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► **To cite this version:**

François Berteloot, Fabrizio Bianchi, Christophe Dupont. Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of $\mathbb{C}P(k)$. 2015. hal-00967542v2

HAL Id: hal-00967542

<https://hal.science/hal-00967542v2>

Preprint submitted on 10 Apr 2015 (v2), last revised 18 Dec 2016 (v3)

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DYNAMICAL STABILITY AND LYAPUNOV EXPONENTS FOR HOLOMORPHIC ENDOMORPHISMS OF \mathbb{P}^k

FRANÇOIS BERTELOOT, FABRIZIO BIANCHI, AND CHRISTOPHE DUPONT

ABSTRACT. We introduce a notion of stability for equilibrium measures in holomorphic families of endomorphisms of \mathbb{P}^k and prove that it is equivalent to the stability of repelling cycles and equivalent to the existence of a measurable holomorphic motion of Julia sets. We characterize the corresponding bifurcations by the strict subharmonicity of the sum of Lyapunov exponents or the instability of critical dynamics and analyze how repelling cycles may bifurcate. Our methods deeply exploit the properties of Lyapunov exponents and are based on ergodic theory and on pluripotential theory.

Key Words: holomorphic dynamics, dynamical stability, positive currents, Lyapunov exponents.

MSC 2010: 32H50, 32U40, 37F45, 37F50, 37H15.

1. INTRODUCTION

1.1. Main definitions and results. In the early 1980's, Mañé, Sad and Sullivan [MSS] and Lyubich [Ly1, Ly2] have independently obtained fundamental results on the stability of holomorphic families $(f_\lambda)_{\lambda \in M}$ of rational maps of the Riemann sphere \mathbb{P}^1 . They proved that the parameter space M splits into an open and dense *stability locus* and its complement, the *bifurcation locus*. They also obtained precise informations on the distribution of hyperbolic parameters which lead to the so-called hyperbolic conjecture. This conjecture asserts that hyperbolic maps are dense in the space of rational maps. The works of Douady and Hubbard on the Mandelbrot set provide a deeper understanding of these questions for the quadratic polynomial family.

In this theory, the finiteness of the critical set and Picard-Montel theorem play a crucial role. They allow to characterize the stability of a parameter $\lambda_0 \in M$ by the stability of the critical orbits of the map f_{λ_0} . Equivalently, λ_0 is in the bifurcation locus if, after an arbitrarily small perturbation, there exists a repelling cycle capturing a critical orbit. The one-dimensional setting also permits, by mean of the λ -lemma, to build holomorphic motions of Julia sets which conjugate the dynamics on connected components of the stability locus. The bifurcation locus also coincides with the closure of the parameters $\lambda \in M$ for which f_λ admits an unpersistent neutral cycle.

This article deals with bifurcations within holomorphic families of endomorphisms of \mathbb{P}^k for $k \geq 1$. Let M be connected complex manifold of dimension m . A holomorphic family of endomorphisms of \mathbb{P}^k can be seen as a holomorphic mapping

$$f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k \quad , \quad (\lambda, z) \mapsto (\lambda, f_\lambda(z))$$

This research is partially supported by the ANR project LAMBDA, ANR-13-BS01-0002. The work of the second author is partially supported by the FIRB2012 grant "Differential Geometry and Geometric Function Theory".

where the algebraic degree d of f_λ is larger than or equal to 2 and does not depend on λ . For instance, M can be the space $\mathcal{H}_d(\mathbb{P}^k)$ of all degree d holomorphic endomorphisms of \mathbb{P}^k , which is a Zariski open subset in some \mathbb{P}^N .

Our main result is Theorem 1.1 below, it asserts that different natural notions of stability are equivalent and leads to a coherent notion of bifurcation for holomorphic families f in \mathbb{P}^k . Our arguments exploit some ergodic and pluripotential tools as those developed in the works of Bedford-Lyubich-Smillie, Fornaess-Sibony, Briend-Duval, Dinh-Sibony on holomorphic dynamics on \mathbb{P}^k or \mathbb{C}^k (see the survey [DS3] for precise references). Let us recall that, for each $\lambda \in M$, we have an ergodic dynamical system $(J_\lambda, f_\lambda, \mu_\lambda)$ where μ_λ is the equilibrium measure of f_λ and J_λ is the topological support of μ_λ called the *Julia set*. The measure μ_λ enjoys a potential interpretation

$$\mu_\lambda = (dd_z^c g(\lambda, z) + \omega_{FS})^k,$$

where g is the Green function of f and ω_{FS} the Fubini-Study form on \mathbb{P}^k . The repelling cycles of f_λ equidistribute the measure μ_λ and hence are dense in J_λ . However, in higher dimension, some repelling cycles may belong to the complement of J_λ . We denote by $L(\lambda) := \int_{\mathbb{P}^k} \log \text{Jac } f \, d\mu_\lambda$ the sum of the Lyapunov exponents of μ_λ . This is a plurisubharmonic function on M which satisfies $L(\lambda) \geq k \frac{\log d}{2}$. Let \mathbf{C}_f denote the current of integration on the critical set of f taking into account the multiplicities of f .

Our main result is as follows. The definitions occurring in (A), (C) and (D) are explained below.

Theorem 1.1. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms where M is a simply connected open subset of the space $\mathcal{H}_d(\mathbb{P}^k)$ of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Then the following assertions are equivalent:*

- (A) *the repelling J -cycles move holomorphically over M ,*
- (B) *the function L is pluriharmonic on M ,*
- (C) *f admits an equilibrium web,*
- (D) *f admits an equilibrium lamination,*
- (E) *any $\lambda_0 \in M$ admits a neighbourhood U such that $\liminf_n d^{-kn} |(f^n)_* \mathbf{C}_f|_U = 0$.*

These equivalences remain true when $k = 2$ for every simply connected manifold M not necessarily included in $\mathcal{H}_d(\mathbb{P}^2)$. Idem for every $k \geq 1$ and for every family whose repelling J -cycles are neither persistently resonant nor persistently undiagonalizable (see Proposition 5.6). It also stays partially true for general families (see Theorem 1.6).

Theorem 1.1 leads us to define the *bifurcation current* of a holomorphic family of endomorphisms of \mathbb{P}^k as the closed positive current $dd_\lambda^c L$, and the *bifurcation locus* as the support of this current. The family is *stable* if its bifurcation locus is empty. This is coherent with the classical one-dimensional definition, due to DeMarco [dM].

Let us now specify the definitions. A central notion is the set

$$\mathcal{J} := \{ \gamma : M \rightarrow \mathbb{P}^k : \gamma \text{ is holomorphic and } \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M \}.$$

The graph $\{(\lambda, \gamma(\lambda)) \mid \lambda \in M\}$ of any element $\gamma \in \mathcal{J}$ is denoted Γ_γ . We endow \mathcal{J} with the topology of local uniform convergence and note that f induces a continuous self-map

$$\mathcal{F} : \mathcal{J} \rightarrow \mathcal{J} \text{ given by } \mathcal{F} \cdot \gamma(\lambda) := f_\lambda(\gamma(\lambda)).$$

Definition 1.2. For every $\lambda \in M$, a repelling J -cycle of f_λ is a repelling cycle which belongs to J_λ . We say that these cycles move holomorphically over M if, for every period n , there exists a finite subset $\{\rho_{n,j}, 1 \leq j \leq N_n\}$ of \mathcal{J} such that $\{\rho_{n,j}(\lambda), 1 \leq j \leq N_n\}$ is precisely the set of n periodic repelling J -cycles of f_λ for every $\lambda \in M$.

Our notions of equilibrium webs and laminations are as follows.

Definition 1.3. An equilibrium web is a probability measure \mathcal{M} on \mathcal{J} such that

- (1) \mathcal{M} is \mathcal{F} -invariant and its support is a compact subset of \mathcal{J} ,
- (2) for every $\lambda \in M$ the probability measure $\mathcal{M}_\lambda := \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma)$ is equal to μ_λ .

This notion is related to Dinh's theory of woven currents and somehow means that the measures $(\mu_\lambda)_{\lambda \in M}$ are holomorphically glued together. In this article we shall also say that $(\mu_\lambda)_{\lambda \in M}$ move holomorphically when such a web exists.

Definition 1.4. An equilibrium lamination is a subset \mathcal{L} of \mathcal{J} such $\mathcal{F}(\mathcal{L}) = \mathcal{L}$ and

- (1) $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ for every distinct $\gamma, \gamma' \in \mathcal{L}$,
- (2) $\mu_\lambda\{\gamma(\lambda), \gamma \in \mathcal{L}\} = 1$ for every $\lambda \in M$,
- (3) Γ_γ does not meet the grand orbit of the critical set of f for every $\gamma \in \mathcal{L}$,
- (4) the map $\mathcal{F} : \mathcal{L} \rightarrow \mathcal{L}$ is d^k to 1.

One can see an equilibrium lamination as a holomorphic motion of the Borel supports of the measures μ_λ . Equilibrium laminations will be extracted from the support of equilibrium webs by using ergodic theory for the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$.

1.2. Sketch of the proofs and further results. The novelty of our approach stays on two specific features. The first one is the use of a formula for the dd^c of the sum L of the Lyapunov exponents to read the interplay between bifurcations and critical dynamics. This formula is

$$(1) \quad dd_\lambda^c L = \pi_{M*} \left((dd_{\lambda,z}^c g(\lambda, z) + \omega_{FS})^k \wedge \mathbf{C}_f \right),$$

it was proved by Bassanelli and the first author [BB1], see also Pham's formula in Theorem 3.3. Like in dimension one, our proofs crucially rely on the links between bifurcations and instability in the critical dynamics. In higher dimensions these interactions cannot be detected by a simple application of Picard-Montel theorem, Formula (1) aims to overcome this problem.

The second feature is the introduction of equilibrium webs to overcome the lack of λ -lemma and build holomorphic motions of Julia sets. This is a weaker, but natural, notion dealing with the measures μ_λ rather than with their supports J_λ . It should be stressed that equilibrium webs are actually obtained as limits of discrete measures by mean of a compactness statement which may be considered as a measurable version of the λ -lemma (see Lemma 2.2).

We now specify our approach and summarize the proof of Theorem 1.1. Simultaneously we state some related results. We obtain $(B) \Leftrightarrow (E)$ by using Formula (1) and the f -invariance of the Green function g . The implication $(A) \Rightarrow (B)$ is proved in Proposition 3.5. We actually establish a stronger statement: we show that $dd_\lambda^c L$ is vanishing if f admits an equilibrium web which is a limit of discrete measures supported on graphs avoiding the critical set of f .

To show that the vanishing of $dd_\lambda^c L$ is a sufficient condition for stability, we exploit the dynamics of the critical set. For that purpose we introduce a suitable notion of Misiurewicz parameters.

Definition 1.5. *One says that $\lambda_0 \in M$ is a Misiurewicz parameter if there exists a holomorphic map γ from a neighbourhood of λ_0 into \mathbb{P}^k such that:*

- 1) $\gamma(\lambda) \in J_\lambda$ and is a repelling p_0 -periodic point of f_λ for some $p_0 \geq 1$,
- 2) $(\lambda_0, \gamma(\lambda_0)) \in f^{n_0}(C_f)$ for some $n_0 \geq 1$,
- 3) the graph Γ_γ of γ is not contained in $f^{n_0}(C_f)$.

We first prove that the pluriharmonicity of L prevents the apparition of such parameters. To do this, we use again Formula (1) and a dynamical rescaling argument. This is done in subsection 3.3. To prove that the absence of Misiurewicz parameters implies the existence of an equilibrium web, we apply our measurable version of the λ -lemma to sequences of discrete measures on pull-backs by f^n of a graph of repelling J -cycles avoiding the post-critical set of f , see Proposition 2.3. The existence of such a graph is involved, we use entropy arguments. These results, which are valid in arbitrary families, are summarized in the following theorem.

Theorem 1.6. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Then the following assertions are equivalent:*

- (a) *the function L is pluriharmonic on M ,*
- (b) *there are no Misiurewicz parameters in M ,*
- (c) *the restriction $f_{B \times \mathbb{P}^k}$, where B is any sufficiently small ball, admits an equilibrium web $\mathcal{M} = \lim_n \mathcal{M}_n$ and the graph of any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$ avoids the critical set of f .*

Among equilibrium webs, those giving no mass to the subset of γ 's in \mathcal{J} whose graphs meet the grand orbit of the critical set of f will play an essential role in the construction of equilibrium laminations. Such webs are called *acritical* (see Definition 2.1). Both Theorem 1.6 and the implication (A) \Rightarrow (B) in Theorem 1.1 are used to get the following important fact.

Corollary 1.7. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. If the repelling J -cycles move holomorphically over M then f admits an ergodic and acritical equilibrium web.*

In section 4 we prove that (A) \Rightarrow (D). We use there the Corollary 1.7 and exploit the stochastic properties of $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ where \mathcal{M} is an acritical and ergodic equilibrium web. We show that the iterated inverse branches of f are exponentially contracting near the graph Γ_γ of \mathcal{M} -almost every $\gamma \in \mathcal{J}$ (see Proposition 4.2). This implies that for \mathcal{M} -almost every $\gamma \in \mathcal{J}$ the graph Γ_γ does not intersect any other graph $\Gamma_{\gamma'}$ where $\gamma \neq \gamma' \in \text{supp } \mathcal{M}$ and allows us to build equilibrium laminations (see Theorem 4.1).

So far we have established that (A) \Rightarrow (B), (A) \Rightarrow (D), (E) \Leftrightarrow (B) and that (B) \Rightarrow (C') were (C') is a local version of (C) (see Theorem 1.6). We prove simultaneously that (C') \Rightarrow (C) \Rightarrow (A). To this purpose, we investigate how the apparition of Siegel discs may affect the continuity of $\lambda \mapsto J_\lambda$ in the Hausdorff topology. The section 5 is mainly devoted to that study (see in particular Proposition 5.3). Finally, by the same argument than in the proof of (b) \Rightarrow (c) in Theorem 1.6 one gets (D) \Rightarrow (C). This completes the proof of Theorem 1.1.

In the last section, we investigate a few properties of bifurcation loci. We first consider the possibility for a bifurcation locus to have a non-empty interior.

Theorem 1.8. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . The set of parameters λ for which \mathbb{P}^k coincides with the closure of the post-critical set of f_λ is dense in any open subset of the bifurcation locus of f .*

We then show that bifurcation loci contain some remarkable elements. Theorem 1.6 says that Misiurewicz parameters are dense in any bifurcation locus. In the same vein, we prove in Theorem 6.6 that the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^k)$ coincides with the closure of the set of endomorphisms which admit repelling J -cycles which bifurcate either by giving Siegel periodic cycles or repelling cycles outside the Julia set. We finally observe in Theorem 6.7 that in any stable family, all elements are Lattès maps as soon as one element is a Lattès map. This follows from the characterization of such maps by their Lyapunov exponents, see [BL, BtDp, Du2].

Let us finally mention that bifurcation phenomena in families of Hénon maps of \mathbb{C}^2 have already been studied by Bedford, Lyubich and Smillie [BLS] and by Dinh and Sibony [DS5], the sharpest achievements are due to Dujardin and Lyubich in their recent work on the two dimensional and dissipative case [DuLy].

Acknowledgements: We would like to thank Eric Bedford, Xavier Buff, Tien-Cuong Dinh, Romain Dujardin, Thomas Gauthier, Michael Lyubich, Jasmin Raissy and Nessim Sibony for helpful discussions or comments on the first draft of this paper.

2. EQUILIBRIUM WEBS

2.1. Definition and construction. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. We recall that M is a connected complex manifold of dimension m and that $f(\lambda, z) = (\lambda, f_\lambda(z))$. Let μ_λ denote the equilibrium measure of f_λ and let J_λ denote the support of μ_λ , this is the Julia set of f_λ . We want here to define a notion of holomorphic motion for the family $(\mu_\lambda)_{\lambda \in M}$. To this purpose we consider the set $\mathcal{O}(M, \mathbb{P}^k)$ of holomorphic maps from M to \mathbb{P}^k , endowed with the metric space topology of local uniform convergence, and the closed subspace

$$\mathcal{J} := \{ \gamma \in \mathcal{O}(M, \mathbb{P}^k) : \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M \}.$$

For any probability measure \mathcal{M} on $\mathcal{O}(M, \mathbb{P}^k)$ and every $\lambda \in M$ we define the measure

$$\mathcal{M}_\lambda := \int \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma).$$

This is a probability measure on \mathbb{P}^k which is actually equal to $p_{\lambda*}\mathcal{M}$, where the mapping $p_\lambda : \mathcal{O}(M, \mathbb{P}^k) \rightarrow \mathbb{P}^k$ is given by $p_\lambda(\gamma) := \gamma(\lambda)$.

Let us recall that an equilibrium web for f is a \mathcal{F} -invariant and compactly supported probability measure \mathcal{M} on \mathcal{J} such that $\mathcal{M}_\lambda = \mu_\lambda$ for every $\lambda \in M$. We shall sometimes say that the measure μ_λ *move holomorphically over M* when f admits an equilibrium structural web. For every probability measure \mathcal{M} on $\mathcal{O}(M, \mathbb{P}^k)$ and in particular for any equilibrium web we can define the current

$$W_{\mathcal{M}} := \int [\Gamma_\gamma] d\mathcal{M}(\gamma).$$

It has bidimension (m, m) on $M \times \mathbb{P}^k$ and is a woven current following Dinh's terminology [Di2].

To construct equilibrium laminations, it will be crucial to deal with equilibrium webs giving no mass to the subset \mathcal{J}_s of \mathcal{J} whose elements have a graph intersecting the grand orbit of the critical set of f . This motivates the following definition.

Definition 2.1. An equilibrium web \mathcal{M} is said acritical if $\mathcal{M}(\mathcal{J}_s) = 0$ where \mathcal{J}_s is given by $\mathcal{J}_s := \{\gamma \in \mathcal{J} : \Gamma_\gamma \cap (\cup_{m \geq 0} f^{-m}(\cup_{n \geq 0} f^n(C_f))) \neq \emptyset\}$.

Equilibrium webs will be obtained as limits of discrete measures on $\mathcal{O}(M, \mathbb{P}^k)$. To this purpose we shall use the following simple tool which somehow plays the role of the classical λ -lemma. We refer to Lemma A.1 for a more general statement.

Lemma 2.2. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let $(\mathcal{M}_n)_{n \geq 1}$ be a sequence of Borel probability measures on $\mathcal{O}(M, \mathbb{P}^k)$ such that:

- 1) $\lim_n (\mathcal{M}_n)_\lambda = \mu_\lambda$ for every $\lambda \in M$,
- 2) $\mathcal{F}_* \mathcal{M}_{n+1} = \mathcal{M}_n$ or $\mathcal{F}_* \mathcal{M}_n = \mathcal{M}_n$ for every $n \geq 1$,
- 3) there exists a compact $\mathcal{K} \subset \mathcal{O}(M, \mathbb{P}^k)$ such that $\mathcal{F}^{-1}(\mathcal{K}) \subset \mathcal{K}$ and $\text{supp } \mathcal{M}_n \subset \mathcal{K}$.

Then any limit of $(\frac{1}{n} \sum_{l=1}^n \mathcal{M}_l)_n$ is an equilibrium web.

Proof. Let $\mathcal{N}_n := \frac{1}{n} \sum_{l=1}^n \mathcal{M}_l$. By Assertion 3) $(\mathcal{N}_n)_{n \geq 1}$ is a sequence of Radon probability measures on the compact metric space \mathcal{K} . Banach-Alaoglu and Riesz-Markov theorems ensure that there exists a subsequence $(\mathcal{N}_{n_k})_{k \geq 1}$ converging weakly to a Radon probability measure \mathcal{M} on \mathcal{K} . By Assertion 2), we have $\mathcal{F}_* \mathcal{N}_{n_k} = \mathcal{N}_{n_k} + \mathcal{E}_k$ where the mass of \mathcal{E}_k is less than $2/n_k$. This implies that $\mathcal{F}_* \mathcal{M} = \mathcal{M}$ as measures on \mathcal{K} . Let us extend \mathcal{M} to a Borel probability measure $\widetilde{\mathcal{M}}$ on $\mathcal{O}(M, \mathbb{P}^k)$ by setting $\widetilde{\mathcal{M}}(A) := \mathcal{M}(A \cap \mathcal{K})$. Let us verify that $\widetilde{\mathcal{M}}$ is an equilibrium web. We still have $\mathcal{F}_* \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}$. Indeed, we deduce from $\mathcal{F}^{-1}(\mathcal{K}) \subset \mathcal{K}$:

$$\mathcal{F}_* \widetilde{\mathcal{M}}(A) = \mathcal{M}(\mathcal{F}^{-1}(A) \cap \mathcal{K}) \geq \mathcal{M}(\mathcal{F}^{-1}(A \cap \mathcal{K})) = \mathcal{M}(A \cap \mathcal{K}) = \widetilde{\mathcal{M}}(A)$$

and the identity follows since $\mathcal{F}_* \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ are probability measures. From $p_{\lambda_*} \widetilde{\mathcal{M}} = p_{\lambda_*} \mathcal{M}$ and $p_{\lambda_*} \mathcal{M} = \lim_k p_{\lambda_*} \mathcal{N}_{n_k} = \mu_\lambda$ provided by Assertion 1), we deduce $p_{\lambda_*} \widetilde{\mathcal{M}} = \mu_\lambda$. It remains to check $\text{supp } \widetilde{\mathcal{M}} \subset \mathcal{J}$. If $\gamma_0 \notin \mathcal{J}$ then $\gamma_0(\lambda_0) \notin \text{supp } \mu_{\lambda_0}$ for some $\lambda_0 \in M$. Let V_0 be a neighbourhood of γ_0 in $\mathcal{O}(M, \mathbb{P}^k)$ such that $p_{\lambda_0}(V_0) \subset \mathbb{P}^k \setminus \text{supp } \mu_{\lambda_0}$. Then

$$\widetilde{\mathcal{M}}(V_0) \leq \widetilde{\mathcal{M}}(p_{\lambda_0}^{-1}(p_{\lambda_0}(V_0))) = p_{\lambda_0_*} \widetilde{\mathcal{M}}(p_{\lambda_0}(V_0)) = \mu_{\lambda_0}(p_{\lambda_0}(V_0)) = 0$$

implies that $\gamma_0 \notin \text{supp } \widetilde{\mathcal{M}}$. □

We now explain how Lemma 2.2 is concretely used to produce equilibrium webs. The proof relies on the equidistribution of preimages of points, see the articles [FS1, BrDv2, DS1] and on the equidistribution of repelling cycles, see [BrDv1].

Proposition 2.3. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree d .

- 1) Assume that M is simply connected and that there exists $\gamma \in \mathcal{O}(M, \mathbb{P}^k)$ such that the graph Γ_γ does not intersect the post-critical set of f . Then an equilibrium web is given by any limit of $(\frac{1}{n} \sum_{i=1}^n \frac{1}{d^{ki}} \sum_{\mathcal{F}^i \cdot \sigma = \gamma} \delta_\sigma)_n$.
- 2) Assume that the repelling J -cycles of f move holomorphically over M . Let $(\rho_{n,j})_{1 \leq j \leq N_n}$ be the elements of \mathcal{J} given by the motions of these n -periodic cycles. Then an equilibrium web is given by any limit of $(\frac{1}{d^{kn}} \sum_{j=1}^{N_n} \delta_{\rho_{n,j}})_n$.

Proof. 1) The map $f^n : (M \times \mathbb{P}^k) \setminus f^{-n}(\cup_{1 \leq p \leq n} f^p(C_f)) \rightarrow (M \times \mathbb{P}^k) \setminus (\cup_{1 \leq p \leq n} f^p(C_f))$ is a covering of degree d^{kn} . Hence, there exist d^{kn} holomorphic graphs $\Gamma_{\sigma_{j,n}}$ such that $f^n(\Gamma_{\sigma_{j,n}}) = \Gamma_\gamma$ i.e.

$\mathcal{F}^n \cdot \sigma_{j,n} = \gamma$. Let us set $\mathcal{M}_n := \frac{1}{d^{kn}} \sum_{j=1}^{d^{kn}} \delta_{\sigma_{j,n}}$. By construction $\mathcal{F}_* \mathcal{M}_{n+1} = \mathcal{M}_n$ and, for every $\lambda \in M$, one has $(\mathcal{M}_n)_\lambda = \frac{1}{d^{kn}} \sum_{j=1}^{d^{kn}} \delta_{\sigma_{j,n}(\lambda)} = \sum_{f_\lambda^n(x)=\gamma(\lambda)} \delta_x \rightarrow \mu_\lambda$, where the limit comes from the fact that $\gamma(\lambda) \notin \cup_{p \geq 1} f_\lambda^p(C_{f_\lambda})$. The family $(\sigma_{j,n})_{j,n}$ is normal, by a theorem of Ueda [Ued, Theorem 2.1], and therefore the closure \mathcal{K} of $\cup_{n \geq 1} \text{supp } \mathcal{M}_n$ is a compact subset of $\mathcal{O}(M, \mathbb{P}^k)$. By construction $\mathcal{F}^{-1}(\mathcal{K}) \subset \mathcal{K}$. The conclusion immediately follows from Lemma 2.2.

2) Let us set $\mathcal{M}_n := \frac{1}{d^{kn}} \sum_{j=1}^{N_n} \delta_{\rho_{j,n}}$. The convergence of $(\mathcal{M}_n)_\lambda$ towards μ_λ follows from the equidistribution of repelling periodic points with respect to the equilibrium measure, see [BrDv2] (note that the repelling cycles produced there are J -cycles). The normality of the family $(\rho_{j,n})_{j,n}$ can be seen by lifting these curves to curves of periodic points of a lift of f to \mathbb{C}^{k+1} . Again, one concludes by using Lemma 2.2. \square

2.2. Elementary properties. As it will turn out, equilibrium webs given by Proposition 2.3 are acritical and this property, combined with ergodicity, will be crucial to build equilibrium laminations. This motivates the following result.

Proposition 2.4. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If f admits an acritical equilibrium web \mathcal{M}_0 then f admits an acritical equilibrium web \mathcal{M}'_0 which is ergodic and such that $\text{supp } \mathcal{M}'_0 \subset \text{supp } \mathcal{M}_0$.*

Proof. Let us consider the convex set $\mathcal{P}_{web}(\mathcal{K})$ of equilibrium webs of f which are supported in \mathcal{K} , where $\mathcal{K} := \text{supp } (\mathcal{M}_0)$. Note that $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$ since \mathcal{M}_0 is \mathcal{F} -invariant. The set $\mathcal{P}_{web}(\mathcal{K})$ is a compact metric space for the topology of weak convergence of measures. It is actually closed in the unit ball $B_{C(\mathcal{K})}$, where $C(\mathcal{K})$ is the separable Banach space of continuous functions on \mathcal{K} endowed with the norm of uniform convergence.

We will use Choquet decomposition theorem to find extremal points \mathcal{M}' in $\mathcal{P}_{web}(\mathcal{K})$ for which $\mathcal{M}'(\mathcal{J}_s) = 0$ and then prove the ergodicity of \mathcal{M}' by showing that these points are also extremal in the set $\mathcal{P}_{inv}(\mathcal{K})$ of \mathcal{F} -invariant probability measures on \mathcal{K} . Let us denote by $\text{Ext}(\mathcal{P}_{web}(\mathcal{K}))$ the set of extremal points of the compact metric space $\mathcal{P}_{web}(\mathcal{K})$. By Choquet theorem, there exists a probability measure ν_0 on $\text{Ext}(\mathcal{P}_{web}(\mathcal{K}))$ such that

$$\mathcal{M}_0 = \int_{\text{Ext}(\mathcal{P}_{web}(\mathcal{K}))} \mathcal{E} \, d\nu_0(\mathcal{E}).$$

Then

$$0 = \mathcal{M}_0(\mathcal{J}_s) = \int_{\text{Ext}(\mathcal{P}_{web}(\mathcal{K}))} \mathcal{E}(\mathcal{J}_s) \, d\nu_0(\mathcal{E})$$

and the set of equilibrium webs $\mathcal{E} \in \text{Ext}(\mathcal{P}_{web}(\mathcal{K}))$ for which $\mathcal{E}(\mathcal{J}_s) = 0$ has full ν_0 -measure.

To conclude the proof we are left to check that any $\mathcal{M}' \in \text{Ext}(\mathcal{P}_{web}(\mathcal{K}))$ is extremal in $\mathcal{P}_{inv}(\mathcal{K})$. Assume that $\mathcal{M}' = \frac{1}{2}\mathcal{M}_1 + \frac{1}{2}\mathcal{M}_2$ where $\mathcal{M}_j \in \mathcal{P}_{inv}(\mathcal{K})$. Then, as \mathcal{M}' is an equilibrium web we have $\mu_\lambda = p_{\lambda*}(\mathcal{M}') = \frac{1}{2}p_{\lambda*}(\mathcal{M}_1) + \frac{1}{2}p_{\lambda*}(\mathcal{M}_2)$ for every $\lambda \in M$. Since $p_\lambda \circ \mathcal{F} = f_\lambda \circ p_\lambda$, the probability measures $p_{\lambda*}(\mathcal{M}_j)$ are f_λ -invariant and therefore the ergodicity of μ_λ implies that $p_{\lambda*}(\mathcal{M}_1) = p_{\lambda*}(\mathcal{M}_2) = \mu_\lambda$. This shows that \mathcal{M}_1 and \mathcal{M}_2 actually belong to $\mathcal{P}_{web}(\mathcal{K})$ and the identity $\mathcal{M}' = \mathcal{M}_1 = \mathcal{M}_2$ then follows from the fact that \mathcal{M}' is extremal in $\mathcal{P}_{web}(\mathcal{K})$. \square

The following simple dynamical properties of the support of an equilibrium web will be very useful. We thank R. Dujardin for pointing us this fact.

Lemma 2.5. *Let M be a connected complex manifold and $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k which admits an equilibrium web \mathcal{M} . Then:*

- 1) *the sequence $(f_\lambda^p(\gamma(\lambda)))_{p \geq 1}$ is normal for every $\gamma \in \text{supp } \mathcal{M}$,*
- 2) *for every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ there exists $\gamma \in \text{supp } \mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$,*
- 3) *for every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ such that z_0 is n -periodic and repelling for f_{λ_0} , there exists $\gamma \in \text{supp } \mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$ and $\gamma(\lambda)$ is n -periodic for f_λ for every $\lambda \in M$.*

Proof. (1) We use $f_\lambda^p(\gamma(\lambda)) = (\mathcal{F}^p \cdot \gamma)(\lambda)$ and the fact that \mathcal{M} is compactly supported and \mathcal{F} -invariant.

(2) As $z_0 \in J_{\lambda_0}$ and $J_{\lambda_0} = \text{supp } \mu_{\lambda_0} = \text{supp } \mathcal{M}_{\lambda_0}$, there exist $(\gamma_n)_n \subset \text{supp } \mathcal{M}$ such that $\gamma_n(\lambda_0) \rightarrow z_0$. Then, since \mathcal{M} is compactly supported, we can take for γ any limit of $(\gamma_n)_n$.

(3) By the implicit function theorem, there exists a neighbourhood V_{λ_0} of λ_0 and a holomorphic map $w : V_{\lambda_0} \rightarrow \mathbb{P}^k$ such that $w(\lambda_0) = z_0$ and $w(\lambda)$ is n -periodic for f_λ . We will show that w coincides on V_{λ_0} with the map γ given by the previous item; the conclusion then follows by analytic continuation. Our argument is local, so we can choose a chart and work on \mathbb{C}^k . Since z_0 is repelling, we can shrink V_{λ_0} and find $A > 1$, $r > 0$ such that

$$(2) \quad \|w(\lambda) - f_\lambda^n(z)\| = \|f_\lambda^n(w(\lambda)) - f_\lambda^n(z)\| \geq A\|w(\lambda) - z\|$$

when $\lambda \in V_{\lambda_0}$ and $\|w(\lambda) - z\| < r$. On the other hand the first item ensures that $(f_\lambda^{pn}(\gamma(\lambda)))_p$ is a normal family, hence we can shrink again V_{λ_0} so that $\|w(\lambda) - f_\lambda^{pn}(\gamma(\lambda))\| < r$ for every $p \geq 1$ and $\lambda \in V_{\lambda_0}$. Combining this with Equation (2) we obtain $r > \|w(\lambda) - f_\lambda^{pn}(\gamma(\lambda))\| \geq A^p\|w(\lambda) - \gamma(\lambda)\|$ for every $p \geq 1$ and $\lambda \in V_{\lambda_0}$. This implies $w(\lambda) = \gamma(\lambda)$ on V_{λ_0} since $A > 1$. \square

To perform certain computations, we will have to explicitly relate equilibrium webs with positive horizontal currents (see Lemma 2.8 below). Before doing this, we recall some basic facts about horizontal currents.

Definition 2.6. *Let M be a complex connected manifold. A current \mathcal{R} on $M \times \mathbb{C}^{k+1}$ is horizontal if $\text{supp } \mathcal{R} \subset M \times K$ for some compact subset $K \subset \mathbb{C}^{k+1}$.*

Let us assume that \mathcal{R} is a closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$ where m is the complex dimension of M . Then the slices $\langle \mathcal{R}, \pi_M, \lambda \rangle$ exist for Lebesgue-almost every $\lambda \in M$ and are positive measures on $M \times \mathbb{C}^{k+1}$ supported on $\{\lambda\} \times \mathbb{C}^{k+1}$. The following *basic slicing formula* holds for every continuous test function ψ on $M \times \mathbb{C}^{k+1}$ and every continuous (m, m) -test form ω on M :

$$(3) \quad \int_M \langle \mathcal{R}, \pi_M, \lambda \rangle \psi \omega(\lambda) = \langle \mathcal{R} \wedge \pi_M^*(\omega), \psi \rangle.$$

Dinh and Sibony have shown that the slices of such currents do actually exist for *every* $\lambda \in M$, see [DS1, theorem 2.1]. Their basic result is as follows, it will be used in the proof of Lemma A.1.

Theorem 2.7. (Dinh-Sibony) *Let M be a m -dimensional complex connected manifold and \mathcal{R} be a closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$. Then the following properties occur:*

- (1) *the slice $\langle \mathcal{R}, \pi_M, \lambda \rangle$ exists for every $\lambda \in M$ and its mass does not depend on $\lambda \in M$,*
- (2) *the function $\lambda \mapsto \int_{\mathbb{C}^{k+1}} \psi(\lambda, z) \langle \mathcal{R}, \pi_M, \lambda \rangle$ is psh or $\equiv -\infty$ on M for any psh function ψ defined on a neighborhood of $\text{supp } \mathcal{R}$.*

Let us now state the announced lemma. Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ be the canonical projection.

Lemma 2.8. *Let B be a ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let \mathcal{K} be a compact subset of $\mathcal{O}(B, \mathbb{P}^k)$. Then, after shrinking B , one may associate to any probability measure \mathcal{N} supported on \mathcal{K} a positive, horizontal (m, m) -bidimensional current $\widetilde{W}_{\mathcal{N}}$ on $B \times \mathbb{C}^{k+1}$ such that $\pi_{\star}(\widetilde{W}_{\mathcal{N}}, \pi_B, \lambda) = \mathcal{N}_{\lambda}$ for every $\lambda \in B$. Moreover, $\widetilde{W}_{\mathcal{N}}$ depends continuously on \mathcal{N} .*

Proof. Let $(\sigma_i)_{1 \leq i \leq N}$ be holomorphic sections of π whose domains of definition Ω_i cover \mathbb{P}^k . Since \mathcal{K} is a normal family, we may shrink B so that for each $\gamma \in \mathcal{K}$ there exists at least one $1 \leq i \leq N$ such that $\Gamma_{\gamma} \subset B \times \Omega_i$. This allows to define a map

$$\begin{aligned} \sigma : \mathcal{K} &\rightarrow \mathcal{O}(B, \mathbb{C}^{k+1}) \\ \gamma &\mapsto \sigma(\gamma) := \sigma_i \circ \gamma \end{aligned}$$

where $l := \min\{1 \leq i \leq N \text{ such that } \Gamma_{\gamma} \subset B \times \Omega_i\}$. Now, for any probability measure \mathcal{N} supported on \mathcal{K} we set

$$\widetilde{W}_{\mathcal{N}} := \int_{\mathcal{J}} [\Gamma_{\sigma(\gamma)}] d\mathcal{N}(\gamma).$$

Then $\pi_{\star}(\widetilde{W}_{\mathcal{N}}, \pi_B, \lambda) = \mathcal{N}_{\lambda}$ for every $\lambda \in B$ by construction. \square

2.3. Continuity of Julia sets and equilibrium webs. In Section 5, we will want to compare the holomorphic motions of the measures $(\mu_{\lambda})_{\lambda \in M}$ with the continuity of their supports J_{λ} in the Hausdorff sense. To this purpose, we recall a few definitions. Let $\text{Comp}^*(\mathbb{P}^k)$ be the set of non-empty compact subsets of \mathbb{P}^k endowed with the Hausdorff distance and let K_{ϵ} denote the ϵ -neighbourhood of $K \in \text{Comp}^*(\mathbb{P}^k)$. A map $E : M \rightarrow \text{Comp}^*(\mathbb{P}^k)$ is said upper semi continuous (*u.s.c*) at $\lambda_0 \in M$ if for every $\epsilon > 0$, one has $E(\lambda) \subset (E(\lambda_0))_{\epsilon}$ when λ is close enough to λ_0 . It is lower semi continuous (*l.s.c*) at λ_0 if for every $\epsilon > 0$, one has $E(\lambda_0) \subset (E(\lambda))_{\epsilon}$ when λ is close enough to λ_0 . For every $A \subset M \times \mathbb{P}^k$ we define $(A)_{\lambda} := A \cap (\{\lambda\} \times \mathbb{P}^k)$.

The starting point about continuity of Julia sets stays on the following observations, see also [DS3, exercises 2.52 and 2.53].

Proposition 2.9. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . The map $\lambda \mapsto J_{\lambda}$ from M to $\text{Comp}^*(\mathbb{P}^k)$ is *l.s.c*. If f admits an equilibrium web \mathcal{M} and $W_{\mathcal{M}}$ is the woven current $\int_{\mathcal{J}} [\Gamma_{\gamma}] d\mathcal{M}(\gamma)$, then $J_{\lambda} \subset (\text{supp } W_{\mathcal{M}})_{\lambda}$ and the map $\lambda \mapsto (\text{supp } W_{\mathcal{M}})_{\lambda}$ from M to $\text{Comp}^*(\mathbb{P}^k)$ is *u.s.c*.*

Proof. The lower semi continuity of J_{λ} is a consequence of the existence of continuous local potentials for μ_{λ} . Assume indeed that $\lambda \mapsto J_{\lambda}$ is not *l.s.c* at λ_0 . Then we may find $\epsilon > 0$ and sequences $\lambda_n \in M$, $z_n \in J_{\lambda_0}$ such that $d_{\mathbb{P}^k}(z_n, J_{\lambda_n}) \geq \epsilon$. After taking a subsequence we may assume that $z_n \rightarrow z_0 \in J_{\lambda_0}$ and $B(z_0, \frac{\epsilon}{4}) \subset B(z_n, \frac{\epsilon}{2}) \subset B(z_0, \epsilon)$. If ϵ is small enough, the projection $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ admits a section σ on $B(z_0, 2\epsilon)$ and the functions $u_{\lambda}(z) := G(\lambda, \sigma(z))$ are local potentials for the equilibrium measures, which means that the restriction of μ_{λ} to $B(z_0, 2\epsilon)$ is the Monge-Ampère mass $(dd_{\bar{z}}^c u_{\lambda}(z))^k$. Observe that, by the continuity of G , the potentials u_{λ_n} converge locally uniformly to u_{λ_0} . This implies that $\liminf_n \mu_{\lambda_n}(B(z_0, \frac{\epsilon}{4})) \geq \mu_{\lambda_0}(B(z_0, \frac{\epsilon}{8}))$. The expected contradiction follows: $0 < \mu_{\lambda_0}(B(z_0, \frac{\epsilon}{8})) \leq \liminf_n \mu_{\lambda_n}(B(z_0, \frac{\epsilon}{4})) \leq \liminf_n \mu_{\lambda_n}(B(z_n, \frac{\epsilon}{2})) = 0$. The inclusion $J_{\lambda} \subset (\text{supp } W_{\mathcal{M}})_{\lambda}$ follows directly from the fact that $J_{\lambda} = \text{supp } \mu_{\lambda}$ and $\mu_{\lambda} = \mathcal{M}_{\lambda} = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma)$. The upper semi continuity of $(\text{supp } W_{\mathcal{M}})_{\lambda}$ is an elementary general topological fact, see [Do, Proposition 2.1]. \square

It is now easy to see that the existence of an equilibrium web implies that the Julia sets depend continuously on the parameter.

Proposition 2.10. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If f admits an equilibrium web then the map $\lambda \mapsto J_\lambda$ from M to $\text{Comp}^*(\mathbb{P}^k)$ is continuous.*

Proof. According to Proposition 2.9, it suffices to show that $(\text{supp } W_{\mathcal{M}})_\lambda \subset J_\lambda$. This follows from the following lemma. \square

Lemma 2.11. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that f admits an equilibrium web \mathcal{M} . If $z_0 \notin J_{\lambda_0}$ then there exist $\epsilon > 0$ and $r_0 > 0$ such that $\mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_\gamma \cap [B(\lambda_0, \epsilon) \times B(z_0, r_0)] \neq \emptyset\} = 0$. Moreover $\mu_\lambda(B(z_0, r_0)) = 0$ for every $\lambda \in B(\lambda_0, \epsilon)$.*

Proof. Pick $r_0 > 0$ such that $\mu_{\lambda_0}(B(z_0, 2r_0)) = 0$. As $\text{supp } \mathcal{M}$ is a normal family, there exists $\epsilon > 0$ such that for any $\gamma \in \text{supp } \mathcal{M}$:

$$\Gamma_\gamma \cap [B(\lambda_0, \epsilon) \times B(z_0, r_0)] \neq \emptyset \Rightarrow \gamma(\lambda) \in B(z_0, 2r_0) \text{ for any } \lambda \in B(\lambda_0, \epsilon).$$

Let $\alpha := \mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_\gamma \cap [B(\lambda_0, \epsilon) \times B(z_0, r_0)] \neq \emptyset\}$. Then, for any $\lambda \in B(\lambda_0, \epsilon)$, we have

$$\alpha \leq \mathcal{M}\{\gamma \in \mathcal{J} : \gamma(\lambda) \in B(z_0, 2r_0)\} = \mu_\lambda(B(z_0, 2r_0)).$$

Applying this to λ_0 yields $\alpha = 0$ as desired. For every $\lambda \in B(\lambda_0, \epsilon)$ we have $\mu_\lambda(B(z_0, r_0)) = \mathcal{M}\{\gamma \in \mathcal{J} : \gamma(\lambda) \in B(z_0, r_0)\} \leq \alpha = 0$. This completes the proof. \square

3. STABILITY AND THE SUM OF LYAPUNOV EXPONENTS

In this section we establish $(A) \Rightarrow (B) \Leftrightarrow (E)$ of Theorem 1.1 and prove Theorem 1.6 and Corollary 1.7. Formulas relating the critical dynamics with the sum of Lyapunov exponents are at the heart of our approach. For a polynomial P of degree d , Przytycki [Prz] proved that the Lyapunov exponent of the equilibrium measure satisfies

$$L(P) = \sum_{c \in C_P} G_P(c) + \log d$$

where $G_P(z) = \lim_n d^{-n} \log^+ |P^n(z)|$ is the dynamical Green function of P . This formula was generalized by DeMarco [dM] for the Lyapunov exponent $L(f)$ of a rational map f . In several complex variables, Bedford-Jonsson [BJ] established an analogous formula for the sum of the Lyapunov exponents of polynomial mappings. We use here an extended formula for holomorphic endomorphisms of \mathbb{P}^k obtained by Bassanelli-Berteloot, see [BB1, Theorem 4.1].

3.1. Formulas for the sum of Lyapunov exponents. To deal with this kind of formulas, the right framework is that of equilibrium currents for holomorphic families of d -homogeneous non-degenerate maps. It has been introduced by Pham [Pha] in the more general context of polynomial like mappings (see also the lecture notes by Dinh and Sibony [DS2, section 2.5]).

Definition 3.1. *Let $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$ be a holomorphic family of d -homogeneous non-degenerate maps where M is some m -dimensional complex connected manifold. Let \mathcal{E} be a closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$. We say that \mathcal{E} is an equilibrium current for F if the slice $\langle \mathcal{E}, \pi_M, \lambda \rangle$ is equal to the equilibrium measure of F_λ for every $\lambda \in M$.*

Equilibrium currents always exist, one may dynamically produce them and they do not detect bifurcations. For instance, Pham proved that the sequence of smooth forms $\left(\frac{1}{d^{(k+1)n}} F^{n*} (\pi_{\mathbb{C}^{k+1}}^* \theta)\right)_n$ converges to such a current for any smooth probability measure θ on \mathbb{C}^{k+1} . Note that such currents are not unique when $k > 1$.

It is also possible to define equilibrium currents for families of endomorphisms of \mathbb{P}^k by means of Green functions. Let us briefly recall their construction. Consider a holomorphic family $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ which admits a lift $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$. The sequence

$$G_n(\lambda, \tilde{z}) := \frac{1}{d^n} \log \|F_\lambda^n(\tilde{z})\|$$

converges locally uniformly on $M \times \mathbb{C}^{k+1} \setminus \{0\}$ to a function G which we call the *Green function of F* . The function G is *psh* and Hölder continuous, see [BB1, section 1.2]. Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ be the canonical projection and ω_{FS} be the Fubini-Study form on \mathbb{P}^k . The functions G_n induce functions $g_n : M \times \mathbb{P}^k \rightarrow \mathbb{R}$ by setting $g_n(\lambda, z) := G_n(\lambda, \tilde{z}) - \log \|\tilde{z}\|$, for every \tilde{z} satisfying $\pi(\tilde{z}) = z$. We have:

$$\frac{1}{d} f^* (dd_{\lambda,z}^c g_n + \omega_{FS}) = dd_{\lambda,z}^c g_{n+1} + \omega_{FS}.$$

We define similarly $g(\lambda, z) := \lim_n g_n(\lambda, z)$, which is equal to $G(\lambda, \tilde{z}) - \log \|\tilde{z}\|$, and set

$$\mathcal{E}_{Green} := (dd_{\lambda,z}^c g + \omega_{FS})^k.$$

This is a current of bidimension (m, m) and, since slicing commutes with the operators d, d^c , the measure $\langle \mathcal{E}_{Green}, \pi_M, \lambda \rangle$ is equal to the equilibrium measure of f_λ for every $\lambda \in M$. The current \mathcal{E}_{Green} will play an important role in our study (see Proposition 3.7). We call it the *Green equilibrium current of f* .

Before stating the results of this subsection, we fix a few notations. Let us set $D := (k+1)(d-1)$. The line bundle $\mathcal{O}_{\mathbb{P}^k}(D)$ over \mathbb{P}^k is seen as the quotient of $(\mathbb{C}^{k+1} \setminus \{0\}) \times \mathbb{C}$ by the relation $(\tilde{z}, x) \equiv (u\tilde{z}, u^D x)$ for every $u \in \mathbb{C}^*$ and its elements are denoted by $[\tilde{z}, x]$. We endow $\mathcal{O}_{\mathbb{P}^k}(D)$ with the canonical metric

$$\|[\tilde{z}, x]\|_0 := e^{-D \cdot \log \|\tilde{z}\|} |x|$$

or, for any $\lambda \in M$, with the metric

$$\|[\tilde{z}, x]\|_\lambda := e^{-D \cdot G(\lambda, \tilde{z})} |x|.$$

Let us set $J_F(\lambda, \tilde{z}) := \det d_{\tilde{z}} F_\lambda$. Then we obtain a family of holomorphic sections of $\mathcal{O}_{\mathbb{P}^k}(D)$ by setting, for every $\tilde{z} \in \mathbb{C}^{k+1} \setminus \{0\}$:

$$J_F^s(\lambda, \pi(\tilde{z})) := [\tilde{z}, J_F(\lambda, \tilde{z})].$$

Observe that

$$(4) \quad \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda = e^{-D \cdot G(\lambda, \tilde{z})} |J_F(\lambda, \tilde{z})|.$$

The current $\mathbf{C}_f := dd_{\lambda,z}^c \log \|J_F^s(\lambda, z)\|_0$ is the current of integration on C_f taking account the topological multiplicities of f , its bidimension is equal to (κ, κ) where $\kappa := k + m - 1$.

Theorem 3.2. (Bassanelli-Berteloot) *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let $L(\lambda)$ be the sum of the Lyapunov exponents of μ_λ . Then*

$$dd_\lambda^c L = \pi_{M*} (\mathcal{E}_{Green} \wedge \mathbf{C}_f).$$

We end this subsection by explaining how Pham [Pha] obtained a more general formula. His result holds for any equilibrium current of any family of polynomial-like maps, we state it in the special case of non-degenerate homogeneous maps for sake of simplicity. Let us recall that for such a family F , the function $\log |J_F(\lambda, \tilde{z})|$ is *psh* on $M \times \mathbb{C}^{k+1}$. Moreover, the sum of Lyapunov exponents of F_λ with respect to its equilibrium measure ν_λ is given by $\int_{\mathbb{C}^{k+1}} \log |J_F(\lambda, \tilde{z})| d\nu_\lambda(\tilde{z})$ and is equal to $L(\lambda) + \log d$ where $L(\lambda)$ is the sum of Lyapunov exponents of (f_λ, μ_λ) .

Theorem 3.3. (Pham) *Let $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$ be a holomorphic family of non-degenerate d -homogeneous maps and let \mathcal{E} be an equilibrium current for F . Then:*

- (1) *the current $\log |J_F| \cdot \mathcal{E}$ has locally finite mass,*
- (2) $dd_\lambda^c L = \pi_{M*} (\mathcal{E} \wedge dd_{\lambda, \tilde{z}}^c \log |J_F|)$.

To prove that $dd_\lambda^c L$ vanishes when repelling J -cycles move holomorphically (subsection 3.2), we shall actually need the following formula for $dd_\lambda^c L$ whose proof follows Pham's arguments.

Proposition 3.4. *Let B be an open ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that f admits an equilibrium web \mathcal{M} . Then*

$$dd_\lambda^c L = \pi_{B*} (\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, z}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda)$$

where $\widetilde{W}_\mathcal{M}$ is the (m, m) -bidimensional current on $M \times \mathbb{C}^{k+1}$ associated to \mathcal{M} by Lemma 2.8.

Proof. We first check that for every $\lambda \in B$ we have

$$(5) \quad \int_{\mathbb{C}^{k+1}} \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \langle \widetilde{W}_\mathcal{M}, \pi_B, \lambda \rangle = L(\lambda) + \log d.$$

Indeed, since $\pi_* \langle \widetilde{W}_\mathcal{M}, \pi_B, \lambda \rangle = \mu_\lambda$, we get

$$\int_{\mathbb{C}^{k+1}} \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \langle \widetilde{W}_\mathcal{M}, \pi_B, \lambda \rangle = \int_{\mathbb{P}^k} \log \|J_F^s(\lambda, z)\|_\lambda \mu_\lambda.$$

On the other hand, by Formula (4) and since G_λ identically vanishes on the support of the equilibrium measure ν_λ of F_λ and $\pi_* \nu_\lambda = \mu_\lambda$, we have

$$\begin{aligned} \int_{\mathbb{P}^k} \log \|J_F^s(\lambda, z)\|_\lambda \mu_\lambda &= \int_{\mathbb{C}^{k+1}} \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \nu_\lambda \\ &= \int_{\mathbb{C}^{k+1}} \log |J_F(\lambda, \tilde{z})| \nu_\lambda = L(\lambda) + \log d, \end{aligned}$$

and the identity (5) follows.

Pham proved that $u \cdot \mathcal{R}$ has locally finite mass for every *psh* function u and every horizontal current \mathcal{R} as soon as $\int_{\mathbb{C}^{k+1}} u(\lambda, \cdot) \langle \mathcal{R}, \pi_B, \lambda \rangle \neq -\infty$ for some $\lambda \in M$, see [Pha, theorem A.2]. It thus follows from (5) that the current $\log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \cdot \widetilde{W}_\mathcal{M}$ is well defined and that its $dd_{\lambda, \tilde{z}}^c$ is equal to $\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda$.

We conclude by simple computation which relies on integration by parts (to make it rigorous one should approximate $\log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda$ by smooth functions). Let φ be a $(m-1, m-1)$ test form on B . Then

$$\begin{aligned} \langle \pi_{B^*} \left(\widetilde{W}_M \wedge dd_{\lambda, \tilde{z}}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \right), \varphi \rangle &= \langle \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \cdot \widetilde{W}_M, dd_{\lambda, \tilde{z}}^c (\pi_B^* \varphi) \rangle \\ &= \langle \widetilde{W}_M \wedge \pi_B^* (dd_\lambda^c \varphi), \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \rangle. \end{aligned}$$

By the basic slicing formula (3) and the identity (5), this is equal to

$$\int_B \left(\langle \widetilde{W}_M, \pi_B, \lambda \rangle \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \right) dd_\lambda^c \varphi = \int_B L dd_\lambda^c \varphi = \langle dd_\lambda^c L, \varphi \rangle.$$

This completes the proof. \square

3.2. Repelling cycles do not move holomorphically on $\text{supp } dd_\lambda^c L$. The following proposition will be used in subsection 3.5 to prove $(A) \Rightarrow (B)$ of Theorem 1.1, namely that $dd_\lambda^c L = 0$ on M if the repelling J -cycles move holomorphically.

Proposition 3.5. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k which admits an equilibrium web \mathcal{M} which is given by $\mathcal{M} = \lim_n \mathcal{M}_n$ where $\Gamma_\gamma \cap C_f = \emptyset$ for any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. Then $dd_\lambda^c L = 0$ on M .*

The fact that the holomorphic motion of the repelling J -cycles over M imply the pluriharmonicity of L on M was proved in [BB1, Theorem 2.2] or [BDM, Theorem 1.5]. Proposition 3.5 actually provides a more general result. The proof needs the following technical lemma.

Lemma 3.6. *Let B be an open ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let Z be a codimension 1 analytic subset of $B \times \mathbb{P}^k$ which does not contain any fiber $\{\lambda\} \times \mathbb{P}^k$. Assume that there exists an equilibrium web satisfying $\mathcal{M} = \lim_n \mathcal{M}_n$, where $\Gamma_\gamma \cap Z = \emptyset$ for every $\gamma \in \cup_n \text{supp } \mathcal{M}_n$ and every $n \geq n_0$. Let B' be a relatively compact ball in B . Then, after shrinking B , there exist $A > 0$ and $0 < a < 1$ such that*

$$\mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \cap Z_\epsilon \neq \emptyset\} \leq A\epsilon^a$$

for every sufficiently small $\epsilon > 0$, where Z_ϵ is the ϵ -neighbourhood of Z .

Proof. We can assume that both B and B' are centered at some λ_0 . After maybe shrinking B we may find a finite collection $(\Omega_i, h_i)_{1 \leq i \leq N}$ where the Ω_i are open and cover $B \times \mathbb{P}^k$, the functions h_i are holomorphic and bounded by 1 on Ω_i and $Z \cap \Omega_i = \{h_i = 0\}$ for any $1 \leq i \leq N$. If ϵ is small enough, we may also assume that $Z_\epsilon \cap \Omega_i \subset \{|h_i| < C_1 \epsilon\}$ and, by Łojasiewicz inequality, that $\{|h_i| < \epsilon\} \subset Z_{C_2 \epsilon^\tau}$ for some constants $C_1, C_2, \tau > 0$. Similarly, one has $Z_\epsilon \cap (\{\lambda_0\} \times \mathbb{P}^k) \subset (Z \cap (\{\lambda_0\} \times \mathbb{P}^k))_{C_3 \epsilon^{\tau_0}}$ for some constants $C_3, \tau_0 > 0$.

Since \mathcal{M} has compact support in \mathcal{J} , we may shrink B again so that for any $\gamma \in \text{supp } \mathcal{M}$ there exists at least one $1 \leq i \leq N$ such that $\Gamma_\gamma \subset \Omega_i$. We shall use the following claim.

Claim: *there exists $0 < \alpha \leq 1$ such that $\sup_{B'} |\phi| \leq |\phi(t_0)|^\alpha$ for every $t_0 \in B'$ and for every holomorphic function $\phi : B \rightarrow \mathbb{D}^*$.*

Let $\gamma \in \text{supp } \mathcal{M}$ such that $\Gamma_\gamma \cap Z = \emptyset$ and $\Gamma_{\gamma|_{B'}} \cap Z_\epsilon \neq \emptyset$. Applying the Claim to $h_i \circ \gamma$ with $\Gamma_\gamma \subset \Omega_i$ we obtain that $\Gamma_{\gamma|_{B'}} \subset Z_{C_4 \epsilon^{\tau\alpha}}$ for some constant $C_4 > 0$.

On the other hand, by our assumption on the approximation by \mathcal{M}_n , Hurwitz lemma implies that either $\Gamma_\gamma \subset Z$ or $\Gamma_\gamma \cap Z = \emptyset$ for any $\gamma \in \text{supp } \mathcal{M}$. We thus have

$$\begin{aligned} \mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \cap Z_\epsilon \neq \emptyset\} &\leq \mathcal{M}\{\gamma \in \mathcal{J} : \Gamma_{\gamma|_{B'}} \subset Z_{C_4\epsilon^{\tau\alpha}}\} \leq \\ \mathcal{M}\{\gamma \in \mathcal{J} : (\lambda_0, \gamma(\lambda_0)) \in Z_{C_4\epsilon^{\tau\alpha}}\} &= \mu_{\lambda_0} (Z_{C_4\epsilon^{\tau\alpha}} \cap (\{\lambda_0\} \times \mathbb{P}^k)) \leq \\ &\mu_{\lambda_0} \left[(Z \cap (\{\lambda_0\} \times \mathbb{P}^k))_{C_3(C_4\epsilon^{\alpha\tau})^{\tau_0}} \right] \leq A\epsilon^a \end{aligned}$$

where the last estimate is due to the fact that μ_{λ_0} has Hölder-continuous local potentials and $Z \cap (\{\lambda_0\} \times \mathbb{P}^k)$ is a proper analytic subset of \mathbb{P}^k .

It remains to prove the Claim. Let $\mathcal{G} := \{\varphi \in \mathcal{O}(B, H) : \varphi(s) = -1 \text{ for some } s \in \overline{B'}\}$ where $H := \{\Re z < 0\}$ is the left half plane. Then \mathcal{G} is compact for the topology of local uniform convergence, and thus the quantity $(-\alpha) := \sup_{\varphi \in \mathcal{G}} \sup_{s \in \overline{B'}} \Re \varphi(s)$ satisfies $-1 \leq -\alpha < 0$. Let $t_0 \in B'$ and $\phi : B \rightarrow \mathbb{D}^*$ be holomorphic. After a rotation in \mathbb{D}^* we may assume that $|\phi(t_0)| = \phi(t_0) \in]0, 1[$. Let $\varphi : B \rightarrow H$ be the lift of ϕ by the exponential map, which satisfies $\varphi(t_0) = \log \phi(t_0) \in]-\infty, 0[$. Then $\varphi_0(t) := -\varphi(t) : \varphi(t_0)$ belongs to \mathcal{G} and thus $\Re(\varphi_0) \leq -\alpha$ on B' . This is the desired estimate since $|\phi| = e^{\Re \varphi} \leq e^{\alpha \log \phi(t_0)} = |\phi(t_0)|^\alpha$. \square

PROOF OF PROPOSITION 3.5: The problem is local and we may therefore take for M a ball $B \subset \mathbb{C}^m$ and assume that $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ admits a lifted family $F : B \times \mathbb{C}^{k+1} \rightarrow B \times \mathbb{C}^{k+1}$ of d -homogeneous non-degenerate maps. We will apply Lemma 3.6 with $Z = C_f$. Let B' be any relatively compact ball contained in B .

After shrinking B we may use Lemma 2.8 and associate to \mathcal{M} the following horizontal current on $B \times \mathbb{C}^{k+1}$

$$\widetilde{W}_{\mathcal{M}} = \int_{\mathcal{J}} [\Gamma_{\sigma(\gamma)}] d\mathcal{M}(\gamma).$$

According to Proposition 3.4, one has

$$dd_{\lambda}^c L = \pi_{B\star} (\widetilde{W}_{\mathcal{M}} \wedge dd_{\lambda, \tilde{z}}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_{\lambda}).$$

Using $\|J_F^s(\lambda, \pi(\tilde{z}))\|_{\lambda} = e^{-D \cdot G(\lambda, \tilde{z})} |J_F(\lambda, \tilde{z})|$ (see Formula (4)), and the fact that the functions L and G are psh , we obtain

$$\begin{aligned} 0 \leq dd_{\lambda}^c L &= \pi_{B\star} (\widetilde{W}_{\mathcal{M}} \wedge dd_{\lambda, \tilde{z}}^c \log |J_F|) - D\pi_{B\star} (\widetilde{W}_{\mathcal{M}} \wedge dd_{\lambda, \tilde{z}}^c G) \\ &\leq \pi_{B\star} (\widetilde{W}_{\mathcal{M}} \wedge dd_{\lambda, \tilde{z}}^c \log |J_F|). \end{aligned}$$

Hence it suffices to show that the current $\log |J_F| \widetilde{W}_{\mathcal{M}}$ restricted to $B' \times \mathbb{C}^{k+1}$ is $dd_{\lambda, \tilde{z}}^c$ closed.

For $\epsilon < 1$ we set $\log_{\epsilon} := \chi_{\epsilon} \circ \log$ where χ_{ϵ} is a convex, smooth, increasing function on \mathbb{R} such that $\chi_{\epsilon}(x) = x$ if $x \geq \log \epsilon$ and $\chi_{\epsilon}(-\infty) = 2 \log \epsilon$. Then $\log_{\epsilon} |J_F|$ is a decreasing family (when $\epsilon \rightarrow 0$) of smooth psh functions which converges to $\log |J_F|$. As $\lim_{\epsilon \rightarrow 0} \log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M}} = \log |J_F| \widetilde{W}_{\mathcal{M}}$ we will actually deal with $\log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M}}$.

To this purpose we set $U_{\epsilon} := \{|J_F| < \epsilon\}$, $S_{\mathcal{M}, \epsilon} := \{\gamma \in \text{supp } \mathcal{M} : \Gamma_{\sigma(\gamma)|_{B'}} \cap U_{\epsilon} \neq \emptyset\}$ and decompose $\widetilde{W}_{\mathcal{M}}$ as:

$$\widetilde{W}_{\mathcal{M}} = \widetilde{W}_{\mathcal{M}, \epsilon} + \widetilde{W}_{\mathcal{M}, \epsilon}^*$$

where $\widetilde{W}_{\mathcal{M},\epsilon} := \int_{\mathcal{J}} [\Gamma_{\sigma(\gamma)}] 1_{S_{\mathcal{M},\epsilon}} d\mathcal{M}(\gamma)$ and $\widetilde{W}_{\mathcal{M},\epsilon}^* := \widetilde{W}_{\mathcal{M}} - \widetilde{W}_{\mathcal{M},\epsilon}$. Then

$$\log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M}} = \log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M},\epsilon} + \log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M},\epsilon}^*$$

and, by construction, the current $\log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M},\epsilon}^*$ on $B' \times \mathbb{C}^{k+1}$ is $dd_{\lambda,\bar{z}}^c$ -closed since $\log_{\epsilon} |J_F| = \log |J_F|$ is pluriharmonic on the graphs Γ_{γ} which do not intersect U_{ϵ} . It thus remains to check that $\lim_{\epsilon} \log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M},\epsilon} = 0$. This follows from the estimate

$$\|\log_{\epsilon} |J_F| \widetilde{W}_{\mathcal{M},\epsilon}\| \lesssim |\log \epsilon| \mathcal{M}(S_{\mathcal{M},\epsilon}) \lesssim \epsilon^a |\log \epsilon|$$

where the last inequality is obtained by observing that there exist $b, \beta > 0$ such that $S_{\mathcal{M},\epsilon} \subset \{\gamma \in \mathcal{J} : \Gamma_{\gamma}|_{B'} \cap (C_f)_{b\epsilon^{\beta}} \neq \emptyset\}$ and applying Lemma 3.6. \square

3.3. Misiurewicz parameters belong to $\text{supp } dd_{\lambda}^c L$. We establish here the following result.

Proposition 3.7. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Then the Misiurewicz parameters belong to the support of $dd_{\lambda}^c L$.*

Proof. If $\lambda_0 \in M$ is a Misiurewicz parameter then, by definition, there exists a holomorphic map γ from a neighbourhood of λ_0 into \mathbb{P}^k such that:

- 1) $\gamma(\lambda) \in J_{\lambda}$ and is a repelling p_0 -periodic point of f_{λ} for some $p_0 \geq 1$,
- 2) $(\lambda_0, \gamma(\lambda_0)) \in f^{n_0}(C_f)$ for some $n_0 \geq 1$,
- 3) the graph Γ_{γ} of γ is not contained in $f^{n_0}(C_f)$.

Without loss of generality we may assume that $p_0 = 1$ and that M is a disc $D_{\rho} \subset \mathbb{C}$ centered at $\lambda_0 = 0$ with radius ρ . Moreover, conjugating by $(\lambda, z) \mapsto (\lambda, T_{\gamma(\lambda)}(z))$ where $T_{\gamma(\lambda)}$ is a suitable family of linear automorphisms of \mathbb{P}^k ensures that γ is constant equal to $z_1 := \gamma(0)$. Let us denote by B_r a ball centered at z_1 and of radius r . Taking ρ and r sufficiently small finally allows us to suppose that:

- (i) f is injective and uniformly expanding on $D_{\rho} \times B_r$: there exists $K > 1$ such that

$$\forall (\lambda, z) \in D_{\rho} \times B_r, d_{\mathbb{P}^k}(f(\lambda, z), f(\lambda, z_1)) \geq K d_{\mathbb{P}^k}(z, z_1)$$

- (ii) $(\lambda, z_1) \in f^{n_0}(C_f) \Leftrightarrow \lambda = 0$.

The fact that $\gamma(\lambda) \in J_{\lambda}$ is crucial but will only be used at the very end of the proof.

We have to show that $\langle dd_{\lambda}^c L, 1_{D_{\epsilon}} \rangle > 0$ for some $0 < \epsilon < \rho$. To this purpose, we will use the formula $dd_{\lambda}^c L = (\pi_{D_{\rho}})_{\star} \left((dd_{\lambda,z}^c g + \omega)^k \wedge \mathbf{C}_f \right)$ given by Theorem 3.2, where $\omega := \omega_{FS}$.

Let $(g_n)_n$ be a sequence of smooth functions on \mathbb{P}^k which converges uniformly to g and satisfies $\frac{1}{d} f^{\star} (dd_{\lambda,z}^c g_n + \omega) = dd_{\lambda,z}^c g_{n+1} + \omega$ (see subsection 3.1). We shall proceed in three steps.

First step: $\langle dd_{\lambda}^c L, 1_{D_{\epsilon}} \rangle \geq d^{-n_0 k} \langle [f^{n_0}(C_f)] \wedge (dd_{\lambda,z}^c g + \omega)^k, 1_{D_{\epsilon} \times B_r} \rangle$.

Pick $(0, z_0) \in C_f$ such that $f^{n_0}(0, z_0) = (0, z_1)$. After reducing ϵ and r , we may find a neighbourhood U of $(0, z_0)$ such that the map $f^{n_0} : U \rightarrow D_{\epsilon} \times B_r$ is proper. According to Theorem 3.2, we have

$$\langle dd_{\lambda}^c L, 1_{D_{\epsilon}} \rangle = \langle (dd_{\lambda,z}^c g + \omega)^k \wedge \mathbf{C}_f, 1_{D_{\epsilon}} \circ \pi_{D_{\rho}} \rangle \geq \langle (dd_{\lambda,z}^c g + \omega)^k \wedge [C_f], 1_U \rangle.$$

Using the smooth approximations g_n , we get

$$\begin{aligned} \langle (dd_{\lambda,z}^c g + \omega)^k \wedge [C_f], 1_U \rangle &= \lim_n \langle (dd_{\lambda,z}^c g_{n+n_0} + \omega)^k \wedge [C_f], 1_U \rangle \\ &= \lim_n d^{-n_0 k} \langle 1_U \cdot [C_f], (f^{n_0})^* (dd_{\lambda,z}^c g_n + \omega)^k \rangle \\ &= \lim_n d^{-n_0 k} \langle (f^{n_0})_* (1_U \cdot [C_f]), (dd_{\lambda,z}^c g_n + \omega)^k \rangle. \end{aligned}$$

Now, as $f^{n_0} : U \rightarrow D_\epsilon \times B_r$ is proper, one has $(f^{n_0})_* (1_U [C_f]) \geq 1_{D_\epsilon \times B_r} [f^{n_0}(C_f)]$ which, since $dd_{\lambda,z}^c g_n + \omega$ is positive, yields

$$\begin{aligned} \langle dd_{\lambda,z}^c L, 1_{D_\epsilon} \rangle &\geq \lim_n d^{-n_0 k} \langle 1_{D_\epsilon \times B_r} [f^{n_0}(C_f)], (dd_{\lambda,z}^c g_n + \omega)^k \rangle \\ &\geq \lim_n d^{-n_0 k} \langle (dd_{\lambda,z}^c g_n + \omega)^k \wedge [f^{n_0}(C_f)], 1_{D_\epsilon \times B_r} \rangle. \end{aligned}$$

The desired estimate follows by uniform convergence of g_n to g .

Second step: Let $A_0 := 1_{D_\epsilon \times B_r} [f^{n_0}(C_f)]$ and $A_{p+1} := 1_{D_\epsilon \times B_r} f_* (A_p)$. Then

$$\begin{aligned} \|A_p \wedge (dd_{\lambda,z}^c g + \omega)^k\| &= d^{pk} \| (1_{D_\epsilon \times B_r} \circ f^p) A_0 \wedge (dd_{\lambda,z}^c g + \omega)^k \| \\ &\leq d^{pk} \|A_0 \wedge (dd_{\lambda,z}^c g + \omega)^k\|. \end{aligned}$$

We use again the smooth approximations g_n . Then:

$$\begin{aligned} \|A_{p+1} \wedge (dd_{\lambda,z}^c g_n + \omega)^k\| &= \langle 1_{D_\epsilon \times B_r} f_* (A_p), (dd_{\lambda,z}^c g_n + \omega)^k \rangle \\ &= \langle A_p, f^* (1_{D_\epsilon \times B_r} (dd_{\lambda,z}^c g_n + \omega)^k) \rangle \\ &= d^k \langle A_p, 1_{D_\epsilon \times B_r} \circ f (dd_{\lambda,z}^c g_{n+1} + \omega)^k \rangle \\ &= d^k \langle A_p \wedge (dd_{\lambda,z}^c g_{n+1} + \omega)^k, 1_{D_\epsilon \times B_r} \circ f \rangle \\ &= d^k \| (1_{D_\epsilon \times B_r} \circ f) A_p \wedge (dd_{\lambda,z}^c g_{n+1} + \omega)^k \|. \end{aligned}$$

Taking the limits when n tends to infinity yields the conclusion.

Third step: $\langle dd_{\lambda,z}^c L, 1_{D_\epsilon} \rangle > 0$.

By combining the two former steps, one gets:

$$(6) \quad d^{(p+n_0)k} \langle dd_{\lambda,z}^c L, 1_{D_\epsilon} \rangle \geq \|A_p \wedge (dd_{\lambda,z}^c g + \omega)^k\|.$$

By (i) and (ii), f is uniformly expanding on $D_\rho \times B_r$ and $(\text{supp } A_0) \cap (D_\rho \times \{z_1\}) = \{(0, z_1)\}$. Thus $\text{supp } A_p \subset D_{\epsilon_p} \times B_r$ for some $\epsilon_p \rightarrow 0$. Let us momentarily admit that there exists $m > 0$ such that

$$(7) \quad A_p \rightarrow m [\{0\} \times B_r].$$

We then deduce from (6) that, for p large enough, one has:

$$d^{(p+n_0)k} \langle dd_{\lambda,z}^c L, 1_{D_\epsilon} \rangle \geq \frac{m}{2} \| [\{0\} \times B_r] \wedge (dd_{\lambda,z}^c g + \omega)^k \|^k.$$

We conclude by using the fact that $z_1 \in J_0$: the right hand side is equal to

$$\frac{m}{2} \int_{B_r} (dd_z^c g(0, z) + \omega)^k = \frac{m}{2} \mu_0(B_r) > 0.$$

To complete the proof it remains to establish (7). Let us denote $V := D_\rho \times B_r$ and $V' := f(V)$. By assumption $f : V \rightarrow V'$ is a biholomorphism whose inverse will be denoted by $h : V' \rightarrow V$. According to (i), $V \subset V'$ and $(h|_V)^P$ converges to $(\lambda, z) \mapsto (\lambda, z_1)$. We now use (ii). After shrinking ρ and r , we may find a Weierstrass polynomial

$$\psi(\lambda, z) := \lambda^m + \alpha_{m-1}(z)\lambda^{m-1} + \dots + \alpha_0(z)$$

such that $\alpha_j(z_1) = 0$ for $0 \leq j \leq m-1$ and $f^{n_0}(C_f) \cap (D_\rho \times B_r) = \{\psi = 0\}$. Observe now that $A_0 = 1_V dd_{\lambda, z}^c \log |\psi|$ and that

$$A_1 = 1_V f_* A_0 = 1_V h^* A_0 = 1_V (1_V \circ h) dd_{\lambda, z}^c \log |\psi \circ h| = 1_V dd_{\lambda, z}^c \log |\psi \circ h|,$$

where the last equality comes from $h(V) \subset V$. Similarly we have $A_p = 1_V dd_{\lambda, z}^c \log |\psi \circ (h|_V)^P|$ and the conclusion follows since $\psi \circ (h|_V)^P(\lambda, z) \rightarrow \lambda^m$. \square

3.4. Misiurewicz parameters are dense in $\text{supp } dd_\lambda^c L$. We start with the following proposition; the statement is local since it is based on holomorphic motion of hyperbolic sets.

Proposition 3.8. *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k where B is a ball centered at the origin in \mathbb{C}^m . If there is no Misiurewicz parameter in B then, after shrinking B , there exists $\gamma \in \mathcal{J}$ such that Γ_γ does not intersect the post-critical set of f .*

Every hyperbolic set admits a holomorphic motion which preserves repelling cycles (see subsection A.2). We need a more precise result concerning the size of such sets and the position of their motions with respect to Julia sets. Here B_r denotes a ball centered at the origin in \mathbb{C}^m and of radius r .

Theorem 3.9. *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . There exist an integer N , a compact hyperbolic set $E_0 \subset J_0$ for f_0^N and a holomorphic motion $h : B_r \times E_0 \rightarrow \mathbb{P}^k$ for some $0 < r < 1$ such that:*

- (1) *the repelling periodic points of f_0^N are dense in E_0 and E_0 is not contained in the post-critical set of f_0^N ,*
- (2) *$h_\lambda(z) \in J_\lambda$ for every $\lambda \in B_r$ and every $z \in E_0$,*
- (3) *if z is periodic repelling for f_0^N then $h_\lambda(z)$ is periodic repelling for f_λ^N .*

The proof of this result requires a few tools. To create hyperbolic sets, we use a classical device based on the following proposition which is a consequence of [BrDv1] (see also [BDM]). For any endomorphism f_0 of \mathbb{P}^k and every $A \subset \mathbb{P}^k$, $n \geq 1$ and $\rho > 0$, we denote by $C_n(A, \rho)$ the set of inverse branches g_i of f_0^n defined on A and satisfying $g_i(A) \subset A$ and $\text{Lip } g_i \leq \rho$.

Proposition 3.10. *Let f_0 be an endomorphism of \mathbb{P}^k of degree d . For every $\rho > 0$ there exist a closed ball $A \subset \mathbb{P}^k$ centered on J_{f_0} and $\alpha > 0$ such that $\text{Card } C_n(A, \rho) \geq \alpha d^{kn}$.*

To control the size of hyperbolic sets, we use an entropy argument. Our key tool is the following result which is due to Briend-Duval [BrDv2], de Thélin [dT] and Dinh [Di3] (see also [DS3] Corollary 1.117).

Theorem 3.11. *Let g be an endomorphism of \mathbb{P}^k of degree d . Let κ be an ergodic g -invariant measure with entropy $h_\kappa > (k-1)\log d$. Then κ gives no mass to analytic subsets of dimension $\leq k-1$ and the support of κ is included in the Julia set of g .*

PROOF OF THEOREM 3.9: Let $\rho < 1$ and A be a closed ball provided by Proposition 3.10. Let us fix N large enough such that $N' := \text{Card } C_N(A, \rho) > d^{(k-1)N}$. We denote by $g_1, \dots, g_{N'}$ the elements of $C_N(A, \rho)$. Let $E_0 := \bigcap_{k \geq 1} E_k$, where

$$E_k := \left\{ g_{i_1} \circ \dots \circ g_{i_k}(A) : (i_1, \dots, i_k) \in \{1, \dots, N'\}^k \right\}.$$

Let $\Sigma := \{1, \dots, N'\}^{\mathbb{N}^*}$ endowed with the product metric and z be a fixed point in $A \cap J_0$, for instance the center of A . The map $\omega : \Sigma \rightarrow E_0$ defined by $(i_1, i_2, \dots) \mapsto \lim_{k \rightarrow \infty} g_{i_1} \circ \dots \circ g_{i_k}(z)$ is a homeomorphism satisfying $f^N \circ \omega = \omega \circ s$, where s is the left shift acting on Σ . We take for κ the image by ω of the uniform product measure on Σ : this is a f^N -invariant ergodic measure with entropy $h_\kappa = \log N' > (k-1)\log d^n$, with support E_0 .

By construction $E_0 \subset J_{f_0}$. Indeed, $E_0 = \left\{ \lim_{k \rightarrow \infty} g_{i_1} \circ \dots \circ g_{i_k}(z) : (i_1, i_2, \dots) \in \Sigma \right\}$ and J_{f_0} is a closed f_0^N -invariant set. Also, repelling cycles of f_0^N are dense in E_0 . According to Theorem 3.11, $E_0 = \text{supp } \kappa$ is not contained in the countable union of analytic subsets $\bigcup_{n \geq 1} f_0^n(C_{f_0})$. The set E_0 is hyperbolic for f_0^N since $|(df_0^N)^{-1}|^{-1} > \frac{1}{\rho} > 1$ on E_0 and thus there exists a holomorphic motion $h : B_r \times E_0 \rightarrow \mathbb{P}^k$ which preserves repelling cycles (see Theorem A.4). It remains to show $h_\lambda(E_0) \subset J_{f_\lambda}$. For that purpose we use the fact that $h_\lambda : E_0 \rightarrow \mathbb{P}^k$ is a continuous injective mapping satisfying $h_\lambda \circ f_0^N = f_\lambda^N \circ h_\lambda$ on E_0 . Then $(h_\lambda)_* \kappa$ is a f_λ^N -invariant ergodic measure whose support coincides with $h_\lambda(E_0)$ and whose metric entropy equals h_κ . Theorem 3.11 yields $h_\lambda(E_0) \subset J_{f_\lambda}$ as desired. \square

We now use Theorem 3.9 to establish Proposition 3.8.

PROOF OF PROPOSITION 3.8: Since f_λ^N and f_λ have same equilibrium measures and post-critical sets, we may assume that $N = 1$. Let $E_0 \subset J_0$ and $r \in]0, 1]$ provided by Theorem 3.9. Let us fix $z \in E_0 \setminus \bigcup_{n \geq 1} f_0^n(C_{f_0})$ (see item 1).

Let us set $\gamma(\lambda) := h_\lambda(z)$. By item 2 we have $\gamma \in \mathcal{J}$. Let us show that

$$(8) \quad \Gamma_\gamma \cap \left(\bigcup_{n \geq 1} f^n(C_f) \right) = \emptyset.$$

Assume to the contrary that there exists $n_0 \geq 1$ such that $\Gamma_\gamma \cap f^{n_0}(C_f) \neq \emptyset$. Note that $\gamma(0) \notin f^{n_0}(C_f)$. By item 1, there exists a sequence $(z_p)_p \subset E_0$ of f_0 -periodic repelling points which converges to z . Items 2 and 3 assert that $h_\lambda(z_p) \in J_\lambda$ and $h_\lambda(z_p)$ is a f_λ -periodic repelling point for every $\lambda \in B_r$. As h is continuous, $\lambda \mapsto h_\lambda(z_p)$ converges locally uniformly to $\lambda \mapsto h_\lambda(z) = \gamma(\lambda)$. Hence, for p large enough, the graph $\{(\lambda, h_\lambda(z_p)) \mid \lambda \in B_r\}$ is not contained in $f^{n_0}(C_f)$ (consider the parameter $\lambda = 0$) and, by Hurwitz's lemma, there exists $\lambda_p \in B_r$ such that $(\lambda_p, h_{\lambda_p}(z_p)) \in f^{n_0}(C_f)$. The parameters λ_p are Misiurewicz, contradicting our assumption. \square

We can now prove Theorem 1.6 which, in particular, says that Misiurewicz parameters are dense in the support of $dd^c L$.

PROOF OF THEOREM 1.6: By Proposition 3.7 there are no Misiurewicz parameters in M if $dd_\lambda^c L \equiv 0$ on M and thus (a) \Rightarrow (b). If there are no Misiurewicz parameters in M then, by Propositions 3.8 and 2.3, for any parameter λ one can find an open ball B centered at λ such

that the restriction $f|_{B \times \mathbb{P}^k}$ admits an equilibrium web $\mathcal{M} = \lim_n \mathcal{M}_n$ satisfying $\Gamma_\gamma \cap C_f = \emptyset$ for any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. Thus (b) \Rightarrow (c). Finally, (c) \Rightarrow (a) follows from Proposition 3.5. \square

3.5. Proofs of part of Theorem 1.1 and Corollary 1.7. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . We first establish the implication (A) \Rightarrow (B) in Theorem 1.1. If the repelling J -cycles of f move holomorphically then, using the second assertion of Proposition 2.3, one gets an equilibrium web \mathcal{M} of $(\mu_\lambda)_{\lambda \in M}$ such that $\mathcal{M} = \lim_n \mathcal{M}_n$ and $\Gamma_\gamma \cap C_f = \emptyset$ for any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. By Proposition 3.5, this implies that $dd_\lambda^c L \equiv 0$ on M . This justifies (A) \Rightarrow (B). The following proposition is in the spirit of the proposition 1.26 of [DS3] concerning the Julia set of a single endomorphism of \mathbb{P}^k , it implies the equivalence (B) \Leftrightarrow (E).

Proposition 3.12. *Let B be an open ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree d . We endow $B \times \mathbb{P}^k$ with the metric $dd_\lambda^c |\lambda|^2 + \omega_{FS}$ and denote $|\cdot|_U$ the mass of currents in $U \times \mathbb{P}^k$. The following properties are equivalent.*

- (1) $\lambda_0 \in \text{supp } dd_\lambda^c L$.
- (2) $|\mathcal{E}_{Green} \wedge \mathbf{C}_f|_U > 0$ for every neighbourhood U of λ_0 .
- (3) $\liminf_n d^{-kn} |(f^n)_* \mathbf{C}_f|_U > 0$ for every neighbourhood U of λ_0 .
- (4) $\limsup_n d^{-(k-1)n} |(f^n)_* \mathbf{C}_f|_U = +\infty$ for every neighbourhood U of λ_0 .

Proof. The equivalence between 1. and 2. follows from Theorem 3.2, which asserts that $dd_\lambda^c L = \pi_{B*}(\mathcal{E}_{Green} \wedge \mathbf{C}_f)$. The equivalence between 2. 3. and 4. come from Lemma 3.13 here below. \square

Lemma 3.13. *There exists $\alpha = \alpha(k, m) > 0$ such that, for every compact subset $U \subset M$:*

$$|(f^n)_* \mathbf{C}_f|_U = \alpha d^{kn} |\mathcal{E}_{Green} \wedge \mathbf{C}_f|_U + O(d^{(k-1)n}).$$

Proof. Let us set $\kappa := k + m - 1$. Then

$$|(f^n)_* \mathbf{C}_f|_U = \int_{U \times \mathbb{P}^k} (f^n)_* \mathbf{C}_f \wedge [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \int_{U \times \mathbb{P}^k} \mathbf{C}_f \wedge (f^n)^* [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa.$$

Using $\omega_{FS}^{k+1} = 0$, we obtain $[\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \sum_{j=0}^k \alpha_j \omega_{FS}^j \wedge (dd_\lambda^c |\lambda|^2)^{\kappa-j}$, where the α_j 's are positive numbers. Since $\pi_M \circ f = \pi_M$, we obtain

$$(f^n)^* [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \sum_{j=0}^k \alpha_j \left((f^n)^* \omega_{FS}^j \right) \wedge (dd_\lambda^c |\lambda|^2)^{\kappa-j}.$$

Let $\mathcal{T} := dd_{\lambda,z}^c g + \omega_{FS}$ so that $\mathcal{T}^k = \mathcal{E}_{Green}$. Using $f^* \mathcal{T} = d\mathcal{T}$ we get $(f^n)^* (\omega_{FS}^j) = (d^n \mathcal{T} - dd_{\lambda,z}^c g \circ f^n)^j$. Now, using the fact that g is bounded, by extracting the k -th term of the preceding sum we obtain:

$$(f^n)^* [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \alpha_k d^{kn} \mathcal{T}^k \wedge (dd_\lambda^c |\lambda|^2)^{m-1} + O(d^{(k-1)n}).$$

We set $\alpha := \alpha_k$. This completes the proof of the lemma. \square

PROOF OF COROLLARY 1.7: By assumption, for every $n \geq 1$ we have subsets $\mathcal{R}_n := \{\rho_{n,j} : 1 \leq j \leq N_n\}$ of \mathcal{J} such that the $\rho_{n,j}(\lambda)$ are repelling n -periodic points of f_λ for every $\lambda \in M$. Note that $\lim_n d^{-kn} N_n = 1$. We define a sequence $(\mathcal{M}_n)_n$ of \mathcal{F} -invariant discrete probability measures on \mathcal{J} by setting $\mathcal{M}_n := \frac{1}{N_n} \sum_{j=1}^{N_n} \delta_{\rho_{n,j}(\lambda)}$. According to the second assertion of Proposition 2.3, $(\mathcal{M}_n)_n$ converges to an equilibrium web \mathcal{M} after taking a subsequence. Moreover, there exists a compact subset \mathcal{K} of \mathcal{J} such that $\mathcal{K} \subset \mathcal{F}(\mathcal{K})$ and $\text{supp } \mathcal{M}_n \subset \mathcal{K}$ for every $n \geq 1$.

Let us now prove that $\mathcal{M}(\mathcal{J}_s) = 0$. By the implication (A) \Rightarrow (B) of Theorem 1.1 we have

$dd^c L = 0$ and then Theorem 1.6 implies that M does not contain Misiurewicz parameters. We can now see that for every $k \in \mathbb{N}$ and every $\gamma \in \text{supp } \mathcal{M}$ one has:

$$\Gamma_\gamma \cap f^k(C_f) \neq \emptyset \Rightarrow \Gamma_\gamma \subset f^k(C_f).$$

Indeed, if this were not the case, by Hurwitz theorem, we could find some $\gamma' \in \cup_n \text{supp } \mathcal{M}_n$ such that $\Gamma_{\gamma'} \cap f^k(C_f) \neq \emptyset$ and $\Gamma_{\gamma'}$ is not contained in $f^k(C_f)$. When $k = 0$ this is clearly impossible since $\gamma'(\lambda)$ is a repelling cycle of f_λ and when $k \geq 1$, this is impossible because M does not contain Misiurewicz parameter.

So, fixing any $\lambda_0 \in M$, we get

$$\begin{aligned} \mathcal{M}(\{\gamma \in \mathcal{J} : \Gamma_\gamma \cap (\cup_{k \geq 0} f^k(C_f)) \neq \emptyset\}) &= \mathcal{M}(\{\gamma \in \mathcal{J} : \Gamma_\gamma \subset (\cup_{k \geq 0} f^k(C_f))\}) \leq \\ &= \mathcal{M}(\{\gamma \in \mathcal{J} : (\lambda_0, \gamma(\lambda_0)) \in (\cup_{k \geq 0} f^k(C_f))\}) = \mu_{\lambda_0}(\cup_{k \geq 0} f^k_{\lambda_0}(C_{f_{\lambda_0}})) = 0 \end{aligned}$$

where the two last equalities come from $p_{\lambda_0 \star}(\mathcal{M}) = \mu_{\lambda_0}$ and the fact that μ_{λ_0} does not charge pluripolar sets in \mathbb{P}^k . The estimate $\mathcal{M}(\mathcal{J}_s) = 0$ follows from the \mathcal{F} -invariance of \mathcal{M} . Finally, Proposition 2.4 shows that there exists an ergodic equilibrium web \mathcal{M}_0 such that $\mathcal{M}_0(\mathcal{J}_s) = 0$. \square

4. FROM EQUILIBRIUM WEBS TO EQUILIBRIUM LAMINATIONS

Our goal here is to establish the implication (A) \Rightarrow (D) in Theorem 1.1. We prove the following more precise result.

Theorem 4.1. *Let M be a simply connected complex manifold and $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. If the repelling J -cycles of f move holomorphically over M or if f admits an acritical and ergodic equilibrium web then there exists an equilibrium lamination \mathcal{L} for f . Moreover, f admits a unique equilibrium web \mathcal{M} and $\mathcal{M}(\mathcal{L}) = 1$.*

Given an acritical and ergodic equilibrium web \mathcal{M} of f , our strategy will consist in first proving that the iterated inverse branches in $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ are exponentially contracting and then exploit this property to extract an equilibrium lamination out of the support of \mathcal{M} . By totally different methods, Berger and Dujardin ([BeDu]) have recently build measurable holomorphic motions in the context of polynomial automorphisms of \mathbb{C}^2 .

4.1. On the rate of contraction of iterated inverse branches in $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. We explain here how certain stochastic properties of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ allow to control the rate of contraction of the iterated inverse branches of \mathcal{F} (see Proposition 4.2). Let us stress that the material presented in this subsection is not original. We simply adapt to the context of $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ the tools which have been first introduced in [BrDv1] by Briend-Duval for the case of a single holomorphic endomorphism of \mathbb{P}^k . New arguments however will be introduced in subsection 4.2.

Since all our statements here are local we may assume that the parameter space M is an open subset of \mathbb{C}^m which we endow with the euclidean norm.

To study the inverse branches of the map \mathcal{F} , it is convenient to transform the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ into an injective one. This is possible using a classical construction called the *natural extension* which we now describe (we refer to [CFS] page 240 for more details).

Recall that $\mathcal{K} := \text{supp } \mathcal{M}$ is a compact subset of \mathcal{J} and that $\mathcal{M}(\mathcal{J}_s) = 0$. Setting $\mathcal{X} := \mathcal{K} \setminus \mathcal{J}_s$, it is not difficult to check that the map $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{X}$ is onto. We may therefore construct the natural extension $(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}})$ of the system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$ in the following way. An element of $\widehat{\mathcal{X}}$

is a bi-infinite sequence $\widehat{\gamma} := (\dots, \gamma_{-j}, \gamma_{-j+1}, \dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots)$ of elements $\gamma_j \in \mathcal{X}$ such that $\mathcal{F}(\gamma_{-j}) = \gamma_{-j+1}$ and one defines the map $\widehat{\mathcal{F}} : \widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}$ by setting

$$\widehat{\mathcal{F}}(\widehat{\gamma}) := (\dots \mathcal{F}(\gamma_j), \mathcal{F}(\gamma_{-j+1}) \dots).$$

The map $\widehat{\mathcal{F}}$ corresponds to the shift operator and is clearly bijective. There exists a unique measure $\widehat{\mathcal{M}}$ on $\widehat{\mathcal{X}}$ such that

$$(\pi_j)_* (\widehat{\mathcal{M}}) = \mathcal{M}$$

for any projection $\pi_j : \widehat{\mathcal{X}} \rightarrow \mathcal{X}$ given by $\pi_j(\widehat{\gamma}) = \gamma_{-j}$. The ergodicity of \mathcal{M} implies the ergodicity of $\widehat{\mathcal{M}}$. We have thus obtained an invertible and ergodic dynamical system $(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}})$.

For every $\gamma \in \mathcal{J}$ whose graph Γ_γ does not meet the critical set of f , we denote by f_γ the injective map which is induced by f on some neighbourhood of Γ_γ and by f_γ^{-1} the inverse branch of f_γ which is defined on some neighbourhood of $\Gamma_{\mathcal{F}(\gamma)}$. Thus, given $\widehat{\gamma} \in \widehat{\mathcal{X}}$ and $n \in \mathbb{N}$ we may define the iterated inverse branch $f_{\widehat{\gamma}}^{-n}$ of f along $\widehat{\gamma}$ and of depth n by

$$f_{\widehat{\gamma}}^{-n} := f_{\gamma_{-n}}^{-1} \circ \dots \circ f_{\gamma_{-2}}^{-1} \circ f_{\gamma_{-1}}^{-1}.$$

Let us stress that $f_{\widehat{\gamma}}^{-n}$ is defined on a neighbourhood of Γ_{γ_0} with values in a neighbourhood of $\Gamma_{\gamma_{-n}}$. Moreover, since only a finite number of components of the grand critical orbit of f are involved for defining $f_{\widehat{\gamma}}^{-n}$, we may always shrink the parameter space M to some $\Omega \Subset M$ so that the domain of definition of $f_{\widehat{\gamma}}^{-n}$ for a fixed $\widehat{\gamma}$ contains a tubular neighbourhood of $\Gamma_{\gamma_0} \cap (\Omega \times \mathbb{P}^k)$ of the form

$$T_\Omega(\gamma_0, \eta) := \{(\lambda, z) \in \Omega \times \mathbb{P}^k : d_{\mathbb{P}^k}(z, \gamma_0(\lambda)) < \eta\}.$$

Our goal is to control the size of $f_{\widehat{\gamma}}^{-n}(T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma})))$ for suitable $\widehat{\eta}_p(\widehat{\gamma}) > 0$ and $U_0 \subset M$. We will now explain how this boils down to estimating some kind of Lyapunov exponent. This requires however to first introduce a few more notations.

To start we need to fix sets of holomorphic charts with bounded distortions on \mathbb{P}^k . For any $\tau > 0$, there exists a covering $\mathbb{P}^k = \cup_{i=1}^N V_i$ by open sets and a collection of holomorphic maps

$$\psi_i : V_i \times B_{\mathbb{C}^k}(0, R_0) \rightarrow \mathbb{P}^k$$

such that $\psi_{i,x} := \psi_i(x, \cdot)$ is a chart of \mathbb{P}^k satisfying $\psi_{i,x}(0) = x$ and

$$(9) \quad e^{-\tau/2}|z - z'| \leq d_{\mathbb{P}^k}(\psi_{i,x}(z), \psi_{i,x}(z')) \leq e^{\tau/2}|z - z'|$$

for every $(x, z) \in V_i \times B_{\mathbb{C}^k}(0, R_0)$ and every $1 \leq i \leq N$.

We will now use these holomorphic charts to express the restrictions of f^n on suitable neighbourhoods of graphs Γ_γ . Let us fix λ_0 in M . Since the family $\mathcal{K} = \text{supp } \mathcal{M}$ is locally equicontinuous, there exists a relatively compact open ball W_0 centered at λ_0 in M such that:

$$\forall \gamma \in \mathcal{K}, \exists i \in \{1, 2, \dots, N\} \text{ such that } \gamma(\lambda) \in V_i \text{ for all } \lambda \in \overline{W_0}.$$

For all $\gamma \in \mathcal{K}$ we set

$$i(\gamma) := \inf\{1 \leq i \leq N : \gamma(\lambda) \in V_i \text{ for all } \lambda \in \overline{W_0}\}.$$

Then, for every $n \geq 1$ there exists $R_n \in]0, R_0]$ such that the maps $F_{\gamma(\lambda)}^n$ given by

$$(10) \quad F_{\gamma(\lambda)}^n := (\psi_{i(\mathcal{F}^n(\gamma)), \mathcal{F}^n(\gamma)(\lambda)})^{-1} \circ f_\lambda^n \circ \psi_{i(\gamma), \gamma(\lambda)}$$

are well defined and holomorphic on a fixed neighbourhood of $\overline{W_0} \times \overline{B_{\mathbb{C}^k}(0, R_n)}$ for every $\gamma \in \mathcal{K}$. This follows immediately from the uniform continuity of f^n on $\overline{W_0} \times \mathbb{P}^k$.

As $F_{\gamma(\lambda)}^n$ is locally invertible at the origin when $\gamma \notin \mathcal{J}_s$, we may now define functions u_n on $\mathcal{X} \times \overline{W_0}$ by setting

$$u_n(\gamma, \lambda) := \log \|(DF_{\gamma(\lambda)}^n(0))^{-1}\|.$$

Let us stress that $(DF_{\gamma(\lambda)}^n(0))^{-1}$ depends holomorphically on $\lambda \in \overline{W_0}$.

From now on we consider three open balls $U_0 \Subset V_0 \Subset W_0$ centered at λ_0 in M . Let us introduce the function r_n on \mathcal{X} and \widehat{u}_n on $\widehat{\mathcal{X}}$ by setting

$$(11) \quad r_n(\gamma) := e^{-2 \sup_{\lambda \in U_0} u_n(\gamma, \lambda)} \quad \text{and} \quad \widehat{u}_n(\widehat{\gamma}) := \sup_{\lambda \in U_0} u_n(\gamma_0, \lambda) = -\frac{1}{2n} \log r_n(\gamma_0).$$

We may now state the announced result.

Proposition 4.2. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$ which admits an acritical and ergodic equilibrium web \mathcal{M} . Assume that the functions \widehat{u}_n are $\widehat{\mathcal{M}}$ -integrable and that $\lim_n \frac{1}{n} \int_{\widehat{\mathcal{X}}} \widehat{u}_n d\widehat{\mathcal{M}} = L$ for some $L \leq -\frac{1}{2} \log d$.*

Then there exist $p \geq 1$, a Borel subset $\widehat{\mathcal{Y}} \subset \widehat{\mathcal{X}}$ such that $\widehat{\mathcal{M}}(\widehat{\mathcal{Y}}) = 1$, a measurable function $\widehat{\eta}_p : \widehat{\mathcal{Y}} \rightarrow]0, 1]$ and a constant $A > 0$ which satisfy the following properties.

For every $\widehat{\gamma} \in \widehat{\mathcal{Y}}$ and every $n \in p\mathbb{N}^$ the iterated inverse branch $f_{\widehat{\gamma}}^{-n}$ is defined on the tubular neighbourhood $T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))$ of $\Gamma_{\gamma_0} \cap (U_0 \times \mathbb{P}^k)$ and*

$$f_{\widehat{\gamma}}^{-n}(T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))) \subset T_{U_0}(\gamma_{-n}, e^{-nA}).$$

Moreover, the map $f_{\widehat{\gamma}}^{-n}$ is Lipschitz with $\text{Lip } f_{\widehat{\gamma}}^{-n} \leq \widehat{l}_p(\widehat{\gamma}) e^{-nA}$ where $\widehat{l}_p(\widehat{\gamma}) \geq 1$.

The proof of Proposition 4.2 follows Briend-Duval [BrDv1] and is given in the Appendix.

4.2. Estimating a Lyapunov exponent. The main result of this subsection is as follows; it asserts that the assumptions of Proposition 4.2 are satisfied.

Proposition 4.3. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$ which admits an acritical and ergodic equilibrium web \mathcal{M} . Then the functions \widehat{u}_n are $\widehat{\mathcal{M}}$ -integrable, there exists a constant $L \leq -\frac{1}{2} \log d$ such that*

$$\lim_n \frac{1}{n} \int_{\widehat{\mathcal{X}}} \widehat{u}_n d\widehat{\mathcal{M}} = L$$

and $\lim_n \frac{1}{n} \widehat{u}_n(\widehat{\gamma}) = L$ for $\widehat{\mathcal{M}}$ -almost every $\widehat{\gamma} \in \mathcal{X}$.

Note that the constant L may be considered as a bound for a Lyapunov exponent of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. The combination of Propositions 4.2 and 4.3 will allow us to prove Theorem 4.1.

We keep here the assumptions and the notations introduced in the previous subsection. In the next Lemma, we list some basic properties of the functions u_n and \widehat{u}_n .

Lemma 4.4. *Let $U_0 \Subset V_0 \Subset W_0$ be open balls centered at λ_0 in M . Let $\chi_1(\lambda)$ be the smallest Lyapunov exponent of the system $(J_\lambda, f_\lambda, \mu_\lambda)$. The functions u_n and \widehat{u}_n satisfy the following properties.*

- 1) $u_n(\gamma, \cdot)$ is psh on W_0 for every $\gamma \in \mathcal{X}$.
- 2) The sequence $(\widehat{u}_n)_n$ is subadditive on $\widehat{\mathcal{X}}$ i.e. $\widehat{u}_{m+n} \leq \widehat{u}_n + \widehat{u}_m \circ \widehat{\mathcal{F}}^n$.
- 3) For any fixed $\lambda \in W_0$, we have $\lim_n \frac{1}{n} u_n(\gamma, \lambda) = -\chi_1(\lambda)$ for \mathcal{M} -almost every $\gamma \in \mathcal{X}$.
- 4) For \mathcal{M} -almost every $\gamma \in \mathcal{X}$ we have $\lim_n \frac{1}{n} u_n(\gamma, \lambda) = -\chi_1(\lambda)$ for Lebesgue-almost every $\lambda \in W_0$.

Proof. 1) When $\gamma \in \mathcal{X}$ is fixed the function $u_n(\gamma, \cdot)$ is clearly continuous on W_0 and $u_n(\gamma, \lambda) = \sup_{\|e\|=1} \log \|(DF_{\gamma(\lambda)}^n(0))^{-1} \cdot e\|$. To see that $u_n(\gamma, \cdot)$ is psh it thus suffices to recall that $\lambda \mapsto \log \|(DF_{\gamma(\lambda)}^n(0))^{-1} \cdot e\|$ is psh for each unit vector $e \in \mathbb{C}^k$.

2) Let $\gamma \in \mathcal{X}$ and $m, n \geq 1$. It follows immediately from (10) that

$$(12) \quad \left(DF_{\gamma(\lambda)}^{m+n}(0)\right)^{-1} = \left(DF_{\gamma(\lambda)}^n(0)\right)^{-1} \circ \left(DF_{\mathcal{F}^n(\gamma)(\lambda)}^m(0)\right)^{-1} \quad \forall \lambda \in W_0.$$

Thus, if $\widehat{\gamma} \in \widehat{\mathcal{X}}$ we have

$$\begin{aligned} \widehat{u}_{m+n}(\widehat{\gamma}) &\leq \log \sup_{\lambda \in U_0} \left(\|(DF_{\gamma_0(\lambda)}^n(0))^{-1}\| \|(DF_{\mathcal{F}^n(\gamma_0)(\lambda)}^m(0))^{-1}\| \right) \\ &\leq \log \sup_{\lambda \in U_0} \|(DF_{\gamma_0(\lambda)}^n(0))^{-1}\| + \log \sup_{\lambda \in U_0} \|(DF_{\mathcal{F}^n(\gamma_0)(\lambda)}^m(0))^{-1}\| = \widehat{u}_n(\widehat{\gamma}) + \widehat{u}_m(\widehat{\mathcal{F}}^n(\widehat{\gamma})). \end{aligned}$$

3) By Oseledec Theorem, the subset $J_{\lambda,1}$ of $J_\lambda \setminus C_{f_\lambda}$ defined by

$$J_{\lambda,1} := \{x \in J_\lambda \setminus C_{f_\lambda} : \lim_n \frac{1}{n} \log \|(Df_\lambda^n)_x^{-1}\| = -\chi_1(\lambda)\}$$

has full μ_λ measure. As $p_{\lambda^*}(\mathcal{M}) = \mu_\lambda$, this implies that $\gamma(\lambda) \in J_{\lambda,1}$ for \mathcal{M} -almost every γ in \mathcal{X} . Then the assertion follows, using (9).

4) Let us denote by \mathcal{L} the Lebesgue measure on M . Let E be the measurable subset of $\mathcal{X} \times W_0$ given by

$$E := \{(\gamma, \lambda) \in \mathcal{X} \times W_0 : \lim_n \frac{1}{n} u_n(\gamma, \lambda) = -\chi_1(\lambda)\}.$$

For every $\lambda \in W_0$ and every $\gamma \in \mathcal{X}$ we set

$$E^\lambda := \{\gamma \in \mathcal{X} : (\gamma, \lambda) \in E\} \quad \text{and} \quad E_\gamma := \{\lambda \in W_0 : (\gamma, \lambda) \in E\}.$$

We have to show that $\mathcal{L}(E_\gamma) = \mathcal{L}(W_0)$ for \mathcal{M} -almost every $\gamma \in \mathcal{X}$. This immediately follows from Tonelli's theorem:

$$\int_{\mathcal{X}} \mathcal{L}(E_\gamma) d\mathcal{M}(\gamma) = \int_{W_0} \mathcal{M}(E^\lambda) d\mathcal{L}(\lambda) = \mathcal{L}(W_0)$$

since, according to the above third assertion, $\mathcal{M}(E^\lambda) = 1$ for every $\lambda \in W_0$. \square

Our strategy is to transfer the estimates known for the system $(J_{\lambda_0}, f_{\lambda_0}, \mu_{\lambda_0})$ to the system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$. This is possible because the graphs Γ_γ for $\gamma \in \mathcal{X}$ must approach the critical set C_f locally uniformly, a phenomenon which simply relies on the compactness of the closure of \mathcal{X} and the following basic property (see the Claim in subsection 3.2).

Fact *There exist $0 < \alpha \leq 1$ such that $\sup_{V_0} |\varphi| \leq |\varphi(\lambda)|^\alpha$ for every $\lambda \in V_0$ and every holomorphic function $\varphi : W_0 \rightarrow \mathbb{C}$ such that $0 < |\varphi| < 1$.*

More specifically, the key uniformity property we need is given by the next lemma. In our proofs, we shall denote the smallest singular value of an invertible linear map L of \mathbb{C}^k by $\delta(L)$. Let us recall that $\delta(L) = \|L^{-1}\|^{-1}$ and that $\delta(L)^k \leq |\det L| \leq \delta(L)\|L\|^{k-1}$.

Lemma 4.5. *Let U_0, V_0, W_0 be as in Lemma 4.4. Then there exist $\alpha > 0$ and $c > 0$ such that $\frac{1}{n}u_n(\gamma, \lambda) \leq \frac{k}{\alpha} \frac{1}{n}u_n(\gamma, \lambda'_0) + \log c$ for every $n \geq 1$, every $\gamma \in \mathcal{X}$ and every $(\lambda'_0, \lambda) \in V_0 \times V_0$.*

Proof. By the compactness of $\overline{\mathcal{X}}$ and $\overline{V_0}$, we get $c_1 := \sup_{\gamma \in \mathcal{X}, \lambda \in V_0} \|DF_{\gamma(\lambda)}^n(0)\|^{k-1} < +\infty$ and thus $|\det(DF_{\gamma(\lambda)}^1(0))| \leq c_1 \delta(DF_{\gamma(\lambda)}(0))$ for every $\lambda \in \overline{V_0}$ and every $\gamma \in \mathcal{X}$.

Then, as $\det DF_{\gamma(\lambda)}^n(0) = \prod_{j=0}^{n-1} \det DF_{\mathcal{F}^j(\gamma)}(0)$ and $\prod_{j=0}^{n-1} \delta(DF_{\mathcal{F}^j(\gamma)}(0)) \leq \delta(DF_{\gamma(\lambda)}^n(0))$ we get

$$(13) \quad |\det DF_{\gamma(\lambda)}^n(0)| \leq c_1^n \delta(DF_{\gamma(\lambda)}^n(0)); \quad \forall \gamma \in \mathcal{X}, \forall \lambda \in \overline{V_0}.$$

Let us set $c_2 := \sup_{\lambda \in \overline{W_0}, \gamma \in \overline{\mathcal{X}}} |\det DF_{\gamma(\lambda)}(0)|$. When $\gamma \in \mathcal{X}$, the holomorphic function $\varphi(\lambda) := \frac{1}{c_2^n} \det DF_{\gamma(\lambda)}^n(0)$ is non vanishing and its modulus is bounded by 1 on W_0 . Applying the above stated Fact to φ , we get $0 < \alpha \leq 1$ (which only depends on V_0 and W_0) such that:

$$(14) \quad \sup_{\lambda \in V_0} |\det DF_{\gamma(\lambda)}^n(0)| \leq c_2^{n(1-\alpha)} |\det DF_{\gamma(\lambda)}^n(0)|^\alpha; \quad \forall n \geq 1, \forall \gamma \in \mathcal{X}, \forall \lambda \in V_0.$$

Using successively (14) and (13) we get for any $\lambda, \lambda'_0 \in V_0$

$$\left[\delta(DF_{\gamma(\lambda'_0)}^n(0)) \right]^k \leq |\det DF_{\gamma(\lambda'_0)}^n(0)| \leq c_2^{n(1-\alpha)} |\det DF_{\gamma(\lambda)}^n(0)|^\alpha \leq c_2^{n(1-\alpha)} c_1^{n\alpha} \left[\delta(DF_{\gamma(\lambda)}^n(0)) \right]^\alpha.$$

Then, applying log and multiplying by $\frac{-1}{n}$ we get

$$k \frac{1}{n} u_n(\gamma, \lambda'_0) \geq \alpha \frac{1}{n} u_n(\gamma, \lambda) - \alpha (\log c_1 + \frac{1-\alpha}{\alpha} \log c_2)$$

which is the desired estimate with $c := c_1 c_2^{(1-\alpha)/\alpha}$. \square

The next Lemma gathers the properties of $(u_n)_n$ which will be crucial to end the proof.

Lemma 4.6. *Let U_0, V_0, W_0 be as in Lemma 4.4. Then the following properties occur.*

- 1) *The sequence $(\frac{1}{n}u_n)_n$ is uniformly bounded from below on $\mathcal{X} \times V_0$.*
- 2) *The sequence $(\frac{1}{n}u_n(\gamma, \cdot))_n$ is uniformly bounded on V_0 for \mathcal{M} -almost every $\gamma \in \mathcal{X}$.*
- 3) *The functions \widehat{u}_n are $\widehat{\mathcal{M}}$ -integrable.*

Proof. 1) Using the properties of the smallest singular value we have

$$\begin{aligned} \frac{1}{n} u_n(\gamma, \lambda) &= -\frac{1}{n} \log \delta \left(DF_{\gamma(\lambda)}^n(0) \right) \geq -\frac{1}{nk} \log |\det \left(DF_{\gamma(\lambda)}^n(0) \right)| \\ &= -\frac{1}{k} \left(\frac{1}{n} \sum_{j=0}^{n-1} \log |\det DF_{\mathcal{F}^j(\gamma)}(0)| \right) \end{aligned}$$

and the assertion follows immediately from the definition and the continuity of $F_{\gamma(\lambda)}$.

2) We have just seen that $\frac{1}{n}u_n(\gamma, \cdot)$ is uniformly bounded from below on V_0 . By the fourth assertion of Lemma 4.4, for \mathcal{M} -almost every $\gamma \in \mathcal{X}$ there exists $\lambda_\gamma \in V_0$ such that $\lim_n \frac{1}{n}u_n(\gamma, \lambda_\gamma) = -\chi_1(\lambda_\gamma)$. On the other hand, by Lemma 4.5, we have $\frac{1}{n}u_n(\gamma, \lambda) \leq \frac{k}{\alpha} \frac{1}{n}u_n(\gamma, \lambda_\gamma) + \log c$ for every

$n \in \mathbb{N}$ and every $\lambda \in V_0$ and thus $\frac{1}{n}u_n(\gamma, \cdot)$ is uniformly bounded from above on V_0 .

3) By the above first assertion, we know that \widehat{u}_n is bounded from below. It thus suffices to show that $\int \widehat{u}_n(\widehat{\gamma}) d\widehat{\mathcal{M}}(\widehat{\gamma}) < +\infty$. By Lemma 4.5 we have

$$\begin{aligned} \int \widehat{u}_n(\widehat{\gamma}) d\widehat{\mathcal{M}}(\widehat{\gamma}) &\leq n \log c + \frac{k}{\alpha} \int u_n((\pi_0(\widehat{\gamma}), \lambda_0)) d\widehat{\mathcal{M}}(\widehat{\gamma}) = n \log c + \frac{k}{\alpha} \int u_n(\gamma, \lambda_0) d\mathcal{M}(\gamma) \\ &= n \log c + \frac{k}{\alpha} \int \log \|(DF_{\gamma(\lambda_0)}^n(0))^{-1}\| d\mathcal{M}(\gamma) = n \log c - \frac{k}{\alpha} \int \log \delta(DF_{\gamma(\lambda_0)}^n(0)) d\mathcal{M}(\gamma). \end{aligned}$$

Using (13), we thus get

$$\begin{aligned} \int \widehat{u}_n(\widehat{\gamma}) d\widehat{\mathcal{M}}(\widehat{\gamma}) &\leq -\frac{k}{\alpha} \int \log |\det(DF_{\gamma(\lambda_0)}^n(0))| d\mathcal{M}(\gamma) + \frac{kn}{\alpha} \log c_1 + n \log c \\ &= -\frac{k}{\alpha} \int \log |\det(Df_{\lambda_0}^n)_{\gamma(\lambda_0)}| d\mathcal{M}(\gamma) + C_n = -\frac{k}{\alpha} \int \log |\det(Df_{\lambda_0}^n)_x| (dp_{\lambda_0 \star} \mathcal{M})(x) + C_n, \end{aligned}$$

the conclusion follows from the integrability of $\log |\det(Df_{\lambda_0}^n)_x|$ for $p_{\lambda_0 \star} \mathcal{M} = \mu_{\lambda_0}$, see [DS3]. \square

We are now ready to establish the main result of this subsection.

PROOF OF PROPOSITION 4.3: We will apply Kingman subadditive ergodic theorem (see [Arn]) to the sequence $(\widehat{u}_n)_n$. This is possible since the system $(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}})$ is ergodic, the sequence $(\widehat{u}_n)_n$ is subadditive (second assertion of Lemma 4.4) and $\widehat{u}_1 \in L^1(\widehat{\mathcal{M}})$ (last assertion of Lemma 4.6). According to this theorem, there exists $L \in \mathbb{R}$ such that $\lim_n \frac{1}{n} \widehat{u}_n(\widehat{\gamma}) = L$ for $\widehat{\mathcal{M}}$ -almost every $\widehat{\gamma} \in \widehat{\mathcal{X}}$ and $\lim_n \frac{1}{n} \int_{\widehat{\mathcal{X}}} \widehat{u}_n d\widehat{\mathcal{M}} = L$. It remains to show that $L \leq -\frac{1}{2} \log d$.

Taking into account the fourth assertion of Lemma 4.4 and the second assertion of Lemma 4.6, we may thus pick $\widehat{\gamma} \in \widehat{\mathcal{X}}$ such that:

- i) $\lim_n \frac{1}{n} \widehat{u}_n(\widehat{\gamma}) = L$,
- ii) $\frac{1}{n} u_n(\gamma_0, \cdot)$ is uniformly bounded on V_0 ,
- iii) $\lim_n \frac{1}{n} u_n(\gamma_0, \lambda) = -\chi_1(\lambda)$ for Lebesgue-almost every $\lambda \in V_0$.

Assuming that $L > \frac{-\log d}{2}$, we will reach a contradiction with the fact that $\chi_1(\lambda) \geq \frac{\log d}{2}$ for all λ (see [BrDv1] or [DS3]). Recalling that $\widehat{u}_n(\widehat{\gamma}) = \sup_{\lambda \in U_0} u_n(\gamma_0, \lambda)$, there exist $\lambda_{n_k} \in U_0$ and $\epsilon > 0$ such that $\lambda_{n_k} \rightarrow \lambda'_0 \in \overline{U_0}$ and $\frac{1}{n_k} u_{n_k}(\gamma_0, \lambda_{n_k}) \geq \frac{-\log d}{2} + \epsilon$. We may pick $r > 0$ such that $B(\lambda_{n_k}, r) \subset V_0$ for all $k \in \mathbb{N}$. Then, by the subharmonicity of $u_{n_k}(\gamma_0, \cdot)$ on V_0 (first assertion of Lemma 4.4) we get:

$$\frac{-\log d}{2} + \epsilon \leq \frac{u_{n_k}(\gamma_0, \lambda_{n_k})}{n_k} \leq \frac{1}{|B(\lambda_{n_k}, r)|} \int_{B(\lambda_{n_k}, r)} \frac{u_{n_k}(\gamma_0, \lambda)}{n_k}$$

which, by Lebesgue dominated convergence theorem, yields

$$\frac{-\log d}{2} + \epsilon \leq \frac{1}{|B(\lambda'_0, r)|} \int_{B(\lambda'_0, r)} -\chi_1(\lambda)$$

and contradicts the fact that $\chi_1(\lambda) \geq \frac{\log d}{2}$ for every λ . \square

4.3. Proof of Theorem 4.1. According to Corollary 1.7, we only need to consider the case where f admits an acritical and ergodic equilibrium web \mathcal{M}_0 . Let $\mathcal{K}_0 := \text{supp } \mathcal{M}_0$.

Consider the set

$$\mathcal{L}_+ := \{\gamma \in \mathcal{K}_0 \setminus \mathcal{J}_s : \forall \gamma' \in \mathcal{K}_0, \forall k \in \mathbb{N}, \Gamma_{\mathcal{F}^k(\gamma)} \cap \Gamma_{\gamma'} \neq \emptyset \Rightarrow \mathcal{F}^k(\gamma) = \gamma'\}$$

and assume that

$$(15) \quad \mathcal{M}_0(\{\gamma \in \mathcal{K}_0 : \exists k \in \mathbb{N}, \exists \gamma' \in \mathcal{K}_0 \text{ s.t. } \Gamma_{\mathcal{F}^k(\gamma)} \cap \Gamma_{\gamma'} \neq \emptyset \text{ and } \mathcal{F}^k(\gamma) \neq \gamma'\}) = 0.$$

By construction we have $\mathcal{M}_0(\mathcal{L}_+) = 1$ and \mathcal{L}_+ satisfies the following properties:

- 1) $\mathcal{L}_+ \subset \mathcal{J} \setminus \mathcal{J}_s$,
- 2) $\mathcal{F}(\mathcal{L}_+) \subset \mathcal{L}_+$,
- 3) $\forall \gamma, \gamma' \in \mathcal{L}_+ : \Gamma_\gamma \cap \Gamma_{\gamma'} \neq \emptyset \Rightarrow \gamma = \gamma'$.

The set $\mathcal{L} := \cup_{m \geq 0} \mathcal{F}^{-m}(\mathcal{L}_+)$ satisfies the same properties and moreover $\mathcal{F} : \mathcal{L} \rightarrow \mathcal{L}$ is d^k -to-1, the existence of an equilibrium lamination \mathcal{L} will thus follow from (15).

To prove (15), it is sufficient to show that for any fixed $k \in \mathbb{N}$ and any $\lambda_0 \in M$ there exists a neighbourhood U_0 of λ_0 such that

$$(16) \quad \mathcal{M}_0(\{\gamma \in \mathcal{K}_0 : \exists \gamma' \in \mathcal{K}_0 \text{ s.t. } \Gamma_{\mathcal{F}^k(\gamma)} \cap \Gamma_{\gamma'} \cap (U_0 \times \mathbb{P}^k) \neq \emptyset \text{ and } \mathcal{F}^k(\gamma) \neq \gamma'\}) = 0.$$

To this purpose, we shall work with the natural extension $(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}}_0)$ of the system $(\mathcal{X}, \mathcal{F}, \mathcal{M}_0)$ and apply Proposition 4.2. We recall that, according to Proposition 4.3, all the assumptions of Proposition 4.2 are satisfied. Let U_0 be a neighbourhood of λ_0 given by that proposition; we may assume that U_0 is simply connected and that $U_0 \Subset M$. We recall that $\mathcal{X} \subset \mathcal{K}_0$.

For any $B \subset U_0$, we define the *ramification* functions R_B by setting

$$R_B(\gamma) := \sup_{\gamma' \in \mathcal{K}_0 : \Gamma_{\gamma'} \cap \Gamma_{\gamma|_B} \neq \emptyset} \sup_B d_{\mathbb{P}^k}(\gamma(\lambda), \gamma'(\lambda)), \quad \forall \gamma \in \mathcal{J}.$$

Let $\widehat{\mathcal{Y}}_\epsilon := \{\widehat{\gamma} \in \widehat{\mathcal{Y}} : R_{U_0}(\gamma_k) > \epsilon\}$, it then suffices to prove that $\widehat{\mathcal{M}}_0(\widehat{\mathcal{Y}}_\epsilon) = 0$ for every $\epsilon > 0$ as it follows from the following observation:

$$\begin{aligned} & \mathcal{M}_0(\{\gamma \in \mathcal{K}_0 : \exists \gamma' \in \mathcal{K}_0 \text{ s.t. } \Gamma_{\gamma_k} \cap \Gamma_{\gamma'} \cap (U_0 \times \mathbb{P}^k) \neq \emptyset \text{ and } \gamma_k \neq \gamma'\}) \\ &= \mathcal{M}_0(\{\gamma \in \mathcal{K}_0 : R_{U_0}(\gamma_k) > 0\}) = \mathcal{M}_0(\{\gamma \in \mathcal{X} : R_{U_0}(\gamma_k) > 0\}) \\ &= \widehat{\mathcal{M}}_0(\{\widehat{\gamma} \in \widehat{\mathcal{Y}} : R_{U_0}(\gamma_k) > 0\}) = \widehat{\mathcal{M}}_0(\cup_{s \in \mathbb{N}^*} \widehat{\mathcal{Y}}_{\frac{\epsilon}{s}}). \end{aligned}$$

Let us proceed by contradiction and assume that $\widehat{\mathcal{M}}_0(\widehat{\mathcal{Y}}_\epsilon) > 0$ for some $\epsilon > 0$. Owing to the equicontinuity of \mathcal{X} (we recall that $\mathcal{X} \subset \text{supp } \mathcal{M}_0$) we may cover U_0 with finitely many open sets $B_i \subset U_0$, say with $1 \leq i \leq N$, such that

$$(17) \quad \forall \gamma, \gamma' \in \mathcal{X}, \forall \lambda_1 \in B_i : \gamma(\lambda_1) = \gamma'(\lambda_1) \Rightarrow \sup_{\lambda \in B_i} d(\gamma(\lambda), \gamma'(\lambda)) < \epsilon.$$

As $R_{U_0}(\gamma) = 0$ when $\max_{1 \leq i \leq N} R_{B_i}(\gamma) = 0$ (by analyticity we have $\gamma = \gamma'$ on U_0 if $\gamma = \gamma'$ on some B_i), there exists $1 \leq j \leq N$ and $\alpha > 0$ such that:

$$\widehat{\mathcal{M}}_0(\{\widehat{\gamma} \in \widehat{\mathcal{Y}} : \widehat{\eta}_p(\widehat{\gamma}_k) > \epsilon \text{ and } R_{B_j}(\gamma_k) > \alpha\}) > 0.$$

Let us set $\widehat{\mathcal{Y}}_{\epsilon,j,\alpha} := \{\widehat{\gamma} \in \widehat{\mathcal{Y}} : \widehat{\eta}_p(\widehat{\gamma}_k) > \epsilon \text{ and } R_{B_j}(\gamma_k) > \alpha\}$. Let p be given by Proposition 4.2, applying Poincaré recurrence theorem to $\widehat{\mathcal{F}}^{-p}$, we find $\widehat{\gamma} \in \widehat{\mathcal{Y}}_{\epsilon,j,\alpha}$ and an increasing sequence of integers $(n_q)_q$ with $n_q \in p\mathbb{N}$ such that $\widehat{\mathcal{F}}^{-n_q}(\widehat{\gamma}) \in \widehat{\mathcal{Y}}_{\epsilon,j,\alpha}$ for every $q \in \mathbb{N}$. In particular $\widehat{\gamma} \in \widehat{\mathcal{Y}}_\epsilon$ and $R_{B_j}(\gamma_{k-n_q}) > \alpha$ for every $q \in \mathbb{N}$. We will reach a contradiction by establishing that

$$(18) \quad \lim_{m \rightarrow +\infty} R_{B_i}(\gamma_{k-mp}) = 0, \quad \forall i \in \{1, \dots, N\}, \quad \forall \widehat{\gamma} \in \widehat{\mathcal{Y}}_\epsilon.$$

To this purpose we shall use Proposition 4.2 to show that $R_{B_i}(\gamma_{k-n}) \leq e^{-nA}$ when $n \in p\mathbb{N}$ and $\widehat{\gamma} \in \widehat{\mathcal{Y}}_\epsilon$. Let $\gamma' \in \mathcal{K}_0$ such that $\gamma'(\lambda_1) = \gamma_{k-n}(\lambda_1)$ for some $\lambda_1 \in B_i$. Then $(\mathcal{F}^n \gamma')(\lambda_1) = \gamma_k(\lambda_1)$ and thus, according to (17), $\sup_{\lambda \in B_i} d((\mathcal{F}^n \gamma')(\lambda), \gamma_k(\lambda)) < \epsilon < \widehat{\eta}_p(\widehat{\gamma}_k)$. This means that

$$(19) \quad \Gamma_{\mathcal{F}^n \gamma'} \cap (B_i \times \mathbb{P}^k) \subset T_{B_i}(\gamma_k, \widehat{\eta}_p(\widehat{\gamma}_k)).$$

Now, by Proposition 4.2, the inverse branch $f_{\widehat{\gamma}_k}^{-n}$ of f^n is defined on the tube $T_{U_0}(\gamma_k, \widehat{\eta}_p(\widehat{\gamma}_k))$ and maps it biholomorphically into $T_{U_0}(\gamma_{k-n}, e^{-nA})$. As $B_i \subset U_0$, this yields:

$$(20) \quad f_{\widehat{\gamma}_k}^{-n}(T_{B_i}(\gamma_k, \widehat{\eta}_p(\widehat{\gamma}_k))) \subset T_{B_i}(\gamma_{k-n}, e^{-nA}).$$

By construction we have $f_{\widehat{\gamma}_k}^{-n}(\Gamma_{\gamma_k}) = \Gamma_{\gamma_{k-n}}$ and thus $f_{\widehat{\gamma}_k}^{-n}((\mathcal{F}^n \gamma')(\lambda_1)) = f_{\widehat{\gamma}_k}^{-n}(\gamma_k(\lambda_1)) = \gamma_{k-n}(\lambda_1) = \gamma'(\lambda_1)$. This implies that $f_{\widehat{\gamma}_k}^{-n}(\Gamma_{\mathcal{F}^n \gamma'}) = \Gamma_{\gamma'}$ which in turns, by (19) and (20), implies that $\sup_{\lambda \in B_i} d_{\mathbb{P}^k}(\gamma'(\lambda), \gamma_{k-n}(\lambda)) \leq e^{-nA}$. Then (18) follows and (16) is proved.

We finally prove the uniqueness assertion. Let us fix $\lambda \in M$ and, for any Borel subset \mathcal{A} of \mathcal{J} , let us set $\mathcal{A}_\lambda := \{\gamma(\lambda) : \gamma \in \mathcal{A}\}$. Then, as $\mathcal{A} \subset p_\lambda^{-1}(\mathcal{A}_\lambda)$ we have

$$\mu_\lambda(\mathcal{A}_\lambda) = (p_{\lambda*} \mathcal{M})(\mathcal{A}_\lambda) = \mathcal{M}(p_\lambda^{-1}(\mathcal{A}_\lambda)) \geq \mathcal{M}(\mathcal{A})$$

for every equilibrium web \mathcal{M} of f . On the other hand, it follows from (16) applied for $k = 0$ that

$$\mu_\lambda(\mathcal{A}_\lambda) = \mathcal{M}_0(p_\lambda^{-1}(\mathcal{A}_\lambda)) = \mathcal{M}_0(\mathcal{A}).$$

We thus have $\mathcal{M}_0(\mathcal{A}) \geq \mathcal{M}(\mathcal{A})$ for any borelian subset \mathcal{A} of \mathcal{J} and this implies that the measures \mathcal{M} and \mathcal{M}_0 must coincide since both are probability measures on \mathcal{J} . \square

5. SIEGEL DISCS AND BIFURCATIONS

As it is well known, the Julia sets of any holomorphic family of rational maps of \mathbb{P}^1 depends continuously on the parameter for the Hausdorff topology if and only if the family is stable. It is worth emphasizing that discontinuities can be explained by the appearance of Siegel discs, see [Do]. We investigate this in higher dimension and, as a consequence, show that the existence of virtually repelling Siegel periodic points in the Julia set (see Definitions 5.1 and 5.2) is an obstruction to the existence of an equilibrium web. We finally exploit this fact to end the proof of Theorem 1.1.

5.1. Siegel discs as obstructions to stability. We define a notion of Siegel disc for endomorphisms of \mathbb{P}^k and investigate how they behave with respect to Julia sets. In this subsection, we endow \mathbb{C}^k with the norm $\|z\| := \sup_i |z_i|$ and set $1 \leq q \leq k-1$. We write $z =: (z', z'')$ where $z' := (z_1, \dots, z_{k-q}) \in \mathbb{C}^{k-q}$ and $z'' := (z_{k-q+1}, \dots, z_k) \in \mathbb{C}^q$. We also set $k' := k-q$, $e^{i\theta_0} := (e^{i\theta_{0,k'+1}}, \dots, e^{i\theta_{0,k}})$ and $e^{i\theta_0} \cdot z'' := (e^{i\theta_{0,k'+1}} z_{k'+1}, \dots, e^{i\theta_{0,k}} z_k)$.

Definition 5.1. Let f_0 be a holomorphic endomorphism of \mathbb{P}^k . One says that $z_0 \in \mathbb{P}^k$ is a Siegel fixed point for f_0 if f_0 is holomorphically linearizable at z_0 and its differential at z_0 is of the form $(A_0 z', e^{i\theta_0} \cdot z'')$ where A_0 is an expanding linear map on $\mathbb{C}^{k'}$ and $\pi, \theta_{0,k'+1}, \dots, \theta_{0,k}$ are linearly independent over \mathbb{Q} . In other words, there exists a local holomorphic chart $\psi_0 : B_R \rightarrow \mathbb{P}^k$ such that $\psi_0(0) = z_0$ and

$$\psi_0^{-1} \circ f_0 \circ \psi_0 = (A_0 z', e^{i\theta_0} \cdot z'')$$

where θ_0 and A_0 are as above. Any set of the form $\psi_0(\{0\} \times B_\rho)$ where $\rho < R$ and B_ρ is a ball centered at the origin in \mathbb{C}^q is called a local Siegel q -disc of f_0 centered at z_0 .

Let us consider a holomorphic family f of endomorphisms of \mathbb{P}^k . If f_0 admits a Siegel fixed point z_0 then, by the implicit function theorem, there exists a unique holomorphic map $z(\lambda)$ defined on some neighbourhood of 0 in M such that $z(0) = z_0$ and $z(\lambda)$ is fixed by f_λ . Moreover, there exist holomorphic functions $w_j(\lambda)$ such that $w_j(0) = e^{i\theta_{0,j}}$ and $w_j(\lambda)$ is an eigenvalue of $d_{z(\lambda)} f_\lambda$ for $k' + 1 \leq j \leq k$. In this context, we coin the following definition.

Definition 5.2. The Siegel fixed point z_0 is called virtually repelling if there exist a holomorphic disc $\sigma : \Delta_{\epsilon_0} \rightarrow M$ and positive constants c_j such that $\sigma(0) = 0$ and $|w_j \circ \sigma(t)| = 1 + c_j t$ for $k' + 1 \leq j \leq k$ and $-t_0 < t < t_0$. If, moreover, $z \circ \sigma(t) \in J_{\sigma(t)}$ for $-t_0 < t < t_0$ the Siegel fixed point z_0 is called virtually J -repelling.

Let us observe that if J_λ is continuous at λ_0 and if f_{λ_0} has a virtually repelling Siegel periodic point outside J_{λ_0} , then λ_0 must be accumulated by parameters λ for which f_λ has periodic repelling points outside J_λ . Examples of such repelling points have been given by Hubbard-Papadopol [HP, section 6, example 2] and Fornæss-Sibony [FS2, section 4.1]. The following proposition discusses the position of Siegel discs with respect to Julia sets. Note that the second item will only be used in Remark 5.8.

Proposition 5.3. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k such that f_{λ_0} admits a virtually repelling Siegel fixed point z_0 .

- 1) If f admits an equilibrium web then every local Siegel q -disc centered at z_0 is contained in $\mathbb{P}^k \setminus J_{\lambda_0}$. In particular $z_0 \notin J_{\lambda_0}$.
- 2) When $q = 1$, if $z_0 \in J_{\lambda_0}$ and if $\lambda \mapsto J_\lambda$ is continuous at λ_0 then any local Siegel q -disc centered at z_0 is contained in J_{λ_0} .

The first item of the preceding proposition immediately yields the following result.

Corollary 5.4. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let U_0 be any neighbourhood of λ_0 in M . If the restriction of f to $U_0 \times \mathbb{P}^k$ admits an equilibrium web then f_{λ_0} has no virtually repelling Siegel periodic point in J_{λ_0} .

The proof of Proposition 5.3 relies on the following technical lemma.

Lemma 5.5. Let $g : \Delta_{r_0} \times B_R \rightarrow \Delta_{r_0} \times B_{R'}$ be a holomorphic map such that $g(\lambda, z) = (\lambda, g_\lambda(z))$, $g_\lambda(0) = 0$ and $g_0(z) = (A_0^{-1} \cdot z', e^{-i\theta_0} \cdot z'')$ where A_0 is an expanding linear map on $\mathbb{C}^{k'}$. Assume that $\frac{\partial g_{\lambda,i}}{\partial z_i}(0) = 0$ for $k' + 1 \leq j \leq k$ and $i \neq j$. Assume moreover that there exists $|u_0| = 1$, $t_0 > 0$ and $c_j > 0$ such that $|\frac{\partial g_{t u_0, j}}{\partial z_j}(0)| = 1 + c_j t$ for $k' + 1 \leq j \leq k$ and $-t_0 < t < t_0$. Then, after taking R smaller, the following properties occur.

- 1) There exists arbitrarily small λ such that $\|g_\lambda(z)\| \leq \alpha_0 \|z\|$ on B_R with $0 < \alpha_0 < 1$.

- 2) Assume $k' = k - 1$. For any $0 < \rho < R_1 < R_2 < R$, there exists arbitrarily small λ such that, for every $a \in B_{R_1}$ which does not belong to the local stable manifold S_λ of g_λ , there exists n_0 such that $g_\lambda^{n_0}(a) \in \{\|z'\| < \rho\} \times \{R_1 < \|z''\| < R_2\}$ and $g_\lambda^k(a) \in B_{R_1}$ for $0 \leq k \leq n_0 - 1$.

Proof. We may write $g_\lambda := (g_{\lambda,j})_{1 \leq j \leq k}$ on the form

$$\begin{aligned} g_{\lambda,j} &= \sum_{i=1}^{k'} (a_{ij} + \lambda \mu_{ij}(\lambda) + \lambda q_{ij}(\lambda, z)) z_i + \lambda \sum_{i=k'+1}^k s_{ij}(\lambda, z) z_i \quad \text{for } 1 \leq j \leq k' \\ g_{\lambda,j} &= (e^{i\theta_j} + \lambda \mu_{jj}(\lambda) + \lambda q_{jj}(\lambda, z)) z_j + \lambda \sum_{i \neq j} s_{ij}(\lambda, z) z_i \quad \text{for } k' + 1 \leq j \leq k \end{aligned}$$

where μ_{ij} , q_{ij} and s_{ij} are holomorphic on $\Delta_{\epsilon_0} \times B_R$ and satisfy $q_{ij}(\lambda, 0) = q_{jj}(\lambda, 0) = 0$. By assumption, we also have $s_{ij}(\lambda, 0) = 0$ for $k' + 1 \leq j \leq k$ and $i \neq j$.

By shrinking ϵ_0 and R , there exists $0 < \alpha_1 < 1$ such that

$$(21) \quad \sup_{1 \leq j \leq k'} |g_{\lambda,j}(z)| \leq \alpha_1 \|z\| \text{ on } \Delta_{\epsilon_0} \times B_R.$$

Let us set $\lambda_t := t u_0$ where $-t_0 < t < t_0$ and $Q_{jt}(z) := e^{i\theta_j} + \lambda_t \mu_{jj}(\lambda_t) + \lambda_t q_{jj}(\lambda_t, z)$ and $R_{jt}(z) := |\lambda_t| \sum_{i \neq j} |s_{ij}(\lambda_t, z)|$ for $k' + 1 \leq j \leq k$. Then, by our assumptions and after taking R smaller, we have

$$(22) \quad |Q_{jt}(z)| \leq 1 + \frac{c_j t}{2} \text{ for } -t_0 < t < 0 \text{ and } z \in B_R$$

$$(23) \quad R_{jt}(z) \leq \frac{c_j |t|}{4} \text{ for } -t_0 < t < t_0 \text{ and } z \in B_R$$

$$(24) \quad 1 + \frac{c_j t}{2} \leq |Q_{jt}(z)| \leq 1 + 2c_j t \text{ for } 0 < t < t_0 \text{ and } z \in B_R.$$

It follows from (22) and (23) that $|g_{\lambda_t,j}(z)| \leq (1 + \frac{tc_j}{4}) \|z\|$ for $k' + 1 \leq j \leq k$, $-t_0 < t < 0$ and $z \in B_R$. This and (21) yields the first assertion of the lemma.

Let us now establish the second one. Fix $0 < t < t_0$ so small that $(1 + \frac{9tc_j}{4})R_1 < R_2$ for $k' + 1 \leq j \leq k$. Let $a \in B_{R_1}$ be outside the local stable manifold of g_{λ_t} . Assume that one cannot find n_0 such that $g_{\lambda_t}^k(a) \in B_{R_1}$ for $0 \leq k \leq n_0 - 1$ and $g_{\lambda_t}^{n_0}(a) \in \{\|z'\| < \rho\} \times \{\|z''\| > R_1\}$. Then, according to (21), the sequence $a_n := g_{\lambda_t}^n(a)$ is well defined and $\|a'_n\| \rightarrow 0$. From (23) and (24) one gets $|a_{n+1,j}| \geq (1 + \frac{c_j t}{2}) |a_{n,j}| - \frac{tc_j}{4} \|a'_n\|$. As $(a_{n,j})_n$ is bounded and $\|a'_n\| \rightarrow 0$, this implies that a_n tends to the origin and contradicts the fact that a does not belong to the local stable manifold of g_{λ_t} . Thus n_0 exists and it remains to check that $\|a'_{n_0}\| < R_2$. From (23) and (24) one gets $|a_{n_0,j}| \leq (1 + 2c_j t) |a_{n_0-1,j}| + \frac{tc_j}{4} \|a'_{n_0-1}\| \leq (1 + \frac{9tc_j}{4}) R_1 < R_2$. \square

PROOF OF PROPOSITION 5.3: We may assume that $M = \Delta_{\epsilon_0}$ and $\lambda_0 = 0$ so that z_0 is a virtually repelling Siegel fixed point of f_0 . Thus there exists a biholomorphism $\psi_0 : B_R \rightarrow \psi_0(B_R)$ such that $\psi_0(0) = z_0$ and $\psi_0^{-1} \circ f_0 \circ \psi_0 = (A_0 \cdot z', e^{i\theta_0} \cdot z'')$ where A_0 is linear and expanding on $\mathbb{C}^{k'}$ and $\pi, \theta_{0,k'+1}, \dots, \theta_{0,k}$ are linearly independant over \mathbb{Q} .

The mapping $\psi_0^{-1} \circ f_\lambda^{-1} \circ \psi_0$ is well defined on $\Delta_{\epsilon_0} \times B_R$ after taking R and ϵ_0 smaller. Since the $e^{i\theta_{0,j}}$ are pairwise distinct for $k' + 1 \leq j \leq k$, we may find q linearly independant vectors

$v_{k'+1}(\lambda), \dots, v_k(\lambda)$ in \mathbb{C}^k and q scalars $w_{k'+1}(\lambda), \dots, w_k(\lambda)$ which depend holomorphically on $\lambda \in \Delta_{\epsilon_0}$ and such that

$$(25) \quad d_{\psi_0^{-1}(z(\lambda))}(\psi_0^{-1} \circ f_\lambda^{-1} \circ \psi_0)(v_j(\lambda)) = w_j(\lambda)v_j(\lambda) \text{ for } k'+1 \leq j \leq k.$$

Using basis like $(v_1, \dots, v_{k'}, v_{k'+1}(\lambda), \dots, v_k(\lambda))$ we may perform change of coordinates of the form $(\lambda, A(\lambda, z))$ where $A(\lambda, \cdot)$ is affine on \mathbb{C}^k which, conjugate by ψ_0 , yield biholomorphisms $\psi_\lambda : B_R \rightarrow \psi_\lambda(B_R)$ such that $g_\lambda := \psi_\lambda^{-1} \circ f_\lambda^{-1} \circ \psi_\lambda$ satisfies the assumptions of Lemma 5.5. The condition $\frac{\partial g_{\lambda,i}}{\partial z_i}(0) = 0$ indeed follows from (25) and the condition $|\frac{\partial g_{\lambda_0,j}}{\partial z_j}(0)| = 1 + c_j t$ follows from the fact that z_0 is virtually repelling. To simplify, we shall denote J_λ the set $\psi_\lambda^{-1}(J_\lambda \cap \psi_\lambda(B_R))$.

1) We proceed by contradiction and assume that $(0', z_0'') \in J_0$ for $0 < \|z_0''\| < r < R$. According to Lemma 2.5, there exists a holomorphic map $\gamma : \Delta_{\epsilon_0} \rightarrow \mathbb{P}^k$ such that $\psi_0^{-1} \circ \gamma(0) = (0', z_0'')$ and $(\mathcal{F}^n \cdot \gamma)_n$ is normal on Δ_{ϵ_0} . We may assume that $\tilde{\gamma}(\lambda) := \psi_\lambda^{-1}(\gamma(\lambda))$ is well defined on Δ_{ϵ_0} . Since $\psi_0^{-1} \circ f_0^n \circ \psi_0(\tilde{\gamma}(0)) = (0', e^{in\theta_0} \cdot z_0'')$ and $(\mathcal{F}^n \cdot \gamma)_n$ is normal, after reducing ϵ_0 , we may suppose that

$$(26) \quad \|\psi_\lambda^{-1} \circ f_\lambda^n \circ \psi_\lambda(\tilde{\gamma}(\lambda))\| \leq r \text{ on } \Delta_{\epsilon_0} \text{ for } n \geq 1.$$

Let us recall that $g_\lambda = \psi_\lambda^{-1} \circ f_\lambda^{-1} \circ \psi_\lambda$. By Lemma 5.5, there exists $\lambda_k \rightarrow 0$ and $0 < \alpha_k < 1$ such that $\|g_{\lambda_k}(z)\| \leq \alpha_k \|z\|$ on B_R . We may thus find a sequence $n_k \rightarrow \infty$ such that

$$(27) \quad \|g_{\lambda_k}^{n_k}(z)\| \leq \frac{1}{k} \|z\| \text{ on } B_r.$$

From (26) and (27) one gets

$$(28) \quad \|\tilde{\gamma}(\lambda_k)\| = \|g_{\lambda_k}^{n_k} \circ \psi_{\lambda_k}^{-1} \circ f_{\lambda_k}^{n_k} \circ \psi_{\lambda_k}(\tilde{\gamma}(\lambda_k))\| \leq \frac{r}{k}$$

which is impossible since $\lim_k \|\tilde{\gamma}(\lambda_k)\| = \|z_0''\| > 0$.

So far we have shown that the punctured q -disc $\{0'\} \times \{0 < \|z''\| < R\}$ is contained in J_0^c . Since J_0 is totally invariant and $g_0 = (A_0^{-1} \cdot z', e^{-i\theta_0} \cdot z'')$ where A_0 is linear and expanding, this implies that $B_R \setminus \{z \in B_R : z'' = 0\} \subset J_0^c$. Finally, as μ_0 does not give mass to analytic sets, we get $B_R \subset J_0^c$.

2) We have to show that $(0', z_{0k}) \in J_0$ if $0 < |z_{0k}| < R$. Assume, to the contrary, that $(0', z_{0k}) \notin J_0$ for some $0 < |z_{0k}| < R$. Then one may pick a neighbourhood V_0 of $(0', z_{0k})$ such that $V_0 \subset (J_0)^c$ and which is of the form

$$V_0 := \{\|z'\| < \rho\} \times \{R_1 < |z_k| < R_2 \text{ and } |\arg z_k - \arg z_{0k}| < \eta\}.$$

Let us now denote by T_{ρ, R_1, R_2} the tube

$$T_{\rho, R_1, R_2} := \{\|z'\| < \rho\} \times \{R_1 < |z_k| < R_2\}.$$

Since A_0 is contracting and θ_0/π irrational, for any $z \in T_{\rho, R_1, R_2}$ there exists an integer n such that $g_0^n(z) \in V_0$. By the invariance of Julia sets we thus have $T_{\rho, R_1, R_2} \subset (J_0)^c$. Let us shrink the tube T_{ρ, R_1, R_2} . By assumption, J_λ is *u.s.c* at 0 and therefore

$$T_{\rho, R_1, R_2} \subset (J_\lambda)^c \text{ when } \lambda \text{ is close enough to } 0.$$

On the other hand, according to the second assertion of Lemma 5.5, we may find parameters λ which are arbitrarily close to 0 and such that $B_{R_1} \setminus S_\lambda \subset \cup_n (g_\lambda^n)^{-1} T_{\rho, R_1, R_2}$ where S_λ denotes the stable manifold of g_λ . As μ_λ gives no mass to analytic sets, this and the inclusion $T_{\rho, R_1, R_2} \subset (J_\lambda)^c$

implies the existence of a sequence of parameters $\lambda_k \rightarrow 0$ such that $B_{R_1} \subset (J_{\lambda_k})^c$. This contradicts the lower semi continuity of J_λ at 0 since $0 \notin (J_{\lambda_k})_{\frac{R_1}{2}}$ but $0 \in J_0$ by our assumption. \square

5.2. End of the proof of Theorem 1.1. In order to obtain Theorem 1.1, it essentially remains to investigate if the repelling J -cycles of f move holomorphically when f admits an equilibrium web. To this purpose we shall use Corollary 5.4 and show how a Siegel disc may appear when a repelling J -cycle fails to move holomorphically.

Proposition 5.6. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family. If f admits an equilibrium web then all repelling J -cycles of f which are neither persistently resonant nor persistently undiagonalizable move holomorphically. When $k = 2$, all repelling J -cycles of f move holomorphically.*

Let us recall that a periodic point is said to be *resonant* if its multipliers w_1, \dots, w_k satisfy a relation of the form $w_1^{m_1} \cdots w_k^{m_k} - w_j = 0$ where the m_j are integers and $m_1 + \cdots + m_k \geq 2$. Note that when $w_j = e^{i\theta_j}$ for $1 \leq j \leq n$ and $n \leq k$ then the absence of resonances forces $\pi, \theta_1, \dots, \theta_n$ to be linearly independant over \mathbb{Q} .

We shall use the following Lemma.

Lemma 5.7. *Let $w_1, \dots, w_k : D(0, R) \rightarrow \mathbb{C}$ be holomorphic functions. Assume that $w_j(0) \neq 0$ and that there exists $\lambda_n \rightarrow 0$ such that $\min_{1 \leq j \leq k} |w_j(\lambda_n)| > 1$. Assume moreover that there exists $1 \leq N \leq k$ such that*

- $|w_j(0)| = 1$ and $w'_j(0) \neq 0$ for $1 \leq j \leq N$,
- $|w_j(0)| \neq 1$ for $N + 1 \leq j \leq k$.

Then, after renumbering, there exist an integer $1 \leq q \leq k$, a disc $D(\lambda_0, r) \subset D(0, R)$ and a partition $D(\lambda_0, r) = D^+(\lambda_0, r) \cup C \cup D^-(\lambda_0, r)$ where C is a real analytic arc through λ_0 and D^+ and D^- are open connected subsets of D such that

- (1) $|w_j| > 1$ on $D^+(\lambda_0, r)$, $|w_j| = 1$ on C and $|w_j| < 1$ on $D^-(\lambda_0, r)$ for $k - q + 1 \leq j \leq k$,
- (2) $|w_j| > 1$ on $D(\lambda_0, r)$ for $1 \leq j \leq k - q$ if $q \leq k - 1$.

Proof. In the sequel we allow to shrink R without specifying it. Let us set $C_j := \{|w_j| = 1\}$ and $U_j^+ := \{|w_j| > 1\}$, $U_j^- := \{|w_j| < 1\}$. Since we can assume that $w'_j(0) \neq 0$ when $\{|w_j| = 1\} \neq \emptyset$ the subset C_j is either empty or a real-analytic arc through 0 in $D(0, R)$. In particular we have

$$C_j = C_l \quad \text{if } C_j \cap C_l \text{ is strictly bigger than } \{0\}.$$

Let us set $U^+ := \bigcap_{j=1}^k U_j^+$. By assumption, $0 \in \overline{U^+}$ and therefore U^+ is a non-empty open subset of $D(0, R)$. It is clear that $\partial U^+ \subset \partial D(0, R) \cup (\bigcup_{j=1}^k C_j)$. On the other hand, we can not have $\partial U^+ \subset \{0\} \cup \partial D(0, R)$ since otherwise $U^+ = D(0, R) \setminus \{0\}$ and the subharmonic function $\psi(\lambda) := \max_{1 \leq j \leq k} |w_j(\lambda)|^{-1}$ would violate the maximum principle (recall that $\psi(0) \geq 1$). We may thus pick $\lambda_0 \neq 0$ such that $\lambda_0 \in C_{j_0} \cap \partial U^+$ for some $1 \leq j_0 \leq k$. Observe that $\lambda_0 \notin U_i^-$ for $1 \leq i \leq k$.

If $C_i \neq C_{j_0}$ for some $1 \leq i \leq k$ then $\lambda_0 \notin C_i$ and thus $\lambda_0 \in U_i^+$. After renumbering we may therefore find $1 \leq q \leq k - 1$ such that

$$\lambda_0 \in C_{k-q+1} = C_{k-q+2} = \cdots = C_k := C \quad \text{and} \quad \lambda_0 \in U_1^+ \cap \cdots \cap U_{k-q}^+.$$

For $r > 0$ sufficiently small we have $D(\lambda_0, r) \cap \bigcap_{i=1}^{k-q} U_i^+$ and $D(\lambda_0, r) \setminus C$ has two connected components Ω_1 and Ω_2 . For each $k - q + 1 \leq i \leq k$, one has $\Omega_1 \subset U_i^+$ and $\Omega_2 \subset U_i^-$ or $\Omega_1 \subset U_i^-$ and $\Omega_2 \subset U_i^+$. Assume for instance that $\Omega_1 \subset U_{k-q+1}^+$. Then, since $\lambda_0 \in \partial U^+$, we must have $\Omega_1 \subset U_i^+$

and $\Omega_2 \subset U_i^-$ for every $k - q + 1 \leq i \leq k$ and we set $D(\lambda_0, r)^+ := \Omega_1$ and $D(\lambda_0, r)^- := \Omega_2$. \square

PROOF OF PROPOSITION 5.6: Let $\lambda_0 \in M$. Assume that z_0 belongs to some p -periodic repelling J -cycle of f_{λ_0} which is not persistently resonant and not persistently undiagonalizable. It suffices to show that the map $\gamma : M \rightarrow \mathbb{P}^k$, which is the element of \mathcal{J} given by Lemma 2.5, enjoys the property that $\gamma(\lambda) \in J_\lambda$ is repelling for every $\lambda \in M$. Let us observe that $\gamma(\lambda)$ is not persistently resonant and not persistently undiagonalizable for any $\lambda \in M$.

Since M is connected, we have to show that the subset $\{\lambda \in M : \gamma(\lambda) \text{ is repelling}\}$ is closed in M . Assume, to the contrary, that this is not true. Then, for arbitrarily small ϵ_0 , one finds a new holomorphic map $\gamma_0 : B_{\epsilon_0} \rightarrow \mathbb{P}^k$ such that $\gamma_0(\lambda) \in J_\lambda$ is fixed by f_λ^p for all $\lambda \in B_{\epsilon_0}$ and $\gamma_0(0)$ is not repelling but $\gamma_0(\lambda_0)$ is repelling for some $\lambda_0 \in B_{\epsilon_0}$. Our aim below is to find $\lambda'_0 \in B_{\epsilon_0}$ such that $\gamma(\lambda'_0)$ is a virtually repelling Siegel fixed point of $f_{\lambda'_0}^p$. Corollary 5.4 then yields a contradiction.

Reducing ϵ_0 allows to use charts and replace \mathbb{P}^k by \mathbb{C}^k . Let us denote $w_1(\lambda), \dots, w_k(\lambda)$ the eigenvalues of $A(\lambda) := (f_\lambda^p)'(\gamma(\lambda))$. There exists a proper analytic subset Z of B_{ϵ_0} such that w_1, \dots, w_k are holomorphic on $B_{\epsilon_0} \setminus Z$. For every $n \in \mathbb{N}$ we define a function ω_n on $B_{\epsilon_0} \setminus Z$:

$$\omega_n(\lambda) := \min_{2 \leq |m| \leq n, 1 \leq j \leq k} |w_1(\lambda)^{m_1} \cdots w_k(\lambda)^{m_k} - w_j(\lambda)|$$

where $|m| := m_1 + \cdots + m_k$ for any $m := (m_1, \dots, m_k) \in \mathbb{N}^k$. Since the cycle $\gamma_0(\lambda)$ is not persistently resonant the functions $\log \omega_n$ are not identically equal to $-\infty$. Moreover, after shrinking ϵ_0 , we have $\log \omega_n(\lambda) \leq \log \omega_2(\lambda) \leq C < +\infty$ on $B_{\epsilon_0} \setminus Z$ and therefore $\log \omega_n$ extends to some *psh* function on B_{ϵ_0} . We now define a function \mathcal{B} on B_{ϵ_0} by setting

$$\mathcal{B}(\lambda) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \log \omega_{2^{n+1}}(\lambda).$$

The interest of this function is that, according to Brjuno's theorem (see [Br]), f_λ^p is holomorphically linearizable at $\gamma(\lambda)$ if $\mathcal{B}(\lambda) > -\infty$ and $A(\lambda)$ is diagonalizable. Let us show that \mathcal{B} is *psh* on B_{ϵ_0} . Since $\mathcal{B}(\lambda) - 2C = \sum_{n=0}^{+\infty} \frac{1}{2^n} (\log \omega_{2^{n+1}}(\lambda) - C)$ is a decreasing limit of *psh* functions, the function \mathcal{B} is either *psh* or identically equal to $-\infty$ on B_{ϵ_0} . Moreover, as $\gamma(\lambda_0)$ is a repelling cycle there exists $n_0 \geq 1$ such that $\log \omega_{2^n} = \log \omega_{2^{n_0}}$ on a neighbourhood V_0 of λ_0 for $n \geq n_0$. We deduce that $\mathcal{B} = \sum_{n=0}^{n_0} \frac{1}{2^n} \log \omega_{2^{n+1}}(\lambda) + \frac{1}{2^{n_0}} \log \omega_{2^{n_0+1}}$ on V_0 , this function is therefore not identically equal to $-\infty$ since $\gamma_0(\lambda)$ is not persistently resonant.

Let us denote by Δ_{ϵ_0} the disc in \mathbb{C} obtained by intersecting B_{ϵ_0} with the complex line through 0 and λ_0 . We may move a little bit λ_0 so that \mathcal{B} is subharmonic on Δ_{ϵ_0} , the set $Z \cap \Delta_{\epsilon_0}$ is discrete and $\gamma_0(\lambda)$ is not persistently undiagonalizable on Δ_{ϵ_0} . In particular, there exists a discrete subset Z_0 of Δ_{ϵ_0} such that on $\Delta_{\epsilon_0} \setminus Z_0$, the cycle $\gamma_0(\lambda)$ is diagonalizable and the functions w_1, \dots, w_k are either constant or holomorphic, non-vanishing and with non-vanishing derivatives.

Let us set

$$\forall \lambda \in \Delta_{\epsilon_0} \setminus Z_0, \quad \varphi(\lambda) := \min(|w_1(\lambda)|, \dots, |w_k(\lambda)|).$$

This extends to a continuous function on Δ_{ϵ_0} . Moreover $\varphi(0) \leq 1$ and $\varphi(\lambda_0) > 1$, in particular φ is not constant. We claim that there exists $\lambda_1 \in \Delta_{\epsilon_0} \setminus Z_0$ such that $\varphi(\lambda_1) < 1$. Indeed, if $\varphi \geq 1$ on $\Delta_{\epsilon_0} \setminus Z_0$, then $\varphi \geq 1$ on Δ_{ϵ_0} and therefore the subharmonic function $\psi := \varphi^{-1}$ violates the maximum principle (indeed $\psi \leq 1 = \psi(0)$ and this function is not constant). Considering a continuous path connecting λ_0 to λ_1 in $\Delta_{\epsilon_0} \setminus Z_0$, one finds $\lambda_2 \in \Delta_{\epsilon_0} \setminus Z_0$ and $\tilde{\lambda}_k \rightarrow \lambda_2$ such that

$\varphi(\lambda_2) = 1$ and $\varphi(\tilde{\lambda}_k) > 1$. Let us pick a small disc $D(\lambda_2, R)$ contained in $\Delta_{\epsilon_0} \setminus Z_0$. Then (after renumbering) the functions w_1, \dots, w_k satisfy the assumptions of Lemma 5.7 on $D(\lambda_2, R)$. Let q be the integer and C be the real analytic arc in $D(\lambda_2, R)$ which are given by this Lemma. Since $|w_j| < 1$ on $D^-(\lambda_2, R)$ for $k - q + 1 \leq j \leq k$ and $\gamma(\lambda) \in J_\lambda$, we must have $1 \leq q \leq k - 1$.

Since \mathcal{B} is subharmonic on Δ_{ϵ_0} , there exists $\lambda'_0 \in C$ such that $\mathcal{B}(\lambda'_0) > -\infty$. Since $\lambda'_0 \in D(\lambda_2, R) \subset \Delta_{\epsilon_0} \setminus Z_0$, the periodic point $\gamma(\lambda'_0)$ is diagonalizable and then, according to Brjuno's theorem, it is holomorphically linearizable. Thus $\gamma(\lambda'_0)$ is a Siegel fixed point of $f_{\lambda'_0}^p$ and, since $\lambda'_0 \in C$, Lemma 5.7 shows that it is virtually repelling as desired. Let us finally explain why we do not need any assumption on the repelling J -cycle in dimension $k = 2$. In that case, the periodic points $\gamma(\lambda)$ for $\lambda \in C$ are diagonalizable and not persistently resonant since one and only one of their two multipliers have modulus 1 and, moreover, is not constant. We thus see that \mathcal{B} is subharmonic on Δ_{ϵ_0} and we can find again some $\lambda'_0 \in C$ such that $\gamma(\lambda'_0)$ is a virtually repelling Siegel fixed point of $f_{\lambda'_0}^p$. \square

It would be interesting to know if the continuity of the map $\lambda \mapsto J_\lambda$ on some open subset of the parameter space is equivalent to the existence of an equilibrium web. This is true when $k = 1$, the following remark summarizes the consequences of the above results on this question in higher dimension.

Remark 5.8. *According to Proposition 5.3 and the proof of Proposition 5.6, when $k = 2$ the Hausdorff continuity of $\lambda \mapsto J_\lambda$ would imply the holomorphic stability if we would know that a local Siegel disc centered at some virtually repelling Siegel periodic point cannot be contained in the Julia set.*

To deduce Theorem 1.1 from Proposition 5.6, we shall use the following Lemma whose proof is left to the reader.

Lemma 5.9. *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family where B is an open ball of the space $\mathcal{H}_d(\mathbb{P}^k)$ of degree d holomorphic endomorphisms of \mathbb{P}^k . Then every repelling J -cycle is neither persistently resonant nor persistently undiagonalizable.*

PROOF OF THEOREM 1.1: In subsection 3.5 we saw that $(A) \Rightarrow (B) \Leftrightarrow (E)$. Theorem 1.6 yields $(B) \Rightarrow (C')$, where (C') is the assertion : "the restriction $f_{B \times \mathbb{P}^k}$ admits an equilibrium web for any sufficiently small ball B ". Assume now that (C') is satisfied. Combining Lemma 5.9 and Proposition 5.6 one sees that, when M satisfies the assumptions of Theorem 1.1, the repelling J -cycles locally move holomorphically. This implies that the set

$$\{(\lambda, z) \in M \times \mathbb{P}^k : z \text{ belongs to some } n\text{-periodic and repelling } J\text{-cycle of } f_\lambda\}$$

is an unramified cover of M . As M is simply-connected, we thus get that the repelling J -cycles move holomorphically over M , hence $(C') \Rightarrow (C)$. Finally proposition 5.6 yields $(C) \Rightarrow (A)$, and therefore the properties (A) , (B) and (C) are equivalent. If (D) is satisfied then by definition any element γ of the equilibrium lamination belongs to \mathcal{J} and satisfies $\Gamma_\gamma \cap PC_f = \emptyset$. Then the first assertion of Proposition 2.3 shows that f admits an equilibrium web. We thus have $(D) \Rightarrow (C)$. Finally, since by Theorem 4.1 $(A) \Rightarrow (D)$, the proof of Theorem 1.1 is completed. \square

6. BIFURCATION LOCI

In view of Theorem 1.1, we define the bifurcation locus and current as follows.

Definition 6.1. Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Let $L(\lambda)$ be the sum of Lyapunov exponents of f_λ with respect to its equilibrium measure. The closed positive current $dd_\lambda^c L$ is called bifurcation current of the family, its support is the bifurcation locus of the family.

We will exploit here our results to get some informations on these loci.

6.1. On the interior of bifurcation loci. In his work on the persistence of homoclinic tangencies, Buzzard [Bu] found open subsets of the space of degree d endomorphisms of \mathbb{P}^2 (for d large enough) in which the maps having infinitely many sinks are dense. This lead us to believe that the bifurcation locus may have a non-empty interior when $k \geq 2$. We investigate here the relations between the presence of open subsets in the support of $dd_\lambda^c L$ and the existence of parameters for which the postcritical set is dense in \mathbb{P}^k .

Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let C denote the critical set of f and let C_λ denote the critical set of f_λ . We set

$$\overline{C^+} := \overline{\bigcup_{n \geq 1} f^n(C)} \quad \text{and} \quad \overline{C_\lambda^+} := \overline{\bigcup_{n \geq 1} f_\lambda^n(C_\lambda)} \quad \text{for every } \lambda \in M.$$

We define $(\overline{C^+})_\lambda := (\{\lambda\} \times \mathbb{P}^k) \cap \overline{C^+}$, observe that $\{\lambda\} \times \overline{C_\lambda^+} \subset (\overline{C^+})_\lambda$. Our aim is to show that if $\text{supp } dd_\lambda^c L$ contains an open subset $\Omega \subset M$, then $\{\lambda \in \Omega : \overline{C_\lambda^+} = \mathbb{P}^k\}$ contains a G_δ -dense subset of Ω . This will prove Theorem 1.8.

As a consequence we recover the fundamental result of Mañé, Sad and Sullivan [MSS] on the density of stable parameters for holomorphic families of rational maps. For such families the bifurcation locus is known to coincide with $\text{supp } dd_\lambda^c L$ ([dM]).

Corollary 6.2. Let $f : M \times \mathbb{P}^1 \rightarrow M \times \mathbb{P}^1$ be a holomorphic family of rational maps. Then $\text{supp } dd_\lambda^c L$ has empty interior.

Proof. Every $\lambda_0 \in \text{supp } dd_\lambda^c L$ can be approximated by parameters λ for which f_λ has an attracting basin, see [B, section 4.3.1], which is an open condition in M . On the other hand, as the critical set is finite, the set $\overline{C_\lambda^+}$ can not be equal to \mathbb{P}^1 when f_λ has an attracting basin. According to Theorem 1.8, this implies that $\text{supp } dd_\lambda^c L$ has empty interior. \square

Remark 6.3. We raise the question, for $k \geq 2$, of the existence of holomorphic families for which $\text{supp } dd_\lambda^c L$ has non empty interior. Note that Theorem 1.8 could be useful for finding families for which $\text{supp } dd_\lambda^c L$ has empty interior.

The proof of Theorem 1.8 relies on a Baire's category argument based on the continuity properties of $\lambda \mapsto \overline{C_\lambda^+}$ and $\lambda \mapsto (\overline{C^+})_\lambda$. The notion of semi continuity with respect to the Hausdorff topology has been discussed in subsection 2.3. We have the following properties, the upper semi continuity can be found in [Do, Proposition 2.1], we give the argument for sake of completeness.

Lemma 6.4. The maps $\lambda \mapsto (\overline{C^+})_\lambda$ and $\lambda \mapsto \overline{C_\lambda^+}$ from M to $\text{Comp}^*(\mathbb{P}^k)$ are respectively upper and lower semi continuous.

Proof. By definition $\{(\lambda, z) \in M \times \mathbb{P}^k, z \in (\overline{C^+})_\lambda\}$ is equal to $\overline{C^+}$, hence is closed in $M \times \mathbb{P}^k$. In particular, for every $\lambda_0 \in M$ and $\epsilon > 0$, the set $F := \{(\lambda, z) \in \overline{C^+}, d_{\mathbb{P}^k}(z, (\overline{C^+})_{\lambda_0}) \geq \epsilon\}$ is a closed subset of $\overline{C^+}$. Let us show that $\pi_M(F)$ is closed in M . Indeed, if $\lambda_n \in \pi_M(F)$ converges to $\lambda \in M$

one may pick $z_n \in (\overline{C^+})_{\lambda_n}$ such that $d_{\mathbb{P}^k}(z_n, (\overline{C^+})_{\lambda_0}) \geq \epsilon$ and $(z_n)_n$ converges to some $z \in \mathbb{P}^k$ after taking a subsequence. Then $(\lambda_n, z_n) \in \overline{C^+}$ converges to $(\lambda, z) \in \overline{C^+}$ satisfying $d_{\mathbb{P}^k}(z, (\overline{C^+})_{\lambda_0}) \geq \epsilon$ and thus $\lambda \in \pi_M(F)$ as desired. Since $\lambda_0 \notin \pi_M(F)$ it follows that $M \setminus \pi_M(F)$ contains an open ball B centered at λ_0 such that $d_{\mathbb{P}^k}(z, (\overline{C^+})_{\lambda_0}) < \epsilon$ for every $z \in (\overline{C^+})_\lambda$ with $\lambda \in B$. This proves the upper semi continuity.

Let us now prove the lower semi continuity of the map $\lambda \mapsto \overline{C^+}_\lambda$. Assume to the contrary that it is not *l.s.c* at $\lambda_0 \in M$. Then there exist $\epsilon > 0$, a sequence $(\lambda_n)_n$ converging to λ_0 and a sequence $(z_n)_n$ in $\overline{C^+}_{\lambda_0}$ such that $d_{\mathbb{P}^k}(z_n, \overline{C^+}_{\lambda_n}) \geq \epsilon$. After taking a subsequence $(z_n)_n$ converges to $z_0 \in \overline{C^+}_{\lambda_0}$. Pick $\xi_0 \in C_{\lambda_0}$ and $p_0 \geq 1$ such that $d_{\mathbb{P}^k}(z_0, f_{\lambda_0}^{p_0}(\xi_0)) < \frac{\epsilon}{4}$. Let also $\xi_n \in C_{\lambda_n}$ such that $\xi_n \rightarrow \xi_0$. Then $d_{\mathbb{P}^k}(z_n, \overline{C^+}_{\lambda_n}) \leq d_{\mathbb{P}^k}(z_n, f_{\lambda_n}^{p_0}(\xi_n)) < \frac{\epsilon}{2}$ for n large, contradicting $d_{\mathbb{P}^k}(z_n, \overline{C^+}_{\lambda_n}) \geq \epsilon$. \square

Lemma 6.4 allows us to prove:

Proposition 6.5. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If $\lambda_0 \in \text{supp } dd_\lambda^c L$ then $(\overline{C^+})_{\lambda_0} = \mathbb{P}^k$.*

Proof. Assume that $B(z_0, r) \cap (\overline{C^+})_{\lambda_0} = \emptyset$ and let us show that $\lambda_0 \notin \text{supp } dd_\lambda^c L$. Since $\lambda \mapsto (\overline{C^+})_\lambda$ is upper semi continuous we deduce that $B(z_0, \frac{r}{2}) \cap (\overline{C^+})_\lambda = \emptyset$ when λ is sufficiently close to λ_0 . In particular, the constant graph $\Gamma_0 := \{(\lambda, z_0) : \lambda \in B(\lambda_0, \epsilon)\}$ does not meet $\cup_{n \geq 1} f^n(C)$ for ϵ small enough. By the first assertion of Proposition 2.3 and Proposition 3.5, we get $dd_\lambda^c L = 0$ on $B(\lambda_0, \epsilon)$. \square

PROOF OF THEOREM 1.8 : The lower semi continuity of $\lambda \mapsto \overline{C^+}_\lambda$ implies that

$$I(B) := \{\lambda \in M : \overline{C^+}_\lambda \cap B \neq \emptyset\}$$

is an open subset of M for every open ball $B \subset \mathbb{P}^k$. Now let Ω be an open subset of M which is contained in the bifurcation locus. Let us show that $I(B)$ is dense in Ω . We may assume that Ω is a ball in \mathbb{C}^m . Let $\lambda_0 \in \Omega$ and $\epsilon > 0$. Since $\lambda_0 \in \text{supp } dd_\lambda^c L$, Proposition 6.5 implies that $(\overline{C^+})_{\lambda_0} \cap B = B$. Thus $(\cup_{n \geq 1} f^n(C)) \cap (B(\lambda_0, \epsilon) \times B) \neq \emptyset$ and there exists $(\lambda_1, z_1) \in f^{n_1}(C) \cap (B(\lambda_0, \epsilon) \times B)$. This shows that $\lambda_1 \in I(B) \cap B(\lambda_0, \epsilon)$ and thus $I(B)$ is open and dense in Ω . Now consider a countable collection $B_i := B(\zeta_i, r_i)$ of balls in \mathbb{P}^k whose centers are dense in \mathbb{P}^k and whose radii tend to 0. According to Baire's theorem $M' := \cap_{i \geq 1} I(B_i)$ is a dense G_δ -subset of Ω . We also have $\overline{C^+}_\lambda = \mathbb{P}^k$ for every $\lambda \in M'$. \square

6.2. Remarkable elements in bifurcation loci. Theorem 1.1 and the proof of Proposition 5.6 immediately yield the following result.

Theorem 6.6. *A degree $d \geq 2$ endomorphism of \mathbb{P}^k belongs to the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^k)$ if and only if it is accumulated by endomorphisms which admit a virtually J -repelling Siegel periodic point or a repelling cycle outside the Julia set which becomes a repelling J -cycle after an arbitrarily small perturbation.*

The next theorem shows that *isolated* Lattès maps belong to the bifurcation locus. We refer to the articles [Di1], [Du1] for an account on Lattès maps of \mathbb{P}^k .

Theorem 6.7. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If the family is stable (i.e. $dd_\lambda^c L = 0$ on M) and f_{λ_0} is a Lattès map for some $\lambda_0 \in M$ then f_λ is a Lattès map for every $\lambda \in M$.*

Proof. By a Theorem of Briend-Duval [BrDv1] we have $L \geq k \frac{\log d}{2}$. The articles of Berteloot, Dupont and Loeb [BL], [BtDp] and [Du2] show that $L(\lambda) = k \frac{\log d}{2}$ if and only if f_λ is a Lattès map. If the family is stable, then the function L is pluriharmonic on M . By the maximum principle (applied to the harmonic function $-L$) we thus have $L(\lambda) = L(\lambda_0) = k \frac{\log d}{2}$ for all $\lambda \in M$ and the conclusion follows. \square

APPENDIX A

A.1. A stronger version of Lemma 2.2.

Lemma A.1. *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that there exists a sequence of Borel probability measures $(\mathcal{M}_n)_{n \geq 1}$ on $\mathcal{O}(M, \mathbb{P}^k)$ such that*

- 1) $\lim_n (\mathcal{M}_n)_\lambda = \mu_\lambda$ for Lebesgue-almost every $\lambda \in M$.
- 2) $\mathcal{F}_* \mathcal{M}_{n+1} = \mathcal{M}_n$ or $\mathcal{F}_* \mathcal{M}_n = \mathcal{M}_n$ for every $n \geq 1$.
- 3) *There exists a compact subset \mathcal{K} of $\mathcal{O}(M, \mathbb{P}^k)$ such that $\text{supp } \mathcal{M}_n \subset \mathcal{K} \subset \mathcal{F}(\mathcal{K})$ for every $n \geq 1$.*

Then any limit \mathcal{M} of $(\frac{1}{n} \sum_{l=1}^n \mathcal{M}_l)_{n \geq 1}$ satisfies $\mathcal{M}_\lambda = \mu_\lambda$ for Lebesgue-almost every $\lambda \in M$.

We shall use the following corollary of Theorem 2.7. It is inspired by [Pha, Proposition 2.1], one can find a more general version in [DS4, remark 2.2.6].

Lemma A.2. *Let $(\mathcal{R}_n)_n$ be a sequence of closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$. Assume that $\lim_n \mathcal{R}_n = \mathcal{R}$ and that $\text{supp } \mathcal{R}_n \subset M \times K$ for some compact subset K of \mathbb{C}^{k+1} . Then, after taking a subsequence, we have $\lim_n \langle \mathcal{R}_n, \pi_M, \lambda \rangle = \langle \mathcal{R}, \pi_M, \lambda \rangle$ for almost every $\lambda \in M$.*

PROOF OF LEMMA A.1 : Let us set $\mathcal{V}_n := \frac{1}{n} \sum_{l=1}^n \mathcal{M}_l$. We may assume that $\mathcal{V}_n \rightarrow \mathcal{V}$. By assumption, $(\mathcal{V}_n)_\lambda \rightarrow \mu_\lambda$ for Lebesgue almost every $\lambda \in M$. Let us show that $\mathcal{V}_\lambda = \mu_\lambda$ for every $\lambda \in M$. The problem being local, we may replace M by any small ball B in \mathbb{C}^m . Let F be a lift of the family f to $M \times \mathbb{C}^{k+1}$. Note that for any test function ϕ on \mathbb{P}^k , the functions $\lambda \mapsto \langle \mathcal{V}_\lambda, \phi \rangle$ and $\lambda \mapsto \langle \mu_\lambda, \phi \rangle$ are both continuous. This follows easily from the facts that \mathcal{V} is supported on \mathcal{K} which is an equicontinuous family of holomorphic maps and that $\mu_\lambda = \pi_* (dd_{\lambda, \tilde{z}}^c G_F(\lambda, \tilde{z}))^{k+1}$ where G_F is the Green function of F which is continuous on $B \times \mathbb{C}^{k+1}$. It is thus enough to show that $\mathcal{V}_\lambda = \mu_\lambda$ for almost every $\lambda \in B$.

For that purpose we use Lemma 2.8 to associate horizontal currents $\widetilde{W}_{\mathcal{V}_n}$ and $\widetilde{W}_{\mathcal{V}}$ to \mathcal{V}_n and \mathcal{V} . As $\widetilde{W}_{\mathcal{V}_n}$ converges towards $\widetilde{W}_{\mathcal{V}}$ as currents, Lemma A.2 implies that $(\mathcal{V}_n)_\lambda \rightarrow \mathcal{V}_\lambda$ for almost every $\lambda \in B$, hence $\mathcal{V}_\lambda = \mu_\lambda$ for almost every $\lambda \in B$. \square

PROOF OF LEMMA A.2: Let ψ be a test function on \mathbb{C}^{k+1} which will be considered as a function on $M \times \mathbb{C}^{k+1}$. We set $u_{\psi, n}(\lambda) := \langle \mathcal{R}_n, \pi_M, \lambda \rangle \psi$, $u_\psi(\lambda) := \langle \mathcal{R}, \pi_M, \lambda \rangle \psi$ for every $\lambda \in M$. According to Theorem 2.7, the slice masses $c_n := |\langle \mathcal{R}_n, \pi_M, \lambda \rangle|$ and $c := |\langle \mathcal{R}, \pi_M, \lambda \rangle|$ do not depend on $\lambda \in M$. Given a (m, m) -test form ω on M , the basic slicing formula (3) gives

$$(29) \quad \lim_n \int_M u_{\psi, n}(\lambda) \omega(\lambda) = \lim_n \langle \mathcal{R}_n \wedge \pi_M^*(\omega), \psi \rangle = \langle \mathcal{R} \wedge \pi_M^*(\omega), \psi \rangle = \int_M u_\psi(\lambda) \omega(\lambda).$$

Applying (29) with $\psi \equiv 1$ on K , we get $c = \lim_n c_n$ and thus $C := \sup_n c_n$ is finite. Taking $C < +\infty$ into account, one sees by using Slutsky's lemma and a diagonal argument that it suffices to prove that $u_{\psi, n} \rightarrow u_\psi$ in $L^1_{loc}(M)$ for every test function ψ . Let us first verify this convergence when ψ is a smooth *psh* function. By Theorem 2.7, $(u_{\psi, n})_n$ is a sequence of *psh* functions, which

is locally uniformly bounded on M since $|u_{\psi,n}| \leq c_n \sup_K |\psi| \leq C \sup_K |\psi|$. As such sequences are relatively compact for the L^1_{loc} topology, it suffices to show that u_ψ is the unique cluster point of $(u_{\psi,n})_n$. Assume that $u_{\psi,n_k} \rightarrow v$ in $L^1_{loc}(M)$. According to (29), we have $\int_M u_\psi \omega = \int_M v \omega$ for every (m, m) -test form ω on M . Hence v and u_ψ coincide in $L^1_{loc}(M)$, as desired. This remains true when ψ is any smooth test function on \mathbb{C}^{k+1} , as one sees by writing it as a difference of two smooth psh functions: $\psi = (\psi + A\|z\|^2) - A\|z\|^2$ with A large enough. \square

A.2. Hyperbolic sets and holomorphic motions.

Definition A.3. Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms where B is a ball centered at the origin in \mathbb{C}^m . Let E_0 be an f_0 -invariant subset of \mathbb{P}^k . A holomorphic motion of E_0 over $B_\rho \subset B$ is a continuous map $h : B_\rho \times E_0 \rightarrow \mathbb{P}^k$ such that :

- (1) $\lambda \mapsto h_\lambda(z)$ is holomorphic on B_ρ for every $z \in E_0$.
- (2) $z \mapsto h_\lambda(z)$ is injective on E_0 for every $\lambda \in B_\rho$.
- (3) $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ on E_0 for every $\lambda \in B_\rho$.

One says that E_0 is a hyperbolic set for f_0 if it is f_0 -invariant and if there exists $K > 1$ such that $|(df_0)^{-1}|^{-1} \geq K$ on E_0 .

Theorem A.4. Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms. Let $E_0 \subset \mathbb{P}^k$ such that $|(df_0)^{-1}|^{-1} \geq K > 3$ on E_0 . Then there exists a holomorphic motion $h : B_\rho \times E_0 \rightarrow \mathbb{P}^k$ which preserves repelling cycles.

The proof is based on classical arguments, we refer to [dMvS, chapter 3, section 2.d] for the one dimensional case. To simplify the exposition we assume that the dilation is larger than 3 on the hyperbolic set.

Proof. Let $\varphi(z) := \inf_{\lambda \in B_\rho} |(d_z f_\lambda)^{-1}|^{-1}$, with the convention $|(d_z f_\lambda)^{-1}|^{-1} = 0$ if $z \in C_{f_\lambda}$. This is a continuous function on \mathbb{P}^k . By taking a smaller ρ , we may assume that

$$(30) \quad \varphi \geq K' > 3 \text{ on a } \tau\text{-neighbourhood } (E_0)_\tau.$$

We shall mainly use the lower estimate on E_0 itself, the lower bound on $(E_0)_\tau$ appears at the end of the proof. Let $\delta = \delta(\rho) := \min\{(1 + \sup_{\lambda \in B_\rho} \|f_\lambda\|_{C^2})^{-1}, \tau\}$.

Lemma A.5. For every $(\lambda, z) \in B_\rho \times E_0$,

- (1) $d_{\mathbb{P}^k}(f_\lambda(z), f_\lambda(w)) \geq (K' - 1)d_{\mathbb{P}^k}(z, w)$ for every $w \in \bar{B}(z, \delta)$,
- (2) $f_\lambda(B(z, c\delta)) \supset B(f_\lambda(z), c\delta)$ for every $0 \leq c \leq 1$,
- (3) if $g_{\lambda,z} : B(f_\lambda(z), \delta) \rightarrow B(z, \delta)$ is the inverse map of f_λ , then $\text{Lip } g_{\lambda,z} \leq (K' - 1)^{-1}$.

Proof. Assertions 2 and 3 follow from the first one (use Jordan's theorem and $K' > 3$ for the second one). So let us prove Assertion 1. We work in local coordinates. For $(\lambda, z) \in B_\rho \times E_0$ and $w \in \bar{B}(z, \delta)$ we have

$$\begin{aligned} |\text{Id}_{\mathbb{C}^k} - (d_z f_\lambda)^{-1} \circ d_w f_\lambda| &\leq |(d_z f_\lambda)^{-1}| \cdot |d_z f_\lambda - d_w f_\lambda| \\ &\leq |(d_z f_\lambda)^{-1}| \cdot |z - w| \cdot \delta^{-1} \leq 1/K'. \end{aligned}$$

That implies $\text{Lip}(\text{Id} - (d_z f_\lambda)^{-1} \circ f_\lambda) \leq 1/K'$ on $\bar{B}(z, \delta)$, which gives in turn

$$|(d_z f_\lambda)^{-1}(f_\lambda(z) - f_\lambda(w))| \geq (1 - 1/K')|z - w|$$

for every $w \in \bar{B}(z, \delta)$. Hence $|f_\lambda(z) - f_\lambda(w)| \geq (K' - 1)|z - w|$ as desired. \square

Lemma A.6. *For every $(\lambda, z) \in B_\rho \times E_0$, we have $B(f_\lambda(z), \delta) \supset B(f_0(z), \delta/2)$ and the inverse map $g_{\lambda, z} : B(f_\lambda(z), \delta) \rightarrow B(z, \delta)$ given by Lemma A.5 satisfies:*

- (1) $g_{\lambda, z}$ is well defined on $B(f_0(z), \delta/2)$,
- (2) it satisfies $\text{Lip } g_{\lambda, z} \leq (K' - 1)^{-1}$ on $B(f_0(z), \delta/2)$,
- (3) $g_{\lambda, z}(B(f_0(z), \delta/2)) \subset B(z, \delta/2)$.

Proof. Let $Q := \max \{ \|d_\lambda f_\lambda(z)\|, (\lambda, z) \in B_\rho \times E_0 \}$. As δ is a continuous function of ρ and $\delta(0) > 0$, we may assume $\delta \geq 2Q\rho$ by taking ρ small enough. For every $\lambda \in B_\rho$ and $z \in E_0$, $d(f_\lambda(z), f_0(z)) \leq Q\rho \leq \delta/2$. That yields $B(f_\lambda(z), \delta) \supset B(f_0(z), \delta/2)$. Items 1 and 2 are obvious from lemma A.5. For item 3, we use $g_{\lambda, z}(B(f_0(z), \delta/2)) \subset g_{\lambda, z}(B(f_\lambda(z), \delta))$, which is included in $B(z, \delta/2)$ by using lemma A.5(3) and $K' > 3$. \square

Let us end the proof of theorem A.4. For $(\lambda, z) \in B_\rho \times E_0$ we set $z_n := f_0^n(z)$ and

$$g_{\lambda, z}^n := g_{\lambda, z} \circ \dots \circ g_{\lambda, z_{n-1}}.$$

This is an inverse branch of f_λ^n . Since $z_1, \dots, z_{n-1} \in E_0$, lemma A.6 yields by induction

$$g_{\lambda, z}^n : B(z_n, \delta/2) \rightarrow B(z, \delta/2) \text{ and } \text{Lip } g_{\lambda, z}^n \leq (K' - 1)^{-n} \text{ on } B(z_n, \delta/2).$$

For $(\lambda, z) \in B_\rho \times E_0$ let us define

$$h_n(\lambda, z) := g_{\lambda, z}^n \circ f_0^n(z) = g_{\lambda, z}^n(z_n).$$

The map h_n is continuous in (λ, z) , holomorphic in λ and $h_n(\lambda, z) \in B(z, \delta/2)$. Moreover

$$(31) \quad f_\lambda \circ h_n(\lambda, z) = h_{n-1}(\lambda, f_0(z)).$$

The sequence $(h_n)_n$ is uniformly Cauchy on $B_\rho \times E_0$. Indeed $h_{n+1}(\lambda, z) - h_n(\lambda, z) = g_{\lambda, z}^n \circ g_{\lambda, z_n}(z_{n+1}) - g_{\lambda, z}^n(z_n)$ and we get $\|h_{n+1} - h_n\|_{B_\rho \times E_0} \leq (\delta/2) \cdot (K' - 1)^{-n}$ since $g_{\lambda, z_n}(z_{n+1}) \in B(z_n, \delta/2)$ by Lemma A.6(3). We define $h_\lambda(z)$ for $(\lambda, z) \in B_\rho \times E_0$ by

$$h_\lambda(z) := \lim_n h_n(\lambda, z) = \lim_n g_{\lambda, z}^n \circ f_0^n(z).$$

The map h is continuous in (λ, z) , holomorphic in λ and $h_\lambda(z) \in \bar{B}(z, \delta/2)$. It also follows from (31) that

$$(32) \quad f_\lambda \circ h_\lambda = h_\lambda \circ f_0.$$

Let us now check that h_λ is injective. Assume $h_\lambda(z) = h_\lambda(z')$. Iterating (32) yields $h_\lambda(f_0^n(z)) = h_\lambda(f_0^n(z'))$. As $h_\lambda(w) \in \bar{B}(w, \delta/2)$ for $w \in E_0$, we get $d(f_0^n(z), f_0^n(z')) \leq \delta$. Then, since $d(f_0^n(z), f_0^n(z')) \geq (K' - 1)^n d(z, z')$ by Lemma A.5(1), we must have $z = z'$.

Finally, h_λ preserves cycles (see (31)) and any periodic $h_\lambda(z)$ must be repelling since $h_\lambda(z) \in \bar{B}(z, \delta/2) \subset (E_0)_\tau$ and $|(df_\lambda)^{-1}|^{-1} > 3$ on $(E_0)_\tau$ (see (30)). This completes the proof of Theorem A.4. \square

A.3. Proof of Proposition 4.2. We work with the notations of Section 4. Let $\tau, \epsilon > 0$ such that $-\frac{\log d}{2} + \tau + 2\epsilon < 0$. Recall that the distortion of the charts is controlled by τ , see Equation (9). Let $p \geq 1$ and $r_p(\gamma) = \inf_{\lambda \in U_0} \|(DF_{\gamma(\lambda)}^p(0))^{-1}\|^{-2}$, see Equation (11). The next lemma shows that r_p measures the size of tubular neighbourhoods of Γ_γ on which f^p is invertible and contracting.

Lemma A.7. *For every small $\epsilon > 0$ there exists $C_p(\epsilon) > 0$ such that for any $\gamma \in \mathcal{X}$ the map f^p admits an inverse branch $(f^p)_\gamma^{-1}$ on the tube $T_{U_0}(\mathcal{F}^p(\gamma), C_p(\epsilon)r_p(\gamma))$ which maps $\Gamma_{\mathcal{F}^p(\gamma)} \cap (U_0 \times \mathbb{P}^k)$ to $\Gamma_\gamma \cap (U_0 \times \mathbb{P}^k)$ and satisfy $\text{Lip}(f^p)_\gamma^{-1} \leq e^{\tau + \epsilon/3} r_p(\gamma)^{-1/2}$.*

Proof. We use a quantitative version of the inverse mapping theorem, see [BrDv2, Lemme 2]. This version is more precise than Lemma A.5. Let $M := \sup_{\lambda \in U_0, \gamma \in \mathcal{X}} \|F_{\gamma(\lambda)}^p\|_{\mathcal{C}^2, \overline{B(0, R_p)}}$ and let $\delta_p(\epsilon) := R_p(1 - e^{-\epsilon/3})/M$. Then for every $(\gamma, \lambda) \in \mathcal{X} \times U_0$:

- $(F_{\gamma(\lambda)}^p)^{-1}$ is defined on $B_{\mathbb{C}^k}(0, \delta_p(\epsilon)) \| (DF_{\gamma(\lambda)}^p(0))^{-1} \|^{-2}$,
- $\text{Lip}(F_{\gamma(\lambda)}^p)^{-1} \leq e^{\frac{\epsilon}{3}} \| (DF_{\gamma(\lambda)}^p(0))^{-1} \|$.

Now we have to consider the distortion due to the charts. Replacing $\delta_p(\epsilon)$ by a smaller constant $C_p(\epsilon)$ and recalling that τ controls this distortion, we obtain for every $\lambda \in M$:

- $(f^p)_{\gamma(\lambda)}^{-1}$ is defined on $B_{\mathbb{P}^k}(f_\lambda^p(\gamma(\lambda)), C_p(\epsilon) \| (DF_{\gamma(\lambda)}^p(0))^{-1} \|^{-2})$,
- $\text{Lip}(f^p)_{\gamma(\lambda)}^{-1} \leq e^{\tau + \frac{\epsilon}{3}} \| (DF_{\gamma(\lambda)}^p(0))^{-1} \|$.

This completes the proof of the Lemma. \square

Let us now prove Proposition 4.2. We recall that $\widehat{u}_p(\widehat{\gamma}) = -\frac{1}{2} \log r_p(\gamma_0)$. By assumption $\lim_n \frac{1}{n} \int_{\widehat{\mathcal{X}}} \widehat{u}_n d\widehat{\mathcal{M}} = L$ with $L \leq -\frac{\log d}{2}$. Let $p \geq 1$ such that $\frac{1}{p} \int_{\widehat{\mathcal{X}}} \widehat{u}_p d\widehat{\mathcal{M}} =: L' \leq L + \epsilon$. By applying Birkhoff Ergodic Theorem there exists $\widehat{\mathcal{Y}} \subset \widehat{\mathcal{X}}$ such that $\widehat{\mathcal{M}}(\widehat{\mathcal{Y}}) = 1$ and

$$(33) \quad \forall \widehat{\gamma} \in \widehat{\mathcal{Y}}, \quad \lim_n \frac{1}{n} \sum_{j=1}^n \widehat{u}_p(\widehat{\mathcal{F}}^{-j}(\widehat{\gamma})) = \int_{\widehat{\mathcal{X}}} \widehat{u}_p d\widehat{\mathcal{M}} = pL'.$$

Since $\widehat{u}_p(\widehat{\mathcal{F}}^{-n}(\widehat{\gamma})) = -\frac{1}{2} \log r_p(\gamma_{-n})$ we deduce from (33) that $\lim_n \frac{1}{n} \log r_p(\gamma_{-n}) = 0$. In particular there exists a measurable function $\widehat{r}_p : \widehat{\mathcal{Y}} \rightarrow]0, 1]$ such that

$$C_p(\epsilon) r_p(\gamma_{-n}) \geq \widehat{r}_p(\widehat{\gamma}) e^{-(n-1)\epsilon/2}.$$

We also deduce from (33) that there exists $\widehat{l}_p : \widehat{\mathcal{Y}} \rightarrow [1, +\infty[$ such that

$$\prod_{j=1}^n (r_p(\gamma_{-j}))^{-1/2} \leq \widehat{l}_p(\widehat{\gamma}) e^{npL' + n\epsilon/6}.$$

Now, setting $\widehat{\eta}_p := \widehat{r}_p / \widehat{l}_p$, one can verify by induction:

- $(f^p)_{\widehat{\gamma}}^{-n}$ is defined on $T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))$,
- $\text{Lip}(f^p)_{\widehat{\gamma}}^{-n} \leq \widehat{l}_p(\widehat{\gamma}) e^{n(pL' + \tau + \epsilon/2)}$,
- $(f^p)_{\widehat{\gamma}}^{-n} [T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))] \subset T_{U_0}(\gamma_{-n}, C_p(\epsilon) r_p(\gamma_{-(n+1)}))$.

See [Du, Section 1.1.6] for more details. This completes the proof of Proposition 4.2.

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