# Orthogonal polynomials associated with the deltoid curve

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# ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE DELTOID CURVE

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Abstract. We study a family of bivariate orthogonal polynomials associated to the deltoid curve. These polynomials arise when classifying bivariate diffusion operators that have discrete spectral decomposition given by orthogonal polynomials with respect to some compactly-supported probability measure on the interior of the deltoid curve.

**Keywords :** orthogonal polynomials, diffusion processes, deltoid, root systems.

**MSC classification :** 47D07, 33C45, 33C50, 33C52, 60H99.

## 1. Introduction

Orthogonal polynomials in the interior of the deltoid curve is one example of the 11 families of orthogonal polynomials on a compact domain in dimension 2 which are at the same time eigenvectors of an elliptic diffusion operators, see [2]. It is also one of the most intriguing one, and have been put forward by Koornwinder [12, 13, 14], (see also [24], [16]). This family of bivariate polynomials depends on a parameter $\alpha$ : if $P(X, Y)$ is the algebraic equation of the boundary of $\Omega$, so that $P(X, Y) > 0$ on $\Omega$, then the measure has density $C_\alpha P(X, Y)^\alpha$ with respect to the Lebesgue measure. Here, $\alpha > -5/6$, as we shall see below (Proposition 4.5).

The two special cases $\alpha = 1/2$ and $\alpha = -1/2$ play a special role and have been particularly investigated, see [3, 7] and also [22, 23] for a spectral point of view. The first one is the image of the Euclidean Laplace operator through the symmetries of the triangular lattice, and the second one is the image of the Casimir operator on $SU(3)$ through the spectral decomposition. It is of course not a surprise, since the root system of $SU(3)$ is $A_2$, which corresponds to the triangular lattice (see [4, 5, 6, 15, 10, 21]). Those two cases are referred below as the geometric cases. The analysis of the geometric cases provides some insight on the general model. Therefore, although this aspect is quite well documented (see [3], [7]), we present them in detail from the point of view of symmetric diffusion operators for the sake of completeness. Moreover, these models will provide us efficient insights towards the general situation, since the study of the general family is more delicate. In the core of the paper, we derive recurrence formulae for the generic measures, which, as is the case in dimension 1, turn out to be a 3 term recurrence formula (although such a simple form is not to be expected

in general in dimension 2). Those recurrence formulae take a particularly simple form in the two geometric cases.

As mentioned above, the study of the two specific geometric cases lead to simple representations of the eigenvectors. In the Euclidean case ($\alpha = -1/2$), we have a very simple presentation of the eigenvectors. The $SU(3)$ case ($\alpha = 1/2$) leads to a representation of the eigenvectors through the representation of the symmetric group, associated with Young diagrams (see [11]). Finally, we derive in the general case partial generating functions, leading to another representation of the orthogonal polynomials, which also provides a complete generating function in the two geometric cases.

The paper is organized as follows : Section 2 is a short presentation of the general setting of symmetric diffusion processes associated with orthogonal polynomials, mostly inspired from [1] and [2]. In Section 3, we give the explicit formulae for the measure and the generator associated to the Deltoid model, and introduce the complex variables in which the operator takes a much simpler form, leading to the explicit values for the eigenvalues of the operator. Section 4 is the presentation of the Euclidean case (that is the case $\alpha = -1/2$), while Section 5 presents the $SU(3)$ case, that is $\alpha = 1/2$. Section 6 concentrates on recurrence formulae in the general case, and Section 7 provides some representation of the eigenvectors, first in the geometric cases and then (but only partially for the generating functions) in the general case.

## 2. Orthogonal polynomials and diffusion generators

Let $\Omega$ be an open bounded domain in $\mathbb{R}^d, d \geq 1$, with piecewise smooth boundary, and let $\mu$ a probability measure on $\overline{\Omega}$. Recall from p. 32 in [24] that a family of polynomials $P_\tau : \mathbb{R}^d \mapsto \mathbb{R}$ is orthogonal in $L^2(\Omega, \mu)$ if

$$\int P_\tau(x) P_{\tau'}(x) \mu(dx) = 0$$

where $\tau = (\tau_1, \cdots, \tau_d) \in \mathbb{N}^d$ is a multi-index, whenever $|\tau| := \tau_1 + \cdots + \tau_d \neq \tau'_1 + \ldots \tau'_d = |\tau'|$. In contrast to the real one variable setting, this family needs not to be unique in higher dimensions due to various orders one may choose when applying the Gram-Schmidt process to the canonical basis $(x_1^{\tau_1} \ldots x_d^{\tau_d})_{\tau \in \mathbb{N}^d}$ (see the bottom of p.31 in [24]). However, in many situations, there are natural choices for this family orthogonal polynomials. In particular, it may happen that they are also eigenvectors of some diffusion differential operator. This is the case for the classical family of orthogonal polynomials in dimension 1, Hermite, Laguerre and Jacobi (although only the last one corresponds to a bounded domain, see [19]).

On the other hand, when solving stochastic differential equations in probability theory, one is often led to consider second order differential operators on $\Omega$ which are symmetric with respect in $\mathcal{L}^2(\mu)$, at least when one restricts it's attention to the set $\mathcal{C}_c^\infty(\Omega)$ of smooth functions compactly supported in

$\Omega$. When $\mu$ has a smooth positive density on $\Omega$, these operators may be represented as

$$\mathrm{L}f := \frac{1}{\rho}\sum_{k,j=1}^{d} \partial_k \left(g_{kj}\rho\partial_j f\right) = \sum_{k,j=1}^{d} g_{kj}\partial^2_{kj} f + \sum_{j=1}^{d} b_j \partial_j f \tag{1}$$

where $g = (g_{kj}(x))_{k,j=1}^{d}, x \in \Omega$ is a symmetric non negative matrix depending smoothly on $x \in \Omega$ and

$$b_j = \frac{1}{\rho}\sum_{k=1}^{d} \partial_k(g_{kj}\rho), \quad j \in \{1,\dots,d\}.$$

The coefficients $b_j(x)$ are called the drift terms of the operator L.

We call such operators symmetric diffusion operators. They are related to Markov diffusion processes $(X_t)$ with values in $\Omega$ through the fact that for any smooth function $f$, the processes $f(X_t) - \int_0^t \mathrm{L}f(X_s)\,ds$ is a (local) martingale. When the operator L is essentially self-adjoint, then this entirely characterizes the law of the processus $(X_t)$ (at least as long as we only consider the finite dimensional marginals). The operator L is called the infinitesimal generator of the process $(X_t)$.

Working with such diffusion operators, it is often convenient to introduce the so called carré du champ operator

$$\Gamma(f,g) = \frac{1}{2}\big(\mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f\big),$$

and observe that L is entirely determined from the knowledge of $\Gamma$ and $\mu$ through the integration by parts formula

$$\int_\Omega f\mathrm{L}g\,d\mu = \int_\Omega g\mathrm{L}f\,d\mu = -\int_\Omega \Gamma(f,g)\,d\mu,$$

valid at least when $f$ and $g$ are smooth and compactly supported in $\Omega$. Moreover From the representation (1), it is immediate that $b_i(x) = \mathrm{L}(x_i)$ and $g_{ij} = \Gamma(x_i,x_j)$.

More generally, the change of variable formula, valid for any smooth $\Phi : \mathbb{R}^k \mapsto \mathbb{R}$, and any $k$-uple $f = (f_1,\cdots,f_k)$ of smooth functions, reads

$$\mathrm{L}(\Phi(f)) = \sum_i \partial_i\Phi(f)\mathrm{L}f_i + \sum_{ij}\partial^2_{ij}\Phi(f)\Gamma(f_i,f_j). \tag{2}$$

In particular, whenever for $i = 1,\cdots,k$, there exist functions $B_i$ and $G_{ij}$ such that $\mathrm{L}f_i = B_i(f)$ and $\Gamma(f_i,f_j) = G_{ij}(f)$, one has

$$\mathrm{L}\big(\Phi(f)\big) = (\mathrm{L}_1\Phi)(f) \tag{3}$$

where $\mathrm{L}_1$ is the new diffusion operator acting on the image of $\Omega$ under the function $f$ (which is not necessary a local diffeomorphism), as

$$\mathrm{L}_1(\Phi) = \sum_{ij} G_{ij}(x)\partial^2_{ij}\Phi + \sum_i B_i(x)\partial_i\Phi,$$

which is called the image of L under the function $f$.

In the probabilistic interpretation, if $(X_t)$ is the stochastic process with generator L, then $f(X_t)$ is again a diffusion Markov process with generator $\mathrm{L}_1$. In particular, the operator $\mathrm{L}_1$ is symmetric with respect to the image measure of $\rho$ under the map $f$, which, following equation (1) may be often a efficient way to determine the image measure. When such situation occurs, we shall say that L projects onto $\mathrm{L}_1$.

In what follows, we restrict for simplicity to the case where the matrix $g(x)$ is positive definite on $\Omega$. It is then natural to raise the question of determining when such L may be extended as a self adjoint operator (see [25]) with spectral decomposition given by a family of orthogonal polynomials with respect to $\mu$. In other words, one wants to determine for which choice of $\rho$ and $g$ there is a complete family of $\mu$-orthogonal polynomials which are at the same time eigenvectors for L. This will produce a natural choice for a basis of orthogonal polynomials.

It turns out that the general answer to this question is the following :

the functions $g_{ij}(x)$ are polynomials with degree at most two, and the boundary $\partial\Omega$ is included in the algebraic set $\{\det(g) = 0\}$. More precisely if $P(x)$ denotes the irreducible equation of the boundary $\partial\Omega$, then there exists a family of degree 1 polynomials $L_i(x)$ such that for any $i$, the algebraic equation

$$\sum_j g_{ij}\partial_j P = L_i P. \tag{4}$$

Moreover, the sets of admissible density measures $\rho$ are entirely described from the algebraic structure of the boundary. In particular, when the determinant $\det(g)$ is irreducible, the only admissible density measures $\rho$ are $C(\lambda)\det(g)^\lambda$, for any real $\lambda$ such that $\rho^\lambda$ is $\mathcal{L}^1(\Omega, dx)$, (see [2]). Once the boundary $\partial\Omega$ is given through it's irreducible equation, the coefficients $(g_{ij})(x)$ are entirely determined from equation (4). It turns out that they are in general unique up to some scaling factor.

## 3. The deltoid model

In dimension 2, up to affine transformations, there are only 11 bounded sets $\Omega$ on which there exist a symmetric diffusion operator for which the associated eigenvectors are orthogonal polynomials with respect to the reversible measure (see [2]) . One of the most intriguing one is the interior of the deltoid curve, which is a degree 4 algebraic curve with equation

$$P(x) = (x_1^2 + x_2^2)^2 + 18(x_1^2 + x_2^2) - 8x_1^3 + 24x_1x_2^2 - 27 = 0.$$

For this particular choice, the matrix $(g_{ij})(x)$ is unique up to some scaling factor, and is given by

$$\begin{cases} g_{11}(x_1,x_2) = -(3x_1^2 - x_2^2 - 6x_1 - 9) \\ g_{12}(x_1,x_2) = -2x_2(2x_1+3) \\ g_{22}(x_1,x_2) = -(3x_2^2 - x_1^2 + 6x_1 - 9) \end{cases} \tag{5}$$

Whence we deduce that $\det(g) = -3P(x)$. Moreover, in this representation, for the measure $\mu(dx) = c(\alpha)|P(x)|^\alpha dx$, the drift terms in the equation read

$$b_1(x_1,x_2) = -2(6\alpha+5)x_1, \quad b_2(x_1,x_2) = -2(6\alpha+5)x_2. \tag{6}$$

The general operator $\mathrm{L}^{(\alpha)}$ on the interior of Deltoid curve for which a family of orthogonal polynomial is formed of eigenvectors of $\mathrm{L}^{(\alpha)}$ is therefore given by

$$\begin{aligned} \mathrm{L}^{(\alpha)} = \quad & g_{11}(x_1,x_2)\partial_1^2 + g_{22}(x_1,x_2)\partial_2^2 + 2g_{12}(x_1,x_2)\partial_{1,2}^2 \\ & -2(6\alpha+5)x_1\partial_1 - 2(6\alpha+5)x_2\partial_2 \end{aligned}$$

with associated measure $c(\alpha)\rho^\alpha dx$, with

$$\rho = \frac{1}{3}\det(g) = -(x_1^2+x_2^2)^2 - 18(x_1^2+x_2^2) + 8x_1^3 - 24x_1x_2^2 + 27.$$

As long as we only deal with polynomials, it turns out that it is simpler to use complex variables. Indeed, let $Z = x_1 + ix_2$ and conjugate $\overline{Z} = x_1 - ix_2$, then the generator is entirely characterized by

$$\begin{cases} \Gamma(Z,Z) = -4Z^2 + 12\overline{Z}, \\ \Gamma(\overline{Z},Z) = -2Z\overline{Z} + 18, \\ \Gamma(\overline{Z},\overline{Z}) = -4\overline{Z}^2 + 12Z, \\ \mathrm{L}^{(\alpha)}Z = -2(6\alpha+5)Z, \mathrm{L}^{(\alpha)}\overline{Z} = -2(6\alpha+5)\overline{Z}. \end{cases} \tag{7}$$

We can simplify the operator by setting $Z = 3Z_1, \bar{Z} = 3\bar{Z}_1$ and multiplying $\mathrm{L}^{(\alpha)}$ by $1/4$, which does not change the eigenvectors and multiply the eigenvalues by $1/4$. This gives

$$\begin{cases} \Gamma(Z,Z) = \overline{Z} - Z^2, \\ \Gamma(\overline{Z},Z) = 1/2(1 - Z\overline{Z}), \\ \Gamma(\overline{Z},\overline{Z}) = Z - \overline{Z}^2, \\ \mathrm{L}^{(\alpha)}Z = -1/2(6\alpha+5)Z, \mathrm{L}^{(\alpha)}\overline{Z} = -1/2(6\alpha+5)\overline{Z}. \end{cases} \tag{8}$$

In particular, giving a particular role to the case $\alpha = -1/2$, one has

$$L^{(\alpha)} = L^{(-1/2)} - \frac{3}{2}(2\alpha+1)(Z\partial_Z + \overline{Z}\partial_{\overline{Z}}). \tag{9}$$

This model has been studied by [12, 13], where the relationship with homogeneous spaces of rank 2 and root system $A_2$ has been put forward. Observe that the case $\lambda = -1/2$ corresponds to the Laplace-Beltrami operator associated with the Riemannian metric $g^{-1}$ associated with the inverse matrix of the matrix $g$.

Our aim here is to study the associated orthogonal polynomials together with the associated eigenvalues, and various representations for it. Indeed, this family belongs to the larger class of Hall polynomials associated with root systems (here the root system $A_2$) (see [18]), and our aim here is to present some properties of these polynomials specific for this model.

## 4. $L^{(-1/2)}$ as a projection of the Euclidean Laplacian.

As already mentioned, the case $\alpha = -1/2$ corresponds to the Laplace-Beltrami operator associated to the inverse matrix $g^{-1}$. If one computes the associated curvature (here, in dimension 2, the scalar curvature is sufficient to characterize the metric), we may observe that it vanishes, and therefore it is not much surprising that the operator is the image, in the sense described in Section 2, of the ordinary Laplace operator in $\mathbb{R}^2$.

In order to perform this identification, we first start by some remarks on the Deltoid curve.

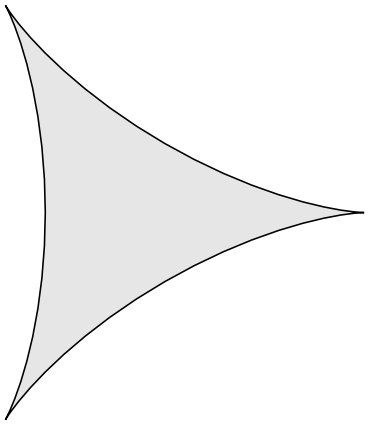


FIGURE 1. The deltoid domain.

It represented by the following parametric equations :

$$x_1(\theta) = 2\cos\theta + \cos 2\theta, \quad x_2(\theta) = 2\sin\theta - \sin 2\theta$$

In complex notations $z(\theta) = x_1(\theta) + ix_2(\theta) = 2e^{i\theta} + e^{-2i\theta}$.

We shall denote $\overline{\Omega}$ the interior of the deltoid curve, and our aim is to identify the above operator $\mathrm{L}^{(-1/2)}$ as the image of the laplace operator $\Delta$ on $\mathbb{R}^2$ through the action of some function $Z : \mathbb{R}^2 \mapsto \overline{\Omega}$ is the sense described above.

To proceed, we shall first use the change of variables formula to see that

$$\mathrm{L}^{(-1/2)}(f)(X,Y) = \Delta[f(X,Y)],$$

where $(X, Y) : \mathbb{R}^2 \mapsto \overline{\Omega}$, or in complex notations

$$\mathrm{L}^{(-1/2)}(f)(Z, \overline{Z}) = \Delta[f(Z, \overline{Z})]$$

where

$$Z = X + iY, \quad \overline{Z} = X - iY.$$

So that we are looking for some functions $Z : \mathbb{R}^2 \mapsto \overline{\Omega}$ satisfying :

$$\Gamma_\Delta(Z, Z) = \nabla Z \cdot \nabla Z = -Z^2 + \overline{Z}, \Gamma_\Delta(\overline{Z}, Z) = \nabla \overline{Z} \cdot \nabla Z = 1/2(1 - Z\overline{Z})$$

$$\Gamma_\Delta(\overline{Z}, \overline{Z}) = \nabla \overline{Z} \cdot \nabla \overline{Z} = -\overline{Z}^2 + Z,$$

$$\Delta Z = -Z, \Delta \overline{Z} = -\overline{Z}.$$

In what follows, for any $\beta \in \mathbb{R}^2 \approx \mathbb{C}$, $e_\beta(x)$ denotes the function $\mathbb{R}^2 \mapsto \mathbb{C}$, $x \mapsto e^{i(\beta \cdot x)}$

**Proposition 4.1.** *Let $\beta_1, \beta_2, \beta_3 \in \mathbb{R}^2$ three vectors satisfying*

$$\beta_k \cdot \beta_k = 1, \quad \beta_k \cdot \beta_l = -\frac{1}{2}, l \neq k.$$

*and let*

$$Z(x) = \sum_{k=1}^{3} e_{\beta_k},$$

*Then, one has*

$$\Delta(Z) = -Z, \ \Delta(\overline{Z}) = -\overline{Z},$$

*and*

$$\Gamma(Z, Z) = -Z^2 + 3\overline{Z}, \Gamma(Z, \overline{Z}) = \frac{1}{2}(9 - Z\overline{Z}), \Gamma(\overline{Z}, \overline{Z}) = -\overline{Z}^2 + 3Z.$$

*Proof.* — Splitting $e_{\beta_k}(x), k \in \{1, 2, 3\}$ into a real and imaginary parts, one derives

$$[\nabla e_{\beta_k} \cdot \nabla e_{\beta_l}] = -(\beta_k \cdot \beta_l) e_{\beta_k + \beta_l}$$

for all $l, k \in \{1, 2, 3\}$. As a result

$$\begin{aligned} \Gamma(Z, Z) &= -\sum_{k,l} (\beta_k \cdot \beta_l) e_{\beta_k + \beta_l} \\ &= -\left[ \sum_{k=1}^{3} e_{2\beta_k} - \sum_{k<l} e_{\beta_k + \beta_l} \right] = -\left[ Z^2 - 3 \sum_{k<l} e_{\beta_k + \beta_l} \right]. \end{aligned}$$

But one easily checks that

$$\sum_{k=1}^{3} \beta_k \cdot \sum_{k=1}^{3} \beta_k = 0$$

so that $\beta_1 + \beta_2 + \beta_3 = 0$ yielding $\Gamma(Z, Z) = -Z^2 + 3\overline{Z}$. Finally

$$\Gamma(Z, \overline{Z}) = 3 - \sum_{k \neq l}^{3} e_{\beta_k - \beta_l} = 3 - (\frac{Z\overline{Z} - 3}{2}).$$

The identification is obtained changing $(Z, \overline{Z})$ into $(Z/3, \overline{Z}/3)$. ∎

**Remark 1.** *The assumption $\beta_k \cdot \beta_l = -1/2, k \neq l$ is by no means a loss of generality. Indeed, if one rather assumes that $\beta_k \cdot \beta_l = c \in (-1, 1)$ then one may take $\beta_1 = 1$ due to the rotation invariance of our conditions. But then $\beta_1 \cdot \beta_2 = \beta_1 \cdot \beta_3$ forces both first coordinates of $\beta_2, \beta_3$ to be equal $c$, while the fact that the vectors have unit length entails*

$$\beta_2 = c + i\sqrt{1-c^2}, \quad \beta_3 = c - i\sqrt{1-c^2}.$$

*Together with $\beta_2 \cdot \beta_3 = c$ show that $c$ to be a root of $2c^2 - c - 1$ yielding finally $c = -1/2$. Hence $\beta_1 = 1, \beta_2 = j, \beta_3 = j^2$, the cubic roots of the unit, up to an orthogonal transformation.*

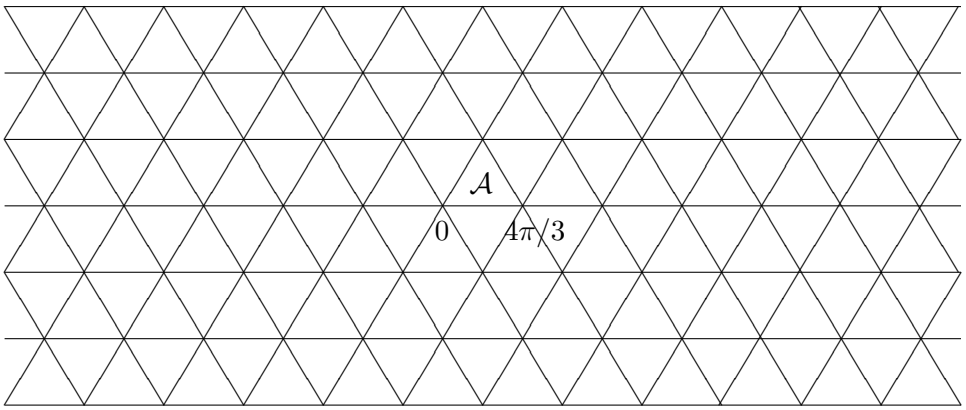
From now on, with no loss of generality, we shall assume that $(\beta_1, \beta_2, \beta_3) = (1, j, j^2)$.

One immediately sees that $Z$ is invariant under the action (by translation) of the lattice $\mathbb{L}$ generated by $4\pi\beta_1, 4\pi\beta_2$,

$$\mathbb{L} = 4\pi\mathbb{Z}\beta_1 + 4\pi\mathbb{Z}\beta_2,$$

by rotation with $2\pi/3$ angles and by symmetry with respect to the lines $\mathbb{R}\beta_k$. One may also observe that it is invariant under the symmetry with respect to the horizontal line $\{y = 2\pi/\sqrt{3}\}$. From this, one sees that $Z$ is also invariant under the symmetries with respect of the lines of the regular triangular lattice $\mathbb{L}_1$ whose fundamental domain is the regular triangle $\mathcal{A}$ whose vertices are $(0,0), (4\pi/3, 0), (4\pi/3)e^{i\pi/3}$ (see below). We shall say in the sequel that a function having those invariance have the symmetries of the lattice $\mathbb{L}_1$.

Equivalently, $Z$ is invariant under the action of the dihedral group $\mathcal{D}_3$ of affine type ([10]). As a matter of fact, $Z$ is uniquely determined by its restriction to $\mathcal{A}$.




The lattice of regular triangles $\mathbb{L}_1$.

**Proposition 4.2.** *$Z$ is a one-to-one map from $\partial\mathcal{A}$ onto $\mathcal{D} = \partial\Omega$ and from $\overline{\mathcal{A}}$ onto $\overline{\Omega}$. In particular, it maps the whole plane onto $\overline{\Omega}$.*

*Proof.* — Recall the parametric equation of $\mathcal{D}$ written in complex notations

$$z(\theta) = 2e^{i\theta} + e^{-2i\theta}$$

so that $z(\theta) = Z(-2\theta, 0)$, where $\theta$ runs over any interval of length $2\pi$. Then the invariance of $Z$ under rotations of angles $\pm 2\pi/3$ shows that the images of the intervals

$$[-4\pi/3, 0], \quad [4\pi/3, 8\pi/3]$$

coincides with the images of the oblique edges of $\mathcal{A}$ (the cusps of $\mathcal{D}$ are the images of $\{\theta = 0, 4\pi/3, 8\pi/3\}$). Thus $Z$ maps $\partial A$ onto $\mathcal{D}$ and it is easy to check from the complex parametrization of $\mathcal{D}$ that $Z$ is one-to-one there. Combined with the compactness of $\mathcal{A}$ and the continuity of $Z$, we deduce that $Z$ maps $\mathcal{A}$ into $\overline{\Omega}$. But then $Z(\mathcal{A}) = \overline{\Omega}$ since otherwise $\overline{\Omega}$ would not be simply connected, which leads to a contradiction. Now, we shall use the following parametrization of $\overline{\Omega}$:

$$Z(x_1, x_2) = e^{ix_1} + 2e^{-i\frac{x_1}{2}} \cos(\frac{\sqrt{3}}{2} x_2),\ x = (x_1, x_2) \in \mathcal{A}.$$

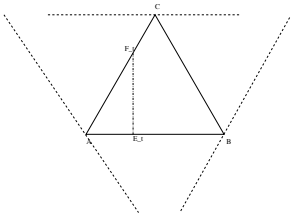
For fixed $x_1 \in [0, 2\pi/3]$, the image by $Z$ of the vertical segments

$$[(x_1, 0), (x_1, \sqrt{3}x_1)] \in \mathcal{A}$$

is the line segment $I(x_1) = [A(x_1), B(x_1)]$ where

$$\begin{aligned} A(x_1) &= (\cos(x_1) + 2\cos(\frac{x_1}{2}), \sin(x_1) - 2\sin(\frac{x_1}{2})) \\ B(x_1) &= (2\cos(x_1) + \cos(2x_1), 2\sin(x_1) - \sin(2x_1)). \end{aligned}$$

Thus, the coordinates of $A(x_1)$ are decreasing as functions of the variable $x_1$ while those of $B(x_1)$ are decreasing and increasing respectively. Equivalently, $A(x_1)$ runs over the half part of the lowest branch of $\mathcal{D}$ starting from $(3, 0)$ while $B(x_1)$ runs over the whole highest one since clearly $B(x_1) = A(-2x_1)$. As a matter of fact, two line segments $I(x_1), I(x_1'), 0 \leq x_1 \neq x_1' \leq 2\pi/3$ never intersect. A similar reasoning applies when $x_1 \in [2\pi/3, 4\pi/3]$ and the segment $[x_1, -\sqrt{3}x_1 + 4\pi/\sqrt{3}]$, $A(x_1)$ runs over the remaining half part of the lowest branch while $B(x_1)$ runs over the whole third one. As a matter of fact, $Z$ is a one-to-one from $\overline{\mathcal{A}}$ onto $\overline{\Omega}$. Finally, $\overline{\mathcal{A}}$ is a fundamental domain for the action of the affine group $\mathcal{D}_3$ on $\mathbb{R}^2$ so that every $x \in \mathbb{R}^2$ is conjugated to a unique element of $\mathcal{A}$. The proposition is proved. ∎




∎

We are now in situation to identify the operator $\mathrm{L}^{(-1/2)}$ as an image of the 2-dimensional Laplace operator acting on functions which are invariant under the symmetries of the lines in the triangular lattice.

**Proposition 4.3.** *A measurable function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ have the symmetries of the lattice $\mathbb{L}_1$ if and only if it may be written $f = g(Z)$, where $g : \Omega \mapsto \mathbb{R}$ is a measurable function.*

*Moreover, when $f \in \mathcal{C}^2$, then we may chose $g \in \mathcal{C}^2$, in which case*

$$\Delta(g(Z)) = \mathrm{L}^{(-1/2)}(g)(Z). \tag{10}$$

*In other words, $\mathrm{L}^{(-1/2)}$ is nothing else than the 2-dimensional Laplace operator acting on set of functions having the symmetries of the lattice $\mathbb{L}_1$.*

*Proof.* — If we denote by $Z^{-1}$ the inverse map $\Omega \mapsto A$ of the restriction of $Z$ to $A$, then we just set $g = f \circ Z^{-1}$.

Moreover, the change of variable formula (2) and formulae given in Proposition 4.1 give immediately (10).

∎

**Remark 4.4.** *It is worth to remark that if we set that $Z = z_1 + z_2 + z_3$, where $z_i$ are complex numbers such that $|z_i| = 1$ and $z_1 z_2 z_3 = 1$, then*

$$\rho = -3(z_1 - z_2)^2(z_2 - z_1)^2(z_3 - z_1)^2.$$

*Indeed, if $\rho(x, y)$ is the determinant of the matrix $(g^{ij})$ written in $(x_1, x_2)$ coordinates, one sees that, in $(Z, \bar{Z})$ coordinates, it may be written as*

$$\frac{1}{4}\Big(\Gamma(Z, \bar{Z})^2 - \Gamma(Z, Z)\Gamma(\bar{Z}, \bar{Z})\Big),$$

*which gives*

$$\rho = 12(Z^3 + \overline{Z}^3) - 3Z^2\overline{Z}^2 - 54Z\overline{Z} + 81 \tag{11}$$

Using remark 4.4 and the above diffeomorphism between the deltoid and the triangle, one obtains

**Proposition 4.5.** *The function $\det(g)^\alpha$ on the deltoid is integrable with respect to the Lebesgue measure if and only if $\alpha > -5/6$.*

In the sequel, we shall always set $\lambda = \frac{1}{2}(6\alpha + 5)$.

*Proof.* — By the change variables formula we have

$$\int_{\mathcal{D}} \det(g)^\alpha dx_1 dx_2 = \int_{\mathcal{A}} \det(g)^{\alpha+\frac{1}{2}} dx_1 dx_2$$
$$= (-3)^{\alpha+\frac{1}{2}} \int_{\mathcal{A}} ((z_1 - z_2)(z_2 - z_3)(z_3 - z_1))^{2\alpha+1} dx_1 dx_2$$

where $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}, z_3 = e^{i\theta 3}$ and $\theta_1 = x_1, \theta_2 = -\frac{x_1}{2} + \frac{\sqrt{3}x_2}{2}, \theta_3 = -(\theta_1 + \theta_2) = -\frac{x_1}{2} - \frac{\sqrt{3}x_2}{2}$

A rapid inspection of the integrability condition for this function on the triangle shows that, near the boundary and outside the corners of the triangle, the integrability condition is $\alpha > -1$, while at the corner of the triangles, the condition is more restrictive. Indeed, for the integrability of the measure near the point $(0,0)$, then $(z_1-z_2)(z_2-z_3)(z_3-z_1)) \simeq -i\frac{3}{4}\sqrt{3}x_2(x_2^2-3x_1^2)$ and if we set $x_2 = \sqrt{3}tx_1, t \in [0,1]$ we have $\det(g)^{\alpha+\frac{1}{2}} \simeq \big(-\frac{27}{4}t^2x_1^6(1-t^2)^2\big)^{\alpha+\frac{1}{2}}$, which is integrable for the measure $tdtdx_1$ if and only if $\alpha > -\frac{5}{6}$. ∎

## 5. The $L^{(1/2)}$ as a projection of the Casimir operator on $SU(3)$

Let $\mathcal{G}$ be a compact semi simple Lie linear group with Lie algebra $\mathcal{L}$ , seen at the tangent space at Id for $G$, with Lie bracket $[A,B]$ (see [9]). On $\mathcal{L}$, the Killing form is a scalar product defined by $\langle A,B\rangle = -\text{trace}\,(A.B)$ . On the other hand, to any $A \in \mathcal{L}$ is associated a vector field $X_A$ on $\mathcal{G}$ defined as $X_A(f)(g) = \partial_t\ |_{t=0}\ f(ge^{tA})$. Given an orthonormal basis $(A_1,\cdots,A_d)$ in $\mathcal{L}$ with respect to the Killing form, the Casimir operator is defined as $\Delta_G = \sum_i X_{A_i}^2$. It is also the Laplace-Beltrami operator on $\mathcal{G}$ when $\mathcal{G}$ inherits the Riemannian structure from the Killing form in $\mathcal{L}$. $\Delta_{\mathcal{G}}$ is a second order differential operator in the sense that it satisfies the change of variable formula (2). We shall denote by $\Gamma_{\mathcal{G}}$ the corresponding carré du champ operator.

The Casimir operator commutes with the Lie group action. More precisely, if, for $g \in \mathcal{G}$ and for any function $f : \mathcal{G} \mapsto \mathbb{R}$, one defines the right action $R_g(f)(k) = f(kg)$, then $\Delta_{\mathcal{G}}R_g = R_g\Delta_{\mathcal{G}}$, and the same holds true for the left action $L_g(f)(k) = f(gk)$.

In order to entirely determine the action of $\Delta_{\mathcal{G}}$ on functions of $\mathcal{G}$, it is enough to compute $\Delta_{\mathcal{G}}(f_i)$ and $\Gamma_{\mathcal{G}}(f_i,f_j)$ for a set of functions which generates all functions on $\mathcal{G}$ (say as $\sigma$-algebras). Once again, it could be helpful to consider complex valued functions, and on $SU(n)$, if one represents $g$ as a matrix $(z_{ij})$ with complex entries, we shall consider the coordinates $g \mapsto z_{ij}$ and $g \mapsto \overline{z}_{ij}$ as generating functions.

When performing the above computations in $SU(n)$, one ends up with the following formulae

$$\Delta_{SU(n)}(z_{kl}) = -2\frac{(n-1)(n+1)}{n}z_{kl},\ \Delta_{SU(n)}(\bar{z}_{kl}) = -2\frac{(n-1)(n+1)}{n}\bar{z}_{kl},$$
$$\Gamma_{SU(n)}(z_{ij},z_{kl}) = -2z_{il}z_{kj} + \frac{2}{n}z_{ij}z_{kl},\ \Gamma_{SU(n)}(\overline{z}_{ij},\overline{z}_{kl}) = -2\overline{z}_{il}\overline{z}_{kj} + \frac{2}{n}\overline{z}_{ij}\overline{z}_{kl},$$
$$\Gamma_{SU(n)}(z_{ij},\overline{z}_{kl}) = 2(\delta_{ik}\delta_{jl} - \frac{1}{n}z_{ij}\overline{z}_{kl}).$$

For any $p \in \mathbb{Z}$, consider the functions $SU(n) \mapsto \mathbb{C}$: $T_p(g) = \text{trace}\,(g^p)$. For $p \geq 1$, one has

$$T_p(g) = \sum_{i_1,\cdots,i_p=1}^{n} z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_p i_1}, \tag{12}$$

while the same formula holds for $p \leq -1$ replacing $z_{ij}$ by $\overline{z}_{ij}$ (and of course $T_{-p} = \overline{T}_p$). From the change of variable formula, one has, for any $m$-uple of functions $(f_1, \cdots, f_m)$ and any diffusion generator L

$$\mathrm{L}(f_1 \cdots f_m) = f_1 \cdots f_m \Big( \sum_{i=1}^{m} \frac{\mathrm{L} f_i}{f_i} + \sum_{i,j=1}^{m} \frac{\Gamma(f_i, f_j)}{f_i f_j} - \sum_{i=1}^{m} \frac{\Gamma(f_i, f_i)}{f_i^2} \Big), \tag{13}$$

and, for any $m$-ulple $(f_1, \cdots, f_m)$ and any $k$-uple $(g_1, \cdots, g_k)$

$$\Gamma(f_1 \cdots f_m, g_1 \cdots g_k) = f_1 \cdots f_m g_1 \cdots g_k \Big( \sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\Gamma(f_i, g_j)}{f_i g_j} \Big).$$

Applying these to the explicit expression (12) of $T_p$, one gets, for $p \geq 1$

$$\Delta_{SU(n)} T_p = -p \Big( 2(\frac{n^2 - p}{n}) T_p + \sum_{i=1}^{p-1} T_i T_{p-i} \Big), \tag{14}$$

with the conjugate formula for $p \leq -1$, while, for any $p, q \in \mathbb{Z}$

$$\Gamma_{SU(n)}(T_p, T_q) = 2|pq| \big( \frac{T_p T_q}{n} - T_{p+q} \big). \tag{15}$$

In particular, if we set $Z = T_1$, $\overline{Z} = T_{-1}$. Then

$$\Delta_{SU(n)} Z = -2 \frac{n^2 - 1}{n} Z, \Delta_{SU(n)} \overline{Z} = -2 \frac{n^2 - 1}{n} \overline{Z}, \tag{16}$$

and

$$\Gamma_{SU(n)}(Z, Z) = 2(\tfrac{Z^2}{n} - T_2),\ \Gamma_{SU(n)}(\overline{Z}, \overline{Z}) = 2(\tfrac{\overline{Z}^2}{n} - \overline{T}_2), \tag{17}$$

$$\Gamma_{SU(n)}(Z, \overline{Z})) = 2(3 - \tfrac{Z\overline{Z}}{n}). \tag{18}$$

Now, consider more precisely the case $n = 3$. For any matrix in $SU(3)$, if $(\mu_1, \mu_2, \mu_3)$ denote its eigenvalues, $T_p = \mu_1^p + \mu_2^p + \mu_3^p$. The $\mu_i$ are complex numbers with $|\mu_i| = 1$ and $\mu_1 \mu_2 \mu_3 = 1$. With $Z = \mu_1 + \mu_2 + \mu_3$, they are solution of the equation

$$X^3 - ZX^2 + \overline{Z}X - 1 = 0,$$

and multiplying this by $X^p$ and summing over the three values $\mu_1, \mu_2, \mu_3$, one gets for any $p \in \mathbb{Z}$

$$T_{p+3} - Z T_{p+2} + \overline{Z} T_{p+1} - T_p = 0, \tag{19}$$

which, for $p=-1$ gives $T_2 = Z^2 - 2\overline{Z}$, and similarly $\overline{T}_2 = \overline{Z}^2 - 2Z$. Replacing this values in (16) and (17) lead to

$$\Delta_{SU(3)} Z = -\frac{16}{3} Z,\ \Delta_{SU(3)}(\overline{Z}) = -\frac{16}{3}\overline{Z}$$

and

$$\Gamma_{SU(3)}(Z,Z) = \frac{4}{3}(3\overline{Z} - Z^2), \Gamma_{SU(3)}(\overline{Z},\overline{Z}) = \frac{4}{3}(3Z - \overline{Z}^2),$$
$$\Gamma_{SU(3)}(Z,\overline{Z}) = \frac{2}{3}(Z\overline{Z} - 9).$$

It remains to replace $Z$ by $Z/3$ to observe that $\frac{3}{4}\Delta_{SU(3)}$, acting on functions of $(Z,\overline{Z})$ is nothing else than $\mathrm{L}^{(1/2)}$ and $\lambda = 4$. Observe also that the functions on $SU(3)$ which depend only on $(Z,\overline{Z})$ are exactly those functions which depend only on the spectrum of the matrix $g \in SU(3)$, that is the functions which are invariant under $g \mapsto h^{-1}gh$, for any $h \in SU(3)$. Indeed, as long as polynomials are concerned, those functions are exactly functions depending only on the traces $T_p, p \in \mathbb{Z}$, and formula (24) shows that these functions are again polynomials in the variables $(Z,\overline{Z})$.

## 6. Eigenvalues and eigenvectors

We proceed now to the determination of the eigenvalues of $\mathrm{L}^{(\alpha)}$, and give a recurrence formula for the corresponding eigenvectors.

In dimension 1, it is well known (and easy to check) that for any probability measure for which the polynomials are dense in $\mathcal{L}^2(\mu)$, the unique (up to the sign) associated sequence of orthogonal polynomials satisfies a 3 term recurrence formula (see [20]), usually written under the form

$$xP_n = a_n P_{n+1} + b_n P_n + a_{n-1} P_{n-1}.$$

It is not the case in dimension 2, since one then would get in general a recurrence formula involving at each step $n$ an increasing number of terms. Indeed, if, for each degree $n$, one denotes by $\mathcal{P}_n$ the space of polynomials of total degree less than $n$ and by $\mathcal{H}_n$ the space of polynomials in $\mathcal{P}_n$ orthogonal to $\mathcal{P}_{n-1}$, one has for any polynomial orthogonal polynomial $P \in \mathcal{H}_n$

$$x_1 P = Q_{n+1} + Q_n + Q_{n-1},\ x_2 P = R_{n+1} + R_n + R_{n-1},$$

where $Q_i$ and $R_i$ belong to $\mathcal{H}_i$. But the spaces $\mathcal{H}_i$ have dimension $i+1$, and one should in general not expect any simple recurrence formula.

However, looking more precisely at the form of the operators $\mathrm{L}^{(\alpha)}$ is the variables $(Z,\overline{Z})$, one should expect for the sequence of eigenvectors of $\mathrm{L}^{(\alpha)}$ a 6 term recurrence formula. It comes as a surprise that indeed one is able to get a 3 term recurrence formula as in dimension 1.

We first start by investigating the eigenvalues. Recall first that we are looking for polynomials $P^{(\alpha)}_{p,q}$ such that

$$\mathrm{L}^{(\alpha)}(P^{(\alpha)}_{p,q}) = -\lambda_{p,q} P^{(\alpha)}_{p,q}$$

where $(p,q) \in \mathbb{N}^2$ is a bi-index whose weight $p+q$ is the degree of $P^{(\alpha)}_{p,q}$.

**Proposition 6.1.** *The eigenvalues of* $\mathrm{L}^{(\alpha)}$ *are*

$$\lambda_{p,q} = (\lambda-1)(p+q) + p^2 + q^2 + pq,$$

*where* $\lambda = \frac{1}{2}(6\alpha+5)$.

*Proof.* — The complex representation easily leads to the eigenvalues. Indeed, if $\mathcal{P}_n$ denotes the space of polynomials (now in the variables $(Z,\overline{Z})$) with total degree at most $n$, one may write any $P \in \mathcal{P}_n$ as

$$P = \sum_{p=0}^{n} a_{p,q} Z^p \overline{Z}^{n-p} + Q = P_n + Q,$$

where $Q \in \mathcal{P}_{n-1}$.

Now, looking at the action of $\mathrm{L}^{(\alpha)}$ on the highest degree term $P_n$ of $P$, one sees that if $\mathrm{L}^{(\alpha)}P = -\mu P$, then the highest degree term $\hat{P}_n$ of $\mathrm{L}^{(\alpha)}P_n$ is equal to $-\mu\hat{P}_n$. It remains to observe the action of $\mathrm{L}^{(\alpha)}$ on those highest terms. Fortunately, in coordinates $(Z,\overline{Z})$, this action is diagonal (which is not the case in coordinates $(x_1,x_2)$).

Indeed, the change of variable formula (2) gives

$$\begin{aligned}\mathrm{L}^{(\alpha)}(Z^p\overline{Z}^q) = \quad & pZ^{p-1}\overline{Z}^q \mathrm{L}^{(\alpha)}Z \\ & + qZ^{q-1}Z^p\mathrm{L}^{(\alpha)}\overline{Z} + p(p-1)Z^{p-2}Z^q\Gamma(Z,Z) \\ & + 2pqZ^{p-1}\overline{Z}^{q-1}\Gamma(Z,\overline{Z}) + q(q-1)\overline{Z}^{q-2}Z^p\Gamma(\overline{Z},\overline{Z}),\end{aligned}$$

whose highest term is

$$-\lambda_{p,q} Z^p \overline{Z}^q,$$

with $\lambda_{p,q} = (\lambda-1)(p+q) + p^2 + q^2 + pq$, where $\lambda = \frac{1}{2}(6\alpha+5)$.

∎

**Remark 6.2.** *When* $\alpha \notin \mathbb{Q}$, *then the eigenspaces associated to the eigenvalues* $\lambda_{p,q}$ *are at most two-dimensional (and exactly two dimensional when* $p \neq q$*). Indeed, writing* $\sigma = p+q$ *and* $\pi = pq$, *with similar notation* $\pi', \sigma'$ *for* $(p',q')$, *we have* $\lambda_{p,q} = (\lambda-1)\sigma + \sigma^2 - \pi$, *and therefore if* $\lambda_{p,q} = \lambda_{p',q'}$, *then either* $\sigma = \sigma'$ *and then* $\pi = \pi'$, *either* $\lambda = 1 - \sigma - \sigma' + \dfrac{\pi - \pi'}{\sigma - \sigma'}$, *whence* $\lambda \in \mathbb{Q}$.

We shall see moreover that for any $(p,q)$ there exists exactly one polynomial $P^{(\alpha)}_{p,q}(Z,\overline{Z}) = Z^p\overline{Z}^q +$ lower degree terms which is an eigenvector of $\mathrm{L}^{(\alpha)}$. Indeed,

**Theorem 6.3.** *Define the family of polynomials* $P^{(\alpha)}_{p,q}(Z,\overline{Z})$ *by induction from*

$$P^{(\alpha)}_{0,0} = 1 \ , P^{(\alpha)}_{0,1} = \overline{Z} \ , P^{(\alpha)}_{1,0} = Z,$$

*and*

$$\begin{cases} P^{(\alpha)}_{p+1,q} &= ZP^{(\alpha)}_{p,q} + a_1(\lambda,p)P^{(\alpha)}_{p-1,q+1} + a_2(\lambda,p,q)P^{(\alpha)}_{p,q-1}, \\ P^{(\alpha)}_{p,q+1} &= \overline{Z}P^{(\alpha)}_{p,q} + a_1(\lambda,q)P^{(\alpha)}_{p+1,q-1} + a_2(\lambda,q,p)P^{(\alpha)}_{p-1,q}, \end{cases} \tag{20}$$

*where*

$$\begin{cases} a_1(\lambda,p) &= -\dfrac{p(3p+2\lambda-5)}{(\lambda+3p-1)(\lambda+3p-4)} \\ a_2(\lambda,p,q) &= -\dfrac{N_{p,q}}{D_{p,q}} \end{cases}$$

*where*

$$\begin{cases} N_{p,q} &= q(3q+2\lambda-5)(\lambda+3(p+q)-1)(\lambda+p+q-2) \\ D_{p,q} &= (\lambda+3q-1)(2\lambda+3(p+q)-5)(2\lambda+3(p+q)-2)(\lambda+3q-4) \end{cases}$$

$$\lambda = \frac{1}{2}(6\alpha+5) > 0 \tag{21}$$

**Remark 6.4.** *The only possible values of $\lambda$ for which the denominators vanishes in the above formulae are $\lambda = 1$ and $\lambda = 4$, which correspond to the $(p,q) \in \{(0,0),(1,0),(0,1)\}$. In those situations, we have to replace $a_1(\lambda,p)$ and $a_2(\lambda,p,q)$ by :*

$$\begin{cases} a_1(\lambda,p) &= \lim_{\epsilon\to 0} a_1(\lambda+\epsilon,p) \\ a_2(\lambda,p,q) &= \lim_{\epsilon\to 0} a_2(\lambda+\epsilon,p,q) \end{cases}$$

*Moreover, $a_1(1,p) = a_1(4,p) = -1/3$ and $a_2(1,p,q) = a_2(4,p,q) = -1/9$ for every $(p,q)$ except for those values of $(p,q) \in \{(0,0),(1,0),(0,1)\}$ . We have indeed*

$$\begin{aligned} a_1(1,1) &= -\frac{2}{3}, a_2(1,0,1) = -\frac{1}{3}, \\ a_1(4,1) &= -\frac{1}{3}, a_2(4,0,1) = -\frac{1}{9}, \\ a_1(\lambda,0) &= a_2(\lambda,p,0) = 0, \end{aligned}$$

*In the case $\alpha = 1/2$ the recurrence formulae simplify and for every $p,q \geq 0$ is :*

$$P^{(1/2)}_{p+1,q} = ZP^{(1/2)}_{p,q} - \frac{1}{3}P^{(1/2)}_{p-1,q+1} - \frac{1}{9}P^{(1/2)}_{p,q-1}, \tag{22}$$

*But in the other case $\alpha = -1/2$, the recurrence formulae is the same except for the values $(p,q) = \{(1,0),(0,1)\}$ corresponding to the polynomials $P^{(-1/2)}_{1,1} = Z\overline{Z} - \frac{1}{3}$, and $P^{(-1/2)}_{2,0} = Z^2 - \frac{2}{3}\overline{Z}$.*

*Therefore, for $\alpha = -1/2, 1/2$, the recurrence formulae for the polynomials are the same, except for the first two coefficients.*

*Then, for the operator $\mathrm{L}^{(\alpha)}$ determined from* (8)*, we have*

$$L^{(\alpha)} P^{(\alpha)}_{p,q} = -\lambda_{p,q} P^{(\alpha)}_{p,q}$$

*where*

$$\lambda_{p,q} = (\lambda - 1)(p+q) + p^2 + q^2 + pq.$$

It is worth to observe that since $a_1(\lambda, 0) = a_2(\lambda, p, 0) = 0$, formula (20) make sense for $p = 0$ and $q = 0$, and defines completely the family $P^{(\alpha)}_{p,q}$ for any $(p,q) \in \mathbb{N}^2$. One observes that $P^{(\alpha)}_{p,q}(Z, \overline{Z}) = Z^p \overline{Z}^q +$lower degree terms, and have real coefficients. It is also easily checked that $P^{(\alpha)}_{q,p} = \overline{P^{(\alpha)}_{p,q}}$.

The proof of Theorem 6.3 is rather tedious. We start with a Lemma:

**Lemma 6.5.** *For the same family of polynomials defined in* (20) *and the $\Gamma$ operator defined in* (8)*, we have*

$$\Gamma(Z, P^{(\alpha)}_{p,q}) = \alpha_0(p,q) P^{(\alpha)}_{p+1,q} + \alpha_1(p,q) P^{(\alpha)}_{p-1,q+1} + \alpha_2(p,q) P^{(\alpha)}_{p,q-1}$$

$$\Gamma(\overline{Z}, P^{(\alpha)}_{p,q}) = \alpha_0(q,p) P^{(\alpha)}_{p,q+1} + \alpha_1(q,p) P^{(\alpha)}_{p+1,q-1} + \alpha_2(q,p) P^{(\alpha)}_{p-1,q}$$

*where*

$$\begin{aligned}
\alpha_0(p,q) &= -\frac{1}{2}(q+2p), \\
\alpha_1(p,q) &= \frac{1}{2}\frac{p(2\lambda+3q-5)(\lambda+p-q-1)}{(\lambda+3p-1)(\lambda+3p-4)} \\
\alpha_2(p,q) &= \frac{1}{2}\frac{N1_{p,q}}{D1_{p,q}}
\end{aligned}$$

*where*

$$\begin{cases} N1_{p,q} = q(3q+2\lambda-5)(\lambda+3(p+q)-1)(\lambda+p+q-2)(2\lambda+p+2q-2). \\ D1_{p,q} = (\lambda+3q-1)(2\lambda+3(p+q)-5)(2\lambda+3(p+q)-2)(\lambda+3q-4). \end{cases}$$

It is worth to observe that although the definition of $\Gamma$ does not involve the parameter $\alpha$ (or equivalently the parameter $\lambda$), the recurrence formula defining $P^{(\alpha)}_{p,q}$ does, and this Lemma is valid whatever the parameter $\alpha$ is. However, it is not clear from the proof below for which family of recurrence formulae on $P^{(\alpha)}_{p,q}$ three terms recurrence formulae for $\Gamma(Z, P^{(\alpha)}_{p,q})$ and $\Gamma(\overline{Z}, P^{(\alpha)}_{p,q})$ are still valid.

*Proof.* In what follows, we remove the parameter $\alpha$ from the formulae, since it shall not change up to end of this Section. Lemma 6.5 proved by induction, from

$$\begin{aligned}\Gamma(Z,P_{p+1,q}) &= \Gamma(Z,ZP_{p,q})+a_1(\lambda,p)\Gamma(Z,P_{p-1,q+1})\\ &\quad +a_2(\lambda,p,q)\Gamma(Z,P_{p,q-1})\\ &= \Gamma(Z)P_{p,q}+Z\Gamma(Z,P_{p,q})+a_1(\lambda,p)\Gamma(Z,P_{p-1,q+1})\\ &\quad +a_2(\lambda,p,q)\Gamma(Z,P_{p,q-1})\end{aligned}$$

and finally

$$\begin{aligned}\Gamma(Z,P_{p+1,q}) &= (\overline{Z}-Z^2)P_{p,q}+Z\Gamma(Z,P_{p,q})\\ &\quad +a_1(\lambda,p)\Gamma(Z,P_{p-1,q+1})+a_2(\lambda,p,q)\Gamma(Z,P_{p,q-1})\end{aligned}$$

And by the definition (6.3) we have

$$\begin{aligned}\overline{Z}P_{p,q} &= P_{p,q+1}-a_1(\lambda,q)P_{p+1,q-1}-a_2(\lambda,q,p)P_{p-1,q}\\ ZP_{p,q} &= P_{p+1,q}-a_1(\lambda,p)P_{p-1,q+1}-a_2(\lambda,p,q)P_{p,q-1}\end{aligned}$$

So that

$$(23)\quad Z^2P_{p,q}=Z(ZP_{p,q})=ZP_{p+1,q}-a_1(\lambda,p)ZP_{p-1,q+1}-a_2(\lambda,p,q)ZP_{p,q-1}$$

Furthermore,

$$\begin{cases}ZP_{p+1,q}=P_{p+2,q}-a_1(\lambda,p+1)P_{p,q+1}-a_2(\lambda,p+1,q)P_{p+1,q-1}\\ ZP_{p-1,q+1}=P_{p,q+1}-a_1(\lambda,p-1)P_{p-2,q+2}-a_2(\lambda,p-1,q+1)P_{p-1,q}\\ ZP_{p,q-1}=P_{p+1,q-1}-a_1(\lambda,p)P_{p-1,q}-a_2(\lambda,p,q-1)P_{p,q-2}\end{cases}$$

which gives

$$\begin{aligned}Z^2P_{p,q} &= P_{p+2,q}-\big(a_1(\lambda,p+1)+a_1(\lambda,p)\big)P_{p,q+1}\\ &\quad -\big(a_2(\lambda,p+1,q)+a_2(\lambda,p,q)\big)P_{p+1,q-1}\\ &\quad +a_1(\lambda,p)\big(a_2(\lambda,p-1,q+1)+a_2(\lambda,p,q)\big)P_{p-1,q}\\ &\quad +a_2(\lambda,p,q)a_2(\lambda,p,q-1)P_{p,q-2}\\ &\quad +a_1(\lambda,p)a_1(\lambda,p-1)P_{p-2,q+2}.\end{aligned}$$

On the other hand, from the induction hypothesis we have

$$
\begin{aligned}
&Z\Gamma(Z,P_{p,q})+a_1(\lambda,p)\Gamma(Z,P_{p-1,q+1})+a_2(\lambda,p,q)\Gamma(Z,P_{p,q-1})=\\
&\alpha_0(p,q)P_{p+2,q}+\\
&\big[\alpha_1(p,q)-\alpha_0(p,q)a_1(\lambda,p+1)+\\
&a_1(\lambda,p)\alpha_0(p-1,q+1)\big]P_{p,q+1}+\\
&\big[\alpha_2(p,q)-\alpha_0(p,q)a_2(\lambda,p+1,q)+\\
&a_2(\lambda,p,q)\alpha_0(p,q-1)\big]P_{p+1,q-1}+\\
&\big[a_1(\lambda,p)\alpha_2(p-1,q+1)+\alpha_1(\lambda,p)a_2(\lambda,p,q)-\\
&\alpha_1(p,q)a_2(\lambda,p-1,q+1)-\alpha_2(p,q)a_1(\lambda,p)\big]P_{p-1,q}+\\
&[a_1(\lambda,p)\alpha_1(p-1,q+1)-a_1(\lambda,p-1)\alpha_1(p,q)]P_{p-2,q+2}+\\
&\big[a_2(\lambda,p,q)\alpha_2(p,q-1)-a_2(\lambda,p,q-1)\alpha_2(p,q)\big]P_{p,q-2},
\end{aligned}
$$

Substituting everything in (23), we get

$$
\begin{aligned}
\Gamma(Z,P_{p+1,q})=\quad&(\alpha_0(p,q)-1)P_{p+2,q}+A_1(p,q)P_{p,q+1}+\\
&A_2(p,q)P_{p+1,q-1}+A_3(p,q)P_{p-1,q}+\\
&A_4(p,q)P_{p-2,q+2}+A_5(p,q)P_{p,q-2},
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(p,q)=\quad&1+\alpha_1(p,q)-\alpha_0(p,q)a_1(\lambda,p+1)+\\
&a_1(\lambda,p)\alpha_0(p-1,q+1)\\
&+a_1(\lambda,p+1)+a_1(\lambda,p)\\
A_2(p,q)=\quad&-a_1(\lambda,q)+a_2(\lambda,p+1,q)+a_2(\lambda,p,q)+\alpha_2(p,q)\\
&-\alpha_0(p,q)a_2(\lambda,p+1,q)+a_2(\lambda,p,q)\alpha_0(p,q-1)\\
A_3(p,q)=\quad&-a_2(\lambda,q,p)-a_1(\lambda,p)a_2(\lambda,p-1,q+1)-a_2(\lambda,p,q)a_1(\lambda,p)\\
&+a_1(\lambda,p)\alpha_2(p-1,q+1)+\alpha_1(p,q-1)a_2(\lambda,p,q)\\
&-\alpha_1(p,q)a_2(\lambda,p-1,q+1)-\alpha_2(p,q)a_1(\lambda,p)\\
A_4(p,q)=\quad&a_1(\lambda,p)\alpha_1(p-1,q+1)-a_1(\lambda,p-1)\alpha_1(p,q)\\
&-a_1(\lambda,p)a_1(\lambda,p-1)\\
A_5(p,q)=\quad&a_2(\lambda,p,q)\alpha_2(p,q-1)-a_2(\lambda,p,q-1)\alpha_2(p,q)\\
&-a_2(\lambda,p,q)a_2(\lambda,p,q-1).
\end{aligned}
$$

A simple calculation shows that

$$
\begin{aligned}
&1+\alpha_1(p,q)=\alpha_1(p+1,q),\\
&A_1(p,q)=\alpha_1(p+1,q),\ \ A_2(p,q)=\alpha_2(p+1,q)\\
&A_3(p,q)=A_4(p,q)=A_5(p,q)=0,
\end{aligned}
$$

which concludes the induction formula for $\Gamma(Z,P_{p+1,q})$. The same method leads to the formula for $\Gamma(\overline{Z},P_{p+1,q})$, and exchanging $p$ and $q$ in the previous amounts to exchange $Z$ and $\overline{Z}$. □

Now , we prove Theorem 6.3 using Lemma 6.5.

*Proof.* Assume by induction that $L^{(\alpha)}P_{p_1,q_1} = -\lambda_{p_1,q_1}P_{p_1,q_1}$ when $p_1+q_1 \leq p+q$, where $\lambda_{p,q} = (\lambda-1)(p+q)+p^2+q^2+pq$. As before, we simply write the change of variable formula

$$\begin{aligned}
\mathrm{L}^{(\alpha)}P_{p+1,q} &= \mathrm{L}^{(\alpha)}(ZP_{p,q}) + a_1(\lambda,p)\mathrm{L}^{(\alpha)}P_{p-1,q+1} + a_2(\lambda,p,q)\mathrm{L}^{(\alpha)}P_{p,q-1} \\
&= Z\mathrm{L}^{(\alpha)}P_{p,q} + P_{p,q}\mathrm{L}^{(\alpha)}Z + 2\Gamma(Z,P_{p,q}) \\
&\quad -a_1(\lambda,p)\lambda_{p-1,q+1}P_{p-1,q+1} - a_2(\lambda,p,q)\lambda_{p,q-1}P_{p,q-1} \\
&= -(\lambda_{p,q}+\lambda)ZP_{p,q} + 2\alpha_0(p,q)P_{p+1,q} + (2\alpha_1(p,q) \\
&\quad -a_1(\lambda,p))P_{p-1,q+1} + (2\alpha_2(p,q) - a_2(\lambda,p,q)\lambda_{p,q-1})P_{p,q-1}.
\end{aligned}$$

But

$$ZP_{p,q} = P_{p+1,q} - a_1(\lambda,p)P_{p-1,q+1} - a_2(\lambda,p,q)P_{p,q-1},$$

so that

$$\mathrm{L}^{(\alpha)}P_{p+1,q} = B_1(p,q)P_{p+1,q} + B_2(p,q)P_{p-1,q+1} + B_3(p,q)P_{p,q-1},$$

where

$$\begin{aligned}
B_1(p,q) &= -(\lambda_{p,q}+\lambda) + 2\alpha_0(p,q) \\
B_2(p,q) &= 2\alpha_1(p,q) + (\lambda_{p,q} - \lambda_{p-1,q+1} + \lambda)a_1(\lambda,p) \\
B_3(p,q) &= 2\alpha_2(p,q) + (\lambda_{p,q} - \lambda_{p,q-1} + \lambda)a_2(\lambda,p,q)
\end{aligned}$$

Everything boils down to the following formulae, which are straightforward to check

$$B_1(p,q) = -\lambda_{p+1,q}, B_2(p,q) = B_3(p,q) = 0.$$

The same proof applies for $\mathrm{L}^{(\alpha)}P_{p,q+1} = -\lambda_{p,q+1}P_{p,q+1}$. The conclusion follows. □

**Remark 6.6.** *From the recurrence formula, it is easily checked that*

$$\begin{aligned}
P_{p,q} = \quad & Z^p\overline{Z}^q + A_{p,q}Z^{p+1}\overline{Z}^{q-2} + B_{p,q}Z^{p-2}\overline{Z}^{q+1} \\
& + C_{p,q}Z^{p-1}\overline{Z}^{q-1} + D_{p,q}Z^{p-4}\overline{Z}^{q+2} + F_{p,q}Z^{p+2}\overline{Z}^{q-4} + R,
\end{aligned}$$

*where* $\mathrm{degree}(R) \leq p+q-3$. *This general form may be easily induced from the form of the operator, and should produce a six term recurrence formula. The fact that the recurrence formula contains only 3 terms (as it is in dimension 1) is indeed quite mysterious.*

## 7. Other representations of eigenpolynomials

In this section, we come back to the two different representations for $\mathrm{L}^{(-1/2)}$ and $\mathrm{L}^{(1/2)}$ which provide new representations for the eigenvectors $P_{p,q}^{(\alpha)}$ in those specific cases. This new representations will allow us to get in those cases linearization formulae for the product together with generating functions.

7.1. **Case** $\alpha = -1/2$**.** Although the $\alpha = -1/2$ case is quite easy, since it comes from an Euclidean Laplace operator, it gives rise to another family of recurrence formulae. On the other hand, the case $\alpha = 1/2$, which comes from the Casimir operator on $SU(3)$, leads to new representations of the eigenvectors $P^{(\alpha)}_{p,q}$ related to the irreducible representations of the symmetric group. In fine, comparing the two cases allows to generalize the $SU(3)$ formulae to the general situation.

With the representation (8) of the operator $\mathrm{L}^{(-1/2)}$, one may represent the function $Z$ as a function $\mathbb{R}^2 \mapsto \mathbb{C}$ as $Z(x_1, x_2) = \frac{1}{3}(e_1 + e_j + e_{\bar{j}})$, where, for $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ $e_\beta = \exp\big(i(\beta_1 x_1 + \beta_2 x_2)\big)$, and $1, j, \bar{j}$ are the third root of unity, that is $1 = (1, 0)$, $j = (-1/2, \sqrt{3}/2)$, $\bar{j} = (-1/2, -\sqrt{3}/2)$. Comparing with the description given in Section 4, the change of normalization comes from the fact that we have divided L by 4 and replaced $Z$ by $Z/3$.

We have already observed that, for any triple $(z_1, z_2, z_3)$ of complex numbers satisfying $|z_i| = 1$, $z_1 z_2 z_3 = 1$, setting for any $p \in \mathbb{Z}$, $T_p = z_1^p + z_2^p + z_3^p$, one has (24)

$$T_{p+2} - 3ZT_{p+1} + 3\overline{Z}T_p - T_{p-1} = 0, \tag{24}$$

with $T_1 = 3Z$ and $T_{-1} = 3\overline{Z}$, $T_0 = 3$. One may observe first that this formula is unchanged if we replace $p$ by $-p$ and $Z$ by $\overline{Z}$. Setting $T_p = 3^{|p|}Q_p$, one gets

$$Q_{p+1} = ZQ_p - \frac{1}{3}\overline{Z}Q_{p-1} + \frac{1}{3^3}Q_{p-2}. \tag{25}$$

From this, it is clear that $Q_p$ is a polynomial with degree less than $p$ in the variables $(Z, \overline{Z})$, of the form $Q_p = Z^p +$ lower degree term. Now, if we replace $z_1, z_2, z_3$ by $e_1, e_j, e_{\bar{j}}$ we see that $Q_p$ is an eigenvector for the Laplace operator $\Delta$ in $\mathbb{R}^2$, with eigenvalue $p^2$. Therefore,

$$\forall p \geq 0,\ Q_p = P^{(-1/2)}_{p,0}, Q_{-p} = \overline{Q_p} = P^{(-1/2)}_{0,p}. \tag{26}$$

Comparing (26) with the recurrence formulae for $P^{(\alpha)}_{p,q}$, the first line in formula (20) gives in this case ($\lambda = 1$)

$$P^{(-1/2)}_{p+1,0} = ZP^{(-1/2)}_{p,0} - \frac{1}{3}P^{(-1/2)}_{p-1,1},$$

which leads to

$$P^{(-1/2)}_{p-1,1} = \overline{Z}P^{(-1/2)}_{p-1,0} - \frac{1}{3^2}P^{(-1/2)}_{p-2,0},$$

and the latter is nothing else than the second line in (20).

On the other hand, coming back to the representation $T_p = z_1^p + z_2^p + z_3^p$, one sees that, for any $(p, q) \in \mathbb{Z}^2$,

$$T_pT_q - T_{p+q} = \sum_{i \neq j} z_i^p z_j^q = \sum_{i \neq j} z_i^{p-q} z_j^{-q},$$

from which we get, for any $(p,q) \in \mathbb{Z}^2$

$$T_pT_q - T_{p+q} = T_{p-q}T_{-q} - T_{p-2q} = T_{q-p}T_{-p} - T_{q-2p}. \tag{27}$$

When $(z_1, z_2, z_3) = (e_1, e_j, e_{\bar{j}})$, this writes as sums of terms of the form $e_{\beta(p,q)}$, where $|\beta_{p,q}|^2 = p^2 + q^2 - pq$. Therefore, for the $\mathrm{L}^{(-1/2)}$ operator, writing $T_pT_{-q} - T_{p-q}$ as a polynomial in $(Z, \overline{Z})$, we see that this is an eigenvector associated with the eigenvalue $p^2 + q^2 + pq$. Looking at the highest degree term, and translating this in terms of the polynomials $Q_p = 3^{-|p|}T_p$, we obtain

$$\forall p, q \geq 1,\ P^{(-1/2)}_{p,q} = Q_pQ_{-q} - 3^{-2\min(p,q)}Q_{p-q}, \tag{28}$$

which gives a representation of $P^{(-1/2)}_{(p,q)}$ in terms of the polynomials $P^{(-1/2)}_{p,0}$ and $P^{(-1/2)}_{0,q}$ which is not easy to obtain directly from the recurrence formula (20).

When $p, q \geq 0, Q_pQ_q - Q_{p+q}$ is also an eigenvector for the Laplace operator associated with the eigenvalue $p^2 + q^2 - pq$. Indeed, using (27) which is valid for any $(p,q) \in \mathbb{Z}^2$, and comparing with (20), we end up, for $p \geq q \geq 0$, with

$$P^{(-1/2)}_{p-q,q} = Q_pQ_q - Q_{p+q},$$

**Proposition 7.1** (Linearization formula)**.**

$$\begin{aligned} P^{(-\frac{1}{2})}_{p,q}P^{(-\frac{1}{2})}_{p',q'} = \quad & P^{(-\frac{1}{2})}_{p+p',q+q'} + \\ & 3^{-\min(q,q')}P^{(-\frac{1}{2})}_{p+p'+\min(q,q'),\max(q,q')-\min(q,q')} \\ & +3^{-\min(p,p')}P^{(-\frac{1}{2})}_{\max(p,p')-\min(p,p'),q+q'+\min(p,p')} \\ & +b_1(p,q,p',q')P^{(-\frac{1}{2}}_{|\gamma|-\max(0,\delta),|\delta|+\min(0,\gamma)} \\ & +b_2(p,q,p',q')P^{(-\frac{1}{2})}_{|\beta|+min(0,\alpha),|\alpha|+min(0,\beta)} \\ & +b_3(p,q,p',q')P^{(-\frac{1}{2})}_{|\beta'|+min(0,\alpha'),|\alpha'|+min(0,\beta')} \end{aligned}$$

*Where*

$$\begin{aligned}
\gamma &= \max(p'-q, p-q') \\
\delta &= \min(p'-q, p-q') \\
\alpha &= p+q-q' \\
\beta &= p'+q'-p \\
\alpha' &= p'+q'-q \\
\beta' &= p+q-p' \\
b_1(p,q,p',q') &= 3^{|\gamma|+|\delta|+\min(0,\gamma)-\max(0,\delta)-(p+q+p'+q')} \\
b_2(p,q,p',q') &= 3^{|\beta|+|\alpha|+min(0,\alpha)+min(0,\beta)-(p+p'+q+q')} \\
b_3(p,q,p',q') &= 3^{|\beta'|+|\alpha'|+min(0,\alpha')+min(0,\beta')-(p+p'+q+q')}
\end{aligned}$$

7.2. **Case $\alpha = 1/2$.** We now turn to the inspection of the family $P^{(1/2)}_{p,q}$. We know that $\frac{4}{3}\mathrm{L}^{(1/2)}$ may be represented as the action of the Casimir operator on $SU(3)$ acting on spectral functions. Comparing with formulae (14) and (15), one sees that, for $p \geq 1$,

$$\mathrm{L}^{(1/2)}(T_p) = -\frac{p}{4}\Big(2(9-p)T_p + 3\sum_{i=1}^{p-1} T_i T_{p-i}\Big), \tag{29}$$

while, for $p, q \geq 0$

$$\Gamma(T_p, T_q) = \frac{pq}{2}\big(T_p T_q - 3T_{p+q}\big), \tag{30}$$

with similar formulae for $p \leq 0, q \leq 0$.

If we remember that $\mathrm{L}^{(1/2)} = \mathrm{L}^{(-1/2)} - 3(Z\partial_Z + \overline{Z}\partial_{\overline{Z}})$, we end up with the formula for $p \geq 0$

$$\big(Z\partial_Z + \overline{Z}\partial_{\overline{Z}}\big)T_p = -\frac{p}{2}(p-3)T_p + \frac{p}{4}\sum_{i=1}^{p-1} T_i T_{p-i}. \tag{31}$$

With the help of formula (9), for $p \geq 1$ we end up with

$$\mathrm{L}^{(\alpha)}T_p = -\frac{p}{4}\big(p(1-6\alpha) + 9(2\alpha+1)\big)T_p - \frac{3p}{8}(2\alpha+1)\sum_{i=1}^{p-1} T_i T_{p-i}. \tag{32}$$

Fix now some integer $n$ and denote by $\Pi_n$ the set of sequences $\pi = (p_1, \cdots, p_k)$ of integers $p_1 \geq p_k \geq 1$ such that $p_1 + \cdots + p_k = n$. For $\pi \in \Pi_n$, denote $T_\pi = \prod_{j=1}^k T_j$. Comparing with formula (30), and the general formula (13), we see that the vector space generated by $T_\pi, \pi \in \Pi_n$, is stable under $\mathrm{L}^{(\alpha)}$. We therefore will be able to diagonalize $\mathrm{L}^{(\alpha)}$ in this vector space.

We first perform a slight change in the normalization of the variables $T_p$, setting $T_p = cS_p$, in order to reduce formulae (32) and (30) to

$$\mathrm{L}^{(\alpha)} S_p = -\mu_{p,\alpha} S_p - \frac{1}{c}\frac{3p}{4}\sum_{i=1}^{p-1} S_i S_{p-i}, \tag{33}$$

and

$$\Gamma(S_p, S_q) = \frac{pq}{2}(S_p S_q - \frac{3}{c} S_{p+q}). \tag{34}$$

With $\mu_{p,\alpha} = \frac{p}{4}\big(p(1-6\alpha) + 9(2\alpha+1)\big)$ and $c = \sqrt{\frac{2}{2\alpha+1}}$,

Following [17], it is easier to introduce the group $\mathcal{S}_n$ of order $n$ permutations. For any $\sigma \in \mathcal{S}_n$, one consider it's cycle decomposition $\sigma = \sigma_1 \cdots \sigma_k$ (ordered in increasing lengths) and denote by $\pi = \pi(\sigma) = (p_1, \cdots, p_k) \in \Pi_n$ the sequence of the lengths of $\sigma_j$. We then denote $S_\sigma = S_{p_1} \cdots S_{p_k}$. It is worth to observe that if $\tau = (ij)$ is a transposition, the cycle decomposition of $\sigma\tau$ splits one cycle in two subcycles when $i$ and $j$ belong to the same cycle and glues together two cycles when $i$ and $j$ belong to two different cycles.

Therefore, if $\mathcal{T}_n$ denotes the set of all transpositions, for a permutation $\sigma$ with $\pi(\sigma) = (p_1, \cdots, p_k)$, through an easy combinatorial argument, one gets

$$\sum_{\tau \in \mathcal{T}_n} S_{\tau\sigma} = \frac{1}{2}\sum_{i=1}^{k} p_i \frac{S_\sigma}{S_{p_i}} \sum_{j=1}^{p_i - 1} S_j S_{p_i - j} + \sum_{i,j=1}^{k} p_i p_j \frac{S_\sigma}{S_{p_i} S_{p_j}} S_{p_i + p_j}.$$

Comparing this and the formula (13), we get

$$\mathrm{L}^{(\alpha)}(S_\sigma) = -\mu_{\sigma,\alpha} S_\sigma - \frac{3}{2c}\sum_{\tau \in \mathcal{T}_n} S_{\sigma\tau}, \tag{35}$$

More precisely

$$\begin{aligned}
\mathrm{L}^{(\alpha)} S_\sigma &= S_\sigma\Big( -\sum_{i=1}^{k} \mu_{p_i,\alpha} + \sum_{i \neq j} \frac{p_i p_j}{2} \\
&\quad -\frac{3}{2c}\big(\sum_{i=1}^{k} \frac{p_i}{2S_{p_i}} \sum_{j=1}^{p_i - 1} S_j S_{p_i - j} + \sum_{i \neq j} p_i p_j \frac{S_{p_i + p_j}}{S_{pi} S_{p_j}}\big)\Big) \\
&= S_\sigma\Big( -\mu_{\sigma,\alpha} - \frac{3}{2c}\big(\sum_{i=1}^{k} \frac{p_i}{2S_{p_i}} \sum_{j=1}^{p_i - 1} S_j S_{p_i - j} + \sum_{i \neq j} p_i p_j \frac{S_{p_i + p_j}}{S_{pi} S_{p_j}}\big)\Big)
\end{aligned}$$

where, for $\pi(\sigma) = (p_1, \cdots, p_k)$,

$$\begin{aligned}
\mu_{\sigma,\alpha} &= \sum_{i=1}^{k} \mu_{p_i,\alpha} - \sum_{i \neq j} \frac{p_i p_j}{2} \\
&= \frac{3}{4}(1-2\alpha)\sum_{i=1}^{k} p_i^2 + \frac{9}{4}(2\alpha+1)n - \frac{n^2}{2}
\end{aligned}$$

Finally

$$\mathrm{L}^{(\alpha)} S_\sigma \quad = \quad -\mu_{\alpha,\sigma} S_\sigma - \frac{3}{2c} \sum_{\tau \in \mathcal{T}_n} S_{\tau\sigma}$$

where $\mathcal{T}_n$ is the set of transpositions in $\mathcal{S}_n$.

It is worth to observe that for $\alpha = 1/2$ (and only in this case), $\mu_{\sigma,\alpha}$ depends on $n$ only, and therefore finding eigenvectors for $\mathrm{L}^{(1/2)}$ amounts to find eigenvectors for the linear operator $S_\sigma \mapsto \sum_{\tau \in \mathcal{T}_n} S_{\sigma\tau}$. But the latter corresponds to the operator $\sum_{\tau \in \mathcal{T}_n} \tau$ in the group algebra of the group $\mathcal{S}_n$, which commutes to every group element. It is therefore diagonal on any irreducible representation. Turning back to our setting, we conclude that for any character $\chi$ of the group $\mathcal{S}_n$, $\sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) S_\sigma$ is an eigenvector for $\mathrm{L}^{(1/2)}$. The characters of the group $\mathcal{S}_n$ are well known and correspond to Young diagrams, which are indeed elements of the set $\Pi_n$ described above. Unfortunately, as we shall see in the following examples, this representation is far from being one to one, and many eigenvectors coming from the $\mathcal{S}_n$ representation has degree less than $n$ in $(Z, \bar{Z})$. The correspondence between the degree and the shape of the Young diagram remains quite mysterious.

The paper [8] describes an elegant method which provides a simple combinatorial way for computing the character table in any symmetric group $S_n$. Since for any character $\chi$, the value of $\chi(\sigma)$ only depends on the conjugacy class of $\sigma$, that is on the Young diagram it belongs to, one has to compute $\chi(\xi)$ for any pair $(\chi, \xi)$ of Young diagrams. This may be achieved through the analysis of the so-called border strips. In what follows, we then give some examples of eigenvectors for $\mathrm{L}^{(1/2)}$ provided by this description, that is $\sum_{\pi \in \Pi_n} |\pi| \xi(\pi) T_\pi$, where $|\pi|$ denotes the size of the conjugacy class $\pi \in \mathcal{S}_n$, that is the number of those elements $\sigma \in \mathcal{S}_n$ such that $\pi(\sigma) = \pi$. For $\pi = (p_1, \cdots, p_k)$, one has $|\pi| = \dfrac{n!}{\prod_{j=1}^n k_j!} \prod_1^k \dfrac{1}{p_j}$, where $k_j$ is the number of cycles with length $j$ in $\pi$.

As an example, we show below the eigenvectors given by this construction for $n = 2, 3, 4$

The group $\mathcal{S}_2$ has two conjugacy classes $\chi_1, \chi_2$, corresponding to the partitions $(2, 0)$ and $(1, 1)$,

$$\chi_1 = \square\square, \ \chi_2 = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}.$$

The character table is then

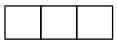
| | $\chi_1$ | $\chi_2$ |
|---|---|---|
| $\chi_1$ | 1 | 1 |
| $\chi_2$ | 1 | -1 |

From his we get the following eigenvectors for $\mathrm{L}^{(1/2)}$:

$$\begin{aligned} Q_1(Z,\overline{Z}) &= \chi_1(\chi_1)T_2 + \chi_1(\chi_2)T_1^2 \\ &= T_2 + T_1^2 \\ &= 18Z^2 - 6\overline{Z} \end{aligned}$$

$$\begin{aligned} Q_2(Z,\overline{Z}) &= \chi_2(\chi_1)T_2 + \chi_2(\chi_2)T_1^2 \\ &= T_1^2 - T_2 \\ &= 6\overline{Z} \end{aligned}$$

For $\mathcal{S}_3$, we have three conjugacy classes $\chi_1, \chi_2, \chi_3$ corresponding to the partitions $(3,0,0), (2,1,0), (1,1,1)$

$\chi_1 =$  , $\chi_2 =$  , $\chi_3 =$  ,

with character table

| | $\chi_1$ | $\chi_2$ | $\chi_3$ |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 2 | 0 | -1 |
| $\chi_3$ | 1 | -1 | 1 |

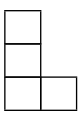
and corresponding eigenvectors

$$\begin{aligned} Q_1(Z,\overline{Z}) &= \chi_1(\chi_1)T_1^3 + \chi_1(\chi_2)T_2T_1 + \chi_1(\chi_3)T_3 \\ &= T_1^3 + T_2T_1 + T_3 \\ &= 3(27Z^3 - 15Z\overline{Z} + 1) \end{aligned}$$

$$\begin{aligned} Q_2(Z,\overline{Z}) &= \chi_2(\chi_1)T_3 + \chi_2(\chi_2)T_1T_2 + \chi_2(\chi_3)T_1^2 \\ &= 2T_3 - 2T_1^3 \\ &= 2(27Z\overline{Z} - 3) \end{aligned}$$

$$\begin{aligned} Q_3(Z,\overline{Z}) &= \chi_3(\chi_1)T_3 + \chi_3(\chi_2)T_1T_2 + \chi_3(\chi_3)T_1^2 \\ &= T_1^3 - 3T_2T_1 + 2T_3 \\ &= 6 \end{aligned}$$

For $\mathcal{S}_4$, we have five conjugacy classes, with Young diagrams $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$, corresponding to $(4,0,0,0), (2,1,1,0), (2,2,0,0), (3,1,0,0), (1,1,1,1)$.

$\chi_1 =$ , $\chi_2 =$  , $\chi_3 =$ , $\chi_4 =$ , $\chi_5 =$  ,

and character table

| | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | -1 | 1 | 1 | -1 |
| $\chi_3$ | 2 | 0 | 2 | -1 | 0 |
| $\chi_4$ | 3 | 1 | -1 | 0 | -1 |
| $\chi_5$ | 3 | -1 | -1 | 0 | 1 |

The corresponding eigenvectors are then:

$$\begin{aligned} Q_1(Z,\overline{Z}) &= \chi_1(\chi_1)T_1^4 + 6\chi_1(\chi_2)T_1^2T_2 + 3\chi_1(\chi_3)T_2^2 + 8\chi_1(\chi_4)T_1T_3 \\ &\quad +6\chi_1(\chi_5)T_4 \\ &= T_1^4 + 6T_1^2T_2 + 3T_2^2 + 8T_1T_3 + 6T_4 \\ &= 72(27Z^4 - 27\overline{Z}Z^2 + 3\overline{Z}^2 + 2Z) \end{aligned}$$

$$\begin{aligned} Q_2(Z,\overline{Z}) &= \chi_2(\chi_1)T_1^4 + 6\chi_2(\chi_2)T_1^2T_2 + 3\chi_2(\chi_3)T_2^2 + 8\chi_2(\chi_4)T_1T_3 \\ &\quad +6\chi_2(\chi_5)T_4 \\ &= T_1^4 - 6T_1^2T_2 + 3T_2^2 + 8T_1T_3 - 6T_4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} Q_3(Z,\overline{Z}) &= \chi_3(\chi_1)T_1^4 + 6\chi_3(\chi_2)T_1^2T_2 + 3\chi_3(\chi_3)T_2^2 + 8\chi_3(\chi_4)T_1T_3 \\ &\quad +6\chi_3(\chi_5)T_4 \\ &= 2T_1^4 + 2.3T_2^2 - 8T_1T_3 \\ &= 72(3\overline{Z}^2 - Z) \end{aligned}$$

$$\begin{aligned} Q_4(Z,\overline{Z}) &= \chi_4(\chi_1)T_1^4 + 6\chi_4(\chi_2)T_1^2T_2 + 3\chi_4(\chi_3)T_2^2 + 8\chi_4(\chi_4)T_1T_3 \\ &\quad +6\chi^4(\chi_5)T_4 \\ &= 3T_1^4 + 6T_1^2T_2 - 3T_2^2 - 6T_4 \\ &= 72(9\overline{Z}Z^2 - 3\overline{Z}^2 - Z) \end{aligned}$$

$$\begin{aligned} Q_5(Z,\overline{Z}) &= \chi_5(\chi_1)T_1^4 + 6\chi_5(\chi_2)T_1^2T_2 + 3\chi_5(\chi_3)T_2^2 + 8\chi_5(\chi_4)T_1T_3 \\ &\quad +6\chi_5(\chi_5)T_4 \\ &= 3T_1^4 - 6T_1^2T_2 - 3T_2^2 + 6T_4 \\ &= 72Z \end{aligned}$$

## 8. Generating functions

In this section, we provide first a partial generating function in the general case for the family $P_{0,n}^{(\alpha)}$ or equivalently for $P_{n,0}^{(\alpha)}$ , which leads to some simple representation of the polynomials $P_{n,m}^{(\alpha)}$ as linear combinations of $P_{p,0}^{(\alpha)}P_{0,q}^{(\alpha)}$. then we turn to the two geometric cases in which a complete generating function may be provided.

8.1. **Partial generating functions in the general case.** In this section, we propose an alternative representation of the eigenvectors in the general case, together with a partial generating function. We start with the following

**Proposition 8.1.** *Let*

$$P(X) = 1 - 3\overline{Z}X + 3ZX^2 - X^3, \tag{36}$$

*Then, still with* $\lambda = \frac{1}{2}(6\alpha + 5)$,

$$\begin{aligned} \mathrm{L}^{(\alpha)}(P(X)) &= -\lambda X P' + \frac{\lambda}{2} X^2 P'', \\ \Gamma(P(X), P(Y)) &= \frac{XY}{2}\big(P'(X)P'(Y) + 3\frac{P'(X)P(Y) - P(X)P'(Y)}{X - Y}\big), \end{aligned}$$

*from which*

$$\Gamma(P(X), P(X)) = \frac{X^2}{2}(3PP'' - 2P'^2),$$

*Also, with* $\bar{P}(Y) = 1 - 3ZY + 3\overline{Z}Y^2 - Y^3 = -Y^3 P(1/Y)$

$$\Gamma(P(X), \bar{P}(Y)) = \frac{XY}{2(XY-1)}\Big(3XP'\bar{P} + 3Y\bar{P}'P - 9P\bar{P} - (XY-1)P'\bar{P}'\Big).$$

*Proof.* — The proof boils down to a simple verification, using the linearity of L and the bilinearity of $\Gamma$. The formula for $\Gamma(P(X), P(X))$ may be obtained directly from $\Gamma(P(X), P(X)) = \lim_{Y \to X} \Gamma(P(X), P(Y))$. ∎

**Proposition 8.2.** *Let* $Q = P^\beta$, $\beta = \frac{1-\lambda}{3} = -\frac{1+2\alpha}{2}$. *Then, for* $\alpha \neq -1/2$, *one has*

$$\mathrm{L}^{(\alpha)}(Q(X)) = -\lambda X Q'(X) - X^2 Q''(X), \tag{37}$$

$$\Gamma(Q(X), Q(Y)) = \frac{XY}{2}\big(Q'(X)Q'(Y) + 3\beta\frac{Q'(X)Q(Y) - Q(X)Q'(Y)}{X - Y}\big). \tag{38}$$

(39)

$$\Gamma(Q(X), \bar{Q}(Y)) = 3\beta^2 \frac{XY}{2(XY-1)} Q(X)\bar{Q}(Y)\Big(XS + Y\bar{S} - 3\Big) - \frac{XY}{2} Q'(X)\bar{Q}'(Y),$$

*where* $S = \frac{P'}{P}(X)$, $\bar{S} = \frac{\bar{P}'}{\bar{P}}(Y)$

$\Big($ *For* $\alpha = -1/2$, *one should replace* $Q = P^\beta$ *by* $Q = \log(P)\Big)$.

*Proof.* — Let us look first at $\mathrm{L}^{(\alpha)}(Q)$. With the formulae (2), we have

$$\mathrm{L}^{(\alpha)}(Q(X)) = \beta P(X)^{\beta-1}\mathrm{L}^{(\alpha)}(P(X)) + \beta(\beta-1)P(X)^{\beta-2}\Gamma(P(X), P(X)),$$

and from Proposition 8.1, we have

$$\begin{aligned} \mathrm{L}^{(\alpha)}(Q(X)) &= -\lambda X Q'(X) + X^2[(\frac{\lambda}{2} + \frac{3}{2}(\beta-1))\beta P''(X)P(X)^{\beta-1} \\ &\quad -\beta(\beta-1)P'^2(X)P^{\beta-2}(X)]. \end{aligned}$$

For the particular value of $\beta = \frac{1-\lambda}{3}$, $\frac{\lambda}{2} + \frac{3}{2}(\beta-1) = -1$, and we get the announced result.

Turning now to formula (38), we write

$$\begin{aligned} \Gamma(Q(X),Q(Y)) &= \beta^2 P^{\beta-1}(X)P^{\beta-1}(Y)\Gamma(P(X),P(Y)) \\ &= \frac{XY}{2}\Big(\beta^2 P'(X)P^{\beta-1}(X)P'(Y)P^{\beta-1}(Y) \\ &\quad + \frac{3\beta}{X-Y}\big(\beta P^{\beta}(Y)P'(X)P^{\beta-1}(X) \\ &\quad -\beta P^{\beta}(X)P'(Y)P^{\beta-1}(Y)\big)\Big) \end{aligned}$$

It remains to write $Q' = \beta P' P^{\beta-1}$ to obtain formula (38).

Formula (39) is obtained in the same way.

∎

**Corollary 8.3.** *$Q(X)$ is a generating function for the family $P_{0,n}^{(\alpha)}$. More precisely, still with $\beta = -(1+2\alpha)/2$,*

$$(1 - 3\bar{Z}X + 3ZX^2 - X^3)^{\beta} = \sum_n c_n P_{0,n}^{(\alpha)} X^n,$$

*where*

$$c_n = (-3)^n \frac{\beta(\beta-1)....(\beta-n+1)}{n!}$$

*$\big($The same remarks as in Proposition 8.2 applies for $\alpha = -1/2\big)$.*

*Proof.* — If we write the asymptotic expansion of $Q(X)$ around $X = 0$ (which is licit since both $Z$ and $\bar{Z}$ are bounded), and writing $Q(X) = \sum_n A_n(Z,\bar{Z})X^n$, equation (37) gives

$$\mathrm{L}^{(\alpha)} A_n = -(\lambda n + 2n(n-1))A_n.$$

As a consequence, then $A_n(Z,\overline{Z})$ are eigenvectors of $\mathrm{L}^{(\alpha)}$. But a simple computation shows that $A_n(Z,\bar{Z})$ is a polynomial in $(Z,\bar{Z})$ with highest degree term $c_n\bar{Z}^n$. Therefore, $A_n = c_n P_{0,n}^{(\alpha)}$.

∎

For example:

$$\begin{aligned} A_0(Z,\overline{Z}) &= 1 \\ A_1(Z,\overline{Z}) &= -3\beta\overline{Z} \\ A_2(Z,\overline{Z}) &= 3\beta(3(\beta-1)\overline{Z}^2 + 2Z) \end{aligned}$$

If we set $\bar{Q} = \bar{P}^\beta$, then $Q(X)\bar{Q}(Y) = \sum_{p,q} c_p c_q P^{(\alpha)}_{0,p} P^{(\alpha)}_{q,0} X^p Y^q$, where the highest degree term in the polynomial $P^{(\alpha)}_{0,p} P^{(\alpha)}_{q,0}$ is $Z^q \bar{Z}^p$. Considering similarly a differential equation satisfied by $Q(X)\bar{Q}(Y)$ will then provide useful informations about the eigenvectors $P^{(\alpha)}_{p,q}$.

Obtaining two variables generating functions for the general family $P^{(\alpha)}_{m,n}$ is not easy. To simplify the calculations, it is simpler to introduce the operator $\hat{L} = \mathrm{L}^{(\alpha)} + L_0$, where

$$L_0 = \lambda(X\partial_X + Y\partial_Y) + X^2\partial^2_X + Y^2\partial^2_Y + XY\partial^2_{X,Y},$$

and associated $\Gamma_0$ operator

$$\Gamma_0(f,g) = X^2\partial_X f \partial_X g + Y^2 \partial_Y f \partial_Y g + \frac{1}{2} XY(\partial_X f \partial_Y g + \partial_X g \partial_Y f)$$

and $\hat{\Gamma} = \Gamma + \Gamma_0$.

Then, from Proposition 8.2, we have $\hat{\mathrm{L}}(Q(X)) = \hat{\mathrm{L}}(\bar{Q}(Y)) = 0$ and therefore

$$\begin{aligned} \hat{L}(Q(X)\bar{Q}(Y)) &= 2\hat{\Gamma}(Q(X), \bar{Q}(Y)) \\ &= 2\big(\Gamma(Q(X), \bar{Q}(Y)) + \frac{XY}{2} Q'(X)\bar{Q}'(Y)\big), \end{aligned}$$

form which we get

**Proposition 8.4.**

$$\hat{L}(Q(X)\bar{Q}(Y)) = \frac{3\beta^2 XY}{XY-1} Q(X)\bar{Q}(Y)\Big(XS + Y\bar{S} - 3\Big),$$

*where as before* $S = \frac{P'}{P}(X)$, $\bar{S} = \frac{\bar{P}'}{\bar{P}}(Y)$.

This proposition leads us to a new representation of the polynomials $P^{(\alpha)}_{n,m}$. Writing for simplicity $Q(X) = \sum_{n\geq 0} A_n X^n$ and $\overline{Q}(Y) = \sum_{m\geq 0} B_m Y^m$ then we deduce that :

$$\mathrm{L}^{(\alpha)}(A_n B_m - A_{n-1}B_{m-1}) \quad = - \quad \lambda_{n,m} A_n B_m - \delta_{n,m} A_{n-1} B_{m-1}$$

where

$$\lambda_{n,m} = (\lambda - 1)(n+m) + n^2 + m^2 + nm.$$

and

$$\delta_{n,m} = (1-\lambda)(\lambda + n + m - 3) - \lambda_{n-1,m-1}.$$

Finally, from an easy induction, one sees that $L^{(\alpha)}(A_n B_m)$ is a linear combination of $A_{n-p}B_{m-p}$, $0 \leq p \leq \min(n,m)$, from which one deduces that $P^{(\alpha)}_{m,n}$ may be written as

$$P^{(\alpha)}_{m,n} = \sum_{p=0}^{\min(m,n)} d_{m,n,p,\alpha} P^{(\alpha)}_{m-p,0} P^{(\alpha)}_{0,n-p}$$

very similar to the representation given for the case $\alpha = -1/2$ in equation (28). Unfortunately, the explicit expression for the constants $d_{m,n,p,\alpha}$ does seem to have any simple form.

From what preceedes, we see that the family $Q(X)\bar{Q}(Y)$ is not a generating function for the family $P^{(\alpha)}_{m,n}(Z,\overline{Z})$, but we may expect that it is the case for some expression of the form $F(XY)Q(X)\bar{Q}(Y)$ for some real valued function $F$. Therefore, we may look at an equation of the form $\hat{L}(F(XY)Q\bar{Q}) = 0$. We may then use the following remark

**Proposition 8.5.**

$$\hat{\Gamma}(XY, Q(X)\bar{Q}(Y)) = \frac{3}{2}\beta XYQ\bar{Q}(XS + Y\bar{S})$$

*where* $S = \frac{P'}{P}(X)$, $\bar{S} = \frac{\bar{P}'}{\bar{P}}(Y)$

*Proof.* —

$$\begin{aligned}
\hat{\Gamma}(XY, Q(X)\bar{Q}(Y)) &= \Gamma_0(XY, Q\bar{Q}) \\
&= X\big(Q\Gamma_0(Y,\bar{Q}) + \bar{Q}\Gamma_0(Y,Q)\big) + \\
&\quad Y\big(Q\Gamma_0(X,\bar{Q}) + \bar{Q}\Gamma_0(X,Q)\big) \\
&= \frac{3}{2}XY\big(YQ(X)\bar{Q}'(Y) + X\bar{Q}(Y)Q'(X)\big) \\
&= \frac{3}{2}\beta XYQ(X)\bar{Q}(Y)\Big(XS + Y\bar{S}\Big)
\end{aligned}$$

∎

Then we have,

$$\begin{aligned}
\hat{L}(F(XY)Q\bar{Q}) &= F(XY)\hat{L}(Q\bar{Q}) + Q\bar{Q}\hat{L}(F(XY)) + 2\hat{\Gamma}(F(XY), Q\bar{Q}) \\
&= F(XY)\hat{L}(Q\bar{Q}) + Q\bar{Q}\big(F'(XY)\hat{L}(XY) + \\
&\quad F''(XY)\hat{\Gamma}(XY,XY)\big) + 2F'(XY)\hat{\Gamma}(XY, Q\bar{Q})
\end{aligned}$$

A simple computation shows that $\hat{L}(XY) = (2\lambda+1)XY$, $\hat{\Gamma}(XY,XY) = 3X^2Y^2$. Finally we have

$$\begin{aligned}
\hat{L}(F(XY)Q\bar{Q}) &= 2\hat{\Gamma}(F(XY), Q\bar{Q})\big(F'(XY) + \frac{\beta}{XY-1}F(XY)\big) \\
&\quad + Q\bar{Q}XY\Big((2\lambda+1)F'(XY) + 3XYF''(XY) - \\
&\quad \frac{9\beta^2}{XY-1}F(XY)\Big)
\end{aligned}$$

In order to get $\hat{L}(F(XY)Q\bar{Q}) = 0$, two are led to solve the two linear differential equations on $F$

$$\begin{cases}
F'(u) + \frac{\beta}{u-1}F(u) = 0 \\
\\
3uF''(u) + (2\lambda+1)F'(u) - \frac{9\beta^2}{u-1}F(u) = 0
\end{cases}$$

and it is easy to see that they are compatible only when $\lambda^2 - 5\lambda + 4 = 0$ that is $\lambda \in \{1, 4\}$, which leads to the study of the two geometric cases.

8.2. **Case $\alpha = -1/2$.** In this case, the previous approach just provides $\log(P(X)) - \log(\bar{P}(Y))$ as a bivariate generating function, which is too degenerate to give useful information on the general polynomials $P_{n,m}^{(-1/2)}$. But the explicit representation of the eigenvectors in this case pearly provides an efficient generating function already obtained in [7].

**Proposition 8.6.** *A generating function in the case $\alpha = -1/2$ for the family $S_{p,q}^{(-\frac{1}{2})} = T_p T_{-q} - T_{p-q}$, $p, q \geq 0$ is defined by :*

$$\begin{aligned} G(X,Y) &= \big(3 - X\frac{\bar{P}'}{\bar{P}}(X)\big)\big(3 - Y\frac{P'}{P}(Y)\big) \\ &\quad + \frac{1}{1-XY}\Big(X\frac{\bar{P}'}{\bar{P}}(X) + Y\frac{P'}{P}(Y) - 3\Big) \end{aligned}$$

*Proof.* — It is quite easy to deduce the generating function for the family $S_{p,q}^{(-\frac{1}{2})}$. Indeed, since $S_{p,q}^{(-\frac{1}{2})} = T_p T_{-q} - T_{p-q}$, $p, q \geq 0$ we get

$$\begin{aligned} G(X,Y) &:= \sum_{p\geq 0, q\geq 0} X^p Y^q S_{p,q} \\ &= (\sum_{p\geq 0} T_p X^p)(\sum_{q\geq 0} T_{-q} Y^q) - \sum_{p\geq 0, p\geq 0} T_{p-q} X^p Y^q, \end{aligned}$$

the series being convergent as soon as $|X| < 1$ and $|Y| < 1$. Using the representation $T_p = z_1^p + z_2^p + z_3^p$, this sums as

$$\Big(\sum_{i=1}^{3}(1 - Xz_i)^{-1}\Big)\Big(\sum_{j=1}^{3}(1 - Y\overline{z_i})^{-1})\Big) - \sum_{i=1}^{3}(1 - Xz_i)^{-1}(1 - Y\overline{z_i})^{-1},$$

But

$$\sum_{i=1}^{3}(1 - Xz_i)^{-1} = \frac{1}{X}\frac{P'}{P}(\frac{1}{X}).$$

On the other hand

$$(1 - Xz_i)^{-1}(1 - Y\overline{z_i})^{-1} = \frac{1}{1-XY}(\frac{1}{1 - Xz_i} + \frac{1}{1 - Y\bar{z}_i} - 1),$$

so that

$$\sum_{i=1}^{3}(1 - Xz_i)^{-1}(1 - Y\overline{z_i})^{-1} = \frac{1}{1-XY}\Big(\frac{1}{X}\frac{P'}{P}(\frac{1}{X}) + \frac{1}{Y}\frac{\bar{P}'}{\bar{P}}(\frac{1}{Y}) - 3\Big).$$

This simplifies using $X^3 P(1/X) = -\bar{P}(X)$, such that

$$\frac{1}{X}\frac{P'}{P}(\frac{1}{X}) = 3 - X\frac{\bar{P}'}{\bar{P}}(X),$$

Then we have the result. Using the notations of the previous subsection 8.1, we may also check directly that $(\mathrm{L}^{(-\frac{1}{2})} + L_0)G(X,Y) = 0$ in this case.

∎

8.3. **Case $\alpha = 1/2$.** Finally, the formulae providing in subsection 8.1 leads us directly to a generating function in the $\alpha = -1/2$ case. This generating function has been proposed in [7] with however a completely different approach, based on the representations of the $SU(3)$ group.

**Proposition 8.7.** *A generating function for the family $P_{m,n}^{(\frac{1}{2})}$ is given by*

$$G(X,Y) \;=\; \frac{1-XY}{(1-3XZ+3X^2\overline{Z}-X^3)(1-3Y\overline{Z}+3Y^2Z-Y^3)}$$

*Proof. — This a a direct application of the computations provided in subsection 8.1 to this particular case.* ∎

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