak{p})}) \end{aligned}$$

if $\mathfrak{l}$ is prime to $\mathfrak{f}$. Moreover, by a simple symmetry argument, the number of descending isogenies is the same for every node lying at the $\mathcal{O}$-level. Indeed, let A and B be two nodes at the $\mathcal{O}$-level. Then there is a projective ideal $\mathfrak{a}$ in $\mathcal{O}_K$ (which may be taken to be prime with both $\mathfrak{l}$ and $\mathfrak{f}$), giving an horizontal isogeny $\lambda_\mathfrak{a} : A \to B$. If A has a descending $\mathfrak{l}$-isogeny towards a variety $A_\mathfrak{l}$ lying at the $\mathcal{O}_\mathfrak{l}$-level $I_\mathfrak{l} : A \to A_\mathfrak{l}$, then there is a variety $B_\mathfrak{l}$ at the $\mathcal{O}_\mathfrak{l}$-level and $\mathfrak{l}$-isogeny $I'_\mathfrak{l} : B \to B$ such that the following diagram is commutative:



where $\lambda'_{\mathfrak{a}}$ is the horizontal isogeny corresponding to the ideal $\mathfrak{a}$.

By comparing the number of abelian varieties at one level and the one below it and taking into account this symmetry, we conclude that the number of of descending isogenies is $\ell - 1$ if $\mathfrak{l}$ is split, ℓ if $\mathfrak{l}$ is ramified and $\ell + 1$ if $\mathfrak{l}$ is inert. If $\mathfrak{l}$ divides $\mathfrak{f}$, we have

$$\#\operatorname{Cl}(\mathcal{O}_{\mathfrak{l}}) = \#\operatorname{Cl}(\mathcal{O})\frac{\#(\mathcal{O}/\mathfrak{f}\mathcal{O}_K)^\times}{\#(\mathcal{O}_{\mathfrak{l}}/\mathfrak{f}\mathfrak{l}\mathcal{O}_K)^\times}$$

which leads to the fact that the number of descending isogenies is ℓ.

Remark 22. Note that the proof stated in Proposition 21 can easily be adapted to the case where ℓ is ramified in $\mathcal{O}_{K_0}$. One can show that the structure of the $\mathfrak{l}$-graph is similar to the one in the split case.

Note that if there exists $\alpha \in K_0^+$ such that $\mathfrak{l} = \alpha O_{K_0}$, then this is an isogeny graph of principally polarizable abelian varieties. The structure of this graph is exactly the one of an ℓ-isogeny graph between elliptic curves, called *volcano* [15,8]. More generally, an $\{\mathfrak{l}_1, \mathfrak{l}_2\}$-isogeny graph can be seen, by the results above, as a direct product of two graphs which share all their characteristics with genus one isogeny volcanoes. In particular the generalization of top rim of the volcano turns into a torus if both $\mathfrak{l}_1$ and $\mathfrak{l}_2$ split. If only one of them splits, the top rim is a circle, and if both are inert we have a single vertex corresponding to a maximal endomorphism ring (since all cyclic isogenies departing from that abelian variety increase both the $\mathfrak{l}_1$- and the $\mathfrak{l}_2$-valuation of the conductor of the endomorphism ring). Proposition 21 gives the following structure of connected components of the non-oriented isogeny graph.

1. At each level, if $\nu_{\mathfrak{l},J}(\pi) > 0$, there are $\ell + 1$ rational isogenies with kernel a cyclic subgroup of $J[\mathfrak{l}]$.
2. If $\mathfrak{l}$ is split in $\mathcal{O}_{K_0}$ then there are two horizontal $\mathfrak{l}$-isogenies at all levels such that the corresponding order is locally maximal at $\mathfrak{l}$. At every intermediary level (i.e. $\nu_{\mathfrak{l},J}(\pi) > 0$), there is one ascending $\mathfrak{l}$-isogeny and ℓ descending ones.
3. If $\nu_{\mathfrak{l},J}(\pi) = 0$, then no smaller order (whose conductor has larger $\mathfrak{l}$-valuation) contains π. There are no rational descending $\mathfrak{l}$-isogeny, and there is exactly one ascending $\mathfrak{l}$-isogeny.

MAGMA experiments. Let J be a Jacobian defined over $\mathbb{F}_q$ with maximal real multiplication. We do not have formulas for computing cyclic isogenies over finite fields (Section 6 works around this difficulty for the computation of endomorphism rings). Instead, we experiment over the complex numbers, and conjecture that there is a graph isomorphism between the $\mathfrak{l}$-isogeny graph having J as a vertex and the graph of its canonical lift.

To draw the graph corresponding to Example 18, it is straightforward to compute the period matrix Ω associated to a complex analytic torus $\mathbb{C}^2/\Lambda_1 + \tau\Lambda_2$, and compute a representative in the fundamental domain for the action of Sp_4 using Gottschling's reduction algorithm[1].

All this can be done symbolically, as the matrix Ω is defined over the reflex field K^r. As a consequence, we may compute isogenies of type (1) and follow the edges of the graph of isogenies between complex abelian surfaces having complex multiplication by an order $\mathcal{O}$ containing

[1] By Gottschling's reduction algorithm, we refer to the reduction algorithm as stated in e.g. [6, chap. 6] or [24, §6.3], and which relies crucially on Gottschling's work [11] for defining the 19 matrices which come into play

$\mathcal{O}_{K_0}[\pi, \bar{\pi}]$. The exploration terminates when outgoing edges from each node have been visited. This yields Figure 3. Violet and orange edges in Figure 3 are α_1 and α_2-isogenies, respectively. Note that since α_1 and α_2 are totally positive, all varieties in the graph are principally polarized. Identification of each variety to its dual, makes the graph of Figure 3 non-oriented.

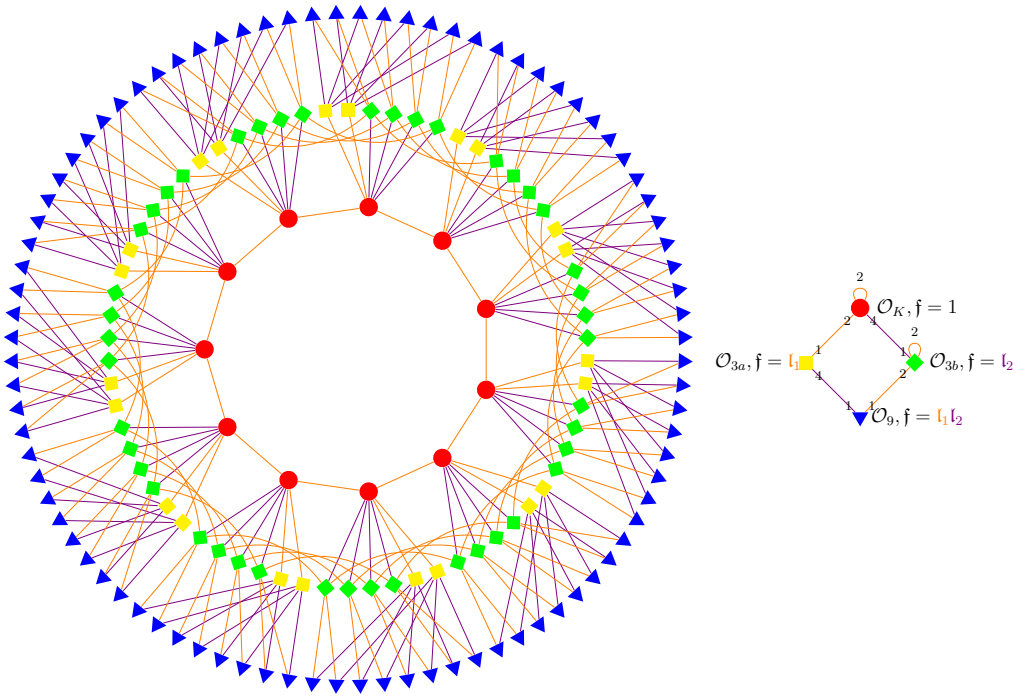


Fig. 3. Graph of ℓ-isogenies preserving real multiplication, for $\ell = 3$, K defined by $\alpha^4+81\alpha^2+1181$, and $\mathcal{O}_{K_0}[\pi]$ defined by the Weil number $\pi = \frac{1}{2}(\alpha^2 + 3\alpha + 45)$, with $p = \mathrm{Norm}_{K/K_0}\, \pi = 211$.

4.1 Isogenies with Weil-isotropic kernel

In a computational perspective, we are interested in ℓ-isogenies, which are accessible to computation using the algorithms developed by [5]. Our description of the $\mathfrak{l}_1$- and $\mathfrak{l}_2$-isogenies is key to understanding the ℓ-isogenies due to the following result.

Proposition 23. *Let $\ell \geq 3$ be a prime number such that $\ell\mathcal{O}_{K_0} = \mathfrak{l}_1\mathfrak{l}_2$. Then all ℓ-isogenies preserving the real multiplication are a composition of an $\mathfrak{l}_1$-isogeny with an $\mathfrak{l}_2$-isogeny.*

Proof. Let $I : A \to B$ be an ℓ-isogeny preserving the real multiplication. Let $\mathcal{O}_A = \mathrm{End}(A)$ and $\mathcal{O}_B = \mathrm{End}(B)$. If the endomorphism rings are equal, then the isogeny corresponds, under the action of the Shimura class group $\mathfrak{C}(K)$ [22], to an ideal class $\mathfrak{a}$ such that $\mathfrak{a}\bar{\mathfrak{a}} = \ell\mathcal{O}_A$. It follows that both $\mathfrak{l}_1$ and $\mathfrak{l}_2$ split in K. Let $\mathfrak{l}_{i,j}$, $i, j \in \{1, 2\}$, be such that $\mathfrak{l}_{i,1}\mathfrak{l}_{i,2} = \mathfrak{l}_i$. Then, we may assume that the isogeny I corresponds to the ideal $\mathfrak{l}_{1,1}\mathfrak{l}_{2,1}$ under the action of the Shimura class group. We conclude that I is a composition of an $\mathfrak{l}_1$-isogeny with an $\mathfrak{l}_2$-isogeny.

Assume now that $\mathcal{O}_A$ and $\mathcal{O}_B$ are not isomorphic. This implies that $\nu_{\mathfrak{l},\mathcal{O}_A}(\pi)$ and $\nu_{\mathfrak{l},\mathcal{O}_B}(\pi)$ differ for some $\mathfrak{l}$, and we may without loss of generality assume $\mathfrak{l} = \mathfrak{l}_1$. By considering the dual isogeny $\hat{I}$ instead of I, we may also assume $\nu_{\mathfrak{l}_1,\mathcal{O}_A}(\pi) > \nu_{\mathfrak{l}_1,\mathcal{O}_B}(\pi)$.

Let $n = \nu_{\mathfrak{l}_1,\mathcal{O}_A}(\pi)$. We then have that any subgroup of $A[\mathfrak{l}_1^n]$ is rational. By Proposition 16, there is a subgroup of $B[\mathfrak{l}_1^n]$ which is not rational. Since $I(A[\mathfrak{l}_1^n]) \subset B[\mathfrak{l}_1^n]$ and the isogeny I

is rational, it follows that $\operatorname{Ker} I$ contains an element $D_1 \in A[\mathfrak{l}_1]$. Let $I_1 : A \to C$ be the isogeny whose kernel is generated by D_1. This isogeny preserves the real multiplication and is an $\mathfrak{l}_1$-isogeny (Proposition 20). By [7, Prop 7], there is an isogeny $I_2 : C \to B$ such that $I = I_2 \circ I_1$. Obviously, I_2 also preserves real multiplication.

Let now $\langle D_1, D_2 \rangle = \operatorname{Ker} I$. Since $\operatorname{Ker} I \subset A[\mathfrak{l}_1] + A[\mathfrak{l}_2]$, we may write $D_2 = D_{2,1} + D_{2,2}$ with $D_{2,i} \in A[\mathfrak{l}_i]$. As $\operatorname{Ker} I$ is Weil-isotropic, we may choose D_2 so that $D_{2,1} = 0$, whence $D_2 \in A[\mathfrak{l}_2]$. He have $I_1(D_2) \neq 0$, so that I_2 is an $\mathfrak{l}_2$-isogeny.

Note that given the $D_2 \in A[\mathfrak{l}_2]$ which we have just defined, we may also consider the $\mathfrak{l}_2$-isogeny $I_2' : A \to C'$ with kernel $\langle D_2 \rangle$, and similarly define the $\mathfrak{l}_1$-isogeny I_1' which is such that $I = I_1' \circ I_2'$. □

The proposition above leads us to consider properties of ℓ-isogenies with regard to the $\mathfrak{l}_i$-isogenies they are composed of. Let $I = I_1 \circ I_2$ be an ℓ-isogeny, with I_i an $\mathfrak{l}_i$-isogeny (for $i = 1, 2$). We say that I is $\mathfrak{l}_1$-ascending (respectively $\mathfrak{l}_1$-horizontal, $\mathfrak{l}_1$-descending) if the $\mathfrak{l}_1$-isogeny I_1 is ascending (respectively horizontal, descending). This is well-defined, since by Lemma 19 there is no interaction of I_2 with the $\mathfrak{l}_i$-valuation of the conductor of the endomorphism ring.

Proposition 23 is a way to interpret Figure 1 as derived from Figure 3 as follows. Vertices are kept, and we use as edges all compositions of one $\mathfrak{l}_1$-isogeny and one $\mathfrak{l}_2$-isogeny. This fact will serve as a basis for our algorithms for computing endomorphism rings, detailed in Section 6.

5 Pairings on the real multiplication isogeny graph

Let J be a Jacobian defined over $\mathbb{F}_q$, with complex multiplication by an $\mathcal{O}_{K_0}$-order. Let $\ell\mathcal{O}_{K_0} = \mathfrak{l}_1\mathfrak{l}_2$. In this Section, $\mathfrak{l}$ denotes any of the ideals $\mathfrak{l}_1, \mathfrak{l}_2$.

We relate some properties of the Tate pairing to the isomorphism class of the endomorphism ring of the Jacobian, by giving a similar result to the one of [14] for genus-1 isogeny graphs. More precisely, we show that the nondegeneracy of the Tate pairing restricted to the kernel of an $\mathfrak{l}$-isogeny determines the type of the isogeny in the graph, at least when $\nu_{\mathfrak{l}}(\pi)$ is below some bound. This result is then exploited to efficiently navigate in isogeny graphs.

Let r be the smallest integer such that $J[\mathfrak{l}] \subset J(\mathbb{F}_{q^r})$. Let n be the largest integer such that $J[\mathfrak{l}^n] \subset J[\mathbb{F}_{q^r}]$. We define $k_{\mathfrak{l},J}$ to be

$$k_{\mathfrak{l},J} = \max_{P \in J[\mathfrak{l}^n]} \{k \mid T_{\ell^n}(P, P) \in \mu_{\ell^k} \backslash \mu_{\ell^{k-1}}\}$$

Definition 24. *Let G be a cyclic group of $J[\mathfrak{l}^n]$. We say that the Tate pairing is $k_{\mathfrak{l},J}$-non-degenerate (or simply non-degenerate) on $G \times G$ if its restriction*

$$T_{\ell^n} : G \times G \to \mu_{\ell^{k_{\mathfrak{l},J}}}$$

is surjective. Otherwise, we say that the Tate pairing is $k_{\mathfrak{l},J}$-degenerate (or simply degenerate) on $G \times G$.

Since $\mathfrak{l}$ is principal in the real quadratic order $\mathcal{O}_{K_0} \subset \operatorname{End}(J)$, it follows that $J[\mathfrak{l}]$ is the kernel of an endomorphism. Since J is ordinary, all endomorphisms are $\mathbb{F}_q$-rational. Consequently, we have that $\pi(J[\mathfrak{l}^n]) \subset J[\mathfrak{l}^n]$, for $n \geq 0$. The following result shows that computing the $\mathfrak{l}$-adic valuation of π is equivalent to computing $k_{\mathfrak{l},J}$.

Proposition 25. *Let r be the smallest integer such that $J[\mathfrak{l}] \subset J(\mathbb{F}_{q^r})$. Let n be the largest integer such that $J[\mathfrak{l}^n] \subset J[\mathbb{F}_{q^r}]$ and that $J[\mathfrak{l}^{n+1}] \not\subset J[\mathbb{F}_{q^r}]$. Then if $\nu_{\mathfrak{l},J}(\pi^r) < 2n$, we have*

$$k_{\mathfrak{l},J} = 2n - \nu_{\mathfrak{l},J}(\pi^r).$$

Proof. Let Q_1, Q_2 form a basis for $J[\mathfrak{l}^{2n}]$. Then $\pi^r(Q_i) = \sum a_{ij} Q_j$, for $i, j = 1, 2$. We have

$$T_{\ell^n}(\ell^n Q_i, \ell^n Q_i) = W_{\ell^{2n}}(\pi(Q_i) - Q_i, Q_i) = W_{\ell^{2n}}(Q_k, Q_i)^{a_{ik}},$$

with $k \equiv i+1 \pmod 2$. By the non-degeneracy of the Weil pairing, this implies $a_{12} \equiv a_{21} \equiv 0 \pmod{\ell^{2n-k_{\mathfrak{l},J}}}$. Moreover, the antisymmetry condition on the Tate pairing says that

$$T_{\ell^n}(\ell^n Q_1, \ell^n Q_2) T_{\ell^n}(\ell^n Q_2, \ell^n Q_1) \in \mu_{\ell^{k_{\mathfrak{l},J}}}.$$

Since $T_{\ell^n}(\ell^n Q_i, \ell^n Q_j) = W_{\ell^{2n}}(Q_i, Q_j)^{a_{jj}-1}$, for $i \neq j$, we have that

$$W_{\ell^{2n}}(Q_1, Q_2)^{a_{11}-1} W_{\ell^{2n}}(Q_2, Q_1)^{a_{22}-1} = W_{\ell^{2n}}(Q_1, Q_2)^{a_{11}-a_{22}} \in \mu_{\ell^{k_{\mathfrak{l},J}}}.$$

We conclude that $\ell^{2n-k_{\mathfrak{l},J}}$ divides all of a_{12}, a_{21}, and $a_{11}-a_{22}$. By Proposition 16, this implies that $2n - k_{\mathfrak{l},J} \leq \nu_{\mathfrak{l},J}(\pi^r)$. Conversely, let $k = 2n - \nu_{\mathfrak{l},J}(\pi^r)$. We know (by Proposition 16) that $\pi = \lambda I_2 + \ell^{2n-k} A$, for $A \in M_2(\mathbb{Z})$ and for some λ coprime to ℓ. Then for $P \in J[\mathfrak{l}^n]$ and $\bar{P}$ such that $\ell^n \bar{P} = P$, we have $T_{\ell^n}(P,P) = W_{\ell^{2n}}(\bar{P}, \lambda\bar{P} + A(\ell^{2n-k}\bar{P})) \in \mu_{\ell^k}$. Hence $k \geq k_{\mathfrak{l},J}$ and this concludes the proof. □

From this proposition, it follows that if $\nu_{\mathfrak{l},J}(\pi) > 2n$, the self-pairings of all kernels of $\mathfrak{l}$-isogenies are degenerate. At a certain level in the isogeny graph, when $\nu_{\mathfrak{l},J}(\pi) < 2n$, there is at least one kernel with non-degenerate pairing (i.e. $k_{\mathfrak{l},J} = 1$). Following the terminology of [13], we call this level *the second stability level*. As we descend to the floor, $k_{\mathfrak{l},J}$ increases. The *first stability level* is the level at which $k_{\mathfrak{l},J}$ equals n.

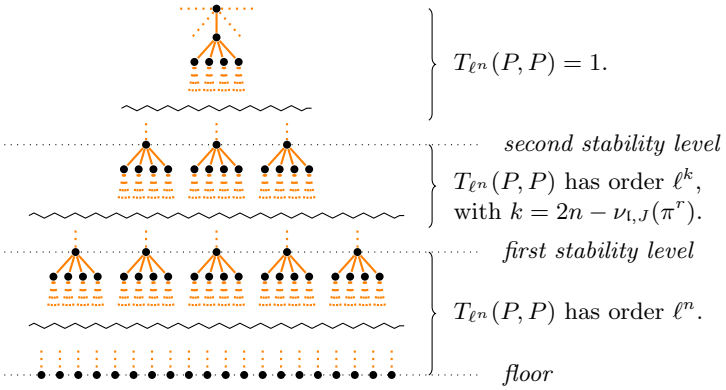


Fig. 4. Stability levels

We now show that from a computation point of view, we can use the Tate pairing to orient ourselves in the $\mathfrak{l}$-isogeny graph. More precisely, cyclic subgroups of the $\mathfrak{l}$-torsion with degenerate self-pairing correspond to kernels of ascending and horizontal isogenies, while subgroups with non-degenerate self pairing are kernels of descending isogenies. Before proving this result, we need the following lemma.

Lemma 26. *If $k_{\mathfrak{l},J} > 0$, then there are at most two subgroups of order ℓ in $J[\mathfrak{l}^n]$ such that points in these subgroups have degenerate self-pairing.*

Proof. We use the shorthand notation $\lambda_{U,V} = \log(T_{\ell^n}(U,V))$ for U, V any two $\mathfrak{l}^n$-torsion points, and where log is a discrete logarithm function in μ_{ℓ^n}.

Suppose that P and Q are two linearly independent $\mathfrak{l}^n$-torsion points. Since all $\mathfrak{l}^n$-torsion points R can be expressed as $R = aP + bQ$, bilinearity of the ℓ^n-Tate pairing gives

$$\lambda_{R,R} = a^2\lambda_{P,P} + ab\,(\lambda_{P,Q} + \lambda_{Q,P}) + b^2\lambda_{Q,Q} \pmod{\ell^n},$$

We now claim that the polynomial

$$S(a,b) = a^2\lambda_{P,P} + ab\,(\lambda_{P,Q} + \lambda_{Q,P}) + b^2\lambda_{Q,Q} \tag{5}$$

is identically zero modulo $\ell^{n-k_{\mathfrak{l},J}-1}$ and nonzero modulo $\ell^{n-k_{\mathfrak{l},J}}$. Indeed, if it were identically zero modulo ℓ^k, with $k > n - k_{\mathfrak{l},J}$, then we would have $T_{\ell^n}(R,R) \in \mu_{\ell^{n-k}}$, which contradicts the definition of $k_{\mathfrak{l},J}$. If it were different from zero modulo $\ell^{n-k_{\mathfrak{l},J}-1}$, then there would be $R \in J[\mathfrak{l}^n]$ such that $T_{\ell^n}(R,R)$ is an $\ell^{k_{\mathfrak{l},J}+1}$-th primitive root of unity, again contradicting the definition of $k_{\mathfrak{l},J}$.

Points with degenerate self-pairing are roots of L. Hence there are at most two subgroups of order ℓ with degenerate self-pairing. □

In the remainder of this paper, we define by

$$S_{\mathfrak{l},J}(a,b) = a^2\lambda_{P,P} + ab(\lambda_{P,Q} + \lambda_{Q,P}) + b^2\lambda_{Q,Q}$$

any polynomial defined by a basis $\{P,Q\}$ of $J[\mathfrak{l}^n]$ in a manner similar to the proof of Lemma 26, and using the same notation λ.

Theorem 27. *Let P be an $\mathfrak{l}$-torsion point and let r be the smallest integer such that $J[\mathfrak{l}] \subset J(\mathbb{F}_{q^r})$. Let n be the largest integer such that $J[\mathfrak{l}^n] \subset J[\mathbb{F}_{q^r}]$. Assume that $k_{\mathfrak{l},J} > 0$. Consider G a subgroup such that $\ell^{n-1}G$ is the subgroup generated by P. Then the isogeny of kernel P is descending if and only if the Tate pairing is non-degenerate on G. It is horizontal or ascending otherwise.*

Proof. We assume $n > 1$ and that $k_{\mathfrak{l},J} > 1$. Otherwise, we consider J' defined over and extension field of $\mathbb{F}_{q^r}$ and apply [12, Lemma 4]. Let $I : J \to J'$ the isogeny of kernel generated by P. Assume that P has non-degenerate self-pairing. Let $\bar{P} \in G$ such that $\ell^{n-1}\bar{P} = P$. Then by [19, Lemma 16.2c] and Lemma 13, we have

$$T_{\ell^{n-1}}(I(\bar{P}), \alpha(I(\bar{P}))) \in \mu_{\ell^{k_{\mathfrak{l},J}-1}} \backslash \mu_{\ell^{k_{\mathfrak{l},J}-2}},$$

where α is a generator of the principal ideal $\mathfrak{l}'$ such that $\mathfrak{l}\mathfrak{l}' = \ell\mathcal{O}_{K_0}$. Since $\mathcal{O}_{K_0}/\alpha\mathcal{O}_{K_0} \simeq \mathbb{Z}/\ell\mathbb{Z}$, then for any $R \in J'[\mathfrak{l}^n]$, we have $\alpha(R) = \lambda R$, for some $\lambda \in \mathbb{Z}/\ell\mathbb{Z}$. Hence we have

$$T_{\ell^{n-1}}(I(\bar{P}), I(\bar{P})) \in \mu_{\ell^{k_{\mathfrak{l},J}-1}} \backslash \mu_{\ell^{k_{\mathfrak{l},J}-2}},$$

There are two possibilities. Either $J'[\mathfrak{l}^n]$ is not defined over $\mathbb{F}_{q^r}$, or $J'[\mathfrak{l}^n]$ is defined over $\mathbb{F}_{q^r}$. In the first case, we have $\nu_{\mathfrak{l},J'}(\pi^r) < \nu_{\mathfrak{l},J}(\pi^r)$ and the isogeny is descending.

Assume now that $J'[\mathfrak{l}^n]$ is defined over $\mathbb{F}_{q^r}$. Then let P_1 such that $I(\bar{P}) = \ell P_1$. Then

$$T_{\ell^n}(P_1, P_1)) \in \mu_{\ell^{k_{\mathfrak{l},J}+1}} \backslash \mu_{\ell^{k_{\mathfrak{l},J}}}.$$

By using Proposition 25, it follows that $\nu_{\mathfrak{l},J'}(\pi^r) < \nu_{\mathfrak{l},J}(\pi^r)$. Hence the isogeny is descending.

Suppose now that the point P has degenerate self-pairing and that the isogeny I is descending. Since there are at most 2 points in $J[\mathfrak{l}^n]$ with degenerate self-pairing, there is at least one point in $J[\mathfrak{l}^n]$ with non-degenerate self-pairing. This point, that we denote by Q, generates the kernel of a descending isogeny $I' : J \to J''$ such that $\mathrm{End}(J') \simeq \mathrm{End}(J'')$. We assume first that $J'[\mathfrak{l}^n]$ and $J''[\mathfrak{l}^n]$ are not defined over $\mathbb{F}_{q^r}$. Then we have

$$T_{\ell^{n-1}}(I(\bar{P}), I(\bar{P}))) \in \mu_{\ell^{k_{\mathfrak{l},J}-2}}, \quad T_{\ell^{n-1}}(\ell I(\bar{Q}), \ell(I(\bar{Q}))) \in \mu_{\ell^{k_{\mathfrak{l},J}-3}}$$
$$T_{\ell^{n-1}}(\ell I'(\bar{P}), \ell I'(\bar{P})) \in \mu_{\ell^{k_{\mathfrak{l},J}-4}}, T_{\ell^{n-1}}(I'(\bar{Q}), I'(\bar{Q}))) \in \mu_{\ell^{k_{\mathfrak{l},J}-1}} \backslash \mu_{\ell^{k_{\mathfrak{l}}-2}}$$

Hence $k_{\mathfrak{l},J'} \neq k_{\mathfrak{l},J''}$, which is a contradiction. The case where $J'[\mathfrak{l}^n]$ and $J''[\mathfrak{l}^n]$ are defined over $\mathbb{F}_{q^r}$ is similar. □

6 Endomorphism ring computation - a depth-first algorithm

We keep the same setting and notations. In particular, ℓ is a fixed odd prime, and we assume that $\ell\mathcal{O}_{K_0} = \mathfrak{l}_1\mathfrak{l}_2$. We intend to compute the endomorphism ring of J a Jacobian defined over $\mathbb{F}_q$, with prior knowledge of the Zeta function of J, and the fact that $\mathrm{End}(J)$ is an $\mathcal{O}_{K_0}$-order. We note that this property holds trivially in the case where $\mathbb{Z}[\pi,\bar{\pi}]$ itself is an $\mathcal{O}_{K_0}$-order, although this is not a necessary condition for the algorithm here to work.

6.1 Description of the algorithm

A consequence of Proposition 23 is that there are at most $(\ell+1)(\ell+1)$ rational ℓ-isogenies preserving the real multiplication. Since we can compute ℓ-isogenies over finite fields [5,2], we use this result to give an algorithm for computing $\nu_{\mathfrak{l},J}(\pi)$, and determine endomorphism rings locally at ℓ, by placing them properly in the order lattice as represented in Figure 2.

We define u_i to be the smallest integer such that $\pi^{u_i} - 1 \in \mathfrak{l}_i\mathcal{O}_K$, and u the smallest integer such that $\pi^u - 1 \in \ell\mathcal{O}_K$. (we have $u = \mathrm{lcm}(u_1, u_2)$). The value of u depends naturally on the splitting of ℓ in K (see [9, Prop. 6.2]). As the algorithm proceeds, the walk on the isogeny graph considers Jacobians over the extension field $\mathbb{F}_{p^u}$.

Idea of the algorithm. As noticed by Lemma 7, we can achieve our goal by considering *separately* the position of the endomorphism ring within the order lattice with respect to $\mathfrak{l}_1$ first, and then with respect to $\mathfrak{l}_2$. The algorithm below is in effect run twice.

Each move in the isogeny graph corresponds to taking an ℓ-isogeny, which is a computationally accessible object. In our prospect to understand the position of the endomorphism ring with respect to $\mathfrak{l}_1$ in Figure 2, we shall not consider what happens with respect to $\mathfrak{l}_2$, and vice-versa. Our input for computing an ℓ-isogeny is a Weil-isotropic kernel. Because we are interested in isogenies preserving the real multiplication, this entails that we consider kernels of the form $K_1 + K_2$, with K_i a cyclic subgroup of $J[\mathfrak{l}_i]$. By Proposition 14, such a group is Weil-isotropic. There are up to $(\ell+1)^2$ such subgroups.

Let $\mathfrak{l}$ be either $\mathfrak{l}_1$ or $\mathfrak{l}_2$. The algorithm computes $\nu_{\mathfrak{l},J}(\pi)$ in two stages.

Our algorithm stops when the floor of rationality has been hit in $\mathfrak{l}$, i.e. the only rational cyclic group in $J[\mathfrak{l}]$ is the one generating the kernel of the ascending $\mathfrak{l}$-isogeny. If $(u, \ell) = 1$, one may prove that testing rationality for the isogenies is equivalent to $J[\mathfrak{l}] \subset J(\mathbb{F}_{q^u})$. Otherwise, in order to test rationality for the isogeny at each step in the algorithm, one has to check whether the kernel of the isogeny is $\mathbb{F}_q$-rational.

Step 1. The idea is to walk the isogeny graph until we reach a Jacobian which is on the second stability level or below (which might already be the case, in which case we proceed to Step 2). If the Jacobian J is above the second stability level, we need to construct several chains of ℓ-isogenies, not backtracking with respect to $\mathfrak{l}$, to make sure at least one of them is descending in the $\mathfrak{l}$-direction. This proceeds exactly as in [8]. The number of chains depends on the number of horizontal isogenies and thus on the splitting of $\mathfrak{l}$ in K (due to the action of the Shimura class group). If $\mathfrak{l}$ is split, one needs three isogeny chains to ensure that one path is descending.

If an isogeny in the chain is descending, then the path continues descending, assuming the isogeny walk does not backtrack with respect to $\mathfrak{l}$ (this aspect is discussed further below). We are done constructing a chain when we have reached the second stability level for $\mathfrak{l}$, which can be checked by computing self-pairing of appropriate ℓ^n-torsion points. The length of the shortest path gives the correct level difference between the second stability level and the Jacobian J.

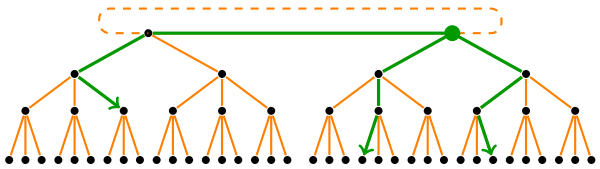


Fig. 5. At least one in three non-backtracking paths has minimum distance to a given level.

Figure 5 represents for $\ell = 3$ a situation where only three non-backtracking paths can guarantee that at least one of them is consistently descending.

Step 2. We now assume that J is on the stability level or below, with respect to $\mathfrak{l}$. We construct a non-backtracking path of ℓ-isogenies, which are consistently descending with respect to $\mathfrak{l}$. In virtue of Theorem 27, this can be achieved by picking Weil-isotropic kernels whose $\mathfrak{l}$-part (which is cyclic) correspond to a non-degenerate self-pairing $T_{\ell^n}(P, P)$. We stop when we have reached the floor of rationality in $\mathfrak{l}$, at which point the valuation $\nu_{\mathfrak{l},J}(\pi)$ is obtained.

Note that at each step taken in the graph, if $J[\mathfrak{l}']$ (where $\mathfrak{l}'$ is the other ideal) is not rational, then we ascend in the $\mathfrak{l}'$-direction, in order to compute an ℓ-isogeny. As said above, this has no impact on the consideration of what happens with respect to $\mathfrak{l}$.

Ensuring isogeny walks are not backtracking As said above, ensuring that the isogeny walk in Step 2 is not backtracking is essentially guaranteed by Theorem 27. Things are more subtle for Step 1. Let J_1 be a starting Jacobian, and $I : J_1 \to J_2$ an ℓ-isogeny whose kernel is $V \subset J[\ell]$. Recall that there are at most $(\ell+1)^2$ Weil-isotropic kernels of the form $K_1 + K_2$ within $J_2[\mathfrak{l}_1] + J_2[\mathfrak{l}_2]$ for candidate isogenies $I' : J_2 \to J_1$. All such isogenies whose kernel has the same component on $J_2[\mathfrak{l}_1]$ as the dual isogeny $\hat{I}$ are backtracking with respect to $\mathfrak{l}_1$ in the isogeny graph. One must therefore identify the dual isogeny $\hat{I}$ and its kernel. Since $\hat{I}$ is such that $\hat{I} \circ I = [\ell]$, we have that $\operatorname{Ker} \hat{I} = I(J_1[\ell])$. If computing $I(J_1[\ell])$ is possible[2], this solves the issue. If not, then enumerating all possible kernels until the dual isogeny is identified is possible, albeit slower.

Algorithm 1 Computing the endomorphism ring: Step 1

INPUT: A Jacobian J of a genus-2 curve defined over $\mathbb{F}_q$ and u the smallest integer s.t. $\pi^u - 1 \equiv 0 \pmod{\ell\mathcal{O}_K}$, the Frobenius $\pi \in K$ where K is a quartic CM field, and $\alpha = a + b(\pi + \bar{\pi})$ such that $\mathfrak{l} = \alpha\mathcal{O}_K$ divides $\ell\mathcal{O}_K$, and $\mathfrak{l}' = \ell/\mathfrak{l}$.
We require that J is above the second stability level with respect to $\mathfrak{l}$.

OUTPUT: A Jacobian J' on or below the second stability level with respect to $\mathfrak{l}$, and the distance from J to this Jacobian.

Let n the largest integer such that $J[\mathfrak{l}^n] \subset J(\mathbb{F}_{q^u})$.
$J_1 \leftarrow J$, $J_2 \leftarrow J$, $J_3 \leftarrow J$.
$\kappa_1 \leftarrow \{0\}$, $\kappa_2 \leftarrow \{0\}$, $\kappa_3 \leftarrow \{0\}$.
length $\leftarrow 0$.
while true do
  length $\leftarrow$ length $+ 1$.
  for all i=1,2,3 **do**
    Compute the matrix of π in $J_i[\ell^\infty](\mathbb{F}_{q^u})$.
    Compute bases for $J_i[\mathfrak{l}](\mathbb{F}_{q^u})$ and $J_i[\mathfrak{l}'](\mathbb{F}_{q^u})$ using $\alpha = a + b(\pi + \bar{\pi})$.
    Pick at random $P_i \in J_i[\mathfrak{l}](\mathbb{F}_{q^u})$ such that $P_i \notin \kappa_i$.
    Pick at random $P_i' \in J_i[\mathfrak{l}'](\mathbb{F}_{q^u})$.
    Compute the ℓ-isogeny $I : J_i \to J_i' = J_i/\langle P_i, P_i' \rangle$.
    $\kappa_i \leftarrow I(J[\mathfrak{l}])$; $J_i \leftarrow J_i'$.
    Compute $S_{\mathfrak{l},J}$.
    if $S_{\mathfrak{l},J} \neq 0$ **then**
      return length.
    end if
  end for
end while

6.2 Complexity analysis

In this Section, we give a complexity analysis of Algorithms 1 and 2 and compare its performance to that of the Eisenträger-Lauter algorithm for computing the endomorphism ring locally at ℓ, for

[2] Computing isogenous Jacobians by isogenies is easier than computing images of divisors. The `avisogenies` software [2] performs the former since its inception, and the latter in its development version, as of 2014.

Algorithm 2 Computing the endomorphism ring: Step 2

INPUT: A Jacobian J of a genus-2 curve defined over $\mathbb{F}_q$ and u the smallest integer s.t. $\pi^u - 1 \equiv 0 \pmod{\ell\mathcal{O}_K}$, the Frobenius $\pi \in K$ where K is a quartic CM field, and $\alpha = a + b(\pi + \bar{\pi})$ such that $\mathfrak{l} = \alpha\mathcal{O}_K$ divides $\ell\mathcal{O}_K$, and $\mathfrak{l}' = \ell/\mathfrak{l}$.

We require that J is on or below the second stability level with respect to $\mathfrak{l}$ (see Algorithm 1).

OUTPUT: The $\mathfrak{l}$-distance from J to the floor.

```
length ← 0.
while true do
    Compute a basis of J[ℓ^∞](F_{q^u}).
    Let n the largest integer such that J[𝔩^n] ⊂ J(F_{q^u}).
    if n = 0 then
        return length.
    end if
    Compute the matrix of π in J_i[ℓ^∞](F_{q^u}).
    Compute bases for J_i[𝔩](F_{q^u}) and J_i[𝔩'](F_{q^u}) using α = a + b(π + π̄).
    Consider P_1, P_2 a basis of J[𝔩^n](F_{q^u})
    Compute S_{𝔩,J} and take x_1, x_2 ∈ P^1(F_ℓ) such that S_{𝔩,J}(x_1, x_2) ≠ 0.
    P ← ℓ^{n-1}(x_1 P_1 + x_1 P_2).
    Pick at random P'_i ∈ J_i[𝔩'](F_{q^u}).
    Compute the ℓ-isogeny I : J' ← J/⟨P, P'⟩
    J ← J'.
    length ← length + 1.
end while
```

small ℓ. If ℓ is large, one should use Bisson's algorithm [1]. Computing a bound on ℓ for which one should switch between the two algorithms and a full complexity analysis of the algorithm for determining the endomorphism ring completely is beyond the scope of this paper.

The Eisenträger-Lauter algorithm For completeness, we briefly recall the Eisenträger-Lauter algorithm [7]. For a fixed order $\mathcal{O}$ in the lattice of orders of K, the algorithm tests whether $\mathcal{O} \subset \mathrm{End}(J)$. This is done by computing a $\mathbb{Z}$-basis of $\mathcal{O}$ and checking whether its elements are endomorphisms of J or not. In order to test if $\alpha \in \mathcal{O}$ is an endomorphism, we write

$$\alpha = \frac{a_0 + a_1\pi + a_2\pi^2 + a_3\pi^3}{N},$$

with a_i integers whose greatest common divisor is coprime to N (N is the smallest integer such that $N\alpha \in \mathbb{Z}[\pi]$). Using [7, Prop. 7], we get $\alpha \in \mathrm{End}(J)$ if and only if $\sum_i a_i\pi^i$ acts as zero on the N-torsion.

Freeman and Lauter [9] work locally modulo prime divisors of N. For all orders such that $\mathbb{Z}[\pi] \subset \mathcal{O} \subset \mathcal{O}_K$, the denominators N considered are divisors of $[\mathcal{O}_K : \mathbb{Z}[\pi]]$ (see [9, Lemma 3.3]). Since $[\mathbb{Z}[\pi, \bar{\pi}] : \mathbb{Z}[\pi]]$ is 1 or p, we have that N divides $[\mathcal{O}_K : \mathbb{Z}[\pi, \bar{\pi}]]$ if $(N, p) = 1$. Moreover, Freeman and Lauter show that if N factors as $\ell_1^{d_1}\ell_2^{d_2}\ldots\ell_r^{d_r}$, it suffices to check if

$$\frac{a_0 + a_1\pi + a_2\pi^2 + a_3\pi^3}{\ell_i^{d_i}},$$

for all i. The advantage of working locally is that instead of working over the extension field generated by the coordinates of the N-torsion points, we may work over the field of definition of the $\ell_i^{d_i}$-torsion, for every prime factor ℓ_i separately. Nevertheless, it should be noted that the exponent d_i can be as large as the ℓ_i-valuation of the conductor $[\mathcal{O}_K : \mathbb{Z}[\pi]]$.

We now set some notations for giving the complexity of algorithms from Section 6 as well as the Eisenträger-Lauter algorithm. We consider the complexity for one odd prime ℓ dividing $[\mathcal{O}_K : \mathbb{Z}[\pi, \bar{\pi}]]$, and assume that $(\ell, p) = 1$. Following the notation on page 9, we denote $h_i = \nu_{\mathfrak{l}_i,\mathcal{O}_K}(\pi)$ for $i = 1, 2$. It follows that $\nu_\ell([\mathcal{O}_K : \mathcal{O}_{K_0}[\pi, \bar{\pi}]]) = h_1 + h_2$. The order $\mathbb{Z}[\pi, \bar{\pi}]$ might be smaller than

$\mathcal{O}_{K_0}[\pi,\bar{\pi}]$, thus we denote $h_0 = \nu_\ell([\mathcal{O}_{K_0}[\pi,\bar{\pi}] : \mathbb{Z}[\pi,\bar{\pi}]])$. Note though that for most practical uses of our algorithm, we expect to gain knowledge that $\mathrm{End}(J)$ has maximal real multiplication from the fact that $\mathbb{Z}[\pi,\bar{\pi}]$ is an $\mathcal{O}_{K_0}$-order itself, which implies $h_0 = 0$. It makes sense to neglect h_0 in this case. Finally, we let as before u be the smallest integer such that $\pi^u \equiv 1 \mod \ell\mathcal{O}_K$, so that the ℓ-torsion on J is defined over $\mathbb{F}_{q^u}$. According to [9, Prop. 6.2], we have $u \in O(\ell^2)$ since ℓ splits in K_0.

We now give the complexity of the algorithm from Section 6. First we compute a basis of the "ℓ^∞-torsion over $\mathbb{F}_{q^u}$", i.e. the ℓ-Sylow subgroup of $J(\mathbb{F}_{q^u})$, which corresponds to $J[\ell^n](\mathbb{F}_{q^u})$ for some integer n. We assume that the zeta function of J and the factorization of $\#J(\mathbb{F}_{q^u}) = \ell^s m$ are given. We denote by $M(u)$ the number of a multiplications in $\mathbb{F}_q$ needed to perform one multiplication in the extension field of degree u. The computation of the Sylow subgroup basis costs $O(M(u)(u\log q + n\ell^2))$ operations in $\mathbb{F}_q$, as described in [3, §3].

Then we compute the matrix of the Frobenius on the ℓ-torsion. Using this matrix, we write down the matrices of α_1 and α_2 in terms of the the matrix of $\pi + \bar{\pi}$. Finally, computing $J[\mathfrak{l}_i]$ for $i = 1, 2$ is just linear algebra and has negligible cost. For each i, the cost of computing the Tate pairing is related to the integers r_i and n_i as defined in Proposition 25. We bound these by $r_i \leq u$, and $n_i \leq n$. Computing the Tate pairing thus costs $O(M(u)(n\log\ell + u\log q))$ operations in $\mathbb{F}_q$, where the first term is the cost of Miller's algorithm and the second one is the cost for the final exponentiation.

The cost of computing an ℓ-isogeny using the algorithm of Cosset and Robert [5] is $O(M(u)\ell^4)$ operations in $\mathbb{F}_q$. We conclude that the cost of Algorithms 1 and 2 is

$$\mathrm{cost}_{\text{algorithms 1+2}} = O(\max(h_1,h_2)M(u)(u\log q + n\ell^2 + \ell^4)).$$

The complexity of Freeman and Lauter's algorithm is dominated by the cost of computing the ℓ-Sylow subgroup of the Jacobian defined over the extension field containing the ℓ^d-torsion, where d is bounded by $\nu_\ell([\mathcal{O}_K : \mathbb{Z}[\pi]]) = \nu_\ell([\mathcal{O}_K : \mathbb{Z}[\pi,\bar{\pi}]]) = h_0 + h_1 + h_2$ (recall that ℓ and π are coprime). The degree of this extension field is $u\ell^{d-1}$ by [9, Prop. 6.3]. This leads to

$$\mathrm{cost}_{\mathrm{EL}} = O(M(u\ell^{d-1})(u\ell^{d-1}\log q + (n+d-1)\ell^2)).$$

Freeman and Lauter	This work (Algorithms 1 and 2)
$O(M(u\ell^{d-1})(u\ell^{d-1}\log q + (n+d-1)\ell^2))$	$O(\max(h_1,h_2)M(u)(u\log q + n\ell^2 + \ell^4))$

Table 1. Cost for computing the endomorphism ring locally at ℓ; we have $u = O(\ell^2)$, $d \leq h_0 + h_1 + h_2$, and $h_0 = 0$ is a typical condition for this work to apply.

6.3 Practical experiments

Let J be the Jacobian of the hyperelliptic curve defined by

$$y^2 = 17422020 + 847562x + 37917221x^2 + 268754x^3 + 4882157x^4 + 14143796x^5 + 50949756x^6$$

over $\mathbb{F}_p$, with $p = 53050573$. The curve has complex multiplication by $\mathcal{O}_K$, with $K = \mathbb{Q}(\zeta)$, defined by the equation $\zeta^4 + 175\zeta^2 + 6925 = 0$. A Weil number for this Jacobian, as well as the corresponding characteristic polynomial, are given as follows:

$$\pi = \frac{1}{15}(45\zeta^3 + 422\zeta^2 + 14940\zeta + 79450),$$
$$\pi^4 - s_1\pi^3 + s_2\pi^2 - s_1 p\pi + p^2 = 0, \text{ with } s_1 = 11340,\ s_2 = 135934954.$$

The real multiplication subfield K_0 has class number 1, and $\ell = 3$ splits in K_0 as $3 = \alpha_1\alpha_2$. The corresponding valuations of the Frobenius are $\nu_{\alpha_1,\mathcal{O}_K}(\pi) = 10$ and $\nu_{\alpha_2,\mathcal{O}_K}(\pi) = 2$. The analogue to Figure 2 is thus a lattice of 20 possible orders to choose from in order to determine $\mathrm{End}(J)$.

Our algorithm computes the 3-torsion group, which is defined over $\mathbb{F}_{p^2}$. Note that in contrast, the Eisenträger-Lauter algorithm computes the 3^{10}-torsion group, defined over $\mathbb{F}_{p^{39366}}$.

We report experimental results of our implementation, using Magma 2.20-6 and `avisogenies` 0.6, on a Intel Core i5-4570 CPU with clock frequency 3.2 GHz. Our computation of $\mathrm{End}(J)$ with Algorithms 1 and 2 goes as follows. Computation shows that the Tate pairing is degenerate on $J[\mathfrak{l}_1]$. We thus use Algorithm 1 to find a shortest path from J, not backtracking with respect to $\mathfrak{l}_1$, and reaching a Jacobian on or above the second stability level. This path is made of ℓ-isogenies defined over $\mathbb{F}_p$, and computed with `avisogenies` from their kernels (here, only what happens with respect to $\mathfrak{l}_1$ is interesting). Such a path with length 3 is found in 20 seconds, where most of the time (15 seconds) is spent on ensuring that the isogeny walks are non-backtracking (see remark on page 18). From there, a consistently descending path of length 5 down to the floor is constructed using Algorithm 2 in 3 seconds. This leads to $\nu_{\mathfrak{l}_1,J}(\pi) = 8$. As for $\mathfrak{l}_2$, the Jacobian J is below the second stability level, so Algorithm 2 applies, and finds $\nu_{\mathfrak{l}_2,J}(\pi) = 1$ in 1 second. In total, the computation $\mathrm{End}(J)$ in this example takes 24 seconds.

7 Conclusion

We have described the structure of the degree ℓ isogeny graph between abelian surfaces with maximal real multiplication. From a computational point of view, we exploited the structure of the graph to describe an algorithm computing locally at ℓ the endomorphism ring of an abelian surface with maximal real multiplication. Further research is needed to extend our results to the general case. Our belief is that the right approach to follow is first to determine the real multiplication and secondly to use an algorithm similar to ours to fully compute the endomorphism ring.

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References

1. G. Bisson. Computing endomorphism rings of abelian varieties of dimension two. `http://eprint.iacr.org/2012/525`.
2. G. Bisson, R. Cosset, and D. Robert. Avisogenies. `http://avisogenies.gforge.inria.fr/`.
3. G. Bisson, R. Cosset, and D. Robert. On the practical computation of isogenies of jacobian surfaces. Manuscript in preparation, available within the source code of [2].
4. R. Bröker, D. Gruenewald, and K. Lauter. Explicit CM theory for level 2-structures on abelian surfaces. *Algebra & Number Theory*, 5(4):495–528, 2011.
5. R. Cosset and D. Robert. Computing (ℓ, ℓ)-isogenies in polynomial time on jacobians of genus 2 curves. *Mathematics of Computation*, 2013.
6. Régis Dupont. *Moyenne arithmético-géométrique, suites de Borchardt et applications.* Thèse, École Polytechnique, 2006. `www.lix.polytechnique.fr/Labo/Regis.Dupont/these_soutenance.pdf`.
7. K. Eisenträger and K. Lauter. A CRT algorithm for constructing genus 2 curves over finite fields. In *Arithmetic, Geometry and Coding Theory (AGCT -10), Séminaires et Congrès 21*, pages 161–176. Société Mathématique de France, 2009.
8. M. Fouquet and F. Morain. Isogeny volcanoes and the SEA algorithm. In C. Fieker and D. R. Kohel, editors, *ANTS-V*, volume 2369 of *Lecture Notes in Computer Science*, pages 276–291. Springer, 2002.

9. D. Freeman and K. Lauter. Computing endomorphism rings of jacobians of genus 2 curves. In *Symposium on Algebraic Geometry and its Applications, Tahiti*, 2006.
10. E. Goren and K. Lauter. The distance between superspecial abelian varieties with real multiplication. *Journal of Number Theory*, 129(6):1562–1578, 2009.
11. E. Gottschling. Explizite Bestimmung der Randflächen des Fundamentalbereiches der Modulgruppe zweiten Grades. *Math. Ann.*, 138:103–124, 1959.
12. S. Ionica. Pairing-based methods for genus 2 jacobians with maximal endomorphism ring. `http://fr.arxiv.org/abs/1204.0222`.
13. S. Ionica and A. Joux. Another approach to pairing computation in Edwards coordinates. In D. R. Chowdhury, V. Rijmen, and A. Das, editors, *Progress in Cryptography- Indocrypt 2008*, volume 5365 of *Lecture Notes in Computer Science*, pages 400–413. Springer, 2008.
14. S. Ionica and A. Joux. Pairing the volcano. *Mathematics of Computation*, 82:581–603, 2013.
15. D. Kohel. *Endomorphism rings of elliptic curves over finite fields.* PhD thesis, University of California, Berkeley, 1996.
16. S. Lichtenbaum. Duality theorems for curves over p-adic fields. *Invent.Math.7*, pages 120–136, 1969.
17. D. Lubicz and D. Robert. Computing isogenies between abelian varieties. *Compositio Mathematica*, 2012.
18. V. Miller. The Weil pairing, and its efficient calculation. *Journal of Cryptology*, 17(4):235–261, September 2004.
19. J. Milne. *Abelian Varieties.*
20. D. Robert. Isogeny graphs in dimension 2. `http://www.normalesup.org/~robert/pro/publications/slides/2014-12-Caen-Isogenies.pdf`, 2014. Cryptography Seminar, Caen.
21. S. L. Schmoyer. The Triviality and Nontriviality of Tate-Lichtenbaum Self-Pairings on Jacobians of curves, 2006. `http://www-users.math.umd.edu/~schmoyer/`.
22. G. Shimura. *Abelian varieties with complex multiplication and modular functions.* Princeton Mathematical Series. Princeton University Press, 1998.
23. G. Shimura and Y. Taniyama. *Complex Multiplication of Abelian Varieties and its Applications to Number Theory.* The Mathematical Society of Japan, 1961.
24. M. Streng. Computing Igusa class polynomials. *Math. Comp.*, 83(285):275–309, 2014.
25. W.C. Waterhouse. Abelian varieties over finite fields. *Annales Scientifiques de l'Ecole Normale Supérieure 2.4*, pages 521–560, 1969.

A Appendix: additional example

We consider the quartic CM field K with defining equation $X^4 + 81X^2 + 1181$. The real subfield is $K_0 = \mathbb{Q}(\sqrt{1837})$, and has class number 1. In the real subfield K_0, we have $3 = \alpha_1\alpha_2$, with $\alpha_1 = \frac{43+\sqrt{1837}}{2}$ and α_2 its conjugate. We consider a Weil number π of relative norm 85201 in $\mathcal{O}_K$. We have that $\nu_{\alpha_1}(\mathfrak{f}_{\mathbb{Z}[\pi,\bar{\pi}]}) = 2$ and $\nu_{\alpha_2}(\mathfrak{f}_{\mathbb{Z}[\pi,\bar{\pi}]}) = 1$. Note that $\mathfrak{l}_1$ is inert and $\mathfrak{l}_2$ is split in K. Our implementation with Magma produced the graph in Figure 6.

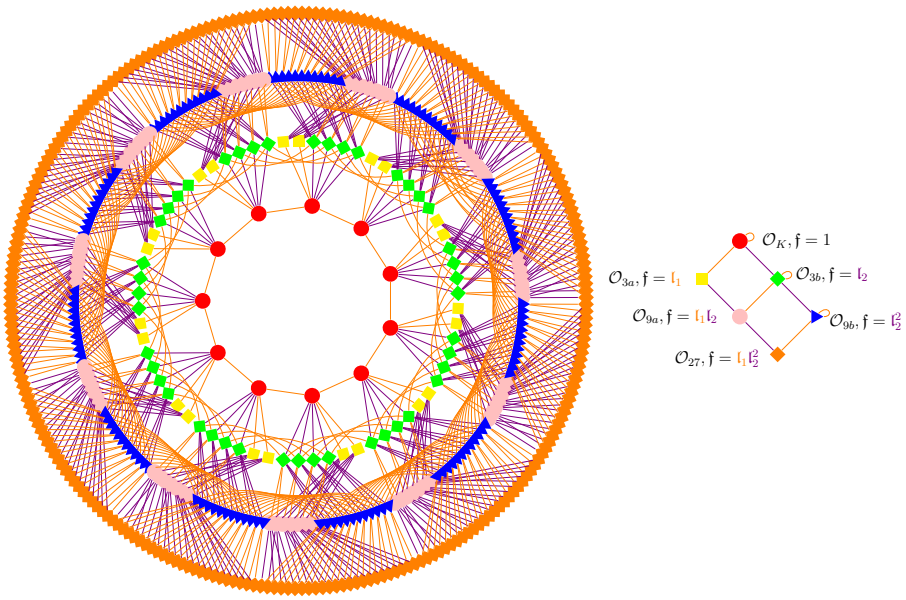

Fig. 6. A larger example