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Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of \mathbb{P}^k

François Berteloot and Christophe Dupont

March 28, 2014

Abstract

We introduce a notion of stability for equilibrium measures in holomorphic families of endomorphisms of \mathbb{P}^k . We characterize the corresponding bifurcations by the strict subharmonicity of the sum of Lyapunov exponents, by the instability of repelling cycles or the instability of critical dynamics. Our methods are based on ergodic theory and on pluripotential theory.

Key Words: holomorphic dynamics, dynamical stability, positive currents, Lyapunov exponents.
MSC 2010: 32H50, 32U40, 37F45, 37F50, 37H15.

1 Introduction and results

In the early 1980's, Mañé, Sad and Sullivan [MSS] and Lyubich [L1, L2] have independently obtained fundamental results on the stability of holomorphic families $(f_\lambda)_{\lambda \in M}$ of rational mappings of the Riemann sphere \mathbb{P}^1 . They discovered that the parameter space M splits into an open and dense *stability locus* and its complement, the *bifurcation locus*. They also obtained precise informations on the distribution of hyperbolic parameters which lead to the so-called hyperbolic conjecture asserting that hyperbolic mappings are dense in the space of degree d rational maps. Thanks to the work of Douady and Hubbard on the Mandelbrot set, we have a much deeper understanding of these questions for the quadratic polynomial family.

The finiteness of the critical set, combined with the use of Picard-Montel's theorem, plays a crucial role in this theory. In particular, it allows to characterize the stability of a parameter $\lambda_0 \in M$ by the stability of the critical orbits of the map f_{λ_0} or, equivalently, a bifurcation at λ_0 by the existence, after an arbitrarily small perturbation, of a repelling cycle capturing a critical orbit. The one-dimensional setting also permits, by mean of the so-called λ -lemma, to build holomorphic motions of Julia sets which conjugate the dynamics on connected components of the stability locus. It should also be stressed that the bifurcation locus coincides with the closure of the subset of parameters $\lambda \in M$ for which the map f_λ admits an unpersistent neutral cycle.

Bifurcation phenomena in families of Hénon maps of \mathbb{C}^2 have already been studied by Bedford, Lyubich and Smillie [BLS] and by Dinh and Sibony [DS5], the sharpest achievements are due to Dujardin and Lyubich in their recent work on the two dimensional and

dissipative case [DL]. However, very little was known up to now for families of endomorphisms of \mathbb{P}^k when $k \geq 2$. Since the classical one-dimensional tools, or even their generalizations to higher dimension, are no longer efficient, one has to adopt a different view-point. Roughly speaking, our approach consists in considering such families as fibered dynamical systems $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ and study them by using ergodic and pluripotential tools as those developed in the works of Bedford-Lyubich-Smillie, Fornæss-Sibony, Briend-Duval, Dinh-Sibony on holomorphic dynamics on \mathbb{P}^k or \mathbb{C}^k (see the survey [DS3] for precise references). Here M is a connected complex manifold and f_λ is the map $f(\lambda, \cdot)$, its algebraic degree is $d \geq 2$ and does not depend on λ . The space $\mathcal{H}_d(\mathbb{P}^k)$ of all degree d holomorphic endomorphisms of \mathbb{P}^k can be identified with the complement of an irreducible complex hypersurface in some \mathbb{P}^N . Every submanifold M of $\mathcal{H}_d(\mathbb{P}^k)$ can be seen as a holomorphic family $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$.

Let us recall that, for each $\lambda \in M$, we have an ergodic dynamical system $(J_\lambda, f_\lambda, \mu_\lambda)$ where μ_λ is the equilibrium measure of f_λ and J_λ its support. The measure μ_λ is also known to be the unique maximal entropy measure of f_λ . The repelling cycles of f_λ are dense in J_λ and even equidistribute the measure μ_λ . We shall call J_λ the *Julia set of f_λ* . Note, however, that J_λ is smaller than the non-normality set of f_λ and that some repelling cycles may belong to the complement of J_λ . The fact that the measure μ_λ enjoys a potential interpretation, $\mu_\lambda = (dd_z^c g(\lambda, z) + \omega_{FS})^k$ where g is the Green function of f and ω_{FS} the Fubini-Study form on \mathbb{P}^k , will also play an important role.

One of our main goals was to relate the stability of repelling cycles with that of Julia sets. However, the lack of λ -lemma lead us to introduce a new notion of stability dealing with the measures μ_λ rather than with their supports J_λ . The measures μ_λ are said to *move holomorphically* over M if there exists a compactly supported probability measure \mathcal{M} on the set

$$\mathcal{J} := \left\{ \gamma : M \rightarrow \mathbb{P}^k / \gamma \text{ is holomorphic and } \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M \right\}$$

such that

$$\mu_\lambda = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma) \text{ for every } \lambda \in M$$

(see Definition 2.1). This notion is related to Dinh's theory of woven currents and somehow means that the measures μ_λ are holomorphically glued together. We call measures as \mathcal{M} *structural webs* of $(\mu_\lambda)_{\lambda \in M}$. They are actually obtained as limits of discrete measures (for instance on graphs of n -periodic repelling cycles) by mean of a compactness statement which may be considered as a measurable version of the λ -lemma (see Theorem 2.3).

Like in dimension one, our proofs crucially rely on the links between bifurcations and instability in the critical dynamics. However, these interactions cannot be detected by a simple application of Picard-Montel's theorem. We will actually read them on a formula, established by Bassanelli and the first author [BB1] (see also Pham's formula in Theorem 2.11) which involves the sum $L(\lambda)$ of Lyapunov exponents of μ_λ and the current of integration \mathbf{C}_f on the critical set C_f taking into account the multiplicities of f :

$$dd_\lambda^c L = \pi_{M*} \left((dd_{\lambda,z}^c g(\lambda, z) + \omega_{FS})^k \wedge \mathbf{C}_f \right).$$

Let us underline that the function L is always *p.s.h* on M .

We will now describe more precisely our approach and state our main results.

Definition 1.1 *The repelling J -cycles are the repelling cycles which belong to the Julia set. We say that these cycles move holomorphically over M if, for every period n , there exists a finite subset $\{\rho_{n,j}, 1 \leq j \leq N_n\}$ of \mathcal{J} such that $\{\rho_{n,j}(\lambda), 1 \leq j \leq N_n\}$ is precisely the set of the n periodic J -cycles of f_λ for each $\lambda \in M$.*

It is known that the holomorphic motion of all the J -repelling cycles over M implies the pluriharmonicity of the function L on M , see [BB1, Theorem 2.2] or [BDM, Theorem 1.5]. We give in Proposition 3.1 a stronger statement. Namely, we show that $dd_\lambda^c L$ is vanishing if the measures μ_λ move holomorphically and admit a structural web which is a limit of discrete measures supported on graphs avoiding the critical set of f . This is done in subsection 3.1. Using the formula for $dd_\lambda^c L$ we characterize its support by a critical growth condition. This leads to the following theorem, where $|\cdot|_U$ denote the mass of currents in $U \times \mathbb{P}^k$.

Theorem 1.2 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. The following properties occur:*

- (1) *if the repelling J -cycles move holomorphically over M then the function L is pluriharmonic on M .*
- (2) *A parameter λ_0 belongs to $\text{supp } dd_\lambda^c L$ if and only if $\liminf_n d^{-kn} |(f^n)_* \mathbf{C}_f|_U > 0$, if and only if $\limsup_n d^{-(k-1)n} |(f^n)_* \mathbf{C}_f|_U = +\infty$ for every neighbourhood U of λ_0 .*

To show that the vanishing of $dd_\lambda^c L$ is a sufficient condition for stability, we exploit the interactions with the critical dynamics. This is where the Misiurewicz parameters enter into the picture.

Definition 1.3 *One says that $\lambda_0 \in M$ is a Misiurewicz parameter if there exists a holomorphic map γ from a neighbourhood of λ_0 into \mathbb{P}^k such that:*

- 1) $\gamma(\lambda) \in J_\lambda$ and is a repelling p_0 -periodic point of f_λ for some $p_0 \geq 1$,
- 2) $(\lambda_0, \gamma(\lambda_0)) \in f^{n_0}(C_f)$ for some $n_0 \geq 1$,
- 3) the graph Γ_γ of γ is not contained in $f^{n_0}(C_f)$.

We first prove that the pluriharmonicity of L prevents the apparition of such parameters. To do this, we use again the formula for $dd_\lambda^c L$ and a dynamical rescaling argument. This is done in subsection 3.2. To prove that the absence of Misiurewicz parameters implies the holomorphic motion of the equilibrium measures, we apply our measurable version of the λ -lemma to sequences of discrete measures on pull-backs by f^n of a graph of J -repelling cycles avoiding the post-critical set of f (see Proposition 2.4). The existence of such a graph is involved, we obtain it through the construction of thick hyperbolic sets. This is done in section 4. These results are summarized in the following theorem.

Theorem 1.4 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Then the following assertions are equivalent:*

- (A) *the function L is pluriharmonic on M .*
- (B) *There are no Misiurewicz parameters in M .*
- (C) *The measures μ_λ locally move holomorphically and admit a structural web $\mathcal{M} = \lim_n \mathcal{M}_n$ such that the graph of any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$ avoids the critical set of f .*

So far, we do not know if the holomorphic motion of measures implies that of J -repelling cycles. To answer this question, we investigate how the apparition of Siegel discs may affect the continuity of $\lambda \mapsto J_\lambda$ in the Hausdorff topology. The section 5 is mainly devoted to that study (see in particular Proposition 5.7). We then obtain the following

Theorem 1.5 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms where M is a simply connected open subset of the space $\mathcal{H}_d(\mathbb{P}^k)$ of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Then the following assertions are equivalent :*

- (A) *the J -repelling cycles move holomorphically over M .*
- (B) *The function L is pluriharmonic on M .*
- (C) *The measures μ_λ move holomorphically over M .*

Let us mention that if M is a simply connected complex manifold, the above result is true for all holomorphic families $f : M \times \mathbb{P}^2 \rightarrow M \times \mathbb{P}^2$ and, more generally, for any family $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ whose repelling J -cycles are neither persistently resonant nor persistently undiagonalizable (see Proposition 6.3).

In view of these results, we define the *bifurcation locus* of a holomorphic family of endomorphisms of \mathbb{P}^k as being the support of the closed positive current $dd_\lambda^c L$. Moreover, we say that $dd_\lambda^c L$ is the *bifurcation current* of the family and that a family is *stable* if its bifurcation locus is empty. This is coherent with the classical one-dimensional definition, as it was first proved by DeMarco [dM].

It follows from Theorem 1.5 that the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^k)$ coincides with the closure of the set of endomorphisms which admit J -repelling cycles which bifurcate either by giving Siegel periodic cycles or repelling cycles outside the Julia set (see Theorem 6.7).

Let us observe that in his work on the persistence of homoclinic tangencies, Buzzard [Bu] found open subsets of the space of degree d endomorphisms of \mathbb{P}^2 (for d large enough) in which the maps having infinitely many sinks are dense. This lead us to believe that the bifurcation locus may have a non-empty interior when $k \geq 2$. In this direction we show that the set of parameters λ for which \mathbb{P}^k coincides with the closure of the post-critical set of f_λ is dense in any open subset of the bifurcation locus (see Theorem 6.11).

Let us finally mention that in any stable family, all elements are Lattès maps as soon as one element is a Lattès map (see Theorem 6.9). This follows from the characterization of such maps by their Lyapunov exponents obtained by Loeb and the authors. It is worth to emphasize that for rational maps of \mathbb{P}^1 , the classical proof of this fact uses the purely one-dimensional tool of orbifolds.

2 Stability and positive currents

2.1 Holomorphic motion of equilibrium measures

In dimension one, the Julia sets of stable holomorphic families of rational maps move holomorphically. The existence of these motions follows from the classical λ -lemma which is not any longer available when $k \geq 2$. Our aim here is to introduce a notion of *holomorphic motion of equilibrium measures* which, for holomorphic families of endomorphisms of \mathbb{P}^k , is devoted to replace that of holomorphic motion of Julia sets.

Our approach is closely related to the basics of the theory of equilibrium currents developed by Dinh-Sibony and Pham (see [DS2, section 2.5] and [Ph]), the theory of horizontal currents of Dinh-Sibony [DS2] and of woven currents of Dinh [Di2]. Some properties of such currents are discussed in the next subsection.

Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. We recall that M is a connected complex manifold of dimension m and that $f(\lambda, z) = (\lambda, f_\lambda(z))$. Let μ_λ denote the equilibrium measure of f_λ and let J_λ denote the support of μ_λ , this is the Julia set of f_λ . We want here to define a notion of holomorphic motion for the family $(\mu_\lambda)_{\lambda \in M}$. To this purpose we consider the set

$$\mathcal{J} := \left\{ \gamma \in \mathcal{O}(M, \mathbb{P}^k) \text{ such that } \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M \right\}.$$

In general this set can be empty, but it can also be large like for instance when $J_\lambda = \mathbb{P}^k$ for every $\lambda \in M$. In our study, \mathcal{J} will always be non empty. Note that \mathcal{J} endowed with the topology of local uniform convergence is a metric space.

For any probability measure \mathcal{M} on $\mathcal{O}(M, \mathbb{P}^k)$ we define

$$W_{\mathcal{M}} := \int [\Gamma_\gamma] d\mathcal{M}(\gamma).$$

This is a current on $M \times \mathbb{P}^k$ of bidimension (m, m) , it is a woven current following Dinh's terminology [Di2]. For every λ in M , we also define

$$\mathcal{M}_\lambda := \int \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma),$$

this is a probability measure on \mathbb{P}^k which is actually equal to $p_{\lambda*}\mathcal{M}$ where the mapping $p_\lambda : \mathcal{O}(M, \mathbb{P}^k) \rightarrow \mathbb{P}^k$ is given by $p_\lambda(\gamma) := \gamma(\lambda)$.

Let $\mathcal{F} : \mathcal{O}(M, \mathbb{P}^k) \rightarrow \mathcal{O}(M, \mathbb{P}^k)$ be defined by $(\mathcal{F} \cdot \gamma)(\lambda) := f_\lambda(\gamma(\lambda))$. We will be interested in situations where \mathcal{M} is an \mathcal{F} -invariant and compactly supported probability measure on \mathcal{J} . Note that in that case we have a dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$.

Definition 2.1 *We say that the equilibrium measures μ_λ move holomorphically over M (or more briefly $(\mu_\lambda)_{\lambda \in M}$ moves holomorphically) if there exists a compactly supported \mathcal{F} -invariant probability measure \mathcal{M} on \mathcal{J} such that $\mathcal{M}_\lambda = \mu_\lambda$ for every $\lambda \in M$. Such a measure \mathcal{M} is called a structural web for $(\mu_\lambda)_{\lambda \in M}$.*

When the equilibrium measures move holomorphically we have the following properties. We thank R. Dujardin for pointing us this fact.

Lemma 2.2 *Let M be a connected complex manifold and $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k whose equilibrium measures $(\mu_\lambda)_{\lambda \in M}$ move holomorphically. Let \mathcal{M} be a structural web for $(\mu_\lambda)_{\lambda \in M}$. Then :*

- 1) *The sequence $(f_\lambda^p(\gamma(\lambda)))_{p \geq 1}$ is normal for every $\gamma \in \text{supp } \mathcal{M}$.*
- 2) *There exists $\gamma \in \text{supp } \mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$ for every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$.*
- 3) *For every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ such that z_0 is n -periodic and repelling for f_{λ_0} , there exists $\gamma \in \text{supp } \mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$ and $\gamma(\lambda)$ is n -periodic for f_λ for every $\lambda \in M$.*

PROOF: (1) This follows from $f_\lambda^p(\gamma(\lambda)) = (\mathcal{F}^p \cdot \gamma)(\lambda)$ and the fact that \mathcal{M} is compactly supported and \mathcal{F} -invariant.

(2) As $z_0 \in J_{\lambda_0}$ and $J_{\lambda_0} = \text{supp } \mu_{\lambda_0} = \text{supp } \mathcal{M}_{\lambda_0}$, there exist $(\gamma_l)_l \subset \text{supp } \mathcal{M}$ such that $\gamma_l(\lambda_0) \rightarrow z_0$. Then, since \mathcal{M} is compactly supported, we can take for γ any limit of $(\gamma_l)_l$.

(3) By the implicit function theorem, there exists a neighbourhood V_{λ_0} of λ_0 and a holomorphic map $w : V_{\lambda_0} \rightarrow \mathbb{P}^k$ such that $w(\lambda_0) = z_0$ and $w(\lambda)$ is n -periodic for f_λ . We will show that w coincides on V_{λ_0} with the map γ given by the previous item, the conclusion then follows by analytic continuation. Our argument is local, so we can choose a chart and work on \mathbb{C}^k . Since z_0 is repelling, we can shrink V_{λ_0} and find $A > 1$, $r > 0$ such that

$$\|w(\lambda) - f_\lambda^n(z)\| = \|f_\lambda^n(w(\lambda)) - f_\lambda^n(z)\| \geq A\|w(\lambda) - z\| \quad (1)$$

when $\lambda \in V_{\lambda_0}$ and $\|w(\lambda) - z\| < r$. On the other hand the first item ensures that $(f_\lambda^{pn}(\gamma(\lambda)))_p$ is a normal family, hence we can shrink again V_{λ_0} so that $\|w(\lambda) - f_\lambda^{pn}(\gamma(\lambda))\| < r$ for every $p \geq 1$ and $\lambda \in V_{\lambda_0}$. Combining this with (1) we obtain $r > \|w(\lambda) - f_\lambda^{pn}(\gamma(\lambda))\| \geq A^p \|w(\lambda) - \gamma(\lambda)\|$ for every $p \geq 1$ and $\lambda \in V_{\lambda_0}$. This implies $w(\lambda) = \gamma(\lambda)$ on V_{λ_0} since $A > 1$. \square

Structural webs will be obtained as limits of discrete measures on $\mathcal{O}(M, \mathbb{P}^k)$. Our basic result in this direction is the following.

Theorem 2.3 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let μ_λ be the equilibrium measure of f_λ . Assume that there exists a sequence of probability measures $(\mathcal{M}_n)_{n \geq 1}$ on $\mathcal{O}(M, \mathbb{P}^k)$ such that*

- 1) $\lim_n (\mathcal{M}_n)_\lambda = \mu_\lambda$ for every $\lambda \in M$.
- 2) $\mathcal{F}_\star \mathcal{M}_{n+1} = \mathcal{M}_n$ or $\mathcal{F}_\star \mathcal{M}_n = \mathcal{M}_n$ for every $n \geq 1$.
- 3) *There exists a compact subset \mathcal{K} of $\mathcal{O}(M, \mathbb{P}^k)$ such that $\text{supp } \mathcal{M}_n \subset \mathcal{K}$ for every $n \geq 1$.*

Then $(\mu_\lambda)_{\lambda \in M}$ move holomorphically and any limit of $(\frac{1}{n} \sum_{l=1}^n \mathcal{M}_l)_n$ is a structural web.

Since it is based on properties of equilibrium currents, the proof will be given in the next subsection. To end this subsection, we explain how Theorem 2.3 is concretely used to produce structural webs. The proof relies on the equidistribution of preimages of points, see the articles [FS1, BD2, DS1] and on the equidistribution of repelling cycles, see [BD1].

Proposition 2.4 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of degree d endomorphisms of \mathbb{P}^k .*

- 1) *Assume that M is simply connected and that there exists $\gamma \in \mathcal{O}(M, \mathbb{P}^k)$ such that the graph Γ_γ does not intersect the post-critical set of f . Then the equilibrium measures $(\mu_\lambda)_{\lambda \in M}$ move holomorphically and a structural web is given by any limit of $\left(\frac{1}{n} \sum_{i=1}^n \frac{1}{d^{ki}} \sum_{\mathcal{F}^i \cdot \sigma = \gamma} \delta_\sigma\right)_n$.*
- 2) *Assume that the repelling J -cycles of f move holomorphically over M and let $(\rho_{n,j})_{1 \leq j \leq N_n}$ be the elements of \mathcal{J} given by the motions of these n -periodic cycles. Then the equilibrium measures $(\mu_\lambda)_{\lambda \in M}$ move holomorphically and a structural web is given by any limit of $\left(\frac{1}{d^{kn}} \sum_{j=1}^{N_n} \delta_{\rho_{n,j}}\right)_n$.*

PROOF: 1) The map $f^n : (M \times \mathbb{P}^k) \setminus f^{-n}(\cup_{1 \leq p \leq n} f^p(C_f)) \rightarrow (M \times \mathbb{P}^k) \setminus (\cup_{1 \leq p \leq n} f^p(C_f))$ is a covering of degree d^{kn} . Hence, there exist d^{kn} holomorphic graphs $\Gamma_{\sigma_{j,n}}$ such that $f^n(\Gamma_{\sigma_{j,n}}) = \Gamma_\gamma$ i.e. $\mathcal{F}^n \cdot \sigma_{j,n} = \gamma$. Let us set $\mathcal{M}_n := \frac{1}{d^{kn}} \sum_{j=1}^{d^{kn}} \delta_{\sigma_{j,n}}$. By construction $\mathcal{F}_* \mathcal{M}_{n+1} = \mathcal{M}_n$ and, for every $\lambda \in M$, one has $(\mathcal{M}_n)_\lambda = \frac{1}{d^{kn}} \sum_{j=1}^{d^{kn}} \delta_{\sigma_{j,n}(\lambda)} = \sum_{f_\lambda^n(x)=\gamma(\lambda)} \delta_x \rightarrow \mu_\lambda$, where the limit comes from the fact that $\gamma(\lambda) \notin \cup_{p \geq 1} f_\lambda^p(C_{f_\lambda})$. The family $(\sigma_{j,n})_{j,n}$ is normal, by a theorem of Ueda [U, Theorem 2.1], and therefore the supports of \mathcal{M}_n are contained in a fixed compact subset of $\mathcal{O}(M, \mathbb{P}^k)$. The conclusion immediately follows from Theorem 2.3.

2) Let us set $\mathcal{M}_n := \frac{1}{d^{kn}} \sum_{j=1}^{N_n} \delta_{\rho_{j,n}}$. The convergence of $(\mathcal{M}_n)_\lambda$ towards μ_λ follows from the equidistribution of repelling periodic points with respect to the equilibrium measure, see [BD2] (note that the repelling cycles produced there are J -cycles). The normality of the family $(\rho_{j,n})_{j,n}$ can be seen by lifting these curves to curves of periodic points of a lift F of f . Again, one concludes by using Theorem 2.3. \square

2.2 Horizontal and equilibrium currents

We gather here the properties of equilibrium currents which will allow us to construct holomorphic motions of equilibrium measures. To start with, we recall some basic facts about horizontal currents.

Definition 2.5 *Let M be a complex connected manifold. A current \mathcal{R} on $M \times \mathbb{C}^{k+1}$ is horizontal if $\text{supp } \mathcal{R} \subset M \times K$ for some compact subset $K \subset \mathbb{C}^{k+1}$.*

Let us assume that \mathcal{R} is a closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$ where m is the complex dimension of M . Then the slices $\langle \mathcal{R}, \pi_M, \lambda \rangle$ exist

for Lebesgue-almost every $\lambda \in M$ and are positive measures on $M \times \mathbb{C}^{k+1}$ supported on $\{\lambda\} \times \mathbb{C}^{k+1}$. The following *basic slicing formula* holds for every continuous test function ψ on $M \times \mathbb{C}^{k+1}$ and every continuous (m, m) -test form ω on M :

$$\int_M \langle \mathcal{R}, \pi_M, \lambda \rangle \psi \omega(\lambda) = \langle \mathcal{R} \wedge \pi_M^*(\omega), \psi \rangle. \quad (2)$$

Dinh and Sibony have shown that the slices of such currents do actually exist for *every* $\lambda \in M$, their fundamental result is as follows, see [DS1, theorem 2.1].

Theorem 2.6 (Dinh-Sibony) *Let M be a m -dimensional complex connected manifold and \mathcal{R} be a closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$. Then the following properties occur.*

1. *The slice $\langle \mathcal{R}, \pi_M, \lambda \rangle$ exists for every $\lambda \in M$ and its mass does not depend on $\lambda \in M$.*
2. *The function $\lambda \mapsto \int_{\mathbb{C}^{k+1}} \psi(\lambda, z) \langle \mathcal{R}, \pi_M, \lambda \rangle$ is psh or $\equiv -\infty$ on M for any psh function ψ defined on a neighborhood of $\text{supp } \mathcal{R}$.*

The following corollary is inspired by [Ph, Proposition 2.1], one can find a more general version in [DS4, remark 2.2.6]. We shall need it to prove Theorem 2.3, namely to produce holomorphic motions of equilibrium measures from motions of approximating discrete measures.

Corollary 2.7 *Let M be as above and $(\mathcal{R}_n)_n$ be a sequence of closed, positive, horizontal current of bidimension (m, m) on $M \times \mathbb{C}^{k+1}$. Assume that $\lim_n \mathcal{R}_n = \mathcal{R}$ and that $\text{supp } \mathcal{R}_n \subset M \times K$ for some compact subset K of \mathbb{C}^{k+1} . Then, after taking a subsequence, we have $\lim_n \langle \mathcal{R}_n, \pi_M, \lambda \rangle = \langle \mathcal{R}, \pi_M, \lambda \rangle$ for almost every $\lambda \in M$.*

PROOF: Let ψ be a test function on \mathbb{C}^{k+1} which will be considered as a function on $M \times \mathbb{C}^{k+1}$. For every $\lambda \in M$, we set

$$u_{\psi, n}(\lambda) := \langle \mathcal{R}_n, \pi_M, \lambda \rangle \psi, \quad u_\psi(\lambda) := \langle \mathcal{R}, \pi_M, \lambda \rangle \psi.$$

According to Theorem 2.6, the slice masses $c_n := |\langle \mathcal{R}_n, \pi_M, \lambda \rangle|$ and $c := |\langle \mathcal{R}, \pi_M, \lambda \rangle|$ do not depend on $\lambda \in M$. Given a (m, m) -test form ω on M , the basic slicing formula (2) gives

$$\lim_n \int_M u_{\psi, n}(\lambda) \omega(\lambda) = \lim_n \langle \mathcal{R}_n \wedge \pi_M^*(\omega), \psi \rangle = \langle \mathcal{R} \wedge \pi_M^*(\omega), \psi \rangle = \int_M u_\psi(\lambda) \omega(\lambda). \quad (3)$$

Applying (3) with $\psi \equiv 1$ on K , we get $c = \lim_n c_n$ and thus $C := \sup_n c_n$ is finite. Taking $C < +\infty$ into account, one sees by using Slutsky's lemma and a diagonal argument that it suffices to prove that

$$u_{\psi, n} \rightarrow u_\psi \text{ in } L^1_{loc}(M) \text{ for every test function } \psi.$$

Let us first prove it when ψ is a smooth *psh* function. By Theorem 2.6, $(u_{\psi,n})_n$ is a sequence of *psh* functions, which is locally uniformly bounded on M since $|u_{\psi,n}| \leq c_n \sup_K |\psi| \leq C \sup_K |\psi|$. As such sequences are relatively compact for the L^1_{loc} topology, it suffices to show that u_ψ is the unique cluster point of $(u_{\psi,n})_n$. Assume that $u_{\psi,n_k} \rightarrow v$ in $L^1_{loc}(M)$. According to (3), we have $\int_M u_\psi \omega = \int_M v \omega$ for every (m,m) -test form ω on M . Hence v and u_ψ coincide in $L^1_{loc}(M)$, as desired. This remains true when ψ is any smooth test function on \mathbb{C}^{k+1} , as one sees by writing it as a difference of two smooth *psh* functions: $\psi = (\psi + A\|z\|^2) - A\|z\|^2$ with A large enough. \square

Let us now define equilibrium currents for holomorphic families of d -homogeneous non-degenerate maps. Such currents always exist, they have been introduced by Pham [Ph] in the more general context of polynomial like mappings (see also the lecture notes by Dinh and Sibony [DS2, section 2.5]).

Definition 2.8 *Let $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$ be a holomorphic family of d -homogeneous non-degenerate maps where M is some m -dimensional complex connected manifold. Let \mathcal{E} be a closed, positive, horizontal current of bidimension (m,m) on $M \times \mathbb{C}^{k+1}$. We say that \mathcal{E} is an equilibrium current for F if the slice $\langle \mathcal{E}, \pi_M, \lambda \rangle$ is equal to the equilibrium measure of F_λ for every $\lambda \in M$.*

One may dynamically produce equilibrium currents. For instance, Pham proved that the sequence of smooth forms $(\frac{1}{d^{(k+1)n}} F^{n*} (\pi_{\mathbb{C}^{k+1}}^* \theta))_n$ converges to such a current for any smooth probability measure θ on \mathbb{C}^{k+1} .

It is also possible to define equilibrium currents for families of endomorphisms of \mathbb{P}^k by mean of Green functions. Let us briefly recall their construction. Consider a holomorphic family $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ which admits a lift $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$. The sequence

$$G_n(\lambda, \tilde{z}) := \frac{1}{d^n} \log \|F_\lambda^n(\tilde{z})\|$$

converges locally uniformly on $M \times \mathbb{C}^{k+1} \setminus \{0\}$ to a function G which we call the *Green function of F* . The function G is *psh* and Hölder continuous, see [BB1, section 1.2]. Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ be the canonical projection, ω_{FS} be the Fubini-Study form on \mathbb{P}^k . The functions G_n induce functions $g_n : M \times \mathbb{P}^k \rightarrow \mathbb{R}$ by setting $g_n(\lambda, z) := G_n(\lambda, \tilde{z}) - \log \|\tilde{z}\|$, for every \tilde{z} satisfying $\pi(\tilde{z}) = z$. We have the

$$\frac{1}{d} f^* (dd_{\lambda,z}^c g_n + \omega_{FS}) = dd_{\lambda,z}^c g_{n+1} + \omega_{FS}.$$

We define similarly $g(\lambda, z) := \lim_n g_n(\lambda, z)$, which is equal to $G(\lambda, \tilde{z}) - \log \|\tilde{z}\|$, and set

$$\mathcal{E}_{Green} := (dd_{\lambda,z}^c g + \omega_{FS})^k.$$

This is a current of bidimension (m,m) and, since slicing commutes with the operators d , d^c and with restriction, the measure $\langle \mathcal{E}_{Green}, \pi_M, \lambda \rangle$ is equal to the equilibrium measure of f_λ for every $\lambda \in M$. The current \mathcal{E}_{Green} will play an important role in our study (see Proposition 3.3), we call it the *Green equilibrium current of f* .

We end this subsection by proving Theorem 2.3. The following lemma allows to exploit the properties of horizontal currents in $M \times \mathbb{C}^{k+1}$.

Lemma 2.9 *Let B be a ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let \mathcal{K} be a compact subset of $\mathcal{O}(B, \mathbb{P}^k)$. Then, after shrinking B , one may associate to any probability measure \mathcal{N} supported on \mathcal{K} a positive, horizontal (m, m) -bidimensional current $\widetilde{W}_{\mathcal{N}}$ on $B \times \mathbb{C}^{k+1}$ such that $\pi_* \langle \widetilde{W}_{\mathcal{N}}, \pi_B, \lambda \rangle = \mathcal{N}_\lambda$ for every $\lambda \in B$. Moreover, $\widetilde{W}_{\mathcal{N}}$ depends continuously on \mathcal{N} .*

PROOF: Let $(\sigma_i)_{1 \leq i \leq N}$ be holomorphic sections of π whose domains of definition Ω_i cover \mathbb{P}^k . Since \mathcal{K} is a normal family, we may shrink B so that for each $\gamma \in \mathcal{K}$ there exists at least one $1 \leq i \leq N$ such that $\Gamma_\gamma \subset B \times \Omega_i$. This allows to define a map

$$\begin{aligned} \sigma : \mathcal{K} &\rightarrow \mathcal{O}(B, \mathbb{C}^{k+1}) \\ \gamma &\mapsto \sigma(\gamma) := \sigma_l \circ \gamma \end{aligned}$$

where $l := \min\{1 \leq i \leq N \text{ such that } \Gamma_\gamma \subset B \times \Omega_i\}$. Now, for any probability measure \mathcal{N} supported on \mathcal{K} we set

$$\widetilde{W}_{\mathcal{N}} := \int_{\mathcal{J}} [\Gamma_{\sigma(\gamma)}] d\mathcal{N}(\gamma).$$

Then $\pi_* \langle \widetilde{W}_{\mathcal{N}}, \pi_B, \lambda \rangle = \mathcal{N}_\lambda$ for every $\lambda \in B$ by construction. \square

PROOF OF THEOREM 2.3: Let us set $\mathcal{V}_n := \frac{1}{n} \sum_{l=1}^n \mathcal{M}_l$ where \mathcal{M}_n satisfies the assumptions of Theorem 2.3. By the second and the third of these assumptions, every limit \mathcal{V} of $(\mathcal{V}_n)_n$ is clearly \mathcal{F} -invariant and supported in \mathcal{K} . By the first one, $(\mathcal{V}_n)_\lambda \rightarrow \mu_\lambda$ for every $\lambda \in M$. Let us show that $\mathcal{V}_\lambda = \mu_\lambda$ for every $\lambda \in M$. We may assume that $\mathcal{V}_n \rightarrow \mathcal{V}$.

The problem being local, we may replace M by any ball B in \mathbb{C}^m . Note that for any test function ϕ on \mathbb{P}^k , the functions $\lambda \mapsto \langle \mathcal{V}_\lambda, \phi \rangle$ and $\lambda \mapsto \langle \mu_\lambda, \phi \rangle$ are both continuous. This follows easily from the facts that \mathcal{V} is supported on \mathcal{K} which is an equicontinuous family and that $\mu_\lambda = \pi_* \left(dd_{\lambda, \tilde{z}}^c G_F(\lambda, \tilde{z}) \right)^{k+1}$ where G_F is the Green function of F which is continuous on $B \times \mathbb{C}^{k+1}$. Thus it suffices to show that $\mathcal{V}_\lambda = \mu_\lambda$ for almost every $\lambda \in B$.

We now use Lemma 2.9 and associate horizontal currents $\widetilde{W}_{\mathcal{V}_n}$ and $\widetilde{W}_{\mathcal{V}}$ to \mathcal{V}_n and \mathcal{V} . As $\widetilde{W}_{\mathcal{V}_n}$ converges towards $\widetilde{W}_{\mathcal{V}}$ as currents, it follows from Corollary 2.7 that $(\mathcal{V}_n)_\lambda \rightarrow \mathcal{V}_\lambda$ for almost every $\lambda \in B$, which implies that $\mathcal{V}_\lambda = \mu_\lambda$ for almost every $\lambda \in B$.

It remains to check that \mathcal{V} is supported on \mathcal{J} . We have to show that if $\gamma_0 \in \mathcal{O}(M, \mathbb{P}^k)$ and $\gamma_0(\lambda_0) \notin J_{\lambda_0}$ for some parameter λ_0 then γ_0 does not belong to the support of \mathcal{V} . Let U_0 be a neighbourhood of $\gamma(\lambda_0)$ such that $\mu_{\lambda_0}(U_0) = 0$ and let V_0 be a neighbourhood of γ_0 in $\mathcal{O}(M, \mathbb{P}^k)$ such that $\{\gamma(\lambda_0) / \gamma \in V_0\} \subset U_0$. Then $0 = \mu_{\lambda_0}(U_0) \geq \mathcal{V}(V_0)$ since $\mu_{\lambda_0} = \mathcal{V}_{\lambda_0}$. \square

2.3 Some fundamental formulas

In [dM], DeMarco proved a formula relating the Lyapunov exponent $L(f)$ of a rational map f with a specific critical data. For a polynomial P of degree d , her formula boils

down to the famous Przytycki's formula, see [Pr]:

$$L(P) = \sum_{c \in C_P} G_P(c) + \log d.$$

Here G_P stands for the dynamical Green function of $P : G_P := \lim_n d^{-n} \log^+ |P^n(z)|$. Bassanelli-Berteloot have obtained a similar formula for the sum of Lyapunov exponents of holomorphic endomorphisms of \mathbb{P}^k , see [BB1, Theorem 4.1].

For holomorphic families $(f_\lambda)_\lambda$, it turns out that the identities between currents obtained by applying dd_λ^c to such formulas enlight bifurcation phenomena (see the survey [Be] or the lecture notes [DS3]). In dimension one, the current $dd_\lambda^c L$ detects the existence of unpersistent neutral cycles while $dd_\lambda^c \sum_{c(\lambda) \in C_{P_\lambda}} G_{P_\lambda}(c(\lambda))$ detects the activity of critical points. In particular, as shown by DeMarco, the support of $dd_\lambda^c L$ coincides with the bifurcation locus. For holomorphic families of endomorphisms of \mathbb{P}^k the identity provided by Theorem 2.10 here below plays a crucial role in our proof (see section 3.2).

Let us set $D := (k+1)(d-1)$. The line bundle $\mathcal{O}_{\mathbb{P}^k}(D)$ over \mathbb{P}^k is seen as the quotient of $(\mathbb{C}^{k+1} \setminus \{0\}) \times \mathbb{C}$ by the relation $(\tilde{z}, x) \equiv (u\tilde{z}, u^D x)$ for every $u \in \mathbb{C}^*$ and its elements are denoted by $[\tilde{z}, x]$. We may endow $\mathcal{O}_{\mathbb{P}^k}(D)$ with the canonical metric

$$\|[\tilde{z}, x]\|_0 := e^{-D \log \|\tilde{z}\|} |x|$$

or, for any $\lambda \in M$, with the metric

$$\|[\tilde{z}, x]\|_\lambda := e^{-DG(\lambda, \tilde{z})} |x|.$$

Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k which admits a lifted family $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$. Let us set $J_F(\lambda, \tilde{z}) := \det d_{\tilde{z}} F_\lambda$. Then we obtain a family of holomorphic sections of $\mathcal{O}_{\mathbb{P}^k}(D)$ by setting for every $\tilde{z} \in \mathbb{C}^{k+1} \setminus \{0\}$:

$$J_F^s(\lambda, \pi(\tilde{z})) = [\tilde{z}, J_F(\lambda, \tilde{z})].$$

The current $\mathbf{C}_f := dd_{\lambda, z}^c \log \|J_F^s(\lambda, z)\|_0$ is the current of integration on C_f taking account the topological multiplicities of f , its bidimension is equal to (κ, κ) where $\kappa := k + m - 1$. The Green equilibrium current $\mathcal{E}_{Green} = (dd_{\lambda, z}^c g + \omega_{FS})^k$ was defined in subsection 2.2.

Theorem 2.10 (Bassanelli-Berteloot) *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let $L(\lambda)$ be the sum of the Lyapunov exponents of μ_λ . Then*

$$dd_\lambda^c L = \pi_{M^*}(\mathcal{E}_{Green} \wedge \mathbf{C}_f).$$

We end this subsection by explaining how Pham [Ph] obtained a more general formula by a rather direct and simple argument. His result holds for any equilibrium current of any family of polynomial-like maps, we state it in the special case of non-degenerate homogeneous maps for sake of simplicity. Let us recall that for such a family F , the function $\log |J_F(\lambda, \tilde{z})|$ is *psh* on $M \times \mathbb{C}^{k+1}$. Moreover, the sum of the Lyapunov exponents of F_λ with respect to its equilibrium measure ν_λ is given by $\int_{\mathbb{C}^{k+1}} \log |J_F(\lambda, \tilde{z})| d\nu_\lambda(\tilde{z})$ and is equal to $L(\lambda) + \log d$ where $L(\lambda)$ is the sum of the Lyapunov exponents of f_λ with respect to μ_λ .

Theorem 2.11 (Pham) *Let $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$ be a holomorphic family of non-degenerate d -homogeneous maps and let \mathcal{E} be an equilibrium current for F . Then:*

1. *The current $\log |J_F| \cdot \mathcal{E}$ has locally finite mass.*
2. $dd_\lambda^c L = \pi_{M\star} \left(\mathcal{E} \wedge dd_{\lambda, \tilde{z}}^c \log |J_F| \right)$.

We shall actually need the following formula for $dd_\lambda^c L$ when $(\mu_\lambda)_\lambda$ move holomorphically. The proof follows Pham's arguments.

Proposition 2.12 *Let B be an open ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that $(\mu_\lambda)_\lambda$ move holomorphically and let \mathcal{M} be a structural web. Then*

$$dd_\lambda^c L = \pi_{B\star} \left(\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, z}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \right)$$

where $\widetilde{W}_\mathcal{M}$ is the (m, m) -bidimensional current on $M \times \mathbb{C}^{k+1}$ associated to \mathcal{M} by Lemma 2.9.

PROOF: We first check that for every $\lambda \in B$ we have

$$\int_{\mathbb{C}^{k+1}} \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \langle \widetilde{W}_\mathcal{M}, \pi_B, \lambda \rangle = L(\lambda) + \log d. \quad (4)$$

Indeed, since $\pi_\star \langle \widetilde{W}_\mathcal{M}, \pi_B, \lambda \rangle = \mu_\lambda$, we get

$$\int_{\mathbb{C}^{k+1}} \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \langle \widetilde{W}_\mathcal{M}, \pi_B, \lambda \rangle = \int_{\mathbb{P}^k} \log \|J_F^s(\lambda, z)\|_\lambda \mu_\lambda.$$

On the other hand, since G_λ identically vanishes on the support of the equilibrium measure ν_λ of F_λ and since $\pi_\star \nu_\lambda = \mu_\lambda$, we have

$$\begin{aligned} \int_{\mathbb{P}^k} \log \|J_F^s(\lambda, z)\|_\lambda \mu_\lambda &= \int_{\mathbb{C}^{k+1}} \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \nu_\lambda \\ &= \int_{\mathbb{C}^{k+1}} \log |J_F(\lambda, \tilde{z})| \nu_\lambda = L(\lambda) + \log d, \end{aligned}$$

and the identity (4) follows.

Pham proved that $u \cdot \mathcal{R}$ has locally finite mass for every psh function u and every horizontal current \mathcal{R} as soon as $\int_{\mathbb{C}^{k+1}} u \langle \mathcal{R}, \pi_B, \lambda \rangle \neq -\infty$ for some $\lambda \in M$, see [Ph, theorem A.2]. It thus follows from (4) that the current $\log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \cdot \widetilde{W}_\mathcal{M}$ is well defined and that its $dd_{\lambda, \tilde{z}}^c$ is equal to $\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda$.

The remainder is a simple computation which relies on integration by parts (to make it rigorous one should approximate $\log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda$ by smooth functions). Let φ be a $(m-1, m-1)$ test form on B . Then

$$\begin{aligned} \langle \pi_{B\star} \left(\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \right), \varphi \rangle &= \langle \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \cdot \widetilde{W}_\mathcal{M}, dd_{\lambda, \tilde{z}}^c (\pi_B^\star \varphi) \rangle \\ &= \langle \widetilde{W}_\mathcal{M} \wedge \pi_B^\star (dd_\lambda^c \varphi), \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda \rangle. \end{aligned}$$

By the basic slicing formula (2) and the identity (4), this is equal to

$$\int_B \left(\langle \widetilde{W}_{\mathcal{M}}, \pi_B, \lambda \rangle \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_{\lambda} \right) dd_{\lambda}^c \varphi = \int_B L dd_{\lambda}^c \varphi = \langle dd_{\lambda}^c L, \varphi \rangle.$$

This completes the proof of the Proposition 2.12. \square

3 The main properties of the current $dd_{\lambda}^c L$

3.1 Repelling cycles do not move holomorphically on $\text{supp } dd_{\lambda}^c L$

Our aim is to establish the first assertion of Theorem 1.2, precisely that $dd_{\lambda}^c L = 0$ on M if the repelling J -cycles move holomorphically. We actually prove here a quite more general result.

Proposition 3.1 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of degree $d \geq 2$ endomorphisms of \mathbb{P}^k whose equilibrium measures μ_{λ} move holomorphically over M . Assume that a structural web \mathcal{M} of $(\mu_{\lambda})_{\lambda \in M}$ is given by $\mathcal{M} = \lim_n \mathcal{M}_n$ where $\Gamma_{\gamma} \cap C_f = \emptyset$ for any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. Then $dd_{\lambda}^c L = 0$ on M .*

Note that, according to the second assertion of Proposition 2.4, the assumptions of Proposition 3.1 are satisfied when the repelling J -cycles move holomorphically. The proof needs the following technical lemma, it will be also useful to show Proposition 6.10.

Lemma 3.2 *Let B be an open ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let Z be a codimension 1 analytic subset of $B \times \mathbb{P}^k$ which does not contain any fiber $\{\lambda\} \times \mathbb{P}^k$. Assume that μ_{λ} move holomorphically and that there exists a structural web satisfying $\mathcal{M} = \lim_n \mathcal{M}_n$, where $\Gamma_{\gamma} \cap Z = \emptyset$ for $\gamma \in \cup_n \text{supp } \mathcal{M}_n$ and $n \geq n_0$. Let B' be a relatively compact ball in B .*

Then, after shrinking B , there exist $A > 0$ and $0 < a < 1$ such that

$$\mathcal{M}\{\gamma \in \mathcal{J} / \Gamma_{\gamma|_{B'}} \cap Z_{\epsilon} \neq \emptyset\} \leq A\epsilon^a$$

for every sufficiently small $\epsilon > 0$, where Z_{ϵ} is the ϵ -neighbourhood of Z .

PROOF: We can assume that both B and B' are centered at some λ_0 . After maybe shrinking B we may find a finite collection $(\Omega_i, h_i)_{1 \leq i \leq N}$ where the Ω_i are open and cover $B \times \mathbb{P}^k$, the functions h_i are holomorphic and bounded by 1 on Ω_i and $Z \cap \Omega_i = \{h_i = 0\}$ for any $1 \leq i \leq N$. If ϵ is small enough, we may also assume that $Z_{\epsilon} \cap \Omega_i \subset \{|h_i| < C_1 \epsilon\}$ and, by Lojasiewicz inequality, that $\{|h_i| < \epsilon\} \subset Z_{C_2 \epsilon^{\tau}}$ for some constants $C_1, C_2, \tau > 0$. Similarly, one has $Z_{\epsilon} \cap (\{\lambda_0\} \times \mathbb{P}^k) \subset (Z \cap (\{\lambda_0\} \times \mathbb{P}^k))_{C_3 \epsilon^{\tau_0}}$ for some constants $C_3, \tau_0 > 0$.

Since \mathcal{M} has compact support in \mathcal{J} , we may shrink B again so that for any $\gamma \in \text{supp } \mathcal{M}$ there exists at least one $1 \leq i \leq N$ such that $\Gamma_{\gamma} \subset \Omega_i$. We shall use the following

Claim: *there exists $0 < \alpha \leq 1$ such that $\sup_{B'} |\phi| \leq |\phi(t_0)|^{\alpha}$ for every $t_0 \in B'$ and for every holomorphic function $\phi : B \rightarrow \mathbb{D}^*$.*

Let $\gamma \in \text{supp } \mathcal{M}$ such that $\Gamma_\gamma \cap Z = \emptyset$ and $\Gamma_{\gamma|_{B'}} \cap Z_\epsilon \neq \emptyset$. Applying the claim to $h_i \circ \gamma$ with $\Gamma_\gamma \subset \Omega_i$ we obtain that $\Gamma_{\gamma|_{B'}} \subset Z_{C_4 \epsilon^\tau}$ for some constant $C_4 > 0$. On the other hand, by our assumption on the approximation by \mathcal{M}_n , Hurwitz lemma implies that either $\Gamma_\gamma \subset Z$ or $\Gamma_\gamma \cap Z = \emptyset$ for any $\gamma \in \text{supp } \mathcal{M}$. We thus have

$$\begin{aligned} \mathcal{M}\{\gamma \in \mathcal{J} / \Gamma_{\gamma|_{B'}} \cap Z_\epsilon \neq \emptyset\} &\leq \mathcal{M}\{\gamma \in \mathcal{J} / \Gamma_{\gamma|_{B'}} \subset Z_{C_4 \epsilon^\tau}\} \leq \\ \mathcal{M}\{\gamma \in \mathcal{J} / (\lambda_0, \gamma(\lambda_0)) \in Z_{C_4 \epsilon^\tau}\} &= \mu_{\lambda_0} \left(Z_{C_4 \epsilon^\tau} \cap \left(\{\lambda_0\} \times \mathbb{P}^k \right) \right) \leq \\ &\mu_{\lambda_0} \left[\left(Z \cap \left(\{\lambda_0\} \times \mathbb{P}^k \right) \right)_{C_3(C_4 \epsilon^\tau)^{\tau_0}} \right] \leq A \epsilon^a \end{aligned}$$

where the last estimate is due to the fact that μ_{λ_0} has Hölder-continuous local potentials and $Z \cap (\{\lambda_0\} \times \mathbb{P}^k)$ is a proper analytic subset of \mathbb{P}^k .

It remains to prove the claim. Let $\mathcal{G} := \{\varphi \in \mathcal{O}(B, H) / \varphi(s) = -1 \text{ for some } s \in \overline{B'}\}$ where $H := \{\Re z < 0\}$ is the left half plane. Then \mathcal{G} is compact for the topology of local uniform convergence, and thus the quantity $(-\alpha) := \sup_{\varphi \in \mathcal{G}} \sup_{s \in \overline{B'}} \Re \varphi(s)$ satisfies $-1 \leq -\alpha < 0$. Let $t_0 \in B'$ and $\phi : B \rightarrow \mathbb{D}^*$ be holomorphic. After a rotation in \mathbb{D}^* we may assume that $|\phi(t_0)| = \phi(t_0) \in]0, 1]$. Let $\varphi : B \rightarrow H$ be the lift of ϕ by the exponential map, which satisfies $\varphi(t_0) = \log \phi(t_0) \in]-\infty, 0[$. Then $\varphi_0(t) := -\varphi(t)/\varphi(t_0)$ belongs to \mathcal{G} and thus $\Re(\varphi_0) \leq -\alpha$ on B' . This is the desired estimate since $|\phi| = e^{\Re \varphi} \leq e^{\alpha \log \phi(t_0)} = |\phi(t_0)|^\alpha$. \square

PROOF OF PROPOSITION 3.1: The problem is local and we may therefore take for M a ball $B \subset \mathbb{C}^m$ and assume that $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ admits a lifted family $F : B \times \mathbb{C}^{k+1} \rightarrow B \times \mathbb{C}^{k+1}$ of d -homogeneous non-degenerate maps. We will apply Lemma 3.2 with $Z = C_f$. Let B' be any relatively compact ball contained in B .

After shrinking B we may use Lemma 2.9 and associate to \mathcal{M} a current of the form

$$\widetilde{W}_\mathcal{M} = \int_{\mathcal{J}} [\Gamma_{\sigma(\gamma)}] d\mathcal{M}(\gamma).$$

According to Proposition 2.12, one has

$$dd_\lambda^c L = \pi_{B^*}(\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c \log \|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda).$$

Using $\|J_F^s(\lambda, \pi(\tilde{z}))\|_\lambda = e^{-DG(\lambda, \tilde{z})} |J_F(\lambda, \tilde{z})|$ as defined in subsection 2.3, and the fact that functions L and G are *psh*, we obtain

$$0 \leq dd_\lambda^c L = \pi_{B^*}(\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c \log |J_F|) - D\pi_{B^*}(\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c G) \leq \pi_{B^*}(\widetilde{W}_\mathcal{M} \wedge dd_{\lambda, \tilde{z}}^c \log |J_F|)$$

hence it suffices to show that the current $\log |J_F| \widetilde{W}_\mathcal{M}$ restricted to $B' \times \mathbb{C}^{k+1}$ is $dd_{\lambda, \tilde{z}}^c$ closed.

For $\epsilon < 1$ we set $\log_\epsilon := \chi_\epsilon \circ \log$ where χ_ϵ is a convex, smooth, increasing function on \mathbb{R} such that $\chi_\epsilon(x) = x$ if $x \geq \log \epsilon$ and $\chi_\epsilon(-\infty) = 2 \log \epsilon$. Then $\log_\epsilon |J_F|$ is a decreasing family (with respect to ϵ) of smooth *psh* functions which converges to $\log |J_F|$. As

$\lim_{\epsilon \rightarrow 0} \log_\epsilon |J_F| \widetilde{W}_\mathcal{M} = \log |J_F| \widetilde{W}_\mathcal{M}$ we will actually deal with $\log_\epsilon |J_F| \widetilde{W}_\mathcal{M}$. To this purpose we set $U_\epsilon := \{|J_F| < \epsilon\}$, $S_{\mathcal{M},\epsilon} := \{\gamma \in \text{supp } \mathcal{M} / \Gamma_{\sigma(\gamma)|_{B'}} \cap U_\epsilon \neq \emptyset\}$ and decompose $\widetilde{W}_\mathcal{M}$ as follows:

$$\widetilde{W}_\mathcal{M} = \widetilde{W}_{\mathcal{M},\epsilon} + \widetilde{W}_{\mathcal{M},\epsilon}^*$$

where $\widetilde{W}_{\mathcal{M},\epsilon} := \int_{\mathcal{J}} [\Gamma_{\sigma(\gamma)}] 1_{S_{\mathcal{M},\epsilon}} d\mathcal{M}(\gamma)$ and $\widetilde{W}_{\mathcal{M},\epsilon}^* := \widetilde{W}_\mathcal{M} - \widetilde{W}_{\mathcal{M},\epsilon}$. Then

$$\log_\epsilon |J_F| \widetilde{W}_\mathcal{M} = \log_\epsilon |J_F| \widetilde{W}_{\mathcal{M},\epsilon} + \log_\epsilon |J_F| \widetilde{W}_{\mathcal{M},\epsilon}^*$$

and, by construction, the current $\log_\epsilon |J_F| \widetilde{W}_{\mathcal{M},\epsilon}^*|_{B' \times \mathbb{C}^{k+1}}$ is $dd_{\lambda,\bar{z}}^c$ -closed since $\log_\epsilon |J_F| = \log |J_F|$ is pluriharmonic on the graphs Γ_γ which do not intersect U_ϵ . It thus remains to check that $\lim_\epsilon \log_\epsilon |J_F| \widetilde{W}_{\mathcal{M},\epsilon} = 0$. This follows from the following estimate

$$\|\log_\epsilon |J_F| \widetilde{W}_{\mathcal{M},\epsilon}\| \lesssim |\log \epsilon| \mathcal{M}(S_{\mathcal{M},\epsilon}) \lesssim \epsilon^a |\log \epsilon|$$

where the last inequality is obtained by observing that there exist $b, \beta > 0$ such that $S_{\mathcal{M},\epsilon} \subset \{\gamma \in \mathcal{J} / \Gamma_{\gamma|_{B'}} \cap (C_f)_{b\epsilon^\beta} \neq \emptyset\}$ and applying Lemma 3.2. \square

3.2 Misiurewicz parameters belong to $\text{supp } dd_\lambda^c L$

Proposition 3.3 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Then the Misiurewicz parameters belong to the support of $dd_\lambda^c L$.*

PROOF: If $\lambda_0 \in M$ is a Misiurewicz parameter then, by definition, there exists a holomorphic map γ from a neighbourhood of λ_0 into \mathbb{P}^k such that:

- 1) $\gamma(\lambda) \in J_\lambda$ and is a repelling p_0 -periodic point of f_λ for some $p_0 \geq 1$,
- 2) $(\lambda_0, \gamma(\lambda_0)) \in f^{n_0}(C_f)$ for some $n_0 \geq 1$,
- 3) the graph Γ_γ of γ is not contained in $f^{n_0}(C_f)$.

To simplify notations we assume that $p_0 = 1$. We may also assume that M is a disc $D_\rho \subset \mathbb{C}$ centered at $\lambda_0 = 0$ with radius ρ . Moreover, conjugating by a suitable "translation" $(\lambda, z) \mapsto (\lambda, T_{\gamma(\lambda)}(z))$ ensures that γ is constant equal to $z_1 := \gamma(0)$. Let us denote by B_r a ball centered at z_1 and of radius r . Taking ρ and r small finally allows us to suppose that

- (i) f is injective and uniformly expanding on $D_\rho \times B_r$: there exists $K > 1$ such that

$$\forall (\lambda, z) \in D_\rho \times B_r, d_{\mathbb{P}^k}(f(\lambda, z), f(\lambda, z_1)) \geq K d_{\mathbb{P}^k}(z, z_1)$$

- (ii) $\forall \lambda \in D_\rho, (\lambda, z_1) \in f^{n_0}(C_f) \Leftrightarrow \lambda = 0$.

The fact that $\gamma(\lambda) \in J_\lambda$ is crucial but will only be used at the very end of the proof.

We have to show that $\langle dd_\lambda^c L, 1_{D_\epsilon} \rangle > 0$ for every $0 < \epsilon < \rho$. To this purpose, we will use the formula $dd_\lambda^c L = (\pi_{D_\rho})_* \left(\left(dd_{\lambda,z}^c g + \omega \right)^k \wedge \mathbf{C}_f \right)$ given by Theorem 2.10, where

$\omega := \omega_{FS}$. Let $(g_n)_n$ be a sequence of smooth functions on \mathbb{P}^k which converges uniformly to g and satisfies $\frac{1}{d}f^*(dd_{\lambda,z}^c g_n + \omega) = dd_{\lambda,z}^c g_{n+1} + \omega$ (see subsection 2.2). We shall proceed in three steps.

First step: $\langle dd_{\lambda}^c L, 1_{D_\epsilon} \rangle \geq d^{-n_0 k} \langle [f^{n_0}(C_f)] \wedge (dd_{\lambda,z}^c g + \omega)^k, 1_{D_\epsilon \times B_r} \rangle$.

Pick $(0, z_0) \in C_f$ such that $f^{n_0}(0, z_0) = (0, z_1)$. After reducing ϵ and r , we may find a neighbourhood U of $(0, z_0)$ such that the map $f^{n_0} : U \rightarrow D_\epsilon \times B_r$ is proper. According to Theorem 2.10, we have

$$\langle dd_{\lambda}^c L, 1_{D_\epsilon} \rangle = \langle (dd_{\lambda,z}^c g + \omega)^k \wedge \mathbf{C}_f, 1_{D_\epsilon} \circ \pi_{D_\rho} \rangle \geq \langle (dd_{\lambda,z}^c g + \omega)^k \wedge [C_f], 1_U \rangle.$$

Using the smooth approximations g_n , we get

$$\begin{aligned} \langle (dd_{\lambda,z}^c g + \omega)^k \wedge [C_f], 1_U \rangle &= \lim_n \langle (dd_{\lambda,z}^c g_{n+n_0} + \omega)^k \wedge [C_f], 1_U \rangle \\ &= \lim_n d^{-n_0 k} \langle 1_U \cdot [C_f], f^{n_0*} (dd_{\lambda,z}^c g_n + \omega)^k \rangle \\ &= \lim_n d^{-n_0 k} \langle (f^{n_0})_* (1_U \cdot [C_f]), (dd_{\lambda,z}^c g_n + \omega)^k \rangle. \end{aligned}$$

Now, as $f^{n_0} : U \rightarrow D_\epsilon \times B_r$ is proper, one has $(f^{n_0})_* (1_U [C_f]) \geq 1_{D_\epsilon \times B_r} [f^{n_0}(C_f)]$ which, since $dd_{\lambda,z}^c g_n + \omega$ is positive, yields

$$\begin{aligned} \langle dd_{\lambda}^c L, 1_{D_\epsilon} \rangle &\geq \lim_n d^{-n_0 k} \langle 1_{D_\epsilon \times B_r} [f^{n_0}(C_f)], (dd_{\lambda,z}^c g_n + \omega)^k \rangle \\ &\geq \lim_n d^{-n_0 k} \langle (dd_{\lambda,z}^c g_n + \omega)^k \wedge [f^{n_0}(C_f)], 1_{D_\epsilon \times B_r} \rangle. \end{aligned}$$

The desired estimate follows by uniform convergence of g_n to g .

Second step: Let $A_0 := 1_{D_\epsilon \times B_r} [f^{n_0}(C_f)]$ and $A_{p+1} := 1_{D_\epsilon \times B_r} f_*(A_p)$. Then

$$\|A_p \wedge (dd_{\lambda,z}^c g + \omega)^k\| = d^{pk} \|(1_{D_\epsilon \times B_r} \circ f^p) A_0 \wedge (dd_{\lambda,z}^c g + \omega)^k\| \leq d^{pk} \|A_0 \wedge (dd_{\lambda,z}^c g + \omega)^k\|.$$

We use again the smooth approximations g_n . Then:

$$\begin{aligned} \|A_{p+1} \wedge (dd_{\lambda,z}^c g_n + \omega)^k\| &= \langle 1_{D_\epsilon \times B_r} f_*(A_p), (dd_{\lambda,z}^c g_n + \omega)^k \rangle \\ &= \langle A_p, f^* (1_{D_\epsilon \times B_r} (dd_{\lambda,z}^c g_n + \omega)^k) \rangle \\ &= d^k \langle A_p, 1_{D_\epsilon \times B_r} \circ f (dd_{\lambda,z}^c g_{n+1} + \omega)^k \rangle \\ &= d^k \langle A_p \wedge (dd_{\lambda,z}^c g_{n+1} + \omega)^k, 1_{D_\epsilon \times B_r} \circ f \rangle \\ &= d^k \|(1_{D_\epsilon \times B_r} \circ f) A_p \wedge (dd_{\lambda,z}^c g_{n+1} + \omega)^k\| \end{aligned}$$

Taking the limits when n tends to infinity yields the conclusion.

Third step: $\langle dd_\lambda^c L, 1_{D_\epsilon} \rangle > 0$.

By combining the two former steps, one gets:

$$d^{(p+n_0)k} \langle dd_\lambda^c L, 1_{D_\epsilon} \rangle \geq \|A_p \wedge (dd_{\lambda,z}^c g + \omega)^k\|. \quad (5)$$

By (i) and (ii), f is uniformly expanding on $D_\rho \times B_r$ and $(\text{supp } A_0) \cap (D_\rho \times \{z_1\}) = \{(0, z_1)\}$. Thus $\text{supp } A_p \subset D_{\epsilon_p} \times B_r$ for some $\epsilon_p \rightarrow 0$. Let us momentarily admit that there exists $m > 0$ such that

$$A_p \rightarrow m [\{0\} \times B_r]. \quad (6)$$

We then deduce from (5) that, for p large enough, one has:

$$d^{(p+n_0)k} \langle dd_\lambda^c L, 1_{D_\epsilon} \rangle \geq \frac{m}{2} \|[\{0\} \times B_r] \wedge (dd_{\lambda,z}^c g + \omega)^k\|.$$

We conclude by using the fact that $z_1 \in J_0$: the right hand side is equal to

$$\frac{m}{2} \int_{B_r} (dd_{z_1}^c g(0, z) + \omega)^k = \frac{m}{2} \mu_0(B_r) > 0.$$

To complete the proof of Proposition 3.3 it remains to establish (6). Let us denote $V := D_\rho \times B_r$ and $V' := f(V)$. By assumption $f : V \rightarrow V'$ is a biholomorphism whose inverse will be denoted $h : V' \rightarrow V$. According to (i), $V \subset V'$ and $(h|_V)^p$ converges to $(\lambda, z) \mapsto (\lambda, z_1)$. We now use (ii). After shrinking ρ and r , we may find a Weierstrass polynomial

$$\psi(\lambda, z) := \lambda^m + \alpha_{m-1}(z)\lambda^{m-1} + \dots + \alpha_0(z)$$

such that $\alpha_j(z_1) = 0$ for $0 \leq j \leq m-1$ and $f^{n_0}(C_f) \cap (D_\rho \times B_r) = \{\psi = 0\}$. Observe now that $A_0 = 1_V dd_{\lambda,z}^c \log |\psi|$ and that

$$A_1 = 1_V f_* A_0 = 1_V h^* A_0 = 1_V (1_V \circ h) dd_{\lambda,z}^c \log |\psi \circ h| = 1_V dd_{\lambda,z}^c \log |\psi \circ h|,$$

where the last equality comes from $h(V) \subset V$. Similarly we have $A_p = 1_V dd_{\lambda,z}^c \log |\psi \circ (h|_V)^p|$ and the conclusion follows from $\psi \circ (h|_V)^p(\lambda, z) \rightarrow \lambda^m$. \square

4 A construction of Misiurewicz parameters

The main goal of this section is to show that Misiurewicz parameters appear when the holomorphic motion of μ_λ fails. The basic idea is as follows. Assume that a graph Γ obtained by the holomorphic motion of a repelling J -cycle stays in Julia sets and is not contained in the post-critical set, then, according to Proposition 2.4, either the equilibrium measures move holomorphically or the preimages of Γ by \mathcal{F}^n must meet the critical set and thus produce a Misiurewicz parameter.

The technical difficulty is to find such a graph of repelling J -cycles. We will obtain it from the holomorphic motion of some sufficiently thick hyperbolic set whose existence is interesting for itself. This is detailed in the first subsection while the second one is devoted to hyperbolic sets and their holomorphic motions.

4.1 Thick hyperbolic sets and Misiurewicz parameters

We prove the following proposition, the statement is local since it is based on holomorphic motion of hyperbolic sets.

Proposition 4.1 *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k where B is a ball centered at the origin in \mathbb{C}^m . If B does not contain any Misiurewicz parameter then, after shrinking B , there exists $\gamma \in \mathcal{J}$ such that $\Gamma_\gamma \cap PC_f = \emptyset$. In particular the equilibrium measures μ_λ move holomorphically over B .*

Let us recall a few definitions concerning holomorphic motions of hyperbolic sets.

Definition 4.2 *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k where B is a ball centered at the origin in \mathbb{C}^m . Let E_0 be some f_0 -invariant subset of \mathbb{P}^k . A holomorphic motion of E_0 over $B_\rho \subset B$ is a continuous map $h : B_\rho \times E_0 \rightarrow \mathbb{P}^k$ such that :*

1. $\lambda \mapsto h_\lambda(z)$ is holomorphic on B_ρ for every $z \in E_0$.
2. $z \mapsto h_\lambda(z)$ is injective on E_0 for every $\lambda \in B_\rho$.
3. $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ on E_0 for every $\lambda \in B_\rho$.

One says that E_0 is a hyperbolic set for f_0 if it is f_0 -invariant and if there exists $K > 1$ such that $|(df_0)^{-1}|^{-1} \geq K$ on E_0 .

We will show in Theorem 4.6 below that every hyperbolic set admits a holomorphic motion which preserves repelling cycles. We need a more precise result concerning the size of such sets and the position of their motions with respect to Julia sets.

Theorem 4.3 *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . There exist an integer N , a compact hyperbolic set $E_0 \subset J_0$ for f_0^N and a holomorphic motion $h : B_r \times E_0 \rightarrow \mathbb{P}^k$ for some $0 < r < 1$ such that :*

1. The repelling periodic points of f_0^N are dense in E_0 and E_0 is not contained in the post-critical set of f_0^N .
2. $h_\lambda(z) \in J_\lambda$ for every $\lambda \in B_r$ and every $z \in E_0$.
3. If z is periodic repelling for f_0^N then $h_\lambda(z)$ is periodic repelling for f_λ^N .

PROOF PROPOSITION 4.1: We use Theorem 4.3. Since f_λ^N and f_λ have same equilibrium measures and post-critical sets, we may assume that $N = 1$. Let $E_0 \subset J_0$ and $r \in]0, 1]$ provided by Theorem 4.3. Let us fix $z \in E_0 \setminus \cup_{n \geq 1} f_0^n(C_{f_0})$ (see item 1). Let Γ_z denote the graph of $\lambda \mapsto h_\lambda(z)$. We want to show that

$$\Gamma_z \cap \left(\cup_{n \geq 1} f^n(C_f) \right) = \emptyset. \quad (7)$$

Assume to the contrary that there exists $n_0 \geq 1$ such that $\Gamma_z \cap f^{n_0}(C_f) \neq \emptyset$. By item 1, there exists a sequence $(z_p)_p \subset E_0$ of f_0 -periodic repelling points which converges to z . Items 2 and 3 assert that $h_\lambda(z_p) \in J_\lambda$ and $h_\lambda(z_p)$ is a f_λ -periodic repelling point for every

$\lambda \in B_r$. As h is continuous, $\lambda \mapsto h_\lambda(z_p)$ converges locally uniformly to $\lambda \mapsto h_\lambda(z)$. Hence, for p large enough, the graph Γ_{z_p} is not contained in $f^{n_0}(C_f)$ (consider the parameter $\lambda = 0$) and, by Hurwitz's lemma, there exists $\lambda_p \in B_r$ such that $(\lambda_p, h_{\lambda_p}(z_p)) \in f^{n_0}(C_f)$. The parameters λ_p are Misiurewicz and this contradicts our assumption. By (7) and the first assertion of Proposition 2.4 we get that the measures μ_λ move holomorphically over B_r . \square

The remainder of the subsection is devoted to the proof of Theorem 4.3. To control the size of hyperbolic sets, we use an entropy argument. Our key tool is the following result which is due to Briend-Duval [BD2], de Thélin [dT] and Dinh [Di3] (see also [DS3] Corollary 1.117).

Theorem 4.4 *Let g be an endomorphism of \mathbb{P}^k of degree d . Let κ be an ergodic g -invariant measure with entropy $h_\kappa > (k-1)\log d$. Then κ gives no mass to analytic subsets of dimension $\leq k-1$ and the support of κ is included in the Julia set of g .*

To create hyperbolic sets, we use a classical device. For that purpose we need the following Proposition which is a consequence of [BD1] (see also [BDM]). For any endomorphism f_0 of \mathbb{P}^k and every $A \subset \mathbb{P}^k$, $n \geq 1$ and $\rho > 0$, we denote by $C_n(A, \rho)$ the set of inverse branches g_n of f_0^n defined on A and satisfying $g_n(A) \subset A$ and $\text{Lip } g_n \leq \rho$.

Proposition 4.5 *Let f_0 be an endomorphism of \mathbb{P}^k of degree d . For every $\rho > 0$ there exist a closed ball $A \subset \mathbb{P}^k$ centered on J_{f_0} and $\alpha > 0$ such that $\text{Card } C_n(A, \rho) \geq \alpha d^{kn}$ for n large.*

PROOF: Let $\mathcal{O} := \{\hat{z} := (z_n)_{n \in \mathbb{Z}}, z_{n+1} = f_0(z_n)\}$ be the set of orbits and let s be the left shift acting on \mathcal{O} . Let $p(\hat{z}) := z_0$ and $\hat{\mu}$ be the unique s -invariant probability measure satisfying $\hat{\mu}(p^{-1}(B)) = \mu(B)$. The measure $\hat{\mu}$ is mixing since μ is mixing. Let $X := \{\hat{z} \in \mathcal{O}, z_n \notin C, \forall n \in \mathbb{Z}\}$ where C is the critical set of f_0 . This subset has full $\hat{\mu}$ -measure since $\mu(C) = 0$. For every $\hat{z} \in X$ we denote $f_{0, \hat{z}}^{-n}$ the inverse branch of f_0^n sending z_0 to z_{-n} . Let λ_1 be the smallest Lyapunov exponent of μ . According to [BD1], for every $\epsilon > 0$ there exist $\hat{H} \subset X$ of positive $\hat{\mu}$ -measure, $r_0 > 0$ and $n_0 \geq 1$ such that: for every $\hat{z} \in \hat{H}$ and $n \geq n_0$,

$$f_{0, \hat{z}}^{-n} \text{ is well defined on } B_{z_0}(r_0) \text{ and } \text{Lip } f_{0, \hat{z}}^{-n} \leq e^{-n\lambda_1 + n\epsilon} \text{ on } B_{z_0}(r_0).$$

Let $(B_i)_{1 \leq i \leq m}$ be a covering of J_{f_0} by closed balls of radius $r < r_0/2$. Let $(B_i^\gamma)_{1 \leq i \leq m}$ be the concentric balls of radius $r + \gamma < r_0/2$. Setting $\hat{B}_i := p^{-1}(B_i)$, let us fix $i_0 \in \{1, \dots, m\}$ satisfying $\hat{\mu}(\hat{H} \cap \hat{B}_{i_0}) > 0$. It remains to verify that $A := B^\gamma$ is convenient, with $B := B_{i_0}$. Let C'_n be the collection of inverse branches $f_{0, \hat{z}}^{-n}$ with $\hat{z} \in \hat{H} \cap \hat{B}^\gamma$ and satisfying $f_{0, \hat{z}}^{-n}(B^\gamma) \cap B \neq \emptyset$. One can verify that $C'_n \subset C_n(B^\gamma, \rho)$. Let us estimate the cardinality of C'_n . The mixing property of $\hat{\mu}$ yields for n large

$$10^{-1} \hat{\mu}(\hat{B} \cap \hat{H}) \hat{\mu}(\hat{B}) \leq \hat{\mu}(s^{-n}(\hat{B} \cap \hat{H}) \cap \hat{B}) \leq \mu(\cup_{C'_n} f_{0, \hat{z}}^{-n}(B^\gamma)). \quad (8)$$

Since $f_0^* \mu = d^k \mu$ the right hand side is less than $\text{Card } C'_n \cdot \mu(B^\gamma) / d^{kn}$. \square

PROOF OF THEOREM 4.3: Let $\rho < 1$ and A be a closed ball provided by proposition 4.5. Let us fix N large enough such that $\text{Card } C_N(A, \rho) > d^{(k-1)N}$. We denote by g_1, \dots, g_M the elements of $C_N(A, \rho)$, let us recall that $g_i : A \rightarrow A$. Let $E_0 := \bigcap_{k \geq 1} E_k$, where

$$E_k := \{ g_{i_1} \circ \dots \circ g_{i_k}(A), (i_1, \dots, i_k) \in \{1, \dots, N'\}^k \}.$$

Let $\Sigma := \{1, \dots, N'\}^{\mathbb{N}^*}$ endowed with the product metric and z be a fixed point in $A \cap J_0$, for instance the center of A . The map $\omega : \Sigma \rightarrow E_0$ defined by $(i_1, i_2, \dots) \mapsto \lim_{k \rightarrow \infty} g_{i_1} \circ \dots \circ g_{i_k}(z)$ is a homeomorphism satisfying $f^N \circ \omega = \omega \circ s$, where s is the left shift acting on Σ . We take for κ the image by ω of the uniform product measure on Σ : this is a f^N -invariant ergodic measure with entropy $h_\kappa = \log N' > (k-1) \log d^n$, with support E_0 .

By construction $E_0 \subset J_{f_0}$. Indeed, $E_0 = \{ \lim_{k \rightarrow \infty} g_{i_1} \circ \dots \circ g_{i_k}(z), (i_1, i_2, \dots) \in \Sigma \}$ and J_{f_0} is a closed f_0^N -invariant set. Also, repelling cycles of f_0^N are dense in E_0 . According to Theorem 4.4, $E_0 = \text{supp } \kappa$ is not contained in the countable union of analytic subsets $\bigcup_{n \geq 1} f_0^n(C_{f_0})$. The set E_0 is hyperbolic for f_0^N since $|(df_0^N)^{-1}|^{-1} > \frac{1}{\rho} > 1$ on E_0 and thus there exists a holomorphic motion $h : B_r \times E_0 \rightarrow \mathbb{P}^k$ which preserves repelling cycles (see Theorem 4.6 below). It remains to show $h_\lambda(E_0) \subset J_{f_\lambda}$. For that purpose we use the fact that $h_\lambda : E_0 \rightarrow \mathbb{P}^k$ is a continuous injective mapping satisfying $h_\lambda \circ f_0^N = f_\lambda^N \circ h_\lambda$ on E_0 . Then $(h_\lambda)_* \kappa$ is a f_λ^N -invariant ergodic measure whose support coincides with $h_\lambda(E_0)$ and whose metric entropy equals h_κ . Theorem 4.4 yields $h_\lambda(E_0) \subset J_{f_\lambda}$ as desired. \square

4.2 Hyperbolic sets and holomorphic motions

This subsection is devoted to theorem 4.6, used to prove theorem 4.3 above. The arguments are classical, we refer to [dMvS, chapter 3, section 2.d] for the one dimensional case. To simplify the exposition we assume that the dilation is larger than 3 on the hyperbolic set.

Theorem 4.6 *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let $E_0 \subset \mathbb{P}^k$ such that $|(df_0)^{-1}|^{-1} \geq K > 3$ on E_0 . Then there exists a holomorphic motion $h : B_\rho \times E_0 \rightarrow \mathbb{P}^k$ which preserves repelling cycles.*

PROOF: Let $\varphi(z) := \inf_{\lambda \in B_\rho} |(d_z f_\lambda)^{-1}|^{-1}$, with the convention $|(d_z f_\lambda)^{-1}|^{-1} = 0$ if $z \in C_{f_\lambda}$. This is a continuous function on \mathbb{P}^k . By taking a smaller ρ , we may assume that

$$\varphi \geq K' > 3 \text{ on a } \tau\text{-neighbourhood } (E_0)_\tau. \quad (9)$$

We shall mainly use the lower estimate on E_0 itself, the lower bound on its τ -neighbourhood appears at the end of the proof. Let $\delta = \delta(\rho) := \min\{(1 + \sup_{\lambda \in B_\rho} \|f_\lambda\|_{C^2})^{-1}, \tau\}$.

Lemma 4.7 *For every $(\lambda, z) \in B_\rho \times E_0$,*

1. $d_{\mathbb{P}^k}(f_\lambda(z), f_\lambda(w)) \geq (K' - 1)d_{\mathbb{P}^k}(z, w)$ for every $w \in \bar{B}(z, \delta)$,
2. $f_\lambda(B(z, c\delta)) \supset B(f_\lambda(z), c\delta)$ for every $0 \leq c \leq 1$,
3. if $g_{\lambda, z} : B(f_\lambda(z), \delta) \rightarrow B(z, \delta)$ is the inverse map of f_λ , then $\text{Lip } g_{\lambda, z} \leq (K' - 1)^{-1}$.

PROOF: Items 2 and 3 follows from item 1 (use Jordan's theorem and $K' > 3$ for the second one). So let us prove item 1. We work in local coordinates. For $(\lambda, z) \in B_\rho \times E_0$ and $w \in \bar{B}(z, \delta)$ we have

$$\begin{aligned} |\text{Id}_{\mathbb{C}^k} - (d_z f_\lambda)^{-1} \circ d_w f_\lambda| &\leq |(d_z f_\lambda)^{-1}| \cdot |d_z f_\lambda - d_w f_\lambda| \\ &\leq |(d_z f_\lambda)^{-1}| \cdot |z - w| \cdot \delta^{-1} \leq 1/K'. \end{aligned}$$

That implies $\text{Lip}(\text{Id} - (d_z f_\lambda)^{-1} \circ f_\lambda) \leq 1/K'$ on $\bar{B}(z, \delta)$, which gives in turn

$$|(d_z f_\lambda)^{-1}(f_\lambda(z) - f_\lambda(w))| \geq (1 - 1/K')|z - w|$$

for every $w \in \bar{B}(z, \delta)$. Hence $|f_\lambda(z) - f_\lambda(w)| \geq (K' - 1)|z - w|$ as desired. \square

Lemma 4.8 *For every $(\lambda, z) \in B_\rho \times E_0$, we have $B(f_\lambda(z), \delta) \supset B(f_0(z), \delta/2)$ and the inverse map $g_{\lambda, z} : B(f_\lambda(z), \delta) \rightarrow B(z, \delta)$ given by Lemma 4.7 satisfies the following properties:*

1. $g_{\lambda, z}$ is well defined on $B(f_0(z), \delta/2)$,
2. it satisfies $\text{Lip } g_{\lambda, z} \leq (K' - 1)^{-1}$ on $B(f_0(z), \delta/2)$,
3. $g_{\lambda, z}(B(f_0(z), \delta/2)) \subset B(z, \delta/2)$.

PROOF: Let $Q := \max\{\|d_\lambda f_\lambda(z)\|, (\lambda, z) \in B_\rho \times E_0\}$. As δ is a continuous function of ρ and $\delta(0) > 0$, we may assume $\delta \geq 2Q\rho$ by taking ρ small enough. For every $\lambda \in B_\rho$ and $z \in E_0$, $d(f_\lambda(z), f_0(z)) \leq Q\rho \leq \delta/2$. That yields $B(f_\lambda(z), \delta) \supset B(f_0(z), \delta/2)$. Items 1 and 2 are obvious from lemma 4.7. For item 3, we use $g_{\lambda, z}(B(f_0(z), \delta/2)) \subset g_{\lambda, z}(B(f_\lambda(z), \delta))$, which is included in $B(z, \delta/2)$ by using lemma 4.7(3) and $K' > 3$. \square

Let us end the proof of theorem 4.6. For $(\lambda, z) \in B_\rho \times E_0$ we set $z_n := f_0^n(z)$ and

$$g_{\lambda, z}^n := g_{\lambda, z} \circ \dots \circ g_{\lambda, z_{n-1}}.$$

This is an inverse branch of f_λ^n . Since $z_1, \dots, z_{n-1} \in E_0$, lemma 4.8 yields by induction

$$g_{\lambda, z}^n : B(z_n, \delta/2) \rightarrow B(z, \delta/2) \text{ and } \text{Lip } g_{\lambda, z}^n \leq (K' - 1)^{-n} \text{ on } B(z_n, \delta/2).$$

For $(\lambda, z) \in B_\rho \times E_0$ let us define

$$h_n(\lambda, z) := g_{\lambda, z}^n \circ f_0^n(z) = g_{\lambda, z}^n(z_n).$$

The map h_n is continuous in (λ, z) , holomorphic in λ and $h_n(\lambda, z) \in B(z, \delta/2)$. Moreover

$$f_\lambda \circ h_n(\lambda, z) = h_{n-1}(\lambda, f_0(z)). \quad (10)$$

The sequence $(h_n)_n$ is uniformly Cauchy on $B_\rho \times E_0$. Indeed $h_{n+1}(\lambda, z) - h_n(\lambda, z) = g_{\lambda, z}^n \circ g_{\lambda, z_n}(z_{n+1}) - g_{\lambda, z}^n(z_n)$ and we get $\|h_{n+1} - h_n\|_{B_\rho \times E_0} \leq (\delta/2) \cdot (K' - 1)^{-n}$ since $g_{\lambda, z_n}(z_{n+1}) \in B(z_n, \delta/2)$ by Lemma 4.8(3). We define $h_\lambda(z)$ for $(\lambda, z) \in B_\rho \times E_0$ by

$$h_\lambda(z) := \lim_n h_n(\lambda, z) = \lim_n g_{\lambda, z}^n \circ f_0^n(z).$$

The map h is continuous in (λ, z) , holomorphic in λ and $h_\lambda(z) \in \bar{B}(z, \delta/2)$. It also follows from (10) that

$$f_\lambda \circ h_\lambda = h_\lambda \circ f_0. \quad (11)$$

Let us now check that h_λ is injective. Assume $h_\lambda(z) = h_\lambda(z')$. Iterating (11) yields $h_\lambda(f_0^n(z)) = h_\lambda(f_0^n(z'))$. As $h_\lambda(w) \in \bar{B}(w, \delta/2)$ for $w \in E_0$, we get $d(f_0^n(z), f_0^n(z')) \leq \delta$. Then, since $d(f_0^n(z), f_0^n(z')) \geq (K' - 1)^n d(z, z')$ by Lemma 4.7(1), we must have $z = z'$. Finally, h_λ preserves cycles (see (10)) and any periodic $h_\lambda(z)$ must be repelling since $h_\lambda(z) \in \bar{B}(z, \delta/2) \subset (E_0)_\tau$ and $|(df_\lambda)^{-1}|^{-1} > 3$ on $(E_0)_\tau$ (see (9)). This completes the proof of theorem 4.6. \square

5 Siegel discs and Hausdorff continuity of Julia sets

As it is well known, the Julia sets of any holomorphic family of rational maps of \mathbb{P}^1 depends continuously on the parameter for the Hausdorff topology if and only if the family is stable. It is worth emphasize that certain semi-continuity properties are always satisfied and that discontinuities might be explained by the appearance of Siegel discs, see [Do]. We will investigate this in higher dimension and from the point of view of equilibrium currents. As a by-product, we will see that the existence of virtually repelling Siegel periodic points in the Julia set (see Definitions 5.5 and 5.6) is an obstruction to the holomorphic motion of equilibrium measures. This fact will play a crucial role in the proof of Theorem 1.5.

5.1 Semi-continuity properties

Let $Comp^*(\mathbb{P}^k)$ be the set of non-empty compact subsets of \mathbb{P}^k endowed with the Hausdorff distance and let K_ϵ denote the ϵ -neighbourhood of $K \in Comp^*(\mathbb{P}^k)$. A map $E : M \rightarrow Comp^*(\mathbb{P}^k)$ is said upper-semi-continuous (*u.s.c*) at $\lambda_0 \in M$ if for every $\epsilon > 0$, one has $E(\lambda) \subset (E(\lambda_0))_\epsilon$ when λ is close enough to λ_0 . It is lower-semi-continuous (*l.s.c*) at λ_0 if for every $\epsilon > 0$, one has $E(\lambda_0) \subset (E(\lambda))_\epsilon$ when λ is close enough to λ_0 . For every $A \subset M \times \mathbb{P}^k$ we define $(A)_\lambda := A \cap (\{\lambda\} \times \mathbb{P}^k)$.

Our starting point is the following observations, see also [DS3, exercises 2.52 and 2.53].

Proposition 5.1 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k whose equilibrium measures μ_λ move holomorphically. Let \mathcal{M} be a structural web and let $W_{\mathcal{M}}$ be the woven current $\int_{\mathcal{J}} [\Gamma_\gamma] d\mathcal{M}(\gamma)$. Then $J_\lambda \subset (\text{supp } W_{\mathcal{M}})_\lambda$ and the maps $\lambda \mapsto J_\lambda$ and $\lambda \mapsto (\text{supp } W_{\mathcal{M}})_\lambda$ from M to $Comp^*(\mathbb{P}^k)$ are respectively *l.s.c* and *u.s.c*.*

PROOF: The inclusion $J_\lambda \subset (\text{supp } W_{\mathcal{M}})_\lambda$ follows directly from the fact that $J_\lambda = \text{supp } \mu_\lambda$ and $\mu_\lambda = \mathcal{M}_\lambda = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma)$. The upper-semi-continuity of $(\text{supp } W_{\mathcal{M}})_\lambda$ is an elementary general topological fact (see [Do, Proposition 2.1], or the proof of lemma 6.14 below). The lower-semi-continuity of J_λ is a consequence of the existence of continuous local potentials for μ_λ . Assume indeed that $\lambda \mapsto J_\lambda$ is not *l.s.c* at λ_0 . Then we may find $\epsilon > 0$ and sequences $\lambda_n \in M$, $z_n \in J_{\lambda_0}$ such that $d(z_n, J_{\lambda_n}) \geq \epsilon$. After taking a subsequence we may assume that $z_n \rightarrow z_0 \in J_{\lambda_0}$ and $B(z_0, \frac{\epsilon}{4}) \subset B(z_n, \frac{\epsilon}{2}) \subset B(z_0, \epsilon)$. If ϵ is small enough,

the projection $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ admits a section σ on $B(z_0, 2\epsilon)$ and the functions $u_\lambda(z) := G(\lambda, \sigma(z))$ are local potentials for the equilibrium measures which means that the restriction of μ_λ to $B(z_0, 2\epsilon)$ is the Monge-Ampère mass $(dd_z^c u_\lambda(z))^k$. Observe that, by the continuity of G , the potentials u_{λ_n} converges locally uniformly to u_{λ_0} . This implies that $\liminf_n \mu_{\lambda_n}(B(z_0, \frac{\epsilon}{4})) \geq \mu_{\lambda_0}(B(z_0, \frac{\epsilon}{8}))$. It is now easy to obtain the expected contradiction : $0 < \mu_{\lambda_0}(B(z_0, \frac{\epsilon}{8})) \leq \liminf_n \mu_{\lambda_n}(B(z_0, \frac{\epsilon}{4})) \leq \liminf_n \mu_{\lambda_n}(B(z_n, \frac{\epsilon}{2})) = 0$. \square

Remark 5.2 *Arguing like in the above Proposition one may show that for a holomorphic family of degree d non-degenerate homogeneous maps $F : M \times \mathbb{C}^{k+1} \rightarrow M \times \mathbb{C}^{k+1}$ with equilibrium current \mathcal{E} , one has $J_{F_\lambda} \subset (\text{supp } \mathcal{E})_\lambda$ and the maps $\lambda \mapsto J_{F_\lambda}$ and $\lambda \mapsto (\text{supp } \mathcal{E})_\lambda$ are respectively l.s.c and u.s.c. Here J_{F_λ} is the support of the equilibrium measure of F_λ .*

We now prove that the existence of a holomorphic motion of equilibrium measures implies that the Julia sets depend continuously on the parameter.

Proposition 5.3 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If the equilibrium measure μ_λ move holomorphically then the map $\lambda \mapsto J_\lambda$ from M to $\text{Comp}^*(\mathbb{P}^k)$ is continuous.*

PROOF: According to Proposition 5.1, it suffices to show that $(\text{supp } W_{\mathcal{M}})_\lambda \subset J_\lambda$. This follows from the following lemma. \square

Lemma 5.4 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that μ_λ move holomorphically and let \mathcal{M} be a structural web. If $z_0 \notin J_{\lambda_0}$ then there exist $\epsilon > 0$ and $r_0 > 0$ such that*

$$\mathcal{M}\{\gamma \in \mathcal{J} / \Gamma_\gamma \cap [B(\lambda_0, \epsilon) \times B(z_0, r_0)] \neq \emptyset\} = 0.$$

Moreover $\mu_\lambda(B(z_0, r_0)) = 0$ for every $\lambda \in B(\lambda_0, \epsilon)$.

PROOF: Pick $r_0 > 0$ such that $\mu_{\lambda_0}(B(z_0, 2r_0)) = 0$. As $\text{supp } \mathcal{M}$ is a normal family, there exists $\epsilon > 0$ such that for any γ in $\text{supp } \mathcal{M}$:

$$\Gamma_\gamma \cap [B(\lambda_0, \epsilon) \times B(z_0, r_0)] \neq \emptyset \Rightarrow \gamma(\lambda) \in B(z_0, 2r_0) \text{ for any } \lambda \in B(\lambda_0, \epsilon).$$

Let $\alpha := \mathcal{M}\{\gamma \in \mathcal{J} / \Gamma_\gamma \cap [B(\lambda_0, \epsilon) \times B(z_0, r_0)] \neq \emptyset\}$. Then, for any $\lambda \in B(\lambda_0, \epsilon)$, we have

$$\alpha \leq \mathcal{M}\{\gamma \in \mathcal{J} / \gamma(\lambda) \in B(z_0, 2r_0)\} = \mu_\lambda(B(z_0, 2r_0)).$$

Applying this to λ_0 yields $\alpha = 0$ as desired. For every $\lambda \in B(\lambda_0, \epsilon)$ we have $\mu_\lambda(B(z_0, r_0)) = \mathcal{M}\{\gamma \in \mathcal{J} / \gamma(\lambda) \in B(z_0, r_0)\} \leq \alpha = 0$. This completes the proof of the lemma. \square

5.2 Siegel discs, discontinuities of J_λ and bifurcations

We define a notion of Siegel disc for endomorphisms of \mathbb{P}^k and investigate how they behave with respect to Julia sets. In this subsection, we endow \mathbb{C}^k with the norm $\|z\| := \sup_i |z_i|$ and set $1 \leq q \leq k-1$. We write $z =: (z', z'')$ where $z' := (z_1, \dots, z_{k-q}) \in \mathbb{C}^{k-q}$ and $z'' := (z_{k-q+1}, \dots, z_k) \in \mathbb{C}^q$. We also set $k' := k-q$, $e^{i\theta_0} := (e^{i\theta_{0,k'+1}}, \dots, e^{i\theta_{0,k}})$ and $e^{i\theta_0} \cdot z'' := (e^{i\theta_{0,k'+1}} z_{k'+1}, \dots, e^{i\theta_{0,k}} z_k)$.

Definition 5.5 *Let f_0 be a holomorphic endomorphism of \mathbb{P}^k . One says that $z_0 \in \mathbb{P}^k$ is a Siegel fixed point for f_0 if f_0 is linearizable at z_0 and its differential at z_0 is of the form $(A_0 z', e^{i\theta_0} \cdot z'')$ where A_0 is an expanding linear map on $\mathbb{C}^{k'}$ and $\pi, \theta_{0,k'+1}, \dots, \theta_{0,k}$ are linearly independent over \mathbb{Q} . In other words, there exists a local holomorphic chart $\psi_0 : B_R \rightarrow \mathbb{P}^k$ such that $\psi_0(0) = z_0$ and*

$$\psi_0^{-1} \circ f_0 \circ \psi_0 = (A_0 z', e^{i\theta_0} \cdot z'')$$

where θ_0 and A_0 are as above. Any set of the form $\psi_0(\{0\} \times B_\rho)$ where $\rho < R$ and B_ρ is a ball centered at the origin in \mathbb{C}^q is called a local Siegel q -disc of f_0 centered at z_0 .

Let us consider a holomorphic family f of endomorphisms of \mathbb{P}^k . If f_0 admits a Siegel fixed point z_0 then, by the implicit function theorem, there exists a unique holomorphic map $z(\lambda)$ defined on some neighbourhood of 0 in M such that $z(0) = z_0$ and $z(\lambda)$ is fixed by f_λ . Moreover, there exists holomorphic functions $w_j(\lambda)$ such that $w_j(0) = e^{i\theta_{0,j}}$ and $w_j(\lambda)$ is an eigenvalue of $d_{z(\lambda)} f_\lambda$ for $k'+1 \leq j \leq k$. In this context, we coin the following definition.

Definition 5.6 *The Siegel fixed point z_0 is called virtually repelling if there exists a holomorphic disc $\sigma : \Delta_{\epsilon_0} \rightarrow M$ and positive constants c_j such that $\sigma(0) = 0$ and $|w_j \circ \sigma(t)| = 1 + c_j t$ for $k'+1 \leq j \leq k$ and $-t_0 < t < t_0$. If, moreover, $z \circ \sigma(t) \in J_{\sigma(t)}$ for $-t_0 < t < t_0$ the Siegel fixed point z_0 is called virtually J -repelling*

Let us observe that if J_λ is continuous at λ_0 and if f_{λ_0} has a virtually repelling Siegel periodic point outside J_{λ_0} , then λ_0 must be accumulated by parameters λ for which f_λ has periodic repelling points outside J_λ . Examples of such repelling points have been given by Hubbard-Papadopol [HP, section 6, example 2] and Fornaess-Sibony [FS2, section 4.1]. The following proposition studies the case when the Siegel periodic point belongs to J_{λ_0} .

Proposition 5.7 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k such that f_{λ_0} admits a virtually repelling Siegel fixed point z_0 .*

- 1) *If μ_λ move holomorphically then every local Siegel q -disc centered at z_0 is contained in $\mathbb{P}^k \setminus J_{\lambda_0}$.*
- 2) *When $q = 1$, if $z_0 \in J_{\lambda_0}$ and if $\lambda \mapsto J_\lambda$ is continuous at λ_0 then any local Siegel q -disc centered at z_0 is contained in J_{λ_0} .*

The first item of the preceding proposition immediately yields:

Corollary 5.8 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If f_{λ_0} has a virtually repelling Siegel periodic point in J_{λ_0} then μ_λ does not move holomorphically near λ_0 .*

The proof of Proposition 5.7 relies on the following technical lemma.

Lemma 5.9 *Let $g : \Delta_{r_0} \times B_R \rightarrow \Delta_{r_0} \times B_{R'}$ be a holomorphic map such that $g(\lambda, z) = (\lambda, g_\lambda(z))$, $g_\lambda(0) = 0$ and $g_0(z) = (A_0^{-1} \cdot z', e^{-i\theta_0} \cdot z'')$ where A_0 is an expanding linear map on $\mathbb{C}^{k'}$. Assume that $\frac{\partial g_{\lambda,j}}{\partial z_i}(0) = 0$ for $k' + 1 \leq j \leq k$ and $i \neq j$. Assume moreover that there exists $|u_0| = 1$, $t_0 > 0$ and $c_j > 0$ such that $|\frac{\partial g_{tu_0,j}}{\partial z_j}(0)| = 1 + c_j t$ for $k' + 1 \leq j \leq k$ and $-t_0 < t < t_0$. Then, after taking R smaller, the following properties occur.*

- 1) *There exists arbitrarily small λ such that $\|g_\lambda(z)\| \leq \alpha_0 \|z\|$ on B_R with $0 < \alpha_0 < 1$.*
- 2) *For any $0 < \rho < R_1 < R_2 < R$, there exists arbitrarily small λ such that, for every $a \in B_{R_1}$ which does not belong to the local stable manifold S_λ of g_λ , there exists n_0 such that $g_\lambda^{n_0}(a) \in \{\|z'\| < \rho\} \times \{R_1 < \|z''\| < R_2\}$ and $g_\lambda^k(a) \in B_{R_1}$ for $0 \leq k \leq n_0 - 1$.*

PROOF: We may write $g_\lambda := (g_{\lambda,j})_{1 \leq j \leq k}$ on the form

$$\begin{aligned} g_{\lambda,j} &= \sum_{i=1}^{k'} (a_{ij} + \lambda \mu_{ij}(\lambda) + \lambda q_{ij}(\lambda, z)) z_i + \lambda \sum_{i=k'+1}^k s_{ij}(\lambda, z) z_i \quad \text{for } 1 \leq j \leq k' \\ g_{\lambda,j} &= \left(e^{i\theta_j} + \lambda \mu_{jj}(\lambda) + \lambda q_{jj}(\lambda, z) \right) z_j + \lambda \sum_{i \neq j} s_{ij}(\lambda, z) z_i \quad \text{for } k' + 1 \leq j \leq k \end{aligned}$$

where μ_{ij} , q_{ij} and s_{ij} are holomorphic on $\Delta_{\epsilon_0} \times B_R$ and satisfy $q_{ij}(\lambda, 0) = q_{jj}(\lambda, 0) = 0$. By assumption, we also have $s_{ij}(\lambda, 0) = 0$ for $k' + 1 \leq j \leq k$ and $i \neq j$. By shrinking ϵ_0 and R , there exists $0 < \alpha_1 < 1$ such that

$$\sup_{1 \leq j \leq k'} |g_{\lambda,j}(z)| \leq \alpha_1 \|z\| \quad \text{on } \Delta_{\epsilon_0} \times B_R. \quad (12)$$

Let us set $\lambda_t := tu_0$ where $-t_0 < t < t_0$ and $Q_{jt}(z) := e^{i\theta_j} + \lambda_t \mu_{jj}(\lambda_t) + \lambda_t q_{jj}(\lambda_t, z)$ and $R_{jt}(z) := |\lambda_t| \sum_{i \neq j} |s_{ij}(\lambda_t, z)|$ for $k' + 1 \leq j \leq k$. Then, by our assumptions and after taking R smaller, we have

$$|Q_{jt}(z)| \leq 1 + \frac{c_j t}{2} \quad \text{for } -t_0 < t < 0 \text{ and } z \in B_R \quad (13)$$

$$R_{jt}(z) \leq \frac{c_j |t|}{4} \quad \text{for } -t_0 < t < t_0 \text{ and } z \in B_R \quad (14)$$

$$1 + \frac{c_j t}{2} \leq |Q_{jt}(z)| \leq 1 + 2c_j t \quad \text{for } 0 < t < t_0 \text{ and } z \in B_R. \quad (15)$$

It follows from (13) and (14) that $|g_{\lambda_t,j}(z)| \leq (1 + \frac{tc_j}{4}) \|z\|$ for $k' + 1 \leq j \leq k$, $-t_0 < t < 0$ and $z \in B_R$. This and (12) yields the first assertion of the lemma.

Let us now establish the second one. Fix $0 < t < t_0$ so small that $(1 + \frac{9tc_j}{4})R_1 < R_2$ for $k' + 1 \leq j \leq k$. Let $a \in B_{R_1}$ be outside the local stable manifold of g_{λ_t} . Assume that one cannot find n_0 such that $g_{\lambda_t}^k(a) \in B_{R_1}$ for $0 \leq k \leq n_0 - 1$ and $g_{\lambda_t}^{n_0}(a) \in \{\|z'\| < \rho\} \times \{\|z''\| > R_1\}$. Then, according to (12), the sequence $a_n := g_{\lambda_t}^n(a)$ is well defined and $\|a'_n\| \rightarrow 0$. From (14) and (15) one gets $|a_{n+1,j}| \geq (1 + \frac{c_j t}{2})|a_{n,j}| - \frac{tc_j}{4}\|a'_n\|$. As $(a_{n,j})_n$ is bounded and $\|a'_n\| \rightarrow 0$, this implies that a_n tends to the origin and contradicts the fact that a does not belong to the local stable manifold of g_{λ_t} . Thus n_0 exists and it remains to check that $\|a''_{n_0}\| < R_2$. From (14) and (15) one gets $|a_{n_0,j}| \leq (1 + 2c_j t)|a_{n_0-1,j}| + \frac{tc_j}{4}\|a'_{n_0-1}\| \leq (1 + \frac{9tc_j}{4})R_1 < R_2$. \square

PROOF OF PROPOSITION 5.7: We may assume that $M = \Delta_{\epsilon_0}$ and $\lambda_0 = 0$ so that z_0 is a virtually repelling Siegel fixed point of f_0 . Thus there exists a biholomorphism $\psi_0 : B_R \rightarrow \psi_0(B_R)$ such that $\psi_0(0) = z_0$ and $\psi_0^{-1} \circ f_0 \circ \psi_0 = (A_0 \cdot z', e^{i\theta_0} \cdot z'')$ where A_0 is linear and expanding on $\mathbb{C}^{k'}$ and $\pi, \theta_{0,k'+1}, \dots, \theta_{0,k}$ are linearly independant over \mathbb{Q} . The mapping $\psi_0^{-1} \circ f_\lambda^{-1} \circ \psi_0$ is well defined on $\Delta_{\epsilon_0} \times B_R$ after taking R and ϵ_0 smaller. Since the $e^{i\theta_{0,j}}$ are pairwise distinct for $k' + 1 \leq j \leq k$, we may find q linearly independant vectors $v_{k'+1}(\lambda), \dots, v_k(\lambda)$ in \mathbb{C}^k and q scalars $w_{k'+1}(\lambda), \dots, w_k(\lambda)$ which depend holomorphically on $\lambda \in \Delta_{\epsilon_0}$ and such that

$$d_{\psi_0^{-1}(z(\lambda))}(\psi_0^{-1} \circ f_\lambda^{-1} \circ \psi_0)(v_j(\lambda)) = w_j(\lambda)v_j(\lambda) \text{ for } k' + 1 \leq j \leq k. \quad (16)$$

Using basis like $(v_1, \dots, v_{k'}, v_{k'+1}(\lambda), \dots, v_k(\lambda))$ we may perform change of coordinates of the form $(\lambda, A(\lambda, z))$ where $A(\lambda, \cdot)$ is affine on \mathbb{C}^k which, conjugate by ψ_0 , yield biholomorphisms $\psi_\lambda : B_R \rightarrow \psi_\lambda(B_R)$ such that $g_\lambda := \psi_\lambda^{-1} \circ f_\lambda^{-1} \circ \psi_\lambda$ satisfies the assumptions of Lemma 5.9. The condition $\frac{\partial g_{\lambda,j}}{\partial z_i}(0) = 0$ indeed follows from (16) and the condition $|\frac{\partial g_{t_0,j}}{\partial z_j}(0)| = 1 + c_j t$ follows from the fact that z_0 is virtually repelling. To simplify, we shall denote J_λ the set $\psi_\lambda^{-1}(J_\lambda \cap \psi_\lambda(B_R))$.

1) We proceed by contradiction and assume that $(0', z''_0) \in J_0$ for $0 < \|z''_0\| < r < R$. According to Lemma 2.2, there exists a holomorphic map $\gamma : \Delta_{\epsilon_0} \rightarrow \mathbb{P}^k$ such that $\psi_0^{-1} \circ \gamma(0) = (0', z''_0)$ and $(\mathcal{F}^n \cdot \gamma)_n$ is normal on Δ_{ϵ_0} (we recall that $\mathcal{F}^n \cdot \gamma(\lambda) = f_\lambda^n(\gamma(\lambda))$). We may assume that $\tilde{\gamma}(\lambda) := \psi_\lambda^{-1}(\gamma(\lambda))$ is well defined on Δ_{ϵ_0} . Since $\psi_0^{-1} \circ f_0^n \circ \psi_0(\tilde{\gamma}(0)) = (0', e^{in\theta_0} \cdot z''_0)$ and $(\mathcal{F}^n \cdot \gamma)_n$ is normal, after reducing ϵ_0 , we may suppose that

$$\|\psi_\lambda^{-1} \circ f_\lambda^n \circ \psi_\lambda(\tilde{\gamma}(\lambda))\| \leq r \text{ on } \Delta_{\epsilon_0} \text{ for } n \geq 1. \quad (17)$$

Let us recall that $g_\lambda = \psi_\lambda^{-1} \circ f_\lambda^{-1} \circ \psi_\lambda$. By Lemma 5.9, there exists $\lambda_k \rightarrow 0$ and $0 < \alpha_k < 1$ such that $\|g_{\lambda_k}(z)\| \leq \alpha_k \|z\|$ on B_R . We may thus find a sequence $n_k \rightarrow \infty$ such that

$$\|g_{\lambda_k}^{n_k}(z)\| \leq \frac{1}{k} \|z\| \text{ on } B_r. \quad (18)$$

From (17) and (18) one gets

$$\|\tilde{\gamma}(\lambda_k)\| = \|g_{\lambda_k}^{n_k} \circ \psi_{\lambda_k}^{-1} \circ f_{\lambda_k}^{n_k} \circ \psi_{\lambda_k}(\tilde{\gamma}(\lambda_k))\| \leq \frac{r}{k} \quad (19)$$

which is impossible since $\lim_k \|\tilde{\gamma}(\lambda_k)\| = \|z''_0\| > 0$.

So far we have shown that the punctured q -disc $\{0'\} \times \{0 < \|z''\| < R\}$ is contained in J_0^c . Since J_0 is totally invariant and $g_0 = (A_0^{-1} \cdot z', e^{-i\theta_0} \cdot z'')$ where A_0 is linear and expanding, this implies that $B_R \setminus \{z \in B_R / z'' = 0\} \subset J_0^c$. Finally, as μ_0 does not give mass to analytic sets, we get $B_R \subset J_0^c$.

2) We have to show that $(0', z_{0k}) \in J_0$ if $0 < |z_{0k}| < R$. Assume, to the contrary, that $(0', z_{0k}) \notin J_0$ for some $0 < |z_{0k}| < R$. Then one may pick a neighbourhood V_0 of $(0', z_{0k})$ such that $V_0 \subset (J_0)^c$ and which is of the form

$$V_0 := \{\|z'\| < \rho\} \times \{R_1 < |z_k| < R_2 \text{ and } |\arg z_k - \arg z_{0k}| < \eta\}.$$

Let us now denote by T_{ρ, R_1, R_2} the tube

$$T_{\rho, R_1, R_2} := \{\|z'\| < \rho\} \times \{R_1 < |z_k| < R_2\}.$$

Since A_0 is contracting and θ_0/π irrational, for any $z \in T_{\rho, R_1, R_2}$ there exists an integer n such that $g_0^n(z) \in V_0$. By the invariance of Julia sets we thus have $T_{\rho, R_1, R_2} \subset (J_0)^c$. Let us shrink the tube T_{ρ, R_1, R_2} . By assumption, J_λ is *u.s.c* at 0 and therefore

$$T_{\rho, R_1, R_2} \subset (J_\lambda)^c \text{ when } \lambda \text{ is close enough to } 0.$$

On the other hand, according to the second assertion of Lemma 5.9, we may find parameters λ which are arbitrarily close to 0 and such that $B_{R_1} \setminus S_\lambda \subset \cup_n (g_\lambda^n)^{-1} T_{\rho, R_1, R_2}$ where S_λ denotes the stable manifold of g_λ . As μ_λ gives no mass to analytic sets, this and the inclusion $T_{\rho, R_1, R_2} \subset (J_\lambda)^c$ implies the existence of a sequence of parameters $\lambda_k \rightarrow 0$ such that $B_{R_1} \subset (J_{\lambda_k})^c$. This contradicts the lower semi-continuity of J_λ at 0 since $0 \notin (J_{\lambda_k})^c$ but $0 \in J_0$ by our assumption. \square

6 Proofs of the main theorems and further results

6.1 Critical growth and the support of $dd_\lambda^c L$

We obtain the following proposition in the spirit of the proposition 1.26 of [DS3] concerning the Julia set of a single endomorphism of \mathbb{P}^k .

Proposition 6.1 *Let B be an open ball in \mathbb{C}^m and let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree d . We endow $B \times \mathbb{P}^k$ with the metric $dd_\lambda^c |\lambda|^2 + \omega_{FS}$ and denote $|\cdot|_U$ the mass of currents in $U \times \mathbb{P}^k$. The following properties are equivalent.*

1. $\lambda_0 \in \text{supp } dd_\lambda^c L$.
2. $|\mathcal{E}_{Green} \wedge \mathbf{C}_f|_U > 0$ for every neighborhood U of λ_0 .
3. $\liminf_n d^{-kn} |(f^n)_* \mathbf{C}_f|_U > 0$ for every neighborhood U of λ_0 .
4. $\limsup_n d^{-(k-1)n} |(f^n)_* \mathbf{C}_f|_U = +\infty$ for every neighborhood U of λ_0 .

PROOF: Items 1. and 2. are equivalent by theorem 2.10, which asserts that $dd_\lambda^c L = \pi_{B*} (\mathcal{E}_{Green} \wedge \mathbf{C}_f)$. The equivalences between 2. 3. and 4. come from lemma 6.2. \square

Lemma 6.2 *There exists $\alpha = \alpha(k, m) > 0$ such that for every compact subset $U \subset M$:*

$$|(f^n)_* \mathbf{C}_f|_U = \alpha d^{kn} |\mathcal{E}_{Green} \wedge \mathbf{C}_f|_U + O(d^{(k-1)n}).$$

PROOF: Let us set $\kappa := k + m - 1$. Then

$$|(f^n)_* \mathbf{C}_f|_U = \int_{U \times \mathbb{P}^k} (f^n)_* \mathbf{C}_f \wedge [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \int_{U \times \mathbb{P}^k} \mathbf{C}_f \wedge (f^n)^* [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa.$$

Using $\omega_{FS}^{k+1} = 0$, we obtain $[\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \sum_{j=0}^k \alpha_j \omega_{FS}^j \wedge (dd_\lambda^c |\lambda|^2)^{\kappa-j}$, where the α_j 's are positive numbers. Since $\pi_M \circ f = \pi_M$, we obtain

$$(f^n)^* [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \sum_{j=0}^k \alpha_j \left((f^n)^* \omega_{FS}^j \right) \wedge (dd_\lambda^c |\lambda|^2)^{\kappa-j}.$$

Let $\mathcal{T} := dd_{\lambda,z}^c g + \omega_{FS}$ so that $\mathcal{T}^k = \mathcal{E}_{Green}$. Using $f^* \mathcal{T} = d\mathcal{T}$ we get $(f^n)^* (\omega_{FS}^j) = (d^n \mathcal{T} - dd_{\lambda,z}^c g \circ f^n)^j$. Now, using the fact that g is bounded, we obtain by extracting the k -th term of the preceding sum:

$$(f^n)^* [\omega_{FS} + dd_\lambda^c |\lambda|^2]^\kappa = \alpha_k d^{kn} \mathcal{T}^k \wedge (dd_\lambda^c |\lambda|^2)^{m-1} + O(d^{(k-1)n}).$$

We set $\alpha := \alpha_k$. That completes the proof of the lemma. \square

6.2 Proofs of the main results

Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k .

PROOF OF THEOREM 1.2: (1) If the repelling J -cycles of f move holomorphically then, using the second assertion of Proposition 2.4, one gets a structural web \mathcal{M} of $(\mu_\lambda)_{\lambda \in M}$ such that $\mathcal{M} = \lim_n \mathcal{M}_n$ and $\Gamma_\gamma \cap C_f = \emptyset$ for any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. By Proposition 3.1, this implies that $dd_\lambda^c L \equiv 0$ on M . (2) This assertion is given by Proposition 6.1. \square

PROOF OF THEOREM 1.4: By Proposition 3.3 there are no Misiurewicz parameters in M if $dd_\lambda^c L \equiv 0$ on M and thus (A) \Rightarrow (B). If there are no Misiurewicz parameters in M then, by Proposition 4.1, any parameter admits an open neighbourhood on which the equilibrium measures μ_λ move holomorphically and admits a structural web $\mathcal{M} = \lim_n \mathcal{M}_n$ such that $\Gamma_\gamma \cap C_f = \emptyset$ for any $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. This implies (B) \Rightarrow (C). Finally, (C) \Rightarrow (A) follows from Proposition 3.1. \square

In order to obtain Theorem 1.5, it remains to investigate if the repelling J -cycles of f move holomorphically when the equilibrium measures μ_λ move holomorphically. To this purpose we shall use the result of Section 5 and show how a Siegel disc may appear when a repelling J -cycle fails to move holomorphically.

Proposition 6.3 *Let M be a connected complex manifold and let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family. If the equilibrium measures μ_λ move holomorphically then all repelling J -cycles of f which are neither persistently resonant nor persistently undiagonalizable move holomorphically. When $k = 2$, all repelling J -cycles of f move holomorphically.*

Let us recall that a periodic point is said to be *resonant* if its multipliers w_1, \dots, w_k satisfy a relation of the form $w_1^{m_1} \dots w_k^{m_k} - w_j = 0$ where the m_j are integers and $m_1 + \dots + m_k \geq 2$. Note that when $w_j = e^{i\theta_j}$ for $1 \leq j \leq n$ and $n \leq k$ then the absence of resonances forces $\pi, \theta_1, \dots, \theta_n$ to be linearly independant over \mathbb{Q} .

We shall use the following Lemma.

Lemma 6.4 *Let $w_1, \dots, w_k : D(0, R) \rightarrow \mathbb{C}$ be holomorphic functions. Assume that $w_j(0) \neq 0$ and that there exists $\lambda_k \rightarrow 0$ such that $\min_{1 \leq j \leq k} |w_j(\lambda_k)| > 1$. Assume moreover that there exists $1 \leq N \leq k$ such that*

- $|w_j(0)| = 1$ and $w_j'(0) \neq 0$ for $1 \leq j \leq N$,
- $|w_j(0)| \neq 1$ for $N + 1 \leq j \leq k$.

Then, after renumbering, there exist a disc $D(\lambda_0, r) \subset D(0, R)$, a real analytic arc C through λ_0 and an integer $1 \leq q \leq k$ such that

1. $D(\lambda_0, r) = D^+(\lambda_0, r) \cup C \cup D^-(\lambda_0, r)$,
2. $|w_j| > 1$ on $D^+(\lambda_0, r)$, $|w_j| = 1$ on C and $|w_j| < 1$ on $D^-(\lambda_0, r)$ for $k - q + 1 \leq j \leq k$,
3. $|w_j| > 1$ on $D(\lambda_0, r)$ for $1 \leq j \leq k - q$ if $q \leq k - 1$.

PROOF: In the sequel we allow to shrink R without specifying it. Let us set $C_j := \{|w_j| = 1\}$ and $U_j^+ := \{|w_j| > 1\}$, $U_j^- := \{|w_j| < 1\}$. Since $w_j'(0) \neq 0$ when $\{|w_j| = 1\} \neq \emptyset$ the subset C_j is either empty or a real-analytic arc through 0 in $D(0, R)$. In particular we have

$$C_j = C_l \quad \text{if } C_j \cap C_l \text{ is strictly bigger than } \{0\}.$$

Let us set $U^+ := \bigcap_{j=1}^k U_j^+$. By assumption, $0 \in \overline{U^+}$ and therefore U^+ is a non-empty open subset of $D(0, R)$. It is clear that $\partial U^+ \subset \partial D(0, R) \cup \left(\bigcup_{j=1}^k C_j \right)$. On the other hand, we can not have $\partial U^+ \subset \{0\} \cup \partial D(0, R)$ since otherwise $U^+ = D(0, R) \setminus \{0\}$ and the subharmonic function $\psi(\lambda) := \max_{1 \leq j \leq k} |w_j(\lambda)|^{-1}$ would violate the maximum principle (recall that $\psi(0) \geq 1$). We may thus pick $\lambda_0 \neq 0$ such that $\lambda_0 \in C_{j_0} \cap \partial U^+$ for some $1 \leq j_0 \leq k$. Observe that $\lambda_0 \notin U_i^-$ for $1 \leq i \leq k$.

If $C_i \neq C_{j_0}$ for some $1 \leq i \leq k$ then $\lambda_0 \notin C_i$ and thus $\lambda_0 \in U_i^+$. After renumbering we may therefore find $1 \leq q \leq k - 1$ such that

$$\lambda_0 \in C_{k-q+1} = C_{k-q+2} = \dots = C_k := C \text{ and } \lambda_0 \in U_1^+ \cap \dots \cap U_{k-q}^+.$$

For $r > 0$ sufficiently small we have $D(\lambda_0, r) \subset \bigcap_1^{k-q} U_i^+$ and $D(\lambda_0, r) \setminus C$ has two connected components Ω_1 and Ω_2 . For each $k - q + 1 \leq i \leq k$, one has $\Omega_1 \subset U_i^+$ and $\Omega_2 \subset U_i^-$ or $\Omega_1 \subset U_i^-$ and $\Omega_2 \subset U_i^+$. Assume for instance that $\Omega_1 \subset U_{k-q+1}^+$. Then, since $\lambda_0 \in \partial U^+$, we must have $\Omega_1 \subset U_i^+$ and $\Omega_2 \subset U_i^-$ for every $k - q + 1 \leq i \leq k$ and we may set $D(\lambda_0, r)^+ := \Omega_1$ and $D(\lambda_0, r)^- := \Omega_2$. \square

PROOF OF PROPOSITION 6.3: Let $\lambda_0 \in M$. Assume that z_0 belongs to some p -periodic repelling J -cycle of f_{λ_0} which is not persistently resonant and not persistently undiagonalizable. It suffices to show that the map $\gamma : M \rightarrow \mathbb{P}^k$ element of \mathcal{J} given by Lemma 2.2

enjoys the property that $\gamma(\lambda) \in J_\lambda$ is repelling for every $\lambda \in M$. Let us observe that $\gamma(\lambda)$ is not persistently resonant and not persistently undiagonalizable for any $\lambda \in M$.

Since M is connected, we have to show that the subset $\{\lambda \in M / \gamma(\lambda) \text{ is repelling}\}$ is closed in M . Assume, to the contrary, that this is not true. Then, for arbitrarily small ϵ_0 , one finds a new holomorphic map $\gamma_0 : B_{\epsilon_0} \rightarrow \mathbb{P}^k$ such that $\gamma_0(\lambda) \in J_\lambda$ is fixed by f_λ^p for all $\lambda \in B_{\epsilon_0}$ and $\gamma_0(0)$ is not repelling but $\gamma_0(\lambda_0)$ is repelling for some $\lambda_0 \in B_{\epsilon_0}$. Our aim below is to find $\lambda'_0 \in B_{\epsilon_0}$ such that $\gamma(\lambda'_0)$ is a virtually repelling Siegel fixed point of $f_{\lambda'_0}^p$. Corollary 5.8 then yields a contradiction.

Reducing ϵ_0 allows to use charts and replace \mathbb{P}^k by \mathbb{C}^k . Let us denote $w_1(\lambda), \dots, w_k(\lambda)$ the eigenvalues of $A(\lambda) := (f_\lambda^p)'(\gamma(\lambda))$. There exists a proper analytic subset Z of B_{ϵ_0} such that w_1, \dots, w_k are holomorphic on $B_{\epsilon_0} \setminus Z$. For every $n \in \mathbb{N}$ we define a function ω_n on $B_{\epsilon_0} \setminus Z$:

$$\omega_n(\lambda) := \min_{2 \leq |m| \leq n, 1 \leq j \leq k} |w_1(\lambda)^{m_1} \cdots w_k(\lambda)^{m_k} - w_j(\lambda)|$$

where $|m| := m_1 + \cdots + m_k$ for any $m := (m_1, \dots, m_k) \in \mathbb{N}^k$. Since the cycle $\gamma_0(\lambda)$ is not persistently resonant the functions $\ln \omega_n$ are not identically equal to $-\infty$. Moreover, after shrinking ϵ_0 , we have $\ln \omega_n(\lambda) \leq \ln \omega_2(\lambda) \leq C < +\infty$ on $B_{\epsilon_0} \setminus Z$ and therefore $\ln \omega_n$ extends to some *p.s.h* function on B_{ϵ_0} . We now define a function \mathcal{B} on B_{ϵ_0} by setting

$$\mathcal{B}(\lambda) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \ln \omega_{2^{n+1}}(\lambda).$$

The interest of this function is that, according to Brjuno's theorem (see [Br]), f_λ^p is holomorphically linearizable at $\gamma(\lambda)$ if $\mathcal{B}(\lambda) > -\infty$ and $A(\lambda)$ is diagonalizable. Let us show that \mathcal{B} is *p.s.h* on B_{ϵ_0} . Since $\mathcal{B}(\lambda) - 2C = \sum_{n=0}^{+\infty} \frac{1}{2^n} (\ln \omega_{2^{n+1}}(\lambda) - C)$ is a decreasing limit of *p.s.h* functions, the function \mathcal{B} is either *p.s.h* or identically equal to $-\infty$ on B_{ϵ_0} . Moreover, as $\gamma(\lambda_0)$ is a repelling cycle there exists $n_0 \geq 1$ such that $\ln \omega_{2^n} = \ln \omega_{2^{n_0}}$ on a neighbourhood V_0 of λ_0 for $n \geq n_0$. We deduce that $\mathcal{B} = \sum_{n=0}^{n_0} \frac{1}{2^n} \ln \omega_{2^{n+1}}(\lambda) + \frac{1}{2^{n_0}} \ln \omega_{2^{n_0+1}}$ on V_0 , this function is therefore not identically equal to $-\infty$ since $\gamma_0(\lambda)$ is not persistently resonant.

Let us denote by Δ_{ϵ_0} the disc in \mathbb{C} obtained by intersecting B_{ϵ_0} with the complex line through 0 and λ_0 . We may move a little bit λ_0 so that \mathcal{B} is subharmonic on Δ_{ϵ_0} , the set $Z \cap \Delta_{\epsilon_0}$ is discrete and $\gamma_0(\lambda)$ is not persistently undiagonalizable on Δ_{ϵ_0} . In particular, there exists a discrete subset Z_0 of Δ_{ϵ_0} such that on $\Delta_{\epsilon_0} \setminus Z_0$, the cycle $\gamma_0(\lambda)$ is diagonalizable and the functions w_1, \dots, w_k are either constant or holomorphic, non-vanishing and with non-vanishing derivatives.

Let us set

$$\forall \lambda \in \Delta_{\epsilon_0} \setminus Z_0, \quad \varphi(\lambda) := \min(|w_1(\lambda)|, \dots, |w_k(\lambda)|).$$

This extends to a continuous function on Δ_{ϵ_0} . Moreover $\varphi(0) \leq 1$ and $\varphi(\lambda_0) > 1$, in particular φ is not constant. We claim that there exists $\lambda_1 \in \Delta_{\epsilon_0} \setminus Z_0$ such that $\varphi(\lambda_1) < 1$. Indeed, if $\varphi \geq 1$ on $\Delta_{\epsilon_0} \setminus Z_0$, then $\varphi \geq 1$ on Δ_{ϵ_0} and therefore the subharmonic function $\psi := \varphi^{-1}$ violates the maximum principle (indeed $\psi \leq 1 = \psi(0)$ and this function is not constant). Considering a continuous path connecting λ_0 to λ_1 in $\Delta_{\epsilon_0} \setminus Z_0$, one finds $\lambda_2 \in \Delta_{\epsilon_0} \setminus Z_0$ and $\tilde{\lambda}_k \rightarrow \lambda_2$ such that $\varphi(\lambda_2) = 1$ and $\varphi(\tilde{\lambda}_k) > 1$. Let us pick a small

disc $D(\lambda_2, R)$ contained in $\Delta_{\epsilon_0} \setminus Z_0$. Then (after renumbering) the functions w_1, \dots, w_k satisfy the assumptions of Lemma 6.4 on $D(\lambda_2, R)$. Let q be the integer and C be the real analytic arc in $D(\lambda_2, R)$ which are given by this Lemma. Since $|w_j| < 1$ on $D^-(\lambda_2, R)$ for $k - q + 1 \leq j \leq k$ and $\gamma(\lambda) \in J_\lambda$, we must have $1 \leq q \leq k - 1$.

Since \mathcal{B} is subharmonic on Δ_{ϵ_0} , there exists $\lambda'_0 \in C$ such that $\mathcal{B}(\lambda'_0) > -\infty$. Since $\lambda'_0 \in D(\lambda_2, R) \subset \Delta_{\epsilon_0} \setminus Z_0$, the periodic point $\gamma(\lambda'_0)$ is diagonalizable and then, according to Brjuno's theorem, it is holomorphically linearizable. Thus $\gamma(\lambda'_0)$ is a Siegel fixed point of $f_{\lambda'_0}^p$ and, since $\lambda'_0 \in C$, Lemma 6.4 shows that it is virtually repelling as desired. Let us finally explain why we do not need any assumption on the repelling J -cycle in dimension $k = 2$. In that case, the periodic points $\gamma(\lambda)$ for $\lambda \in C$ are diagonalizable and not persistently resonant since one and only one of their two multipliers have modulus 1 and, moreover, is not constant. We thus see that \mathcal{B} is subharmonic on Δ_{ϵ_0} and we can find again some $\lambda'_0 \in C$ such that $\gamma(\lambda'_0)$ is a virtually repelling Siegel fixed point of $f_{\lambda'_0}^p$. \square

To deduce Theorem 1.5 from Proposition 6.3, we shall use the following Lemma whose proof is left to the reader.

Lemma 6.5 *Let $f : B \times \mathbb{P}^k \rightarrow B \times \mathbb{P}^k$ be a holomorphic family where B is an open ball of the space $\mathcal{H}_d(\mathbb{P}^k)$ of degree d holomorphic endomorphisms of \mathbb{P}^k . Then every repelling J -cycle is neither persistently resonant nor persistently undiagonalizable.*

PROOF OF THEOREM 1.5 : Theorem 1.2 yields $(A) \Rightarrow (B)$ and Theorem 1.4 yields $(B) \Rightarrow (C) \Rightarrow (C')$, where (C') is the assertion : "the equilibrium measures (μ_λ) locally move holomorphically". Assume now that (C') is satisfied. When M is an open ball of $\mathcal{H}_d(\mathbb{P}^k)$, the repelling cycles are neither persistently resonant nor persistently undiagonalizable, see lemma 6.5. Proposition 6.3 thus shows that, when M satisfies the assumptions of Theorem 1.5, the repelling J -cycles locally move holomorphically. This implies that

$$\{(\lambda, z) \in M \times \mathbb{P}^k, z \text{ is } n\text{-periodic and } J\text{-repelling for } f_\lambda\}$$

is an unramified cover of M . If M is simply-connected, we thus get that the repelling J -cycles move holomorphically over M , hence $(C') \Rightarrow (C)$. Finally proposition 6.3 yields $(C) \Rightarrow (A)$, completing the proof of theorem 1.5. \square

In view of Theorem 1.5, we may now define the bifurcation locus and current as follows.

Definition 6.6 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Let $L(\lambda)$ be the sum of Lyapunov exponents of f_λ with respect to its equilibrium measure. The closed positive current $dd_\lambda^c L$ is called bifurcation current of the family, its support is the bifurcation locus of the family.*

The following result is a consequence of Theorem 1.5 and the proof of Proposition 6.3.

Theorem 6.7 *A degree $d \geq 2$ endomorphism of \mathbb{P}^k belongs to the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^k)$ if and only if it is accumulated by endomorphisms which admit a virtually J -repelling Siegel periodic point or a repelling cycle outside the Julia set which becomes a repelling J -cycle after an arbitrarily small perturbation.*

It would be interesting to know if the continuity of the map $\lambda \mapsto J_\lambda$ on some open subset of the parameter space of a holomorphic family of endomorphisms of \mathbb{P}^k is equivalent to the existence of a holomorphic motion of the equilibrium measures. This is true when $k = 1$, the following remark summarizes the consequences of our work on this question in higher dimension.

Remark 6.8 *According to Proposition 5.7 and the proof of Proposition 6.3, when $k = 2$ the Hausdorff continuity of $\lambda \mapsto J_\lambda$ would imply the holomorphic stability if we would know that a local Siegel disc centered at some virtually repelling Siegel periodic point cannot be contained in the Julia set.*

As a consequence of our results, we may prove that Lattès maps belongs to the bifurcation locus. We refer to [Di1], [Du1] for an account on Lattès maps of \mathbb{P}^k .

Theorem 6.9 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If the family is stable (i.e. $dd_\lambda^c L = 0$ on M) and f_{λ_0} is a Lattès map for some $\lambda_0 \in M$ then f_λ is a Lattès map for every $\lambda \in M$.*

PROOF: By a Theorem of Briend-Duval [BD1] we have $L \geq k \times \frac{\log d}{2}$. The articles of Berteloot, Dupont and Loeb [BL], [BDu] and [Du2] show that $L(\lambda) = k \times \frac{\log d}{2}$ if and only if f_λ is a Lattès map. If the family is stable, then the function L is pluriharmonic on M . By the maximum principle (applied to the harmonic function $-L$) we thus have $L(\lambda) = L(\lambda_0) = k \times \frac{\log d}{2}$ for all $\lambda \in M$ and the conclusion follows. \square

Let us end with a remark on the intersection between structural webs and critical sets.

Proposition 6.10 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that μ_λ move holomorphically. Then for every $\lambda_0 \in M$ we may find neighbourhood U_0 of λ_0 such that the induced family $f : U_0 \times \mathbb{P}^k \rightarrow U_0 \times \mathbb{P}^k$ admits a structural web \mathcal{M} for which $\mathcal{M}\{\gamma / \Gamma_\gamma \cap PC_f \neq \emptyset\} = 0$.*

PROOF: According to Theorem 1.4 there are no Misiurewicz parameters in M . Thus, using Proposition 4.1, one may find a simply-connected neighbourhood U_0 of λ_0 and a holomorphic graph Γ_{γ_0} over U_0 such that $\Gamma_{\gamma_0} \cap PC_f = \emptyset$ and $\gamma_0(\lambda) \in J_\lambda$ for every $\lambda \in U_0$. Then, using the first assertion of Proposition 2.4, one gets a structural web \mathcal{M} for $f : U_0 \times \mathbb{P}^k \rightarrow U_0 \times \mathbb{P}^k$ such that $\mathcal{M} = \lim_n \mathcal{M}_n$ and $\Gamma_\gamma \cap PC_f = \emptyset$ for all $\gamma \in \cup_n \text{supp } \mathcal{M}_n$. The conclusion then follows from Lemma 3.2. \square

6.3 A remark on the interior of bifurcation loci

We investigate the relations between the presence of open subsets in the support of $dd_\lambda^c L$ and existence of parameters for which the postcritical set is dense in \mathbb{P}^k .

Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Let C denote the critical set of f and let C_λ denote the critical set of f_λ . We set

$$\overline{C^+} := \overline{\cup_{n \geq 1} f^n(C)} \quad \text{and} \quad \overline{C_\lambda^+} := \overline{\cup_{n \geq 1} f_\lambda^n(C_\lambda)} \quad \text{for every } \lambda \in M.$$

We define $(\overline{C^+})_\lambda := (\{\lambda\} \times \mathbb{P}^k) \cap \overline{C^+}$, let us observe that $\{\lambda\} \times \overline{C^+} \subset (\overline{C^+})_\lambda$. Our aim is to prove the following result.

Theorem 6.11 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that $\text{supp } dd_\lambda^c L$ contains an open subset $\Omega \subset M$. Then $\{\lambda \in \Omega, \overline{C^+}_\lambda = \mathbb{P}^k\}$ contains a G_δ -dense subset of Ω .*

As a consequence we recover a fundamental result of Mañé, Sad and Sullivan [MSS] on the density of stable parameters for holomorphic families of rational maps. For such families the bifurcation locus is known to coincide with $\text{supp } dd_\lambda^c L$.

Corollary 6.12 *Let $f : M \times \mathbb{P}^1 \rightarrow M \times \mathbb{P}^1$ be a holomorphic family of rational maps. Then $\text{supp } dd_\lambda^c L$ has empty interior.*

PROOF: Every $\lambda_0 \in \text{supp } dd_\lambda^c L$ can be approximated by parameters λ for which f_λ has an attracting basin, see [Be, section 4.3.1], which is an open condition in M . On the other hand, as the critical set is finite, the set $\overline{C^+}_\lambda$ can not be equal to \mathbb{P}^1 when f_λ has an attracting basin. According to Theorem 6.11, this implies that $\text{supp } dd_\lambda^c L$ has empty interior. \square

Remark 6.13 *In the introduction we raised the question for $k \geq 2$ of the existence of holomorphic families for which $\text{supp } dd_\lambda^c L$ has non empty interior. Note that Theorem 6.11 could be useful for finding families for which $\text{supp } dd_\lambda^c L$ has empty interior.*

The proof of Theorem 6.11 relies on a Baire's category argument based on the semi-continuity properties of $\lambda \mapsto \overline{C^+}_\lambda$ or $\lambda \mapsto (\overline{C^+})_\lambda$. Let us recall that the notion of semi-continuity with respect to the Hausdorff topology has been discussed in subsection 5.1. We have the following properties, the upper semi-continuity can be found in [Do, Proposition 2.1], we give the argument for sake of completeness.

Lemma 6.14 *The maps $\lambda \mapsto (\overline{C^+})_\lambda$ and $\lambda \mapsto \overline{C^+}_\lambda$ from M to $\text{Comp}^*(\mathbb{P}^k)$ are respectively upper and lower-semi-continuous.*

PROOF: Observe that $\{(\lambda, z) \in M \times \mathbb{P}^k, z \in (\overline{C^+})_\lambda\}$ is equal to $\overline{C^+}$, hence is closed in $M \times \mathbb{P}^k$. In particular, for every $\lambda_0 \in M$ and $\epsilon > 0$, the set $F := \{(\lambda, z) \in \overline{C^+}, d(z, (\overline{C^+})_{\lambda_0}) \geq \epsilon\}$ is a closed subset of $\overline{C^+}$. Let us show that $\pi_M(F)$ is closed in M . Indeed, if $\lambda_n \in \pi_M(F)$ converges to $\lambda \in M$ one may pick $z_n \in (\overline{C^+})_{\lambda_n}$ such that $d(z_n, (\overline{C^+})_{\lambda_0}) \geq \epsilon$ and $(z_n)_n$ converges to some $z \in \mathbb{P}^k$ after taking a subsequence. Then $(\lambda_n, z_n) \in \overline{C^+}$ converges to $(\lambda, z) \in \overline{C^+}$ satisfying $d(z, (\overline{C^+})_{\lambda_0}) \geq \epsilon$ and thus $\lambda \in \pi_M(F)$ as desired. Since $\lambda_0 \notin \pi_M(F)$ it follows that $M \setminus \pi_M(F)$ contains an open ball B centered at λ_0 such that $d(z, (\overline{C^+})_{\lambda_0}) < \epsilon$ for every $z \in (\overline{C^+})_\lambda$ with $\lambda \in B$. This proves the upper semi-continuity.

Now let us prove the lower semi-continuity of the map $\lambda \mapsto \overline{C^+}_\lambda$. Assume to the contrary that it is not *l.s.c* at $\lambda_0 \in M$. Then there exist $\epsilon > 0$, a sequence $(\lambda_n)_n$ converging to λ_0 and a sequence $z_n \in \overline{C^+}_{\lambda_n}$ such that $d(z_n, \overline{C^+}_{\lambda_0}) \geq \epsilon$. After taking a subsequence $(z_n)_n$ converges to $z_0 \in \overline{C^+}_{\lambda_0}$. Pick $\xi_0 \in C_{\lambda_0}$ and $p_0 \geq 1$ such that $d(z_0, f_{\lambda_0}^{p_0}(\xi_0)) < \frac{\epsilon}{4}$. Let also

$\xi_n \in C_{\lambda_n}$ such that $\xi_n \rightarrow \xi_0$. Then $d(z_n, \overline{C_{\lambda_n}^+}) \leq d(z_n, f_{\lambda_n}^{p_0}(\xi_n)) < \frac{\epsilon}{2}$ for n large, contradicting $d(z_n, \overline{C_{\lambda_n}^+}) \geq \epsilon$. \square

The proof of theorem 6.11 will also rely on the following key fact.

Proposition 6.15 *Let $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of \mathbb{P}^k . If $\lambda_0 \in \text{supp } dd_{\lambda}^c L$ then $(\overline{C^+})_{\lambda_0} = \mathbb{P}^k$.*

PROOF: Assume that $B(z_0, r) \cap (\overline{C^+})_{\lambda_0} = \emptyset$ and let us show that $\lambda_0 \notin \text{supp } dd_{\lambda}^c L$. Since $\lambda \mapsto (\overline{C^+})_{\lambda}$ is upper-semi-continuous we deduce that $B(z_0, \frac{r}{2}) \cap (\overline{C^+})_{\lambda} = \emptyset$ when λ is sufficiently close to λ_0 . In particular, the constant graph $\Gamma_0 := \{(\lambda, z_0) / \lambda \in B(\lambda_0, \epsilon)\}$ does not meet $\cup_{n \geq 1} f^n(C)$ for ϵ small enough. By the first assertion of Proposition 2.4 and Proposition 3.1, we get $dd_{\lambda}^c L = 0$ on $B(\lambda_0, \epsilon)$. \square

PROOF OF THEOREM 6.11 : The lower semi-continuity of $\lambda \mapsto \overline{C_{\lambda}^+}$ implies that

$$I(B) := \{\lambda \in M / \overline{C_{\lambda}^+} \cap B \neq \emptyset\}$$

is an open subset of M for every open ball $B \subset \mathbb{P}^k$. Now let us show that $I(B)$ is dense in Ω . Let $\lambda_0 \in \Omega$ and $\epsilon > 0$. Since $\lambda_0 \in \text{supp } dd_{\lambda}^c L$, Proposition 6.15 implies that $(\overline{C^+})_{\lambda_0} \cap B = B$. Thus $(\cup_{n \geq 1} f^n(C)) \cap (B(\lambda_0, \epsilon) \times B) \neq \emptyset$ and there exists $(\lambda_1, z_1) \in f^{n_1}(C) \cap (B(\lambda_0, \epsilon) \times B)$. This shows that $\lambda_1 \in I(B) \cap B(\lambda_0, \epsilon)$. Now consider a countable collection $B_i := B(\zeta_i, r_i)$ of balls in \mathbb{P}^k whose centers are dense in \mathbb{P}^k and whose radii tend to 0. According to Baire's theorem $M' := \cap_{i \geq 1} I(B_i)$ is a dense G_{δ} -subset of M . We also have $\overline{C_{\lambda}^+} = \mathbb{P}^k$ for every $\lambda \in M'$. \square

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