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On Hypothesis Testing for Poisson Processes. Regular Case

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Abstract

We consider the problem of hypothesis testing in the situation when the first hypothesis is simple and the second one is local one-sided composite. We describe the choice of the thresholds and the power functions of the Score Function test, of the General Likelihood Ratio test, of the Wald test and of two Bayes tests in the situation when the intensity function of the observed inhomogeneous Poisson process is smooth with respect to the parameter. It is shown that almost all these tests are asymptotically uniformly most powerful. The results of numerical simulations are presented.

MSC 2010 Classification: 62M02, 62F03, 62F05.

Key words: Hypothesis testing, inhomogeneous Poisson processes, asymptotic theory, composite alternatives, regular situation.

1 Introduction

The hypothesis testing theory is a well developed branch of mathematical statistics [12]. The asymptotic approach allows to find satisfactory solutions

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in many different situations. The simplest problems, like the testing of two simple hypotheses, have well known solutions. Recall that if we fix the first type error and seek the test which maximizes the power, then we obtain immediately (by Neyman-Pearson lemma) the most powerful test based on the likelihood ratio statistic. The case of composite alternative is more difficult to treat and here the asymptotic solution is available in the regular case. It is possible, using, for example, the Score Function test (SFT), to construct the asymptotically (locally) most powerful test. Moreover, the General Likelihood Ratio Test (GLRT) and the Wald test (WT) based on the maximum likelihood estimator are asymptotically most powerful in the same sense. In the non regular cases the situation became much more complex. First of all, there are different non regular (singular) situations. Moreover, in all these situations, the choice of the asymptotically best test is always an open question.

This work is an attempt to study all these situations on the model of inhomogeneous Poisson processes. This model is sufficiently simple to allow us to realize the construction of the well known tests (SFT, GLRT, WT) and to verify that these test are asymptotically most powerful also for this model, in the case when it is regular. In the next paper we study the behavior of these tests in the case when the model is singular. The "evolution of the singularity" of the intensity function is the following: regular case (finite Fisher information, this paper), continuous but not differentiable (cusp-type singularity, [4]), discontinuous (jump-type singularity, [4]). In all the three cases we describe the tests analytically. More precisely, we describe the test statistics, the choice of the thresholds and the behavior of the power functions for local alternatives.

Note that the notion of *local alternatives* is different following the type of regularity/singularity. Suppose we want to test the simple hypothesis $\vartheta = \vartheta_1$ against the one-sided alternative $\vartheta > \vartheta_1$. In the regular case, the local alternatives are usually given by $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$, $u > 0$. In the case of a cusp-type singularity, the local alternatives are introduced by $\vartheta = \vartheta_1 + u\, n^{-\frac{1}{2\kappa+1}}$, $u > 0$. As to the case of a jump-type singularity, the local alternatives are $\vartheta = \vartheta_1 + \frac{u}{n}$, $u > 0$. In all these problems, the most interesting for us question is the comparison of the power functions of different tests. In singular cases, the comparison is done with the help of numerical simulations. The main results concern the limit likelihood ratios in the non-regular situations. Let us note, that in many other models of observations (i.i.d., time series, diffusion processes etc.) the likelihood ratios have the same limits as here (see, for example, [6] and [2]). Therefore, the results presented here are of more universal nature and are valid for any other (non necessarily Poissonian)

model having one of considered here limit likelihood ratios.

We recall that $X=(X_t,\ t\geq 0)$ is an inhomogeneous Poisson process with intensity function $\lambda(t)$, $t\geq 0$, if $X_0=0$ and the increments of X on disjoint intervals are independent and distributed according to the Poisson law

$$\mathbf{P}\left\{X_t-X_s=k\right\}=\frac{\left(\int_s^t\lambda(t)\,\mathrm{d}t\right)^k}{k!}\exp\left\{-\int_s^t\lambda(t)\,\mathrm{d}t\right\}.$$

In all statistical problems considered in this work, the intensity functions are periodic with some known period τ and depend on some one-dimensional parameter, that is, $\lambda(t)=\lambda(\vartheta,t)$. The basic hypothesis and the alternative are always the same: $\vartheta=\vartheta_1$ and $\vartheta>\vartheta_1$. The diversity of statements corresponds to different types of regularity/singularity of the function $\lambda(\vartheta,t)$. The case of unknown period τ needs a special study.

The hypothesis testing problems (or closely related properties of the likelihood ratio) for inhomogeneous Poisson processes were studied by many authors (see, for example, Brown [1], Kutoyants [7], Léger and Wolfson [11], Liese and Lorz [14], Sung *et al.* [16], Fazli and Kutoyants [5], Dachian and Kutoyants [3] and the references therein). Note finally, that the results of this study will appear later in the work [9].

2 Auxiliary results

For simplicity of exposition we consider the model of n independent observations of an inhomogeneous Poisson process: $X^n=(X_1,\ldots,X_n)$, where $X_j=(X_j(t),\ 0\leq t\leq\tau)$, $j=1,\ldots,n$, are Poisson processes with intensity function $\lambda(\vartheta,t)$, $0\leq t\leq\tau$. Here $\vartheta\in\Theta=[\vartheta_1,b)$, $b<\infty$, is a one-dimensional parameter. We have

$$\mathbf{E}_\vartheta X_j(t)=\Lambda(\vartheta,t)=\int_0^t\lambda(\vartheta,s)\ \mathrm{d}s$$

where $\mathbf{E}_\vartheta$ is the mathematical expectation in the case when the true value is ϑ. Note that this model is equivalent to the one, where we observe an inhomogeneous Poisson process $X^T=(X_t,\ 0\leq t\leq T)$ with periodic intensity $\lambda(\vartheta,t+j\tau)=\lambda(\vartheta,t)$, $j=1,2,\ldots,n-1$, and $T=n\tau$ (the period τ is supposed to be known). Indeed, if we put $X_j(s)=X_{s+\tau(j-1)}-X_{\tau(j-1)}$, $s\in[0,\tau]$, $j=1,\ldots,n$, then the observation of one trajectory X^T is equivalent to n independent observations $X_1,\ldots,X_n$.

The intensity function is supposed to be separated from zero on $[0,\tau]$. The measures corresponding to Poisson processes with different values of ϑ are

equivalent. The likelihood function is defined by the equality (see Liese [13])

$$L(\vartheta, X^n) = \exp\left\{\sum_{j=1}^{n}\int_0^\tau \ln\lambda\left(\vartheta, t\right)\mathrm{d}X_j\left(t\right) - n\int_0^\tau\left[\lambda\left(\vartheta, t\right) - 1\right]\mathrm{d}t\right\}$$

and the likelihood ratio function is

$$L\left(\vartheta, \vartheta_1, X^n\right) = L\left(\vartheta, X^n\right)/L\left(\vartheta_1, X^n\right).$$

We have to test the following two hypotheses

$$\begin{array}{lcl} \mathscr{H}_1 & : & \vartheta = \vartheta_1, \\ \mathscr{H}_2 & : & \vartheta > \vartheta_1. \end{array}$$

A test $\bar{\psi}_n = \bar{\psi}_n\left(X^n\right)$ is defined as the probability to accept the hypothesis $\mathscr{H}_2$. Its power function is $\beta\left(\bar{\psi}_n, \vartheta\right) = \mathbf{E}_\vartheta\,\bar{\psi}_n(X^n)$, $\vartheta > \vartheta_1$.

Denote $\mathcal{K}_\varepsilon$ the class of tests $\bar{\psi}_n$ of asymptotic size $\varepsilon \in [0, 1]$:

$$\mathcal{K}_\varepsilon = \left\{\bar{\psi}_n \quad : \quad \lim_{n\to\infty}\mathbf{E}_{\vartheta_1}\,\bar{\psi}_n\left(X^n\right) = \varepsilon\right\}.$$

Our goal is to construct tests which belong to this class and have some proprieties of asymptotic optimality.

The comparison of tests can be done by comparison of their power functions. It is known that for any reasonable test and for any fixed alternative the power function tends to 1. To avoid this difficulty, as usual, we consider *close* or *contiguous* alternatives. We put $\vartheta = \vartheta_1 + \varphi_n u$, where $u \in \mathbb{U}_n^+ = \left[0, \varphi_n^{-1}\left(b - \vartheta_1\right)\right)$, $\varphi_n = \varphi_n\left(\vartheta_1\right) > 0$ and $\varphi_n \to 0$. The rate of convergence $\varphi_n \to 0$ must be chosen so that the normalized likelihood ratio

$$Z_n\left(u\right) = \frac{L\left(\vartheta_1 + \varphi_n u, X^n\right)}{L\left(\vartheta_1, X^n\right)}, \qquad u \geq 0,$$

has a non degenerate limit. In the regular case this rate is usually $\varphi_n = n^{-1/2}$.

Then the initial problem of hypotheses testing can be rewritten as

$$\begin{array}{lcl} \mathscr{H}_1 & : & u = 0, \\ \mathscr{H}_2 & : & u > 0. \end{array}$$

The power function of a test $\bar{\psi}_n$ is now denoted as

$$\beta\left(\bar{\psi}_n, u\right) = \mathbf{E}_{\vartheta_1+\varphi_n u}\,\bar{\psi}_n, \qquad u > 0.$$

The asymptotic optimality of tests is introduced with the help of the following definition (see [15]).

Definition 1. *We call a test $\psi_n^\star(X^n) \in \mathcal{K}_\varepsilon$ locally asymptotically uniformly most powerful (LAUMP) in the class $\mathcal{K}_\varepsilon$ if its power function $\beta(\psi_n^\star, u)$ satisfies the relation: for any test $\bar{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$ and any $K > 0$ we have*

$$\varliminf_{n\to\infty} \inf_{0<u\leq K} \left[\beta(\psi_n^\star, u) - \beta\left(\bar{\psi}_n, u\right)\right] \geq 0.$$

Below we show that in the regular case many tests are LAUMP. In the next paper [4], where we consider some singular situations, a "reasonable" definition of asymptotic optimality of tests is still an open question. That is why we use numerical simulations to compare the tests in [4].

We assume that the following *Regularity conditions* are satisfied.

Smoothness. *The intensity function $\lambda(\vartheta, t)$, $0 \leq t \leq \tau$, of the observed Poisson process X^n is two times continuously differentiable w.r.t. ϑ, is separated from zero uniformly on $\vartheta \geq \vartheta_1$, and the Fisher information is positive:*

$$\mathrm{I}(\vartheta) = \int_0^\tau \frac{\dot{\lambda}(\vartheta, t)^2}{\lambda(\vartheta, t)} \mathrm{d}t, \qquad \inf_{\vartheta\in\Theta} \mathrm{I}(\vartheta) > 0.$$

Here $\dot{\lambda}$ denotes the derivative of λ w.r.t. ϑ and, at the point ϑ_1, the derivative is from the right.

Distinguishability. *For any $\nu > 0$, we have*

$$\inf_{\vartheta_*\in\Theta} \inf_{|\vartheta-\vartheta_*|>\nu} \left\| \sqrt{\lambda(\vartheta, \cdot)} - \sqrt{\lambda(\vartheta_1, \cdot)} \right\|_{L^2} > 0.$$

Here

$$\|h(\cdot)\|_{L^2}^2 = \int_0^\tau h(t)^2 \, \mathrm{d}t.$$

In this case, the natural normalization function is $\varphi_n = n^{-1/2}$ and the change of variables is $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$.

The key propriety of statistical problems in the regular case is the *local asymptotic normality* (LAN) of the family of measures of corresponding inhomogeneous Poisson processes at the point ϑ_1. This means that the normalized likelihood ratio

$$\tilde{Z}_n(u) = L\left(\vartheta_1 + \frac{u}{\sqrt{n}}, \vartheta_1, X^n\right)$$

admits the representation

$$\tilde{Z}_n(u) = \exp\left\{u\,\widetilde{\Delta}_n(\vartheta_1, X^n) - \frac{u^2}{2}\mathrm{I}(\vartheta_1) + r_n\right\},$$

where (using the central limit theorem) we have

$$\widetilde{\Delta}_n\left(\vartheta_1, X^n\right) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_0^\tau \frac{\dot{\lambda}\left(\vartheta_1, t\right)}{\lambda\left(\vartheta_1, t\right)} \left[\mathrm{d}X_j\left(t\right) - \lambda\left(\vartheta_1, t\right)\mathrm{d}t\right]$$
$$\Longrightarrow \widetilde{\Delta} \sim \mathcal{N}\left(0, \mathrm{I}\left(\vartheta_1\right)\right)$$

(convergence in distribution under ϑ_1), and $r_n = r_n\left(\vartheta_1, u, X^n\right) \xrightarrow{p} 0$ (convergence in probability under ϑ_1). Moreover, the last convergence is uniform on $0 \leq u < K$ for any $K > 0$.

Let us now briefly recall how this representation was obtained in [7]. Denoting $\lambda_0 = \lambda\left(\vartheta_1, t\right)$ and $\lambda_u = \lambda\left(\vartheta_1 + \frac{u}{\sqrt{n}}, t\right)$, with the help of the Taylor series expansion we can write

$$\begin{aligned}
\ln Z_n\left(u\right) &= \sum_{j=1}^{n} \int_0^\tau \ln \frac{\lambda_u}{\lambda_0} \left[\mathrm{d}X_j\left(t\right) - \lambda_0 \mathrm{d}t\right] - n \int_0^\tau \left[\lambda_u - \lambda_0 - \lambda_0 \ln \frac{\lambda_u}{\lambda_0}\right] \mathrm{d}t \\
&= \frac{u}{\sqrt{n}} \sum_{j=1}^{n} \int_0^\tau \frac{\dot{\lambda}_0}{\lambda_0} \left[\mathrm{d}X_j\left(t\right) - \lambda_0 \mathrm{d}t\right] - \frac{u^2}{2} \int_0^\tau \frac{\dot{\lambda}_0^2}{\lambda_0} \mathrm{d}t + r_n \\
&= u\widetilde{\Delta}_n\left(\vartheta_1, X^n\right) - \frac{u^2}{2}\mathrm{I}\left(\vartheta_1\right) + r_n \Longrightarrow \widetilde{\Delta} - \frac{u^2}{2}\mathrm{I}\left(\vartheta_1\right).
\end{aligned}$$

In the sequel, we choose reparametrizations which lead to *universal* in some sense limits. For example, in the regular case, we put

$$\varphi_n = \varphi_n\left(\vartheta_1\right) = \frac{1}{\sqrt{n\mathrm{I}\left(\vartheta_1\right)}}, \qquad u \in \mathbb{U}_n^+ = \left[0, \varphi_n^{-1}\left(b - \vartheta_1\right)\right).$$

With such change of variables, we have

$$Z_n\left(u\right) = L\left(\vartheta_1 + u\varphi_n, \vartheta_1, X^n\right) = \exp\left\{u\,\Delta_n\left(\vartheta_1, X^n\right) - \frac{u^2}{2} + r_n\right\},$$

where

$$\Delta_n\left(\vartheta_1, X^n\right) = \frac{1}{\sqrt{\mathrm{I}\left(\vartheta_1\right)}}\,\widetilde{\Delta}_n \Longrightarrow \Delta \sim \mathcal{N}\left(0, 1\right).$$

The LAN families have many remarkable properties and some of them will be used below.

Let us remind here one general result which is valid in a more general situation. We suppose only that the normalized likelihood ratio $Z_n\left(u\right)$ converges to some limit $Z\left(u\right)$ in distribution. Note that this is the case in all our regular and singular problems. The following property allows us to calculate the distribution under local alternative when we know the distribution under the null hypothesis. Moreover, it gives an efficient algorithm for calculating power functions in numerical simulations.

Lemma 1 (Le Cam's Third Lemma)**.** *Suppose that* $(Z_n(u), Y_n)$ *converges in distribution under* ϑ_1*:*

$$(Z_n(u), Y_n) \Longrightarrow (Z(u), Y).$$

Then, for any bounded continuous function $g(\cdot)$*, we have*

$$\mathbf{E}_{\vartheta_1+\varphi_n u}[g(Y_n)] \longrightarrow \mathbf{E}[Z(u)\, g(Y)].$$

For the proof see [10].

In the regular case, the limit of $Z_n(\cdot)$ is the random function

$$Z(u) = \exp\left\{u\,\Delta - \frac{u^2}{2}\right\}, \qquad u \geq 0.$$

So, for any fixed $u > 0$, we have the convergence

$$Z_n(u) \Longrightarrow Z(u).$$

According to this lemma, we can write the following relations for the characteristic function of $\Delta_n = \Delta_n(\vartheta_1, X^n)$:

$$\mathbf{E}_{\vartheta_1+\varphi_n u}\, e^{i\mu\Delta_n} \longrightarrow \mathbf{E}\, Z(u)\, e^{i\mu\Delta} = e^{-\frac{u^2}{2}}\, \mathbf{E}\, e^{u\Delta+i\mu\Delta} = e^{i\mu u-\frac{\mu^2}{2}} = \mathbf{E}\, e^{i\mu(u+\Delta)},$$

which yields the asymptotic distribution of the statistic Δ_n under the alternative $\vartheta_1 + \varphi_n u$:

$$\Delta_n(\vartheta_1, X^n) \Longrightarrow u + \Delta \;\sim\; \mathcal{N}(u, 1).$$

3 Weak convergence

All the tests considered in this paper are functionals of the normalized likelihood ratio $Z_n(\cdot)$. For each of them, we have to evaluate two quantities. The first one is the threshold, which guarantees the desired asymptotic size of the test, and the second one is the limit power function, which has to be calculated under alternative. Our study is based on the weak convergence of the likelihood ratio $Z_n(\cdot)$ under hypothesis (to calculate the threshold) and under alternative (to calculate the limit power function). Note that the test statistics of all the tests are continuous functionals of $Z_n(\cdot)$. That is why the weak convergence of $Z_n(\cdot)$ allows us to obtain the limit distributions of these statistics.

We denote $\mathbf{P}_\vartheta$ the distribution that the observed inhomogeneous Poisson processes X^n induce on the measurable space of their realizations. The

measures in the family $\{\mathbf{P}_\vartheta,\ \vartheta \geq \vartheta_1\}$ are equivalent, and the normalized likelihood ratio is

$$\ln Z_n(u) = \sum_{j=1}^{n} \int_0^\tau \ln \frac{\lambda(\vartheta_1 + \varphi_n(\vartheta_1) u, t)}{\lambda(\vartheta_1, y)} \mathrm{d}X_j(t)$$
$$- n \int_0^\tau \left[\lambda(\vartheta_1 + \varphi_n(\vartheta_1) u, t) - \lambda(\vartheta_1, t)\right] \mathrm{d}t,$$

where $u \in \mathbb{U}_n^+ = \left[0, \varphi_n^{-1}(b - \vartheta_1)\right)$. We define $Z_n(u)$ to be linearly decreasing to zero on the interval $\left[\varphi_n^{-1}(b - \vartheta_1), \varphi_n^{-1}(b - \vartheta_1) + 1\right]$ and we put $Z_n(u) = 0$ for $u > \varphi_n^{-1}(b - \vartheta_1) + 1$. Now the random function $Z_n(u)$ is defined on $\mathbb{R}_+$ and belongs to the space $\mathscr{C}_0(\mathbb{R}_+)$ of continuous on $\mathbb{R}_+$ functions such that $z(u) \to 0$ as $u \to \infty$. Introduce the uniform metric in this space and denote $\mathcal{B}$ the corresponding Borel sigma-algebra. The next theorem describes the weak convergence under the alternative $\vartheta = \vartheta_1 + \varphi_n u_*$ (with fixed $u_* > 0$) of the stochastic process $(Z_n(u),\ u \in \mathbb{R}_+)$ to the process

$$Z(u, u_*) = \exp\left\{u\Delta + u u_* - \frac{u^2}{2}\right\}, \quad u \in \mathbb{R},$$

in the measurable space $(\mathscr{C}_0(\mathbb{R}_+), \mathcal{B})$. Note that in [8] this theorem was proved for a fixed true value ϑ. In the hypothesis testing problems considered here, we need this convergence both under hypothesis $\mathscr{H}_1$, that is, for fixed true value $\vartheta = \vartheta_1$ $(u_* = 0)$, and under alternative $\mathscr{H}_2$ with "moving" true value $\vartheta = \vartheta_{u_*} = \vartheta_1 + \varphi_n u_*$.

Theorem 1. *Let us suppose that the Regularity conditions are fulfilled. Then, under alternative ϑ_{u_*}, we have the weak convergence of the stochastic process $Z_n = (Z_n(u),\ u \geq 0)$ to $Z = (Z(u, u_*),\ u \geq 0)$.*

According to [6, Theorem 1.10.1], to prove this theorem it is sufficient to verify the following three properties of the process $Z_n(\cdot)$.

1. The finite-dimensional distributions of $Z_n(\cdot)$ converge, under alternative ϑ_{u_*}, to the finite-dimensional distributions of $Z(\cdot, u_*)$.

2. The inequality

$$\mathbf{E}_{\vartheta_{u_*}} \left|Z_n^{1/2}(u_2) - Z_n^{1/2}(u_1)\right|^2 \leq C\left|u_2 - u_1\right|^2$$

holds for every $u_1, u_2 \in \mathbb{U}_n^+$ and some constant $C > 0$.

3. There exists $d>0$, such that for some $n_0>0$ and all $n\geq n_0$ we have the estimate

$$\mathbf{P}_{\vartheta_{u_*}}\left\{Z_n(u)>e^{-d|u-u_*|^2}\right\}\leq e^{-d|u-u_*|^2}.$$

Let us rewrite the random function $Z_n(\cdot)$ as follows:

$$\begin{aligned}Z_n(u)&=L\left(\vartheta_1+u\varphi_n,\vartheta_1,X^n\right)\\&=L\left(\vartheta_1+u_*\varphi_n,\vartheta_1,X^n\right)L\left(\vartheta_1+u\varphi_n,\vartheta_1+u_*\varphi_n,X^n\right).\end{aligned}$$

For the first term we have

$$L\left(\vartheta_1+u_*\varphi_n,\vartheta_1,X^n\right)=L\left(\vartheta_{u_*}-u_*\varphi_n,\vartheta_{u_*},X^n\right)^{-1}\Longrightarrow\exp\left\{u_*\Delta+\frac{u_*^2}{2}\right\}.$$

Therefore we only need to check the conditions 2–3 for the term

$$Z_n(u,u_*)=L\left(\vartheta_1+u\varphi_n,\vartheta_1+u_*\varphi_n,X^n\right)=L\left(\vartheta_{u_*}+(u-u_*)\varphi_n,\vartheta_{u_*},X^n\right).$$

Lemma 2. *The finite-dimensional distributions of $Z_n(\cdot)$ converge, under alternative ϑ_{u_*}, to the finite-dimensional distributions of $Z(\cdot,u_*)$.*

Proof. The limit process for $Z_n(\cdot,u_*)$ is

$$\exp\left\{(u-u_*)\Delta-\frac{(u-u_*)^2}{2}\right\},\quad u\in\mathbb{R}_+.$$

Hence

$$Z_n(u)\Longrightarrow\exp\left\{u_*\Delta+\frac{u_*^2}{2}\right\}\exp\left\{(u-u_*)\Delta-\frac{(u-u_*)^2}{2}\right\}=Z(u,u_*).$$

For the details see, for example, [8]. □

Lemma 3. *Let the Regularity conditions be fulfilled. Then there exists a constant $C>0$, such that*

$$\mathbf{E}_{\vartheta_{u_*}}\left|Z_n^{1/2}(u_1,u_*)-Z_n^{1/2}(u_2,u_*)\right|^2\leq C\left|u_1-u_2\right|^2$$

for all $u_*,u_1,u_2\in\mathbb{U}_n^+$ and sufficiently large values of n.

Proof. According to [8, Lemma 1.1.5], we have:

$$\begin{aligned}
&\mathbf{E}_{\vartheta_{u_*}}\left|Z_n^{1/2}(u_1,u_*)-Z_n^{1/2}(u_2,u_*)\right|^2\\
&\quad\le n\int_0^\tau\left(\frac{\lambda^{1/2}(\vartheta_1+u_1\varphi_n,t)}{\lambda^{1/2}(\vartheta_1+u_*\varphi_n,t)}-\frac{\lambda^{1/2}(\vartheta_1+u_2\varphi_n,t)}{\lambda^{1/2}(\vartheta_1+u_*\varphi_n,t)}\right)^2\lambda(\vartheta_1+u_*\varphi_n,t)\,\mathrm{d}t\\
&\quad= n\int_0^\tau\left(\lambda^{1/2}(\vartheta_1+u_1\varphi_n,t)-\lambda^{1/2}(\vartheta_1+u_2\varphi_n,t)\right)^2\mathrm{d}t\\
&\quad=\frac{n}{4}\,\varphi_n^2\,(u_2-u_1)^2\int_0^\tau\frac{\dot\lambda\,(\vartheta_v,t)^2}{\lambda\,(\vartheta_v,t)}\,\mathrm{d}t\le C\,(u_2-u_1)^2\,,
\end{aligned}$$

where v is some intermediate point between u_1 and u_2. □

Lemma 4. *Let the Regularity conditions be fulfilled. Then there exists a constant $d>0$, such that*

$$\mathbf{P}_{\vartheta_{u_*}}\left\{Z_n(u,u_*)>e^{-d|u-u_*|^2}\right\}\le e^{-d|u-u_*|^2}\tag{1}$$

for all $u_*,u\in\mathbb{U}_n^+$ and sufficiently large value of n.

Proof. Using the Markov inequality, we get

$$\mathbf{P}_{\vartheta_{u_*}}\left\{Z_n(u,u_*)>e^{-d|u-u_*|^2}\right\}\le e^{\frac12 d|u-u_*|^2}\,\mathbf{E}_{\vartheta_{u_*}}\,Z_n^{1/2}(u,u_*).$$

According to [8, Lemma 1.1.5], we have

$$\begin{aligned}
&\mathbf{E}_{\vartheta_{u_*}}Z_n^{1/2}(u,u_*)\\
&\quad=\exp\left\{-\frac12\,n\int_0^\tau\left(\frac{\lambda^{1/2}(\vartheta_1+u\varphi_n,t)}{\lambda^{1/2}(\vartheta_1+u_*\varphi_n,t)}-1\right)^2\lambda(\vartheta_1+u_*\varphi_n,t)\,\mathrm{d}t\right\}\\
&\quad=\exp\left\{-\frac12\,n\int_0^\tau\left(\lambda^{1/2}\big(\vartheta_1+u\varphi_n,t\big)-\lambda^{1/2}(\vartheta_1+u_*\varphi_n,t)\right)^2\mathrm{d}t\right\}.
\end{aligned}$$

Using the Taylor expansion we get

$$\lambda^{1/2}\big(\vartheta_1+u\varphi_n,t\big)=\lambda^{1/2}\big(\vartheta_1+u_*\varphi_n,t\big)+\frac{\varphi_n(u-u_*)}{2}\,\frac{\dot\lambda\,(\vartheta_v,t)}{\lambda^{1/2}\,(\vartheta_v,t)}\,,$$

where v is some intermediate point between u_* and u. Hence, for sufficiently large n providing $\varphi_n\,|u-u_*|\le\gamma$, we have the inequality $\mathrm{I}\,(\vartheta_v)\ge\frac12\mathrm{I}\,(\vartheta_1)$, and we obtain

$$\begin{aligned}
\mathbf{E}_{\vartheta_{u_*}}\,Z_n^{1/2}(u,u_*)&\le\exp\left\{-\frac{1}{8\mathrm{I}\,(\vartheta_1)}\,|u-u_*|^2\,\mathrm{I}\,(\vartheta_v)\right\}\\
&\le\exp\left\{-\frac{|u-u_*|^2}{16}\right\}.
\end{aligned}\tag{2}$$

By Distinguishability condition, we can write

$$g(\gamma)=\inf_{\varphi_n|u-u_*|>\gamma}\int_0^\tau\Big(\lambda^{1/2}\big(\vartheta_1+u\varphi_n,t\big)-\lambda^{1/2}(\vartheta_1+u_*\varphi_n,t)\Big)^2\,\mathrm{d}t>0,$$

and hence

$$\int_0^\tau\Big(\lambda^{1/2}\big(\vartheta_1+u\varphi_n,t\big)-\lambda^{1/2}(\vartheta_1+u_*\varphi_n,t)\Big)^2\,\mathrm{d}t\geq g(\gamma)\geq g(\gamma)\frac{\varphi_n^2(u-u_*)^2}{(b-\vartheta_1)^2}$$

and

$$\mathbf{E}_{\vartheta_{u_*}}\,Z_n^{1/2}(u,u_*)\leq\exp\left\{-\frac{g(\gamma)\,|u-u_*|^2}{2\mathrm{I}\,(\vartheta_1)\,(b-\vartheta_1)^2}\right\}. \tag{3}$$

So, putting

$$d=\frac{2}{3}\min\left(\frac{1}{16}\,,\,\frac{g(\gamma)}{2\mathrm{I}\,(\vartheta_1)\,(b-\vartheta_1)^2}\right),$$

the estimate (1) follows from (2) and (3). □

The weak convergence of $Z_n\,(\cdot)$ now follows from [6, Theorem 1.10.1].

4 Hypothesis testing

In this section, we construct the Score Function test, the General Likelihood Ratio test, the Wald test and two Bayes tests. For all these tests we describe the choice of the thresholds and evaluate the limit power functions for local alternatives.

4.1 Score Function test

Let us introduce the *Score Function test* (SFT)

$$\psi_n^\star\,(X^n)=\mathbb{1}_{\{\Delta_n(\vartheta_1,X^n)>z_\varepsilon\}},$$

where z_ε is the $(1-\varepsilon)$-quantile of the standard normal distribution $\mathcal{N}\,(0,1)$ and the statistic $\Delta_n\,(\vartheta_1,X^n)$ is

$$\Delta_n\,(\vartheta_1,X^n)=\frac{1}{\sqrt{n\mathrm{I}\,(\vartheta_1)}}\sum_{j=1}^n\int_0^\tau\frac{\dot\lambda\,(\vartheta_1,t)}{\lambda\,(\vartheta_1,t)}\left[\mathrm{d}X_j\,(t)-\lambda\,(\vartheta_1,t)\,\mathrm{d}t\right].$$

The SFT has the following well-known properties (one can see, for example, [12, Theorem 13.3.3] for the case of i.i.d. observations).

Proposition 1. *The test $\psi_n^\star(X^n) \in \mathcal{K}_\varepsilon$ and is LAUMP. For its power function the following convergence hold:*

$$\beta(\psi_n^\star, u_*) \longrightarrow \beta^\star(u_*) = \mathbf{P}(\Delta > z_\varepsilon - u_*), \quad \Delta \sim \mathcal{N}(0,1).$$

Proof. The property $\psi_n^\star(X^n) \in \mathcal{K}_\varepsilon$ follows immediately from the asymptotic normality (under hypothesis $\mathscr{H}_1$)

$$\Delta_n(\vartheta_1, X^n) \Longrightarrow \Delta.$$

Further, we have (under alternative $\vartheta_{u_*} = \vartheta_1 + u_*\varphi_n$) the convergence

$$\beta(\psi_n^\star, u_*) \longrightarrow \mathbf{P}(\Delta + u_* > z_\varepsilon) = \beta^\star(u_*).$$

This follows from the Le Cam's Third Lemma and can be shown directly as follows. Suppose that the intensity of the observed Poisson process is $\lambda(\vartheta_1 + u_*\varphi_n, t)$, then we can write

$$\begin{aligned}
\Delta_n(\vartheta_1, X^n) &= \frac{1}{\sqrt{n\mathrm{I}(\vartheta_1)}} \sum_{j=1}^n \int_0^\tau \frac{\dot\lambda(\vartheta_1,t)}{\lambda(\vartheta_1,t)} \left[\mathrm{d}X_j(t) - \lambda(\vartheta_1 + u_*\varphi_n, t)\,\mathrm{d}t\right] \\
&\quad + \frac{1}{\sqrt{n\mathrm{I}(\vartheta_1)}} \sum_{j=1}^n \int_0^\tau \frac{\dot\lambda(\vartheta_1,t)}{\lambda(\vartheta_1,t)} \left[\lambda(\vartheta_1 + u_*\varphi_n, t) - \lambda(\vartheta_1,t)\right]\mathrm{d}t \\
&= \Delta_n^*(\vartheta_1, X^n) + \frac{u_*}{n\mathrm{I}(\vartheta_1)} \sum_{j=1}^n \int_0^\tau \frac{\dot\lambda(\vartheta_1,t)^2}{\lambda(\vartheta_1,t)}\mathrm{d}t + o(1) \\
&= \Delta_n^*(\vartheta_1, X^n) + u_* + o(1) \Longrightarrow \Delta + u_*.
\end{aligned}$$

To show that the SFT is LAUMP, it is sufficient to verify that the limit of its power function coincides (for each fixed value $u_* > 0$) with the limit of the power of the corresponding likelihood ratio (Neyman-Person) test (N-PT) $\psi_n^*(X^n)$. Remind that the N-PT is the most powerful for each fixed (simple) alternative (see, for example, Theorem 13.3 in Lehman and Romano [12]). Of course, the N-PT is not a real test (in our one-sided problem), since for its construction one needs to know the value u_* of the parameter u under alternative.

The N-PT is defined by

$$\psi_n^*(X^n) = \mathbb{1}_{\{Z_n(u_*) > d_\varepsilon\}} + q_\varepsilon \mathbb{1}_{\{Z_n(u_*) = d_\varepsilon\}},$$

where the threshold d_ε and the probability q_ε are chosen from the condition $\psi_n^*(X^n) \in \mathcal{K}_\varepsilon$, that is,

$$\mathbf{P}_{\vartheta_1}\{Z_n(u_*) > d_\varepsilon\} + q_\varepsilon \mathbf{P}_{\vartheta_1}\{Z_n(u_*) = d_\varepsilon\} = \varepsilon.$$

Of course, we can put $q_\varepsilon = 0$ because the limit random variable $Z(u_*)$ has continuous distribution function.

The threshold d_ε can be found as follows. The LAN of the family of measures at the point ϑ_1 allows us to write

$$\mathbf{P}_{\vartheta_1}\left(Z_n(u_*) > d_\varepsilon\right) = \mathbf{P}_{\vartheta_1}\left(u_*\Delta_n(\vartheta_1, X^n) - \frac{u_*^2}{2} + r_n > \ln d_\varepsilon\right)$$
$$\longrightarrow \mathbf{P}\left(u_*\Delta - \frac{u_*^2}{2} > \ln d_\varepsilon\right) = \mathbf{P}\left(\Delta > \frac{\ln d_\varepsilon}{u_*} + \frac{u_*}{2}\right) = \varepsilon.$$

Hence, we have

$$\frac{\ln d_\varepsilon}{u_*} + \frac{u_*}{2} = z_\varepsilon \qquad \text{and} \qquad d_\varepsilon = \exp\left\{u_* z_\varepsilon - \frac{u_*^2}{2}\right\}.$$

Therefore the N-PT

$$\psi_n^*(X^n) = \mathbb{1}_{\left\{Z_n(u_*) > \exp\left\{u_* z_\varepsilon - \frac{u_*^2}{2}\right\}\right\}}$$

belongs to $\mathcal{K}_\varepsilon$.

For the power of the N-PT we have (denoting as usually $\vartheta_{u_*} = \vartheta_1 + u_*\varphi_n$)

$$\beta(\psi_n^*, u_*) = \mathbf{P}_{\vartheta_{u_*}}\left(Z_n(u_*) > d_\varepsilon\right) = \mathbf{P}_{\vartheta_{u_*}}\left(u_*\Delta_n(\vartheta_1, X^n) + r_n > u_* z_\varepsilon\right)$$
$$= \mathbf{P}_{\vartheta_{u_*}}\left(\Delta_n(\vartheta_1, X^n) + \frac{r_n}{u_*} > z_\varepsilon\right) \longrightarrow \mathbf{P}\left(\Delta + u_* > z_\varepsilon\right) = \beta^\star(u_*).$$

Therefore the limits of the powers of the tests $\psi_n^\star$ and ψ_n^* coincide, that is, the Score Function test is asymptotically as good as the Neyman-Pearson optimal one. Note that the limits are valid for any sequence of $0 \le u_* \le K$. So, for any $K > 0$, we can choose a sequence $\hat{u}_n \in [0, K]$ such that

$$\sup_{0 \le u_* \le K} \left|\beta(\psi_n^\star, u_*) - \beta(\psi_n^*, u_*)\right| = \left|\beta(\psi_n^\star, \hat{u}_n) - \beta(\psi_n^*, \hat{u}_n)\right| \longrightarrow 0,$$

which represents the asymptotic coincidence of the two tests and concludes the proof. □

4.2 GLRT and Wald test

Let us remind that the maximum likelihood estimator (MLE) $\hat{\vartheta}_n$ is defined by the equation:

$$L\left(\hat{\vartheta}_n, \vartheta_1, X^n\right) = \sup_{\vartheta \in [\vartheta_1, b)} L(\vartheta, \vartheta_1, X^n),$$

where the likelihood ratio function is

$$L\left(\vartheta,\vartheta_1,X^n\right)=\exp\left\{\sum_{j=1}^n\int_0^\tau\ln\frac{\lambda\left(\vartheta,t\right)}{\lambda\left(\vartheta_1,t\right)}\,\mathrm{d}X_j\left(t\right)\right.$$
$$\left.-n\int_0^\tau\left[\lambda\left(\vartheta,t\right)-\lambda\left(\vartheta_1,t\right)\right]\mathrm{d}t\right\},\qquad\vartheta\in\left[\vartheta_1,b\right).$$

The GLRT is

$$\hat{\psi}_n\left(X^n\right)=\mathbb{1}_{\{Q(X^n)>h_\varepsilon\}},$$

where

$$Q\left(X^n\right)=\sup_{\vartheta\in[\vartheta_1,b)}L\left(\vartheta,\vartheta_1,X^n\right)=L\left(\hat{\vartheta}_n,\vartheta_1,X^n\right)\quad\text{and}\quad h_\varepsilon=\exp\{z_\varepsilon^2/2\}.$$

The Wald's test is based on the MLE $\hat{\vartheta}_n$ and is defined as follows:

$$\psi_n^\circ\left(X^n\right)=\mathbb{1}_{\left\{\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)>z_\varepsilon\right\}}.$$

The properties of these tests are given in the following Proposition.

Proposition 2. *The tests $\hat{\psi}_n\left(X^n\right)$ and $\psi_n^\circ\left(X^n\right)$ belong to $\mathcal{K}_\varepsilon$, their power functions $\beta(\hat{\psi}_n,u_*)$ and $\beta\left(\psi_n^\circ,u_*\right)$ converge to $\beta^\star\left(u_*\right)$, and therefore they are LAUMP.*

Proof. Let us put $\vartheta=\vartheta_1+u\varphi_n$ and denote $\hat{u}_n=\varphi_n^{-1}\left(\hat{\vartheta}_n-\vartheta_1\right)$. We have

$$\mathbf{P}_{\vartheta_1}\left\{\sup_{\vartheta\in[\vartheta_1,b)}L\left(\vartheta,\vartheta_1,X^n\right)>h_\varepsilon\right\}=\mathbf{P}_{\vartheta_1}\left\{\sup_{u\in\mathbb{U}_n^+}L\left(\vartheta_1+u\varphi_n,\vartheta_1,X^n\right)>h_\varepsilon\right\}$$
$$=\mathbf{P}_{\vartheta_1}\left\{\sup_{u\in\mathbb{U}_n^+}Z_n\left(u\right)>h_\varepsilon\right\}.$$

According to Theorem 1 (with $u_*=0$), we have the weak convergence (under ϑ_1) of the measure of the stochastic processes $\left(Z_n\left(u\right),\ u\geq 0\right)$ to those of the process $\left(Z\left(u\right),\ u\geq 0\right)$. This provides us the convergence of the distributions of all continuous in uniform metric functionals. Hence

$$Q\left(X^n\right)=\sup_{u>0}Z_n\left(u\right)\Longrightarrow\sup_{u>0}Z\left(u\right)$$
$$=\sup_{u>0}\exp\left\{u\Delta-\frac{u^2}{2}\right\}=\exp\left\{\frac{\Delta^2}{2}\mathbb{1}_{\{\Delta\geq 0\}}\right\},$$

which yields (we suppose that $\varepsilon \leq \frac{1}{2}$)

$$\mathbf{E}_{\vartheta_1} \hat{\psi}_n (X^n) \longrightarrow \mathbf{P} \left\{ \Delta \mathbb{1}_{\{\Delta \geq 0\}} > z_\varepsilon \right\} = \mathbf{P} \left\{ \Delta > z_\varepsilon \right\} = \varepsilon.$$

Using the same weak convergence we obtain the asymptotic normality of the MLE (see [6] or [8]):

$$\hat{u}_n = \frac{\hat{\vartheta}_n - \vartheta_1}{\varphi_n} \Longrightarrow \hat{u} = \Delta \mathbb{1}_{\{\Delta \geq 0\}},$$

and hence $\mathbf{E}_{\vartheta_1} \psi_n^\circ \longrightarrow \varepsilon$. So both $\hat{\psi}_n$ and ψ_n° belong to $\mathcal{K}_\varepsilon$.

Now, let us fix some $u_* > 0$ and study the limit behavior of the power functions of the tests.

Using the weak convergence of the likelihood ratio process under the alternative $\vartheta_{u_*} = \vartheta_1 + u_* \varphi_n$, we have

$$\begin{aligned} Q(X^n) = \sup_{u>0} Z_n(u) \Longrightarrow \sup_{u>0} Z(u, u_*) &= \sup_{u>0} \exp \left\{ u\Delta + u u_* - \frac{u^2}{2} \right\} \\ &= \exp \left\{ \frac{(\Delta + u_*)^2}{2} \mathbb{1}_{\{\Delta + u_* \geq 0\}} \right\}. \end{aligned}$$

Hence (we suppose again that $\varepsilon \leq \frac{1}{2}$),

$$\begin{aligned} \beta \left(\hat{\psi}_n, u_* \right) &= \mathbf{P}_{\vartheta_{u_*}} \left\{ Q(X^n) > h_\varepsilon \right\} \\ &\longrightarrow \mathbf{P} \left\{ (\Delta + u_*) \mathbb{1}_{\{\Delta + u_* \geq 0\}} > z_\varepsilon \right\} = \mathbf{P} \left\{ \Delta > z_\varepsilon - u_* \right\} = \beta^\star (u_*). \end{aligned}$$

Similarly we have

$$\beta (\psi_n^\circ, u_*) \longrightarrow \mathbf{P} \left\{ (\Delta + u_*) \mathbb{1}_{\{\Delta + u_* \geq 0\}} > z_\varepsilon \right\} = \beta^\star (u_*).$$

Therefore the tests are LAUMP. □

Example 1. As the family of measures is LAN and the problem is asymptotically equivalent to the corresponding hypothesis testing problem for a Gaussian model, we propose here a similar test for Gaussian observations.

Suppose that the random variable $X \sim \mathcal{N}(u, 1)$ and we have to test the hypothesis $\mathcal{H}_1 \ : \ u = 0$ against $\mathcal{H}_2 \ : \ u > 0$. Then the SFT $\hat{\psi}(X) = \mathbb{1}_{\{X > z_\varepsilon\}}$ is the uniformly most powerful in the class of tests of size ε. Its power function is $\beta \left(\hat{\psi}, u_* \right) = \mathbf{P} (\Delta > z_\varepsilon - u_*)$. The log-likelihood function is

$$L(u, X) = -\frac{1}{2} \ln (2\pi) - \frac{1}{2} (X - u)^2$$

The one-sided MLE $\hat{u}$ is given by

$$\hat{u} = \underset{u \geq 0}{\operatorname{argmax}}\, L(X, u) = \max\{X, 0\},$$

and it is easy to see that the test $\hat{\psi}\,(X)$ and the Wald test $\psi^{\circ}(X) = \mathbb{1}_{\{\hat{u} > z_{\varepsilon}\}}$ have identical power functions.

Let us note, that the asymptotic equivalence to the SFT and the optimality is a well known property of these tests in regular statistical experiments (see, for example, [12] and [9]). We present these properties here in order to compare the asymptotics of these tests in regular and singular situations (see [4]). In particular, we will see that the asymptotic properties of these tests in singular situations will be essentially different.

4.3 Bayes tests

Suppose now that the unknown parameter ϑ is a random variable with *a priori* density $p\,(\theta)$, $\vartheta_1 \leq \theta < b$. Here $p\,(\cdot)$ is a known continuous function satisfying $p\,(\vartheta_1) > 0$. We consider two approaches. The first one is based on the Bayes estimator and the second one on the averaged likelihood ratio function.

The first test, wich we call BT1, is a Wald type test but based on the Bayes estimator (BE) $\tilde{\vartheta}_n$:

$$\tilde{\psi}_n\,(X^n) = \mathbb{1}_{\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) > k_{\varepsilon}\right\}}.$$

Remind that the BE for quadratic loss function is

$$\tilde{\vartheta}_n = \int_{\vartheta_1}^{b} \theta\, p\,(\theta | X^n)\, \mathrm{d}\theta = \frac{\int_{\vartheta_1}^{b} \theta\, p\,(\theta)\, L\,(\theta, \vartheta_1, X^n)\, \mathrm{d}\theta}{\int_{\vartheta_1}^{b} p\,(\theta)\, L\,(\theta, \vartheta_1, X^n)\, \mathrm{d}\theta}.$$

After the change of variables $\theta = \vartheta_1 + v\varphi_n$ in the integrals, we obtain the relation

$$\varphi_n^{-1}\left(\tilde{\vartheta}_n - \vartheta_1\right) = \frac{\int_{\mathbb{U}_n^+} v p\,(\vartheta_1 + v\varphi_n)\, Z_n\,(v)\, \mathrm{d}v}{\int_{\mathbb{U}_n^+} p\,(\vartheta_1 + v\varphi_n)\, Z_n\,(v)\, \mathrm{d}v}.$$

The properties of $Z_n\,(\cdot)$ established in the proof of Theorem 1 yield the

following convergence in distribution under the hypothesis $\mathscr{H}_1$ (see [6] or [8])

$$\begin{aligned}
\varphi_n^{-1}\left(\tilde{\vartheta}_n-\vartheta_1\right) \Longrightarrow \tilde{u} &= \frac{\int_0^\infty vZ\left(v\right)\mathrm{d}v}{\int_0^\infty Z\left(v\right)\mathrm{d}v} \\
&= \frac{\int_0^\infty \left(v-\Delta\right)\exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\mathrm{d}v}{\int_0^\infty \exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\mathrm{d}v}+\Delta \\
&= \frac{-\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\Big|_{v=0}^{+\infty}}{\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left\{-\frac{(v-\Delta)^2}{2}\right\}\mathrm{d}v}+\Delta \\
&= \frac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\Delta^2}{2}\right\}}{\left(1-F\left(-\Delta\right)\right)}+\Delta = \frac{f\left(\Delta\right)}{F\left(\Delta\right)}+\Delta,
\end{aligned}$$

where $f\left(\cdot\right)$ and $F\left(\cdot\right)$ are the density and the distribution function of the standard normal Gaussian random variable Δ. Hence, if we take k_ε to be solution of the equation

$$\mathbf{P}\left\{\frac{f\left(\Delta\right)}{F\left(\Delta\right)}+\Delta > k_\varepsilon\right\} = \varepsilon,$$

then the BT1 $\tilde{\psi}_n$ belongs to $\mathcal{K}_\varepsilon$.

A similar calculation under the alternative $\vartheta_{u_*} = \vartheta + u_*\varphi_n$ allows us to evaluate the limit power function of the BT1 as follows:

$$\begin{aligned}
\beta\left(\tilde{\psi}_n, u_*\right) &= \mathbf{P}_{\vartheta_{u_*}}\left\{\varphi_n^{-1}\left(\tilde{\vartheta}_n-\vartheta_1\right) > k_\varepsilon\right\} \\
&\longrightarrow \mathbf{P}\left\{\frac{\int_0^\infty vZ\left(v,u_*\right)\mathrm{d}v}{\int_0^\infty Z\left(v,u_*\right)\mathrm{d}v} > k_\varepsilon\right\} \\
&= \mathbf{P}\left\{\frac{f\left(\Delta+u_*\right)}{F\left(\Delta+u_*\right)}+\Delta+u_* > k_\varepsilon\right\}.
\end{aligned}$$

Another possibility in Bayesian approach is to define the test as the test with the minimal mean error of the second kind. For a test $\bar{\psi}_n$, let us denote $\alpha\left(\bar{\psi}_n,\theta\right) = 1-\beta\left(\bar{\psi}_n,\theta\right)$ the error of the second kind and introduce the mean error of the second kind:

$$\alpha\left(\bar{\psi}_n\right) = \int_{\vartheta_1}^b \alpha\left(\bar{\psi}_n,\theta\right)\, p\left(\theta\right)\mathrm{d}\theta.$$

The Bayes test $\tilde{\psi}_n^\star(X^n)$ is defined as the test which minimizes this mean error:

$$\alpha\left(\tilde{\psi}_n^\star\right) = \inf_{\bar{\psi}_n \in \mathcal{K}_\varepsilon} \alpha\left(\bar{\psi}_n\right).$$

We can rewrite the above integral as follows

$$\begin{aligned}\int_{\vartheta_1}^b \left(1 - \mathbf{E}_\theta\, \bar{\psi}_n\left(X^n\right)\right) p\left(\theta\right) \mathrm{d}\theta &= \int_{\vartheta_1}^b \int \left(1 - \bar{\psi}_n\left(x^n\right)\right) \mathrm{d}\mathbf{P}_\theta\, p\left(\theta\right) \mathrm{d}\theta \\ &= \int \left(1 - \bar{\psi}_n\left(x^n\right)\right) \mathrm{d}\mathbb{P} = \mathbb{E}\left(1 - \bar{\psi}_n\left(X^n\right)\right),\end{aligned}$$

where we denoted $\mathbf{P}_\theta$ the distribution of the sample X^n and

$$\mathbb{P}\left(X^n \in A\right) = \int_{\vartheta_1}^b \mathbf{P}_\theta\left(X^n \in A\right) p\left(\theta\right) \mathrm{d}\theta.$$

The averaged power $\beta\left(\bar{\psi}_n\right) = \mathbb{E}\,\bar{\psi}_n\left(X^n\right)$ is the same as if we have two simple hypothesis. Under $\mathscr{H}_1$ we observe a Poisson process of intensity function $\lambda\left(\vartheta_1, \cdot\right)$, and under the alternative $\mathscr{H}_2$ the observed point process has random intensity and its measure is $\mathbb{P}$. This process is a mixture $\big($according to the density $p(\theta)\big)$ of inhomogeneous Poisson processes with intensities $\lambda\left(\theta, \cdot\right)$, $\theta \in \left(\vartheta_1, b\right)$. This means that we have two simple hypotheses and the most powerful (Neyman-Pearson) test is of the form

$$\tilde{\psi}_n^* = \mathbb{1}_{\left\{\tilde{L}(X^n) > \tilde{m}_\varepsilon\right\}}, \qquad \mathbf{E}_{\vartheta_1}\, \tilde{\psi}_n^*\left(X^n\right) = \varepsilon,$$

where the averaged likelihood ratio

$$\tilde{L}\left(X^n\right) = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbf{P}_{\vartheta_1}}\left(X^n\right) = \int_{\vartheta_1}^b \frac{\mathrm{d}\mathbf{P}_\theta}{\mathrm{d}\mathbf{P}_{\vartheta_1}}\left(X^n\right) p\left(\theta\right) \mathrm{d}\theta.$$

To study this test under hypothesis we change the variables:

$$\tilde{L}\left(X^n\right) = \int_{\vartheta_1}^b L\left(\theta, \vartheta_1, X^n\right) p\left(\theta\right) \mathrm{d}\theta = \varphi_n \int_0^{\varphi_n^{-1}(b-\vartheta_1)} Z_n\left(v\right) p\left(\vartheta_1 + v\varphi_n\right) \mathrm{d}v.$$

The limit of the last integral was already described above and this allow us to write

$$\begin{aligned}R_n\left(X^n\right) &= \frac{\tilde{L}\left(X^n\right)}{p\left(\vartheta_1\right)\varphi_n} = \frac{1}{p\left(\vartheta_1\right)} \int_0^{\varphi_n^{-1}(b-\vartheta_1)} e^{v\Delta_n - \frac{v^2}{2} + r_n}\, p\left(\vartheta_1 + v\varphi_n\right) \mathrm{d}v \\ &\Longrightarrow \int_0^\infty e^{v\Delta - \frac{v^2}{2}} \mathrm{d}v = e^{\frac{\Delta^2}{2}} \int_{-\Delta}^\infty e^{-\frac{y^2}{2}} \mathrm{d}y = \sqrt{2\pi}\, e^{\frac{\Delta^2}{2}} \left(1 - F\left(-\Delta\right)\right) = \frac{F\left(\Delta\right)}{f\left(\Delta\right)},\end{aligned}$$

where $F(\cdot)$ and $f(\cdot)$ are again the distribution function and the density of the standard Gaussian random variable Δ. Hence, if we take m_ε to be solution of the equation

$$\mathbf{P}\left\{\frac{F(\Delta)}{f(\Delta)} > m_\varepsilon\right\} = \varepsilon,$$

then the test $\tilde{\psi}_n^\star(X^n) = \mathbb{1}_{\{R_n > m_\varepsilon\}}$, which we call BT2, belongs to $\mathcal{K}_\varepsilon$ and coincides with the test $\tilde{\psi}_n^*(X^n)$ if we put $\tilde{m}_\varepsilon = m_\varepsilon\, p(\vartheta_1)\, \varphi_n$.

A similar calculation under the alternative $\vartheta_{u_*} = \vartheta + u_*\varphi_n$ allows us to evaluate the limit power function of the BT2 as follows:

$$\beta\left(\tilde{\psi}_n^\star, u_*\right) = \mathbf{P}_{\vartheta_{u_*}}\{R_n > m_\varepsilon\} \longrightarrow \mathbf{P}\left\{\frac{F(\Delta + u_*)}{f(\Delta + u_*)} > m_\varepsilon\right\}.$$

5 Simulations

Below we present the results of numerical simulations for the power functions of the tests. We observe n independent realizations $X_j = (X_j(t),\ t \in [0,3])$, $j = 1, ..., n$, of inhomogeneous Poisson process of intensity function

$$\lambda(\vartheta, t) = 3\cos^2(\vartheta t) + 1, \qquad 0 \le t \le 3, \qquad \vartheta \in [3, 7),$$

where $\vartheta_1 = 3$. The Fisher information at the point ϑ_1 is $\mathrm{I}(\vartheta_1) \approx 19.82$. Recall that all our tests (except Bayes tests) in regular case are LAUMP. Therefore they have the same limit power function. Our goal is to study the power functions of different tests for finite n.

The normalized likelihood ratio $Z_n(u)$ is given by the expression

$$Z_n(u) = \exp\Bigg\{\varphi_n \sum_{j=1}^n \int_0^3 \ln \frac{3\cos^2((3 + u\varphi_n)\, t) + 1}{3\cos^2(3t) + 1}\, \mathrm{d}X_j(t) - \frac{3n}{4(3 + u\varphi_n)} \sin(6(3 + u\varphi_n)) + \frac{n}{4}\sin(18)\Bigg\},$$

where $\varphi_n = (19.82\, n)^{-1/2}$.

The numerical simulation of the observations allows us to obtain the power functions presented in Figures 1 and 2. For example, the computation of the numerical values of the power function of the SFT was done as follows. We define an increasing sequence of u beginning at $u = 0$. Then, for every u, we simulate N i.i.d. observations of n-tuples of inhomogeneous Poisson processes $X^{n,i}$, $i = 1, ..., N$, with the intensity function $\lambda(3 + u\varphi_n, t)$

and calculate the corresponding statistics $\Delta_n(3, X_{n,i})$, $i = 1, ..., N$. The empirical frequency of acceptation of the alternative gives us an estimate of the power function:

$$\beta\left(\psi_n^\star, u\right) \approx \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\Delta_n(3, X_{n,i}) > z_\varepsilon\}}.$$

We repeat this procedure for different values of u until the values of $\beta\left(\psi_n^\star, u\right)$ become close to 1.

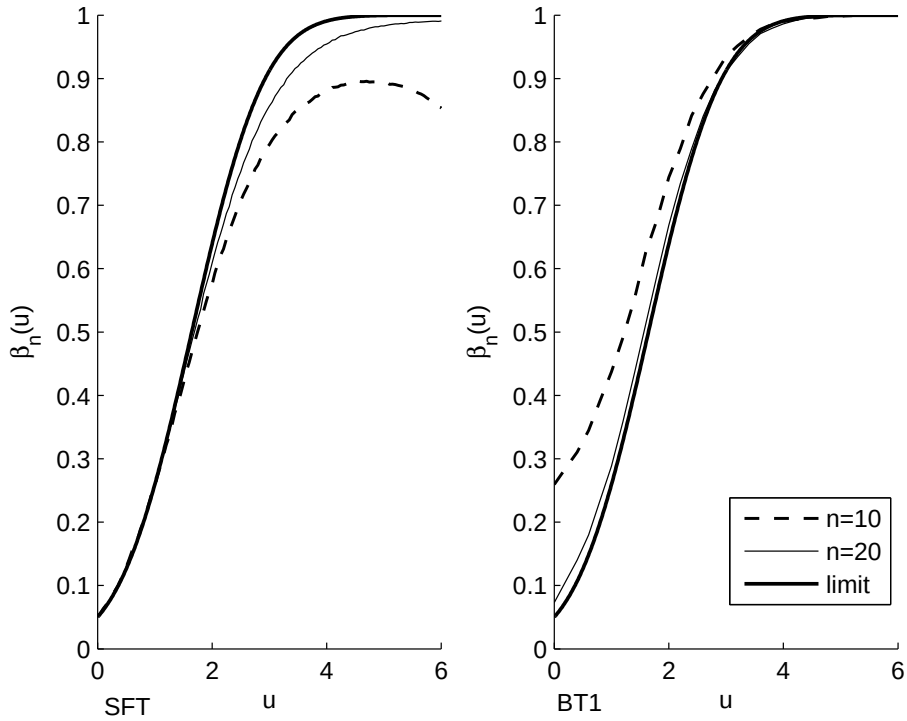

Figure 1: Power functions of SFT and BT1

In the computation of the power function of the Bayes test BT1, we take as *a priori* law the uniform distribution, that is, $\vartheta \sim \mathcal{U}([3, 7])$. The thresholds of the BT1 are obtained by simulating $M = 10^5$ random variables $\Delta_i \sim \mathcal{N}(0, 1)$, $i = 1, \ldots, M$, calculating for each of them the quantity

$$\frac{f\left(\Delta_i\right)}{F\left(\Delta_i\right)} + \Delta_i, \qquad i = 1, \ldots, M,$$

and taking the $(1 - \varepsilon)\, M$-th greatest between them. Some of the thresholds are presented in Table 1.

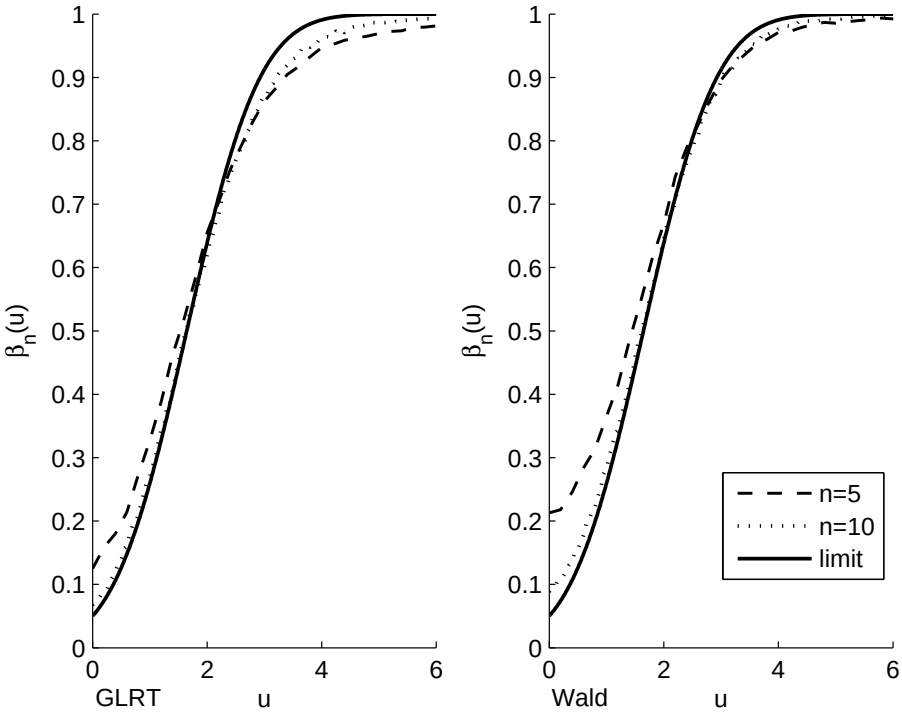

Figure 2: Power functions of GLRT and WT

Note that for the small values of n, under alternative, the power function of SFT starts to decrease (see Figure 2). This interesting fact can be explained by the strongly non linear dependence of the likelihood ratio on the

ε	0.01	0.05	0.10	0.2	0.4	0.5
k_ε	2.325	1.751	1.478	1.193	0.895	0.794

Table 1: Thresholds of BT1

parameter. The test statistic $\Delta_n = \Delta_n\left(3, X^n\right)$ can be rewritten as follows:

$$\begin{aligned}
\Delta_n &= \varphi_n \sum_{j=1}^{n} \int_0^T \frac{\dot{\lambda}\left(\vartheta_1, t\right)}{\lambda\left(\vartheta_1, t\right)} \left[\mathrm{d}X_j(t) - \lambda\left(\vartheta_1 + u\varphi_n, t\right) \mathrm{d}t\right] \\
&\quad + \sqrt{\frac{n}{\mathrm{I}\left(\vartheta_1\right)}} \int_0^T \frac{\dot{\lambda}\left(\vartheta_1, t\right)}{\lambda\left(\vartheta_1, t\right)} \left[\lambda\left(\vartheta_1 + u\varphi_n, t\right) - \lambda\left(\vartheta_1, t\right)\right] \mathrm{d}t \\
&= -3\varphi_n \sum_{j=1}^{n} \int_0^3 \frac{t\sin(6t)}{3\cos^2(3\,t) + 1} \left[\mathrm{d}X_j\left(t\right) - \left(3\cos^2((3 + u\varphi_n)\,t) + 1\right) \mathrm{d}t\right] \\
&\quad + 9\sqrt{\frac{n}{\mathrm{I}\left(\vartheta_1\right)}} \int_0^3 \frac{t\sin(6t)}{3\cos^2(3\,t) + 1} \times \left[\cos^2(3\,t) - \cos^2\left((3 + u\varphi_n\,)\,t\right)\right] \mathrm{d}t.
\end{aligned}$$

The last integral becomes negative for some values of u, which explains the loss of power of the SFT (for $n = 10$).

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References

[1] Brown, M. Discrimination of Poisson processes. *Ann. Math. Statist.*, 1971, 42, 773–776.

[2] Dachian, S., Kutoyants, Yu. A. On cusp estimation of ergodic diffusion process. *J. Statist. Plann. Inference*, 2003, 117, 153-199.

[3] Dachian S., Kutoyants Yu. A., Hypotheses testing : Poisson versus stress-release, *J. Statist. Plan. Inference,* 2009, 139, 1668-1684.

[4] Dachian S., Kutoyants Yu. A., Yang, L. On hypothesis testing for Poisson processes. Singular case. Submitted, 2014.

[5] Fazli K., Kutoyants Yu. A., Two simple hypotheses testing for Poisson process, *Far East J. Theor. Stat.*, 2005, 15, 2, 251-290.

[6] Ibragimov, I.A. and Hasminskii, R.Z., "*Statistical Estimation. Asymptotic Theory*", Springer, New York, 1981.

[7] Kutoyants Yu. A. Asymptotically optimal algorithm of detecting Poisson signals. *Radiotehnika i Elektronika*, 1977, 22, 3, 621-625.

[8] Kutoyants, Yu. A., *Statistical Inference for Spatial Poisson Processes*, Lect. Notes Statist. **134**, Springer-Verlag, New York, 1998.

[9] Kutoyants, Yu. A., *Introduction to Statistics of Poisson Processes*, to appear, 2014.

[10] Le Cam, L. *Asymptotic Methods in Statistical Decision Theory.* Springer, New York, 1986.

[11] Léger, C. and Wolfson, D. B. Hypothesis testing for a non-homogeneous Poisson process. *Comm. Statist. Theory Methods,* 1987, 3, 3, 439-455.

[12] Lehmann, E. and Romano, J., *Testing Statistical Hypotheses*, Springer, Heidelberg, 2005.

[13] Liese, F. Eine informationstheoretische Bedingung fűr die Äquivalenz unbegrenzt teilbarer Punktprozesse. *Math. Nachr.* 1975, 70, 183–196.

[14] Liese, F. and Lorz, U. *Contiguity and LAN-property of sequences of Poisson processes. Kybernetika*, 1999, 35, 3, 281-308.

[15] Roussas, G. G. *Contiguity of Probability Measures.* Cambridge University Press, 1972.

[16] Sung,Y., Tong, L., Swami, A. Asymptotic locally optimal detector for large-scale sensor networks under the Poisson regime. *IEEE Trans. Signal Proces.*, 2005, 53, 6.