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# Noise-induced phase slips, log-periodic oscillations, and the Gumbel distribution

Nils Berglund

## Abstract

When two synchronised phase oscillators are perturbed by weak noise, they display occasional losses of synchrony, called phase slips. The slips can be characterised by their location in phase space and their duration. We show that when properly normalised, their location converges, in the vanishing noise limit, to the sum of an asymptotically geometric random variable and a Gumbel random variable. The duration also converges to a Gumbel variable with different parameters. We relate these results to recent works on the phenomenon of log-periodic oscillations and on links between transition path theory and extreme-value theory.

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## 1 Introduction

The aim of this work is to explore connections between the concepts given in the title:

1. *Noise-induced phase slips* are occasional losses of synchrony of two coupled phase oscillators, due to stochastic perturbations [50]. The problem of finding the distribution of their location and length can be formulated as a stochastic exit problem, which involves the exit through a so-called *characteristic boundary* [20, 21].
2. *Log-periodic oscillations* designate the periodic dependence of a quantity of interest, such as a power law exponent, on the logarithm of a system parameter. They often occur in systems presenting a discrete scale invariance [53]. In the context of the stochastic exit problem, they are connected to the phenomenon of *cycling* of the exit distribution through an unstable periodic orbit [22, 11, 14].
3. The *Gumbel distribution* is one of the max-stable distributions known from extreme-value theory [39]. This distribution has been known to occur in the exit distribution through an unstable periodic orbit [48, 11, 14]. More recently, the Gumbel distribution has also been found to govern the length of *reactive paths* in one-dimensional exit problems [17, 6, 7].

In this work, we review a number of prior results on exit distributions, and build on them to derive properties of the phase slip distributions. We start in Section 2 by recalling the classical situation of two coupled phase oscillators, and the phenomenology of noise-induced phase slips. In Section 3, we present the mathematical set-up for the systems we will consider, and introduce different tools that are used for the study of the stochastic exit problem.

Section 4 contains our main results on the distribution of the phase slip location. These results are mainly based on those of [14] on the exit distribution through an unstable planar periodic orbit, slightly reformulated in the context of limit distributions. We also discuss links to the concept of log-periodic oscillations.

In Section 5, we discuss a number of connections to extreme-value theory. After summarizing properties of the Gumbel distribution relevant to our problem, we give a short review of recent results by C erou, Guyader, Leli evre and Malrieu [17] and by Bakhtin [6, 7] on the appearance of the Gumbel distribution in transition path theory for one-dimensional problems.

Section 6 presents our results on the duration of phase slips, which build on the previous results from transition path theory. Section 7 contains a summary and some open questions, while the proofs of the main theorems are contained in the Appendix.

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## 2 Synchronization of phase oscillators

In this section, we briefly recall the setting of two coupled phase oscillators showing synchronization, following mainly [50].

### 2.1 Deterministic phase locking

Consider two oscillators, whose dynamics is governed by ordinary differential equations (ODEs) of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) , \\ \dot{x}_2 &= f_2(x_2) ,\end{aligned}\tag{2.1}$$

where  $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$  with  $n_1, n_2 \geq 2$ . A classical example of a system displaying oscillations is the Van der Pol oscillator [54, 55, 56]

$$\ddot{\theta}_i - \gamma_i(1 - \theta_i^2)\dot{\theta}_i + \theta_i = 0 ,\tag{2.2}$$

which can be transformed into a first-order system by setting  $x_i = (\theta_i, \dot{\theta}_i) \in \mathbb{R}^2$ . The precise form of the vector fields  $f_i$ , however, does not matter. What is important is that each system admits an asymptotically stable periodic orbit, called a *limit cycle*. These limit cycles can be parametrised by angular variables  $\phi_1, \phi_2 \in \mathbb{R}/\mathbb{Z}$  in such a way that

$$\begin{aligned}\dot{\phi}_1 &= \omega_1 , \\ \dot{\phi}_2 &= \omega_2 ,\end{aligned}\tag{2.3}$$

where  $\omega_1, \omega_2 \in \mathbb{R}$  are constant angular frequencies [50, Section 7.1]. Note that the product of the two limit cycles forms a two-dimensional invariant torus in phase space  $\mathbb{R}^{n_1+n_2}$ .

Consider now a perturbation of System (2.1) in which the oscillators interact weakly, given by

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + \varepsilon g_1(x_1, x_2) , \\ \dot{x}_2 &= f_2(x_2) + \varepsilon g_2(x_1, x_2) .\end{aligned}\tag{2.4}$$

The theory of normally hyperbolic invariant manifolds (see for instance [40]) shows that the invariant torus persists for sufficiently small nonzero  $\varepsilon$  (for stronger coupling, new phenomena such as oscillation death can occur [50, Section 8.2.2]). For small  $\varepsilon$ , the reduced equations (2.3) for the dynamics on the torus take the form

$$\begin{aligned}\dot{\phi}_1 &= \omega_1 + \varepsilon Q_1(\phi_1, \phi_2) , \\ \dot{\phi}_2 &= \omega_2 + \varepsilon Q_2(\phi_1, \phi_2) ,\end{aligned}\tag{2.5}$$

where  $Q_{1,2}$  can be computed perturbatively in terms of  $f_{1,2}$  and  $g_{1,2}$ . Assume that the natural frequencies  $\omega_1, \omega_2$  are different, but that the *detuning*  $\nu = \omega_2 - \omega_1$  is small. Introducing new variables  $\psi = \phi_1 - \phi_2$  and  $\varphi = (\phi_1 + \phi_2)/2$  yields a system of the form

$$\begin{aligned}\dot{\psi} &= -\nu + \varepsilon q(\psi, \varphi) , \\ \dot{\varphi} &= \omega + \mathcal{O}(\varepsilon) ,\end{aligned}\tag{2.6}$$

where  $\omega = (\omega_1 + \omega_2)/2$  is the mean frequency. Note that the phase difference  $\psi$  evolves more slowly than the mean phase  $\varphi$ , so that the theory of averaging applies [16, 58]. For small  $\nu$  and  $\varepsilon$ , solutions of (2.6) are close to those of the averaged system

$$\omega \frac{d\psi}{d\varphi} = -\nu + \varepsilon \bar{q}(\psi) , \quad \bar{q}(\psi) = \int_0^1 q(\psi, \varphi) d\varphi\tag{2.7}$$

(recall our convention that the period is equal to 1). In particular, solutions of the equation  $-\nu + \varepsilon \bar{q}(\psi) = 0$  correspond to stationary solutions of the averaged equation (2.7), and to periodic orbits of the original equation (2.6) (and thus also of (2.5)).

For example, in the case of *Adler's equation*,  $\bar{q}(\psi) = \sin(2\pi\psi)$ , there are two stationary points whenever  $|\nu| < |\varepsilon|$ . They give rise to one stable and one unstable periodic orbit. The stable periodic orbit corresponds to a synchronized state, because the phase difference  $\psi$  remains bounded for all times. This is the phenomenon known as *phase locking*.

**Remark 2.1.** Similar phase locking phenomena appear when the ratio  $\omega_2/\omega_1$  is close to any rational number  $m/n \in \mathbb{Q}$ . Then for small  $\varepsilon$  the quantity  $n\phi_1 - m\phi_2$  may stay bounded for all times (*n:m frequency locking*). The sets of parameter values  $(\varepsilon, \nu)$  for which frequency locking with a specific ratio occurs are known as *Arnold tongues* [2].

## 2.2 Noise-induced phase slips

Consider now what happens when noise is added to the system. This is often done (see e.g. [50, Chapter 9]) by looking at the effect of noise on the averaged system (2.7), which becomes

$$\omega \frac{d\psi}{d\varphi} = -\nu + \varepsilon \bar{q}(\psi) + \text{noise} ,\tag{2.8}$$

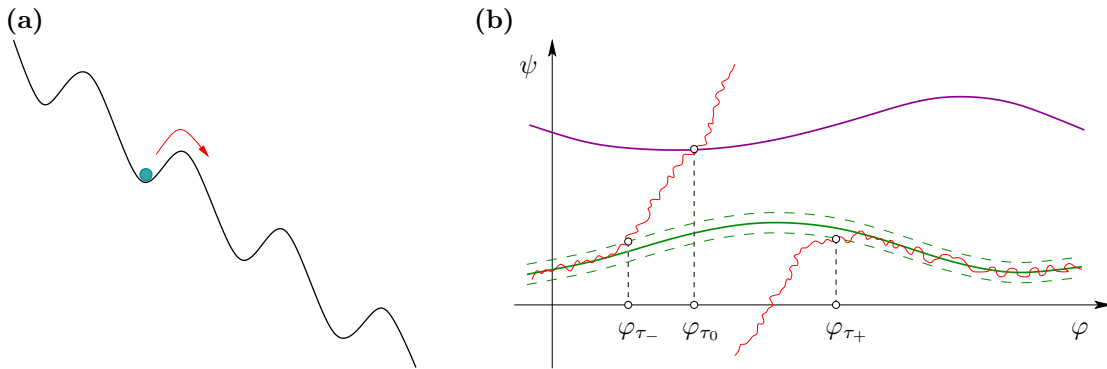


FIGURE 1. **(a)** Washboard potential  $V(\psi)$  of the averaged system (2.8), in a case where  $\nu < 0$ . Phase slips correspond to transitions over a local potential maximum. **(b)** For the unaveraged system (2.10), phase slips involve crossing the unstable periodic orbit delimiting the basin of attraction of the synchronized state.

where we will specify in the next section what kind of noise we consider. The first two terms on the right-hand side of (2.8) can be written as

$$-\frac{\partial}{\partial \psi} V(\psi), \quad \text{where } V(\psi) = \nu\psi - \varepsilon \int_0^\psi \bar{q}(x) dx. \quad (2.9)$$

In the synchronization region, the potential  $V(\psi)$  has the shape of a tilted periodic, or *washboard* potential (Figure 1a). The local minima of the potential represent the synchronized state, while the local maxima represent an unstable state delimiting the basin of attraction of the synchronized state. In the absence of noise, trajectories are attracted exponentially fast to the synchronized state and stay there. When weak noise is added, solutions still spend most of the time in a neighbourhood of a synchronized state. However, occasional transitions through the unstable state may occur, meaning that the system temporarily desynchronizes, before returning to synchrony. This behaviour is called a *phase slip*. Transitions in both directions may occur, that is,  $\psi$  can increase or decrease by 1 per phase slip. When detuning and noise are small, however, transitions over the lower local maximum of the washboard potential are more likely.

In reality, however, we should add noise to the unaveraged system (2.6), which becomes

$$\begin{aligned} \dot{\psi} &= -\nu + \varepsilon q(\psi, \varphi) + \text{noise}, \\ \dot{\varphi} &= \omega + \mathcal{O}(\varepsilon) + \text{noise}. \end{aligned} \quad (2.10)$$

Phase slips are now associated with transitions across the unstable orbit (Figure 1b). Two important random quantities characterising the phase slips are

1. the value of the phase  $\varphi_{\tau_0}$  at the time  $\tau_0$  when the unstable orbit is crossed, and
2. the duration of the phase slip, which can be defined as the phase difference between the time  $\tau_-$  when a suitably defined neighbourhood of the stable orbit is left, and the time  $\tau_+$  when a neighbourhood of (a translated copy of) the stable orbit is reached.

Unless the system (2.10) is independent of the phase  $\varphi_{\tau_0}$ , there is no reason for the slip phases to have a uniform distribution. Our aim is to determine the weak-noise asymptotics of the phase  $\varphi_{\tau_0}$  and of the phase slip duration  $\varphi_{\tau_+} - \varphi_{\tau_-}$ .

### 3 The stochastic exit problem

Let us now specify the mathematical set-up of our analysis. The stochastically perturbed systems that we consider are Itô stochastic differential equations (SDEs) of the form

$$dx_t = f(x_t) dt + \sigma g(x_t) dW_t, \quad (3.1)$$

where  $x_t$  takes values in  $\mathbb{R}^2$ , and  $W_t$  denotes  $k$ -dimensional standard Brownian motion, for some  $k \geq 2$ . Physically, this describes the situation of Gaussian white noise with a state-dependent amplitude  $g(x)$ . Of course, one may consider more general types of noise, such as time-correlated noise, but such a setting is beyond the scope of the present analysis.

The drift term  $f$  and the diffusion term  $g$  are assumed to satisfy the usual regularity assumptions guaranteeing the existence of a unique strong solution for all square-integrable initial conditions  $x_0$  (see for instance [49, Section 5.2]). In addition, we assume that  $g$  satisfies the uniform ellipticity condition

$$c_1 \|\xi\|^2 \leq \langle \xi, D(x)\xi \rangle \leq c_2 \|\xi\|^2 \quad \forall x, \xi \in \mathbb{R}^2, \quad (3.2)$$

where  $c_2 \geq c_1 > 0$ . Here  $D(x) = gg^T(x)$  denotes the *diffusion matrix*.

We finally assume that the drift term  $f(x)$  results from a system of the form (2.6), in a synchronized case where there is one stable and one unstable orbit. It will be convenient to choose coordinates  $x = (\varphi, r)$  such that the unstable periodic orbit is given by  $r = 0$  and the stable orbit is given by  $r = 1/2$ . The original system is defined on a torus, but we will unfold everything to the plane  $\mathbb{R}^2$ , considering  $f$  (and  $g$ ) to be periodic with period 1 in both variables  $\varphi$  and  $r$ . The resulting system has the form

$$\begin{aligned} dr_t &= f_r(r_t, \varphi_t) dt + \sigma g_r(r_t, \varphi_t) dW_t, \\ d\varphi_t &= f_\varphi(r_t, \varphi_t) dt + \sigma g_\varphi(r_t, \varphi_t) dW_t, \end{aligned} \quad (3.3)$$

and admits unstable orbits of the form  $\{r = n\}$  and stable orbits of the form  $\{r = n/2\}$  for any integer  $n$ . In particular,  $f_r(n/2, \varphi) = 0$  for all  $n \in \mathbb{Z}$  and all  $\varphi \in \mathbb{R}$ . Using a so-called equal-time parametrisation of the periodic orbits, it is also possible to assume that  $f_\varphi(0, \varphi) = 1/T_+$  and  $f_\varphi(1/2, \varphi) = 1/T_-$  for all  $\varphi \in \mathbb{R}$ , where  $T_\pm$  denote the periods of the unstable and stable orbit [14, Proposition 2.1]<sup>1</sup>. The instability of the orbit  $r = 0$  means that the characteristic exponent

$$\lambda_+ = \int_0^1 \partial_r f_r(0, \varphi) d\varphi \quad (3.4)$$

is strictly positive. The similarly defined exponent  $-\lambda_-$  of the stable orbit is negative. It is then possible to redefine  $r$  in such a way that

$$\begin{aligned} f(0, \varphi) &= \lambda_+ r + \mathcal{O}(r^2), \\ f(1/2, \varphi) &= -\lambda_- (r - 1/2) + \mathcal{O}((r - 1/2)^2) \end{aligned} \quad (3.5)$$

for all  $\varphi \in \mathbb{R}$  (see again [14, Proposition 2.1]). It will be convenient to assume that  $f_\varphi(r, \varphi)$  is positive, bounded away from zero, for all  $(r, \varphi)$ .

---

<sup>1</sup>Because of second-order terms in Itô's formula, the periodic orbits of the reparametrized system may not lie exactly on horizontal lines  $r = n/2$ , but be shifted by a small amount of order  $\sigma^2$ .

Finally, for definiteness, we assume that the system is asymmetric, in such a way that it is easier for the system starting with  $r$  near  $-1/2$  to reach the unstable orbit in  $r = 0$  rather than its translate in  $r = -1$ . This corresponds intuitively to the potential in Figure 1 tilting to the right, and can be formulated precisely in terms of large-deviation rate functions introduced in Section 3.2 below.

### 3.1 The harmonic measure

Fix an initial condition  $(r_0 \in (-1, 0), \varphi_0 = 0)$  and let

$$\tau_0 = \inf\{t > 0: r_t = 0\} \quad (3.6)$$

denote the first-hitting time of the unstable orbit. Note that  $\tau_0$  can also be viewed as the first-exit time from the set  $\mathcal{D} = \{r < 0\}$ . The crossing phase  $\varphi_{\tau_0}$  is equal to the exit location from  $\mathcal{D}$ , and its distribution is also known as the *harmonic measure* associated with the infinitesimal generator

$$L = \sum_{i \in \{r, \varphi\}} f_i(x) \frac{\partial}{\partial x_i} + \frac{\sigma^2}{2} \sum_{i, j \in \{r, \varphi\}} D_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \quad (3.7)$$

of the diffusion process. It is known that the harmonic measure admits a smooth density for sufficiently smooth  $f, g$  and  $\partial\mathcal{D}$  [9].

It follows from Dynkin's formula [49, Section 7.4] that for any continuous bounded test function  $b : \partial\mathcal{D} \rightarrow \mathbb{R}$ , the function  $h(x) = \mathbb{E}^x \{b(\varphi_{\tau_0})\}$  satisfies the boundary value problem<sup>2</sup>

$$\begin{cases} Lh(x) = 0 & x \in \mathcal{D}, \\ h(x) = b(x) & x \in \partial\mathcal{D}. \end{cases} \quad (3.8)$$

One may think of the case of a sequence  $b_n$  converging to the indicator function  $1_{\{\varphi \in B\}}$ . Then the associated  $h_n$  converge to  $h(x) = \mathbb{P}^x \{\varphi_{\tau_0} \in B\}$  gives the harmonic measure of  $B \subset \partial\mathcal{D}$ . While it is in general difficult to solve the equation (3.8) explicitly, the fact that  $Lh = 0$  ( $h$  is said to be *harmonic*) yields some useful information. In particular,  $h$  satisfies a maximum principle and Harnack inequalities [38, Chapter 9].

### 3.2 Large deviations

The theory of large deviations has been developed in the context of general SDEs of the form (3.1) by Freidlin and Wentzell [35]. With a path  $\gamma : [0, T] \rightarrow \mathbb{R}^2$  it associates the *rate function*

$$I_{[0, T]}(\gamma) = \frac{1}{2} \int_0^T (\dot{\gamma}_s - f(\gamma_s))^T D(\gamma_s)^{-1} (\dot{\gamma}_s - f(\gamma_s)) ds. \quad (3.9)$$

Roughly speaking, the probability of the stochastic process tracking a particular path  $\gamma$  on  $[0, T]$  behaves like  $e^{-I_{[0, T]}(\gamma)/\sigma^2}$  as  $\sigma \rightarrow 0$ .

In the case of the stochastic exit problem from a domain  $\mathcal{D}$ , containing a unique attractor  $\mathcal{A}$ , the theory of large deviations yields in particular the following information. For  $y \in \partial\mathcal{D}$  let

$$V(y) = \inf_{T > 0} \inf_{\gamma: \mathcal{A} \rightarrow y} I_{[0, T]}(\gamma), \quad (3.10)$$

---

<sup>2</sup>Several tools will require  $\mathcal{D}$  to be a bounded set. This does not create any problems, because our assumptions on the deterministic vector field imply that probabilities are only affected by a negligible amount if  $\mathcal{D}$  is replaced by its intersection with some large compact set.

be the *quasipotential*, where the second infimum runs over all paths connecting  $\mathcal{A}$  to  $y$  in time  $T$ . Then for  $x_0 \in \mathcal{A}$

$$\lim_{\sigma \rightarrow 0} \sigma^2 \log \mathbb{E}^{x_0} \{\tau_0\} = \inf_{y \in \partial \mathcal{D}} V(y). \quad (3.11)$$

Furthermore, if the quasipotential reaches its infimum at a unique isolated point  $y^* \in \partial \mathcal{D}$ , then

$$\lim_{\sigma \rightarrow 0} \mathbb{P}^{x_0} \{\|x_{\tau_0} - y^*\| > \delta\} = 0 \quad (3.12)$$

for all  $\delta > 0$ . This means that exit locations concentrate in points where the quasipotential is minimal.

If we try to apply this last result to our problem, however, we realise that it does not lead any useful information. Indeed, the quasipotential  $V$  is constant on the unstable orbit  $\{r = 0\}$ , because any two points on the orbit can be connected at zero cost, just by tracking the orbit.

Nevertheless, the theory of large deviations provides some useful information, since it allows to determine most probable exit paths. The rate function (3.9) can be viewed as a Lagrangian action. Minimizing the action via Euler–Lagrange equations is equivalent to solving Hamilton equations with Hamiltonian

$$H(\gamma, \eta) = \frac{1}{2} \eta^T D(\gamma) \eta + f(\gamma)^T \eta, \quad (3.13)$$

where  $\eta = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma)) = (p_r, p_\varphi)$  is the moment conjugated to  $\gamma$ . This is a two-degrees-of-freedom Hamiltonian, whose orbits live in a four-dimensional space, which is, however, foliated into three-dimensional hypersurfaces of constant  $H$ .

Writing out the Hamilton equations (cf. [14, Section 2.2]) shows that the plane  $\{p_r = p_\varphi = 0\}$  is invariant. It corresponds to deterministic motion, and contains in particular the periodic orbits of the original system. These turn out to be hyperbolic periodic orbits of the three-dimensional flow on the set  $\{H = 0\}$ , with characteristic exponents  $\pm \lambda_+ T_+$  and  $\mp \lambda_- T_-$ . Typically, the unstable manifold of the stable orbit and the stable manifold of the unstable orbit will intersect transversally, and the intersection will correspond to a minimiser  $\gamma_\infty$  of the rate function, connecting the two orbits in infinite time. In the sequel, we will assume that this is the case, and that  $\gamma_\infty$  is unique up to shifts  $\varphi \mapsto \varphi + n$  (cf. [14, Assumption 2.3 and Figure 2.2] and Figure 6).

### 3.3 Random Poincaré maps

The periodicity in  $\varphi$  of the system (3.3) yields useful information on the distribution of the crossing phase  $\varphi_{\tau_0}$ . We fix an initial condition  $(r_0, \varphi_0 = 0)$  with  $-1 < r_0 < 0$  and define for every  $n \in \mathbb{N}$

$$\tau_n = \inf\{t > 0: \varphi_t = n\}. \quad (3.14)$$

In addition, we kill the process at the first time  $\tau_0$  it hits the unstable orbit at  $r = 0$ , and set  $\tau_n = \infty$  whenever  $\varphi_{\tau_0} < n$ . The sequence  $(R_0, R_1, \dots, R_N)$  defined by  $R_k = r_{\tau_k}$  and  $N = \lfloor \tau_0 \rfloor$  defines a substochastic Markov chain on  $E = \mathbb{R}_-$ , which records the successive values of  $r$  whenever  $\varphi$  reaches for the first time the vertical lines  $\{\varphi = k\}$  (Figure 2). This Markov chain has a transition kernel with density  $k(x, y)$ , that is,

$$\mathbb{P}\{R_{n+1} \in B | r_{\tau_n} = R_n\} =: K(R_n, B) = \int_B k(R_n, y) dy, \quad B \subset E \quad (3.15)$$



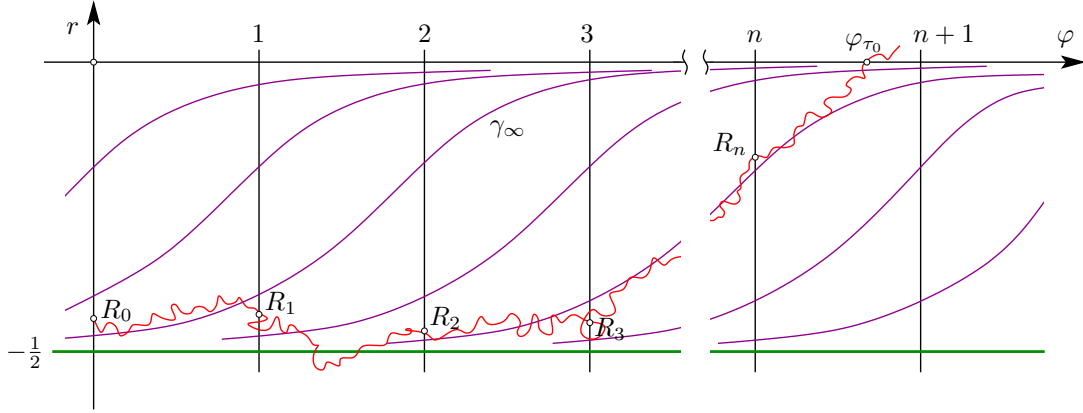


FIGURE 2. Definition of the random Poincaré map. The sequence  $(R_0, R_1, \dots, R_{\lfloor \tau_0 \rfloor})$  forms a Markov chain, killed at  $\lfloor \tau_0 \rfloor$ , where  $\tau_0$  is the first time the process hits the unstable periodic orbit in  $r = 0$ . The process is likely to track a translate of the path  $\gamma_\infty$  minimizing the rate function.

for all  $n \geq 0$ . In fact,  $k(x, y)$  is obtained<sup>3</sup> by restricting to  $\{\varphi = 1\}$  the harmonic measure for exit from  $\{\varphi < 1, r < 0\}$ , for a starting point  $(0, x)$ . We denote by  $K^n$  the  $n$ -step transition probabilities defined recursively by

$$K^n(R_0, B) := \mathbb{P}^{R_0}\{R_n \in B\} = \int_E K^{n-1}(R_0, dy)K(y, B). \quad (3.16)$$

If we decompose  $\varphi = n + s$  into its integer part  $n$  and fractional part  $s$ , we can write

$$\mathbb{P}^{0, R_0}\{\varphi_{\tau_0} \in n + ds\} = \int_E K^n(R_0, dy)\mathbb{P}^{0, y}\{\varphi_{\tau_0} \in ds\}. \quad (3.17)$$

Results by Fredholm [34] and Jentzsch [43], extending the well-known Perron–Frobenius theorem, show that  $k$  admits a spectral decomposition. In particular,  $k$  admits a *principal eigenvalue*  $\lambda_0$ , which is real, positive, and larger than the module of all other eigenvalues  $\lambda_k$ . The substochastic nature of the Markov chain, due to the killing, implies that  $\lambda_0 < 1$ . If we can obtain a bound  $\rho < 1$  on the ratio  $|\lambda_k|/\lambda_0$  valid for all  $k \geq 1$  (*spectral gap estimate*), then we can write

$$K^n(R_0, B) = \lambda_0^n \pi_0(B)[1 + \mathcal{O}(\rho^n)] \quad (3.18)$$

as  $n \rightarrow \infty$ . Here  $\pi_0$  the probability measure defined by the right eigenfunction of  $K$  corresponding to  $\lambda_0$  [43, 46, 15]. Since

$$\mathbb{P}^{R_0}\{R_n \in B | N > n\} = \frac{K^n(R_0, B)}{K^n(R_0, E)} = \pi_0(B)[1 + \mathcal{O}(\rho^n)], \quad (3.19)$$

the measure  $\pi_0$  represents the asymptotic probability distribution of the process conditioned on having survived. It is called the *quasistationary distribution* of the process [59, 52]. Plugging (3.18) into (3.17), we see that

$$\mathbb{P}^{0, R_0}\{\varphi_{\tau_0} \in n + ds\} = \lambda_0^n \int_E \pi_0(dy)\mathbb{P}^{0, y}\{\varphi_{\tau_0} \in ds\}[1 + \mathcal{O}(\rho^n)]. \quad (3.20)$$

This implies that the distribution of crossing phases  $\varphi_{\tau_0}$  asymptotically behaves like a periodically modulated geometric distribution: it satisfies  $P(\varphi + 1) = \lambda_0 P(\varphi)$  for large  $\varphi$ .

<sup>3</sup>Again, for technical reasons, one has to replace the set  $\{\varphi < 1, r < 0\}$  by a large bounded set, but this modifies probabilities by exponentially small errors that will be negligible.

## 4 Log-periodic oscillations

In this section, we formulate our main result on the distribution of crossing phases  $\varphi_{\tau_0}$  of the unstable orbit, which describe the position of phase slips. This result is based on the work [14], but we will reformulate it in order to allow comparison with related results.

### 4.1 The distribution of crossing phases

Before stating the results applying to general nonlinear equations of the form (3.3), let us consider an system approximating it near the unstable orbit at  $r = 0$ , given by

$$\begin{aligned} dr_t &= \lambda_+ r_t dt + \sigma g_r(0, \varphi_t) dW_t, \\ d\varphi_t &= \frac{1}{T_+} dt. \end{aligned} \quad (4.1)$$

This system can be transformed into the simpler form

$$dy_s = \lambda_+ y_s ds + \sigma \tilde{g}_r(0, s) dW_s, \quad \tilde{g}_r(0, s) = \frac{g_r(0, \varphi_t)}{\sqrt{D_{rr}(0, \varphi_t)}}, \quad (4.2)$$

in which the effective noise intensity is constant, i.e.  $\tilde{D}_{rr}(0, s) = \tilde{g}_r(0, s)\tilde{g}_r(0, s)^T = 1$ . To achieve this, one combines a scaling  $r = [2\lambda_+ T_+ h^{\text{per}}(\varphi)]^{1/2} y$  with the random time change  $ds = [\theta'(\varphi_t)/(\lambda_+ T_+)] dt$ , where  $h^{\text{per}}(\varphi)$  is the periodic solution of

$$\frac{dh}{d\varphi} = 2\lambda_+ T_+ h - D_{rr}(0, \varphi), \quad (4.3)$$

and

$$\theta(\varphi) = \lambda_+ T_+ \varphi - \frac{1}{2} \log \left( \frac{h^{\text{per}}(\varphi)}{2h^{\text{per}}(0)^2} \right). \quad (4.4)$$

This follows from a direct computation, using  $\theta'(\varphi) = D_{rr}(0, \varphi)/(2h^{\text{per}}(\varphi)) > 0$ . The function  $\theta(\varphi)$  should be thought of as a parametrisation of the unstable orbit that makes the stochastic dynamics as simple as possible. Note that  $\theta(\varphi + 1) = \theta(\varphi) + \lambda_+ T_+$ .

In order to formulate the main result of this section, we set

$$\theta_\delta(\varphi) = \theta(\varphi) - \log \delta + \log \left( \frac{h^{\text{per}}(s_\delta^*)}{h^{\text{per}}(0)} \right), \quad (4.5)$$

where  $s_\delta^* \in [0, 1)$  is such that  $(s_\delta^*, -\delta)$  belongs to a translate of the optimal path  $\gamma_\infty$ . A real-valued random variable  $Z$  is said to follow the *standard Gumbel law* if

$$\mathbb{P}\{Z \leq t\} = e^{-e^{-t}} \quad \forall t \in \mathbb{R}. \quad (4.6)$$

Figure 3 shows the density  $e^{-t-e^{-t}}$  of a standard Gumbel law.

**Theorem 4.1** ([14, Theorem 2.4]). *Fix an initial condition  $(r_0, \varphi_0 = 0)$  of the nonlinear system (3.3) with  $r_0$  sufficiently close to the stable orbit in  $r = -1/2$ . There exist  $\beta, c > 0$  such that for any sufficiently small  $\delta, \Delta > 0$ , there exists  $\sigma_0 > 0$  such that for  $0 < \sigma < \sigma_0$ ,*

$$\begin{aligned} \mathbb{P}^{r_0, 0} \left\{ \frac{\theta_\delta(\varphi_{\tau_0})}{\lambda_+ T_+} \in [t, t + \Delta] \right\} &= \Delta [1 - \lambda_0(\sigma)] \lambda_0(\sigma)^t Q_{\lambda_+ T_+} \left( \frac{|\log \sigma|}{\lambda_+ T_+} - t + \mathcal{O}(\delta) \right) \\ &\quad \times \left[ 1 + \mathcal{O}(e^{-c\varphi/|\log \sigma|}) + \mathcal{O}(\delta |\log \delta|) + \mathcal{O}(\Delta^\beta) \right]. \end{aligned} \quad (4.7)$$

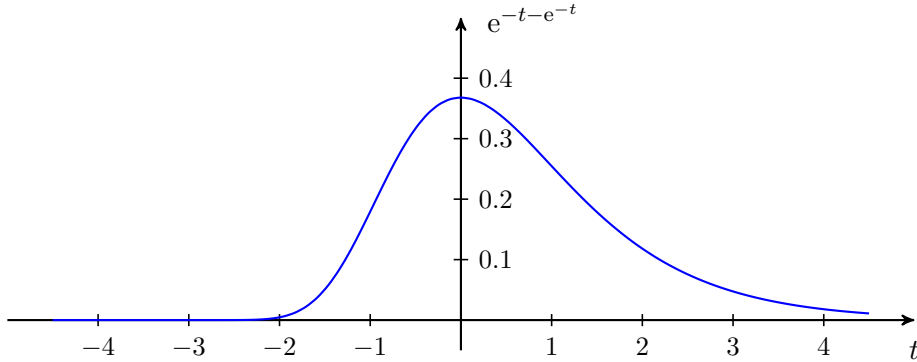


FIGURE 3. Density of a standard Gumbel random variable.

Here  $\lambda_0(\sigma)$  is the principal eigenvalue of the Markov chain, and  $1 - \lambda_0(\sigma)$  is of order  $e^{-I_\infty/\sigma^2}$ , where  $I_\infty = I(\gamma_\infty)$  is the value of the rate function for the path  $\gamma_\infty$ . Furthermore,  $Q_{\lambda_+T_+}(x)$  is the periodic function, with period 1, given by

$$Q_{\lambda_+T_+}(x) = \sum_{n \in \mathbb{Z}} A(\lambda_+T_+(n - x)), \quad (4.8)$$

where

$$A(x) = \exp\left\{-2x - \frac{1}{2}e^{-2x}\right\} \quad (4.9)$$

is the density of  $(Z - \log 2)/2$ , with  $Z$  a standard Gumbel variable.

We will discuss various implications of this result in the next sections. The periodic dependence on  $\log \sigma$  will be addressed in Section 4.2, and we will say more on the Gumbel law in Section 5. For now, let us give a reformulation of the theorem, which will ease comparison with other related results. Following [41, 42], we say that an integer-valued random variable  $Y$  is *asymptotically geometric* with success probability  $p$  if

$$\lim_{n \rightarrow \infty} \mathbb{P}\{Y = n + 1 | Y > n\} = p. \quad (4.10)$$

We use the notation  $\lim_{n \rightarrow \infty} \text{Law}(X_n) = \text{Law}(X)$  to denote convergence in distribution of a sequence of random variables  $X_n$  to a random variable  $X$ .

**Theorem 4.2.** *There exists a family  $(Y_m^\sigma)_{m \in \mathbb{N}, \sigma > 0}$  of asymptotically geometric random variables such that*

$$\lim_{m \rightarrow \infty} \left[ \lim_{\sigma \rightarrow 0} \text{Law}(\theta(\varphi_{\tau_0}) - |\log \sigma| - \lambda_+T_+Y_m^\sigma) \right] = \text{Law}\left(\frac{Z}{2} - \frac{\log 2}{2}\right), \quad (4.11)$$

where  $Z$  is a standard Gumbel random variable independent of the  $Y_m^\sigma$ . The success probability of  $Y_m^\sigma$  is of the form  $p_{m,\sigma} = e^{-I_m/\sigma^2}$ , where  $I_m = I_\infty + \mathcal{O}(e^{-2m\lambda_+T_+})$ .

This theorem is almost a corollary of Theorem 4.1, but a little work is required to control the limit  $m \rightarrow \infty$ , which corresponds to the limit  $\delta \rightarrow 0$ . We give the details in Appendix A.

The interpretation of (4.11) is as follows. To reach the unstable orbit at  $\varphi_{\tau_0}$ , the system will track, with high probability, a translate  $\gamma_\infty(\cdot + n)$  of the optimal path  $\gamma_\infty$

(Figure 2). The random variable  $Y_m^\sigma$  is the index  $n$  of the chosen translate. This index follows an approximately geometric distribution of parameter  $1 - \lambda_0(\sigma) \simeq e^{-I_\infty/\sigma^2}$ , which also manifests itself in the factor  $(1 - \lambda_0)\lambda_0^k$  in (4.7). The distribution of  $\varphi_{\tau_0}$  conditional on the event  $\{Y_m^\sigma = n\}$  converges to a shifted Gumbel distribution — we will come back to this point in Section 5.

We may not be interested in the integer part of the crossing phase  $\varphi_{\tau_0}$ , which counts the number of rotations around the orbit, but only in its fractional part  $\hat{\varphi}_{\tau_0} = \varphi_{\tau_0} \pmod{1}$ . Then it follows immediately from (4.11) and the fact that  $Y_m^\sigma$  is integer-valued that

$$\lim_{\sigma \rightarrow 0} \text{Law}(\theta(\hat{\varphi}_{\tau_0}) - |\log \sigma|) = \text{Law}\left(\left[\frac{Z}{2} - \frac{\log 2}{2}\right] \pmod{\lambda_+ T_+}\right). \quad (4.12)$$

This result can also be derived directly from [14, Corollary 2.5], by the same procedure as the one used in Section A.1.

**Remark 4.3.**

1. The index  $m$  in (4.11) seems artificial, and one would like to have a similar result for the law of  $\theta(\varphi_{\tau_0}) - |\log \sigma| - \lambda_+ T_+ Y_\infty^\sigma$ . Unfortunately, the convergence as  $\sigma \rightarrow 0$  is not uniform in  $m$ , so that the two limits in (4.11) have to be taken in that particular order.
2. The speed of convergence in (4.10) depends on the spectral gap of the Markov chain. In [14, Theorem 6.14], we proved that this spectral gap is bounded by  $e^{-c/|\log \sigma|}$  for some constant  $c > 0$ , though we expect that the gap can be bounded uniformly in  $\sigma$ . We expect, but have not proved, that the constant  $c$  is uniform in  $m$  (i.e. uniform in the parameter  $\delta$ ).

## 4.2 The origin of oscillations

A striking aspect of the expression (4.7) for the distribution of  $\varphi_{\tau_0}$  is that it depends periodically on  $|\log \sigma|$ . This means that as  $\sigma \rightarrow 0$ , the distribution does not converge, but is endlessly shifted around the unstable orbit proportionally to  $|\log \sigma|$ . This phenomenon has been discovered by Martin Day, who called it *cycling* [20, 21, 22, 24]. See also [48, 11, 36, 37] for related work.

The intuitive explanation of cycling is as follows. We have seen that the large-deviation rate function is minimized by a path  $\gamma_\infty$  (and its translates). The path approaches the unstable orbit as  $\varphi \rightarrow \infty$ , and the stable one as  $\varphi \rightarrow -\infty$ . The distance between  $\gamma_\infty$  and the unstable orbit satisfies

$$|r(\varphi)| \simeq c e^{-\theta(\varphi)} \quad \text{as } \varphi \rightarrow \infty. \quad (4.13)$$

This implies

$$|r(\varphi)| = \sigma \quad \Leftrightarrow \quad \theta(\varphi) \simeq |\log \sigma| + \log c. \quad (4.14)$$

Thus everything behaves as if the unstable orbit has an “effective thickness” equal to the standard deviation  $\sigma$  of the noise. Escape becomes likely once the optimal path  $\gamma_\infty$  touches the thickened unstable orbit.

It is interesting to note that the periodic dependence on the logarithm of a parameter, or *log-periodic oscillations*, appear in many systems presenting discrete-scale invariance [53]. These include for instance hierarchical models in statistical physics [27, 18] and self-similar networks [28], diffusion through fractals [1, 29] and iterated maps [25, 26].

One link with the present situation is that (4.13) implies a discrete-scale invariance, since scaling  $r$  by a factor  $e^{-\theta(1)} = e^{-\lambda_+ T_+}$  is equivalent to scaling the noise intensity by the same factor. There might be deeper connections due to the fact that certain key functions, as the Gumbel distribution in our case, obey functional equations — see for instance the similar behaviour of the example in [26, Remark 3.1].

**Remark 4.4.** The periodic “cycling profile”  $Q_{\lambda_+ T_+}$  admits the Fourier series representation

$$Q_{\lambda_+ T_+}(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}, \quad a_k = \frac{2^{-\pi i k / (\lambda_+ T_+)}}{\lambda_+ T_+} \Gamma\left(1 - \frac{\pi i k}{\lambda_+ T_+}\right), \quad (4.15)$$

where  $\Gamma$  is Euler’s Gamma function.  $Q_{\lambda_+ T_+}$  is also an *elliptic function*, since in addition to being periodic in the real direction, it is also periodic in the imaginary direction. Indeed, by definition of  $A(x)$ , we have

$$Q_{\lambda_+ T_+}\left(z + \frac{\pi i}{\lambda_+ T_+}\right) = Q_{\lambda_+ T_+}(z) \quad \forall z \in \mathbb{C}. \quad (4.16)$$

Being non-constant and doubly periodic,  $Q_{\lambda_+ T_+}$  necessarily admits at least two poles in every translate of the unit cell  $(0, 1) \times (0, \pi i / (\lambda_+ T_+))$ .

## 5 The Gumbel distribution

### 5.1 Extreme-value theory

Let  $X_1, X_2, \dots$  be a sequence of independent, identically distributed (i.i.d.) real random variables, with common distribution function  $F(x) = \mathbb{P}\{X_1 \leq x\}$ . Extreme-value theory is concerned with deriving the law of the maximum

$$M_n = \max\{X_1, \dots, X_n\} \quad (5.1)$$

as  $n \rightarrow \infty$ . It is immediate to see that the distribution function of  $M_n$  is  $F(t)^n$ . We will say that  $F$  belongs to the basin of attraction of a distribution function  $\Phi$ , and write  $F \in D(\Phi)$ , if there exist sequences of real numbers  $a_n > 0$  and  $b_n$  such that

$$\lim_{n \rightarrow \infty} F(a_n x + b_n)^n = \Phi(x) \quad \forall x \in \mathbb{R}. \quad (5.2)$$

This is equivalent to the sequence of random variables  $(M_n - b_n)/a_n$  converging in distribution to a random variable with distribution function  $\Phi$ . Clearly, if  $F \in D(\Phi)$ , then one also has  $F \in D(\Phi(ax + b))$  for all  $a > 0, b \in \mathbb{R}$ , so it makes sense to work with equivalence classes  $\{\Phi(ax + b)\}_{a,b}$ .

Any possible limit of (5.2) should satisfy the functional equation

$$\Phi(ax + b)^2 = \Phi(x) \quad \forall x \in \mathbb{R} \quad (5.3)$$

for some constants  $a > 0, b \in \mathbb{R}$ . Fréchet [33], Fischer and Tippett [32] and Gnedenko [39] have shown that if one excludes the degenerate case  $F(x) = 1_{\{x \geq c\}}$ , then the only possible solutions of (5.3) are in one of the following three classes, where  $\alpha > 0$  is a parameter:

$$\begin{aligned} \Phi_\alpha(x) &= e^{-x^{-\alpha}} 1_{\{x > 0\}} && \text{Fréchet law,} \\ \Psi_\alpha(x) &= e^{-(-x)^\alpha} 1_{\{x \leq 0\}} + 1_{\{x > 0\}} && \text{Weibull law,} \\ \Lambda(x) &= e^{-e^{-x}} && \text{Gumbel law.} \end{aligned} \quad (5.4)$$

In [39], Gnedenko gives precise characterizations on when  $F$  belongs to the basin of attraction of each of the above laws. Of particular interest to us is the following result. Let

$$R(t) = 1 - F(t) = \mathbb{P}\{X_1 > t\} \quad (5.5)$$

denote the tail probabilities of the i.i.d. random variables  $X_i$ .

**Lemma 5.1** ([39, Lemma 4]). *A nondegenerate distribution function  $F$  belongs to the basin of attraction of  $\Phi$  if and only if there exist sequences  $a_n > 0$  and  $b_n$  such that*

$$\lim_{n \rightarrow \infty} nR(a_n x + b_n) = -\log \Phi(x) \quad \forall x \text{ such that } \Phi(x) > 0. \quad (5.6)$$

The sequences  $a_n$  and  $b_n$  are not unique, but [39, Theorem 6] shows that in the case of the Gumbel distribution  $\Phi = \Lambda$ ,

$$b_n = \inf \left\{ x : F(x) > 1 - \frac{1}{n} \right\}, \quad a_n = \inf \left\{ x : F(x) + b_n > 1 - \frac{1}{ne} \right\} \quad (5.7)$$

is a possible choice. In this way it is easy to check that the normal law is attracted to the Gumbel distribution.

Another related characterization of  $F$  being in the basin of attraction of the Gumbel law is the following.

**Theorem 5.2** ([39, Theorem 7]). *Let  $x_0 = \inf\{x : F(x) = 1\} \in \mathbb{R} \cup \{\infty\}$ . Then  $F \in D(\Lambda)$  if and only if there exists a continuous function  $A(z)$  such that  $\lim_{z \rightarrow x_0^-} A(z) = 0$  and*

$$\lim_{z \rightarrow x_0^-} \frac{R(z(1 + A(z)x))}{R(z)} = -\log \Lambda(x) = e^{-x} \quad \forall x \in \mathbb{R}. \quad (5.8)$$

The function  $A(z)$  can be chosen such that  $A(b_n) = a_n/b_n$  for all  $n$ , where  $a_n$  and  $b_n$  satisfy (5.6).

The quantity on the left-hand side of (5.8) can be rewritten as

$$\mathbb{P}\{X_1 > z(1 + A(z)x) \mid X_1 > z\}, \quad (5.9)$$

that is, it represents a *residual lifetime*. See also [8].

## 5.2 Length of reactive paths

The Gumbel distribution also appears in the context of somewhat different exit problems (which, however, will turn out not to be so different after all). In [17], Cérou, Guyader, Lelièvre and Malrieu consider one-dimensional SDEs of the form

$$dx_t = -V'(x_t) dt + \sigma dW_t, \quad (5.10)$$

where  $V(x)$  is a double-well potential (Figure 4).

Assume, without loss of generality, that the local maximum of  $V$  is in 0. Denote the local minima of  $V$  by  $x_-^* < 0 < x_+^*$ , and assume  $\lambda = -V''(0) > 0$ . Pick an initial condition  $x_0 \in (x_-^*, 0)$ . A classical question is to determine the law of the first-hitting time  $\tau_b$  of a point  $b \in (0, x_+^*]$ . The expected value of  $\tau_b$  obeys the so-called *Eyring-Kramers law* [3, 31, 45]

$$\mathbb{E}^{x_0} \{\tau_b\} = \frac{2\pi}{\sqrt{V''(x_-^*)|V''(0)|}} e^{2[V(0) - V(x_-^*)]/\sigma^2} [1 + \mathcal{O}(\sigma)]. \quad (5.11)$$

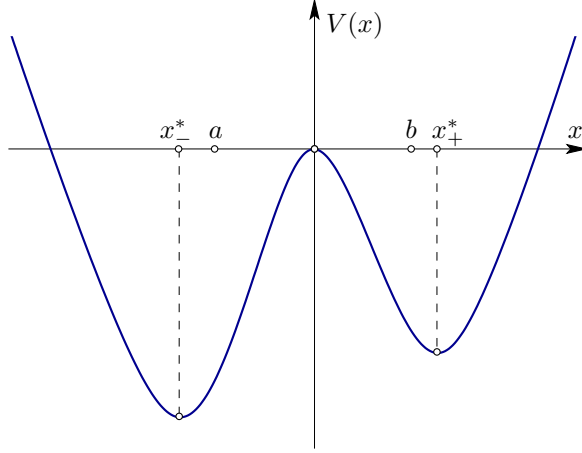


FIGURE 4. An example of double-well potential occurring in (5.10).

In addition, Day [19] has proved (in a more general context) that the distribution of  $\tau_b$  is asymptotically exponential:

$$\lim_{\sigma \rightarrow 0} \mathbb{P}^{x_0} \{ \tau_b > s \mathbb{E}^{x_0} \{ \tau_b \} \} = e^{-s} . \quad (5.12)$$

The picture is that sample paths spend an exponentially long time near the local minimum  $x_-^*$ , with occasional excursions away from  $x_-^*$ , until ultimately managing to cross the saddle. See for instance [10] for a recent review.

In transition-path theory [30, 57], by contrast, one is interested in the very last bit of the sample path, between its last visit to  $x_-^*$  and its first passage in  $b$ . The length of this transition is considerably shorter than  $\tau_b$ . A way to formulate this is to fix a point  $a \in (x_-^*, x_0)$ , and to condition on the event that the path hits  $b$  before hitting  $a$ . The result can be formulated as follows (note that our  $\sigma$  corresponds to  $\sqrt{2\varepsilon}$  in [17]):

**Theorem 5.3** ([17, Theorem 1.4]). *For any fixed  $a < x_0 < 0 < b$  in  $(x_-^*, x_+^*)$ ,*

$$\lim_{\sigma \rightarrow 0} \mathbf{Law}(\lambda \tau_b - 2|\log \sigma| \mid \tau_b < \tau_a) = \mathbf{Law}(Z + T(x_0, b)) , \quad (5.13)$$

where  $Z$  is a standard Gumbel variable, and

$$T(x_0, b) = \log(|x_0|b\lambda) + \int_{x_0}^0 \left( \frac{\lambda}{V'(y)} + \frac{1}{y} \right) dy - \int_0^b \left( \frac{\lambda}{V'(y)} + \frac{1}{y} \right) dy . \quad (5.14)$$

The proof is based on Doob's  $h$ -transform, which allows to replace the conditioned problem by an unconditioned one, with a modified drift term. The new drift term becomes singular as  $x \rightarrow a_+$ . See also [47] for other uses of Doob's  $h$ -transform in the context of reactive paths.

As shown in [17, Section 4],  $2|\log \sigma| + T(x_0, b)/\lambda$  is the sum of the deterministic time needed to go from  $\sigma$  to  $b$ , and of the deterministic time needed to go from  $-\sigma$  to  $a$  (in the one-dimensional setting, paths minimizing the large-deviation rate function are time-reversed deterministic paths).

### 5.3 Bakhtin's approach

Yuri Bakhtin has recently provided some interesting insights into the question of why the Gumbel distribution governs the length of reactive paths [6, 7]. They apply to linear equations of the form

$$dx_t = \lambda x_t dt + \sigma dW_t, \quad (5.15)$$

where  $\lambda > 0$ . However, we will see in Section 6 below that they can be extended to the nonlinear setting by using the technique outlined in Appendix A.1.

The solution of (5.15) is an “explosive Ornstein–Uhlenbeck process”

$$x_t = e^{\lambda t} \left( x_0 + \sigma \int_0^t e^{-\lambda s} dW_s \right), \quad (5.16)$$

which can also be represented in terms of a time-changed Brownian motion,

$$x_t = e^{\lambda t} \tilde{x}_t, \quad \tilde{x}_t = x_0 + \widetilde{W}_{\sigma^2(1-e^{-2\lambda t})/(2\lambda)} \quad (5.17)$$

(this follows by evaluating the variance of  $\tilde{x}_t$  using Itô's isometry). Thus  $\tilde{x}_t - x_0$  is equal in distribution to  $\sigma \sqrt{(1 - e^{-2\lambda t})/(2\lambda)} N$ , where  $N$  is a standard normal random variable.

Assume  $x_0 < 0$  and denote by  $\tau_0$  the first-hitting time of  $x = 0$ . Then André's reflection principle allows to write

$$\mathbb{P}\{\tau_0 < t \mid \tau_0 < \infty\} = \frac{\mathbb{P}\{\tau_0 < t\}}{\mathbb{P}\{\tau_0 < \infty\}} = \frac{2\mathbb{P}\{\tilde{x}_t > 0\}}{2\mathbb{P}\{\tilde{x}_\infty > 0\}} = \mathbb{P}\{\tilde{x}_t > 0 \mid \tilde{x}_\infty > 0\}. \quad (5.18)$$

Now we observe that

$$\begin{aligned} \mathbb{P}\left\{\tau_0 < t + \frac{1}{\lambda} \mid \log \sigma \mid \tau_0 < \infty\right\} &= \mathbb{P}\left\{\tilde{x}_{t+\frac{1}{\lambda} \mid \log \sigma} > 0 \mid \tilde{x}_\infty > 0\right\} \\ &= \mathbb{P}\left\{N > \frac{|x_0|}{\sigma} \sqrt{\frac{2\lambda}{1 - \sigma^2 e^{-2\lambda t}}} \mid N > \frac{|x_0|}{\sigma} \sqrt{2\lambda}\right\}, \end{aligned} \quad (5.19)$$

where  $N$  is a standard normal random variable. It follows that

$$\lim_{\sigma \rightarrow 0} \mathbb{P}\left\{\tau_0 < t + \frac{1}{\lambda} \mid \log \sigma \mid \tau_0 < \infty\right\} = \exp\{-x_0^2 \lambda e^{-2\lambda t}\}. \quad (5.20)$$

This can be checked by a direct computation, using tail asymptotics of the normal law. However, it is more interesting to view the last expression in (5.19) as a residual lifetime, given by the expression (5.9) with  $z = (|x_0|/\sigma)\sqrt{2\lambda}$ ,  $A(z) = z^{-2}$  and  $x = x_0^2 \lambda e^{-2\lambda t}$ . The right-hand side of (5.20) is the distribution function of  $(Z + \log(x_0^2 \lambda))/(2\lambda)$ , where  $Z$  is a standard Gumbel variable. Building on this computation, Bakhtin provided a new proof of the following result, which was already obtained by Day in [20].

**Theorem 5.4** ([20] and [7, Theorem 3]). *Fix  $a < 0$  and an initial condition  $x_0 \in (a, 0)$ . Then*

$$\lim_{\sigma \rightarrow 0} \text{Law}(\lambda \tau_0 - |\log \sigma| \mid \tau_0 < \tau_a) = \text{Law}\left(\frac{Z}{2} + \frac{\log(x_0^2 \lambda)}{2}\right). \quad (5.21)$$

Observe the similarity with Theorem 4.2 (and also with Proposition A.4). The proof in [7] uses the fact that conditioning on  $\{\tau_0 < \tau_a\}$  is asymptotically equivalent to conditioning on  $\{\tau_0 < \infty\}$ . Note that we use a similar argument in the proof of Theorem 4.2 in Appendix A.3.



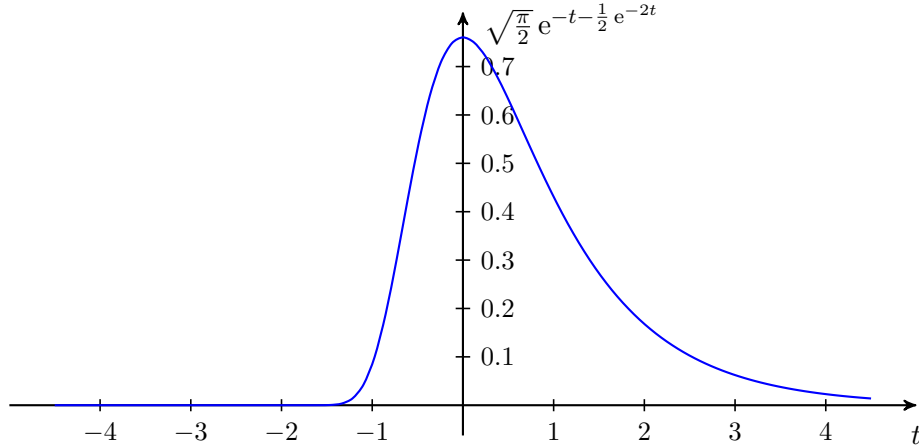


FIGURE 5. Density of the random variable  $\Theta = -\log |N|$ .

The expression (5.21) differs from (5.13) in some factors 2. This is due to the fact that Theorem 5.4 considers the first-hitting time  $\tau_0$  of the saddle, while Theorem 5.3 considers the first-hitting time  $\tau_b$  of a point  $b$  in the right-hand potential well. We will come back to this point in Section 6.2.

The observations presented here provide a connection between first-exit times and extreme-value theory, via the reflection principle and residual lifetimes. As observed in [6, Section 4], the connection depends on the seemingly accidental property

$$-\log \Lambda(e^{-x}) = \Lambda(x), \quad (5.22)$$

or  $\Lambda(e^{-x}) = e^{-\Lambda(x)}$ , of the Gumbel distribution function. Indeed, the right-hand side in (5.20) is identified with  $-\log \Lambda(x)$ , evaluated in a point  $x$  proportional to  $-e^{-2\lambda t}$ .

## 6 The duration of phase slips

### 6.1 Leaving the unstable orbit

Consider again, for a moment, the linear equation (5.15). Now we are interested in the situation where the process starts in  $x_0 = 0$ , and hits a point  $b > 0$  before hitting a point  $a < 0$ . In this section,  $\Theta$  will denote the random variable  $\Theta = -\log |N|$ , where  $N$  is a standard normal variable. Its density is given by

$$\frac{d}{dt} 2\mathbb{P}\{N < -e^{-t}\} = \sqrt{\frac{2}{\pi}} e^{-t-\frac{1}{2}} e^{-2t}, \quad (6.1)$$

which is similar to, but different from, the density of a Gumbel distribution, see Figure 5.

**Theorem 6.1** ([23, 4, 5]). *Fix  $a < 0 < b$  and an initial condition  $x_0 = 0$ . Then the linear system (5.15) satisfies*

$$\lim_{\sigma \rightarrow 0} \text{Law}(\lambda\tau_b - |\log \sigma| \mid \tau_b < \tau_a) = \text{Law}\left(\Theta + \frac{\log(2b^2\lambda)}{2}\right). \quad (6.2)$$

The intuition for this result is as follows. Consider first the symmetric case where  $a = -b$  and let  $\tau = \inf\{t > 0: |x_t| = b\} = \tau_a \wedge \tau_b$ . The solution of (5.15) starting in 0 can be written  $x_t = e^{\lambda t} \tilde{x}_t$ , where  $\tilde{x}_t = \sigma \sqrt{(1 - e^{-2\lambda t})/(2\lambda)} N$  and  $N$  is a standard normal random variable, cf. (5.17). The condition  $|x_\tau| = b$  yields

$$b = e^{\lambda\tau} \sigma \sqrt{\frac{1 - e^{-2\lambda\tau}}{2\lambda}} |N| \simeq e^{\lambda\tau} \sigma \frac{1}{\sqrt{2\lambda}} |N|. \quad (6.3)$$

Solving for  $\tau$  yields  $\lambda\tau - |\log \sigma| \simeq \log(2\lambda b^2)/2 - \log |N|$ . One can also show [23, Theorem 2.1] that  $\text{sign}(x_\tau)$  converges to a random variable  $\nu$ , independent of  $N$ , such that  $\mathbb{P}\{\nu = 1\} = \mathbb{P}\{\nu = -1\} = 1/2$ . This implies (6.2) in the symmetric case, and the asymmetric case is dealt with in [4, Theorem 1].

Let us return to the nonlinear system (3.3) governing the coupled oscillators. We seek a result similar to Theorem 6.1 for the first exit from a neighbourhood of the unstable orbit. The scaling argument given at the beginning of Section 4.1 indicates that simpler expressions will be obtained if this neighbourhood has a non-constant width of size proportional to  $\sqrt{2\lambda_+ T_+ h^{\text{per}}(\varphi)}$ . This is also consistent with the discussion in [13, Section 3.2.1]. Let us thus set

$$\tilde{\tau}_\delta = \inf\left\{t > 0: r_t = \delta \sqrt{2\lambda_+ T_+ h^{\text{per}}(\varphi)}\right\}. \quad (6.4)$$

**Theorem 6.2.** *Fix an initial condition  $(\varphi_0, r_0 = 0)$  on the unstable periodic orbit. Then the system (3.3) satisfies*

$$\lim_{\sigma \rightarrow 0} \text{Law}\left(\theta(\varphi_{\tilde{\tau}_\delta}) - \theta(\varphi_0) - |\log \sigma| \mid \tilde{\tau}_\delta < \tilde{\tau}_{-\delta}\right) = \text{Law}\left(\Theta + \frac{\log(2\lambda_+ \delta^2)}{2} + \mathcal{O}(\delta)\right) \quad (6.5)$$

as  $\delta \rightarrow 0$ .

We give the proof in Appendix B.1. Note that this result is indeed consistent with (6.2), if we take into account the fact that  $h^{\text{per}}(\varphi) \equiv 1/(2\lambda_+ T_+)$  in the case of a constant diffusion matrix  $D_{rr} \equiv 1$ .

## 6.2 There and back again

A nice observation in [7] is that Theorems 5.4 and 6.1 imply Theorem 5.3 on the length of reactive paths in the linear case. This follows immediately from the following fact.

**Lemma 6.3** ([7]). *Let  $Z$  and  $\Theta = -\log |N|$  be independent random variables, where  $Z$  follows a standard Gumbel law, and  $N$  a standard normal law. Then*

$$\text{Law}\left(\frac{1}{2}Z + \Theta\right) = \text{Law}\left(Z + \frac{\log 2}{2}\right). \quad (6.6)$$

PROOF: This follows directly from the expressions

$$\mathbb{E}\{e^{itZ}\} = \Gamma(1 - it) \quad \text{and} \quad \mathbb{E}\{e^{it\Theta}\} = \mathbb{E}\{|N|^{-i\theta}\} = \frac{2^{-it/2}}{\sqrt{\pi}} \Gamma\left(\frac{1 - it}{2}\right) \quad (6.7)$$

for the characteristic functions of  $Z$  and  $\Theta$ , and the duplication formula for the Gamma function,  $\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2})$ .  $\square$

Let us now apply similar ideas to the nonlinear system (3.3) in order to derive information on the duration of phase slips. In order to define this duration, consider two families of continuous curves  $\Gamma_-^s$  and  $\Gamma_+^s$ , depending on a parameter  $s \in \mathbb{R}$ , periodic in the  $\varphi$ -direction, and such that each  $\Gamma_-^s$  lies in the set  $\{-1/2 < r < 0\}$  and each  $\Gamma_+^s$  lies in  $\{0 < r < 1/2\}$ . We set

$$\tau_{\pm}^s = \inf\{t > 0: (r_t, \varphi_t) \in \Gamma_{\pm}^s\}, \quad (6.8)$$

while  $\tau_0$  is defined as before by (3.6). Given an initial condition  $(r_0 = -1/2, \varphi_0)$ , let us call a *successful phase slip* a sample path that does not return to the stable orbit  $\{r = -1/2\}$  between  $\tau_-^s$  and  $\tau_0$ , and that does not return to  $\Gamma_-^s$  between  $\tau_0$  and  $\tau_+^s$  (see Figure 1). Then we have the following result.

**Theorem 6.4.** *There exist families of curves  $\{\Gamma_{\pm}^s\}_{s \in \mathbb{R}}$  such that conditionally on a successful phase slip,*

$$\lim_{\sigma \rightarrow 0} \text{Law}(\theta(\varphi_{\tau_0}) - \theta(\varphi_{\tau_-^s}) - |\log \sigma|) = \text{Law}\left(\frac{Z}{2} - \frac{\log(2)}{2} + s\right), \quad (6.9)$$

$$\lim_{\sigma \rightarrow 0} \text{Law}(\theta(\varphi_{\tau_+^s}) - \theta(\varphi_{\tau_0}) - |\log \sigma|) = \text{Law}(\Theta + s), \quad (6.10)$$

$$\lim_{\sigma \rightarrow 0} \text{Law}(\theta_+(\varphi_{\tau_+^s}) - \theta_-(\varphi_{\tau_-^s}) - 2|\log \sigma|) = \text{Law}(Z + 2s), \quad (6.11)$$

where  $Z$  denotes a standard Gumbel variable,  $\Theta = -\log |N|$ , and  $N$  is a standard normal random variable. The curves  $\Gamma_{\pm}^s$  are ordered in the sense that if  $s_1 < s_2$ , then  $\Gamma_{\pm}^{s_1}$  lies below  $\Gamma_{\pm}^{s_2}$ . Furthermore,  $\Gamma_+^s$  converges to the unstable orbit  $\{r = 0\}$  as  $s \rightarrow -\infty$ , and to the stable orbit  $\{r = 1/2\}$  as  $s \rightarrow \infty$ . Similarly,  $\Gamma_-^s$  converges to the unstable orbit  $\{r = 0\}$  as  $s \rightarrow \infty$ , and to the stable orbit  $\{r = -1/2\}$  as  $s \rightarrow -\infty$ .

We give the proof in Appendix B.2, along with more details on how to construct the curves  $\Gamma_{\pm}^s$ . In a nutshell, they are obtained by letting evolve under the deterministic flow the curves  $\{r = \pm \delta \sqrt{2\lambda_+ T_+ h^{\text{per}}(\varphi)}\}$  introduced in the previous section. The parameter  $s$  plays an analogous role as  $T(x_0, b)$  in (5.14).

## 7 Conclusion and outlook

Let us restate our main results in an informal way. Theorem 4.2 shows that in the weak-noise limit, the position of the center  $\varphi_{\tau_0}$  of a phase slip, defined by the crossing location of the unstable orbit, behaves like

$$\theta(\varphi_{\tau_0}) \simeq |\log \sigma| + \lambda_+ T_+ Y^\sigma + \frac{Z}{2} - \frac{\log 2}{2}, \quad (7.1)$$

where  $Y^\sigma$  is an asymptotically geometric random variable with success probability of order  $e^{-I_\infty/\sigma^2}$ , and  $Z$  is a standard Gumbel random variable. This expression is dominated by the term  $Y^\sigma$ , which accounts for exponentially long waiting times between phase slips. The term  $|\log \sigma|$  is responsible for the cycling phenomenon, and the term  $(Z - \log(2))/2$  determines the shape of the cycling profile.

Theorem 6.4 shows in particular that the duration of a phase slip behaves like

$$\theta(\varphi_{\tau_+}) - \theta(\varphi_{\tau_-}) \simeq 2|\log \sigma| + Z + 2s, \quad (7.2)$$

where  $s$  is essentially the deterministic time required to travel between  $\sigma$ -neighbourhoods of the orbits, while the other two terms account for the time spent near the unstable orbit. The dominant term here is  $2|\log \sigma|$ , which reflects the intuitive picture that noise enlarges the orbit to a thickness of order  $\sigma$ , outside which the deterministic dynamics dominates. The phase slip duration is split into two contributions from before and after crossing the unstable orbit, of respective size  $|\log \sigma| + (Z - \log 2)/2 + s$  and  $\Theta + s$ .

Decreasing the noise intensity has two main effects. The first one is to increase the duration of phase slips by an amount  $2|\log \sigma|$ , which is due to the longer time spent near the unstable orbit. The second effect is to the shift of the phase slip location by an amount  $|\log \sigma|$ , which results in log-periodic oscillations. Note that other quantities of interest can be deduced from the above expressions, such as the distribution of residence times, which are the time spans separating phase slips when the system is in a stationary state. The residence-time distribution is given by the sum of an asymptotically geometric random variable and a variable with density proportional to  $1/\cosh^2(\theta)$  [12].

The connection between first-exit distributions and extreme-value theory is partially understood in the context as residual lifetimes, as summarized in Section 5.3. It is probable that other connections remain to be discovered. For instance, functional equations satisfied by the Gumbel distribution seem to play an important rôle. One of them is the equation

$$\Lambda(x - \log 2)^2 = \Lambda(x) \quad (7.3)$$

which results from the Gumbel law being max-stable. Another one is the equation

$$\Lambda(e^{-x}) = e^{-\Lambda(x)} \quad (7.4)$$

which appears in the context of the residual-lifetime interpretation. These functional equations may prove useful to establish other connections with critical phenomena and discrete scale invariance.

## A Proof of Theorem 4.2

### A.1 Dynamics near the unstable orbit

To prove convergence in law of certain random variables, we will work with characteristic functions. The following lemma allows to compare characteristic functions of random variables that are only known in a coarse-grained sense, via probabilities to belong to small intervals of size  $\Delta$ .

**Lemma A.1.** *Let  $X, X_0$  be real-valued random variables. Assume there exist constants  $a < b \in \mathbb{R}$ ,  $\alpha, \beta > 0$  such that as  $\Delta \rightarrow 0$ ,*

1.  $\mathbb{P}\{X_0 \notin [a, b]\} = \mathcal{O}(\Delta^\alpha)$ ,
2. for any  $k \in \mathbb{Z}$  such that  $I_k = [k\Delta, (k+1)\Delta]$  intersects  $[a, b]$ ,

$$\mathbb{P}\{X \in I_k\} = \mathbb{P}\{X_0 \in I_k\} [1 + \mathcal{O}(\Delta^\beta)] . \quad (A.1)$$

Then

$$|\mathbb{E}\{e^{i\eta X}\} - \mathbb{E}\{e^{i\eta X_0}\}| \leq 4 \sin\left(\frac{|\eta|\Delta}{2}\right) + \mathcal{O}(\Delta^{\alpha \wedge \beta}) \quad (A.2)$$

holds for all  $\eta \in \mathbb{R}$ .

PROOF: We start by noting that (A.1) implies

$$\mathbb{P}\{X \in [a, b]\} = \mathbb{P}\{X_0 \in [a, b]\} [1 + \mathcal{O}(\Delta^\beta)] = [1 - \mathcal{O}(\Delta^\alpha)] [1 + \mathcal{O}(\Delta^\beta)], \quad (\text{A.3})$$

and thus  $\mathbb{P}\{X \notin [a, b]\} = \mathcal{O}(\Delta^{\alpha \wedge \beta})$ . It follows that

$$\begin{aligned} |\mathbb{E}\{e^{i\eta X} 1_{\{X \notin [a, b]\}}\} - \mathbb{E}\{e^{i\eta X_0} 1_{\{X_0 \notin [a, b]\}}\}| &\leq \mathbb{P}\{X \notin [a, b]\} + \mathbb{P}\{X_0 \notin [a, b]\} \\ &= \mathcal{O}(\Delta^{\alpha \wedge \beta}). \end{aligned} \quad (\text{A.4})$$

Next, using the triangular inequality, we obtain for all  $k$  such that  $I_k \cap [a, b] \neq \emptyset$

$$|\mathbb{E}\{e^{i\eta X} 1_{\{X \in I_k\}}\} - \mathbb{E}\{e^{i\eta X_0} 1_{\{X_0 \in I_k\}}\}| \leq A_k + B_k + C_k, \quad (\text{A.5})$$

where

$$\begin{aligned} A_k &= |\mathbb{E}\{e^{i\eta X} 1_{\{X \in I_k\}}\} - e^{i\eta k \Delta} \mathbb{P}\{X \in I_k\}|, \\ B_k &= |e^{i\eta k \Delta} (\mathbb{P}\{X \in I_k\} - \mathbb{P}\{X_0 \in I_k\})|, \\ C_k &= |e^{i\eta k \Delta} \mathbb{P}\{X_0 \in I_k\} - \mathbb{E}\{e^{i\eta k X_0} 1_{\{X_0 \in I_k\}}\}|. \end{aligned} \quad (\text{A.6})$$

Using the fact that  $|e^{i\eta(x-k\Delta)} - 1| \leq 2 \sin(|\eta|\Delta/2)$  for all  $x \in I_k$ , we obtain

$$\begin{aligned} A_k &\leq \int_{I_k} |e^{i\eta x} - e^{i\eta k \Delta}| \mathbb{P}\{X \in dx\} \leq 2 \sin\left(\frac{|\eta|\Delta}{2}\right) \mathbb{P}\{X \in I_k\}, \\ C_k &\leq \int_{I_k} |e^{i\eta k \Delta} - e^{i\eta x}| \mathbb{P}\{X_0 \in dx\} \leq 2 \sin\left(\frac{|\eta|\Delta}{2}\right) \mathbb{P}\{X_0 \in I_k\}. \end{aligned} \quad (\text{A.7})$$

In addition, (A.1) implies that  $B_k \leq \mathcal{O}(\Delta^\beta) \mathbb{P}\{X_0 \in I_k\}$ . Hence the result follows by summing (A.5) over all  $k$  and adding (A.4).  $\square$

Consider now the solution  $(r_t, \varphi_t)$  of (3.3), starting in  $(0, -\delta)$ . Recall that  $\tau_0$  denotes the first-hitting time of the unstable orbit at  $r = 0$ . In addition, we denote by  $\tau_{-2\delta}$  the first-hitting time of the line  $\{r = -2\delta\}$ .

**Proposition A.2.** *For sufficiently small  $\delta > 0$ ,*

$$\lim_{\sigma \rightarrow 0} \mathbb{E}^{0, -\delta} \{e^{i\eta(\theta_\delta(\varphi_{\tau_0}) - |\log \sigma|)} \mid \tau_0 < \tau_{-2\delta}\} = 2^{-i\eta/2} \Gamma\left(1 - i\frac{\eta}{2}\right) + \mathcal{O}(\delta), \quad (\text{A.8})$$

where  $\Gamma$  is Euler's Gamma function.

PROOF: We will use several relations from [14, Sections 6 and 7], which apply to the process killed whenever  $r_t$  leaves the interval  $(-2\delta, 0)$ . For  $\ell \in \mathbb{N}$  and  $s \in [0, 1)$ , let

$$\begin{aligned} Q_{\tilde{\Delta}}(0, \ell + s) &= \mathbb{P}^{0, -\delta} \{\varphi_{\tau_0} \in [\ell + s, \ell + s + \tilde{\Delta}]\} \\ &= \mathbb{P}^{0, -\delta} \{\theta_\delta(\varphi_{\tau_0}) \in [t, t + \Delta]\}, \end{aligned} \quad (\text{A.9})$$

where  $t = \theta_\delta(\ell + s) = \ell \lambda_+ T_+ + \theta_\delta(s)$  and  $\Delta = \theta'_\delta(s) \tilde{\Delta} + \mathcal{O}(\tilde{\Delta}^2)$ .

By [14, (7.14) and (7.18)], we also have

$$\begin{aligned} Q_{\tilde{\Delta}}(0, \ell + s) &= C(\sigma) \tilde{\Delta} \theta'_\delta(s) A(t - |\log \sigma| + \mathcal{O}(\delta \ell)) [1 + \mathcal{O}(\tilde{\Delta}^\beta)] \\ &= C(\sigma) \Delta A(t - |\log \sigma| + \mathcal{O}(\delta \ell)) [1 + \mathcal{O}(\Delta^\beta)] \end{aligned} \quad (\text{A.10})$$

for some constants  $C(\sigma), 0 < \beta < 1$ , where  $A(t) = e^{-2t - \frac{1}{2}e^{-2t}}$ . It follows that

$$\mathbb{P}^{0, -\delta} \{ \theta_\delta(\varphi_{\tau_0}) - |\log \sigma| \in [t, t + \Delta] \} = C(\sigma) \Delta A(t + \mathcal{O}(\delta\ell)) [1 + \mathcal{O}(\Delta^\beta)]. \quad (\text{A.11})$$

We now apply Lemma A.1, where  $X_0$  is a random variable with density proportional to  $A(t - |\log \sigma| + \mathcal{O}(\delta\ell)) [1 + \mathcal{O}(\Delta^\beta)]$ , and  $X$  is the random variable  $\theta_\delta(\varphi_{\tau_0}) - |\log \sigma|$ , conditional on  $\tau_0 < \tau_{-2\delta}$ , i.e.  $\mathbb{P}\{X \in B\} = \mathbb{P}\{\theta_\delta(\varphi_{\tau_0}) - |\log \sigma| \in B | \tau_0 < \tau_{-2\delta}\}$ . We choose cut-offs  $a = -\frac{1}{2} \log(\log \Delta^{-1})$  and  $b = \log(\Delta^{-1})$ . This guarantees, on the one hand, that  $\mathbb{P}\{X_0 \notin [a, b]\} = \mathcal{O}(\Delta^{1/2})$ . On the other hand,  $|A'(x)/A(x)|$  is bounded above by  $\mathcal{O}(\log \Delta^{-1})$  on  $[a, b]$ , which allows to check that Condition (A.1) holds. It follows that

$$\begin{aligned} \mathbb{E}^{0, -\delta} \{ e^{i\eta(\theta_\delta(\varphi_{\tau_0}) - |\log \sigma|)} | \tau_0 < \tau_{-2\delta} \} &= \tilde{C}(\sigma) \int_{\theta_\delta(0) - |\log \sigma|}^{\infty} e^{i\eta t} A(t) dt [1 + \mathcal{O}(\delta) + \mathcal{O}(\Delta^\beta)] \\ &\quad + \mathcal{O}(\Delta^{1/2 \wedge \beta}) + \mathcal{O}\left(\frac{|\eta| \Delta}{2}\right), \end{aligned} \quad (\text{A.12})$$

where  $\tilde{C}(\sigma) = C(\sigma) / \mathbb{P}\{\tau_0 < \tau_{-2\delta}\}$ . The change of variables  $v = \frac{1}{2} e^{-2t}$  yields

$$\begin{aligned} \int_{\theta_\delta(0) - |\log \sigma|}^{\infty} e^{i\eta t} A(t) dt &= 2^{-i\eta/2} \int_0^{e^{-2\theta_\delta(0)}/2\sigma^2} v^{-i\eta/2} e^{-v} dv \\ &= 2^{-i\eta/2} \Gamma\left(1 - \frac{i\eta}{2}\right) + \mathcal{O}(e^{-\mathcal{O}(\delta^2/\sigma^2)}). \end{aligned} \quad (\text{A.13})$$

Plugging this into (A.12), taking first the limit  $\sigma \rightarrow 0$  and then the limit  $\Delta \rightarrow 0$  shows that  $\tilde{C}(0) = 1 + \mathcal{O}(\delta)$  (by evaluating in  $\eta = 0$ ) and proves (A.8).  $\square$

Note that the limit as  $\delta \rightarrow 0$  of the right-hand side of (A.8) is the characteristic function of  $(Z - \log 2)/2$ , where  $Z$  is a standard Gumbel random variable.

## A.2 Large deviations and dynamics far from the unstable orbit

We consider in this section the dynamics up to the time

$$\tau_{-\delta} = \inf\{t > 0: r_t = -\delta\} \quad (\text{A.14})$$

the process enters a  $\delta$ -neighbourhood of the unstable periodic orbit. We will choose a sequence  $(\delta_m)_{m \in \mathbb{N}}$ , converging to zero as  $m \rightarrow \infty$ , such that the optimal path  $\gamma_\infty$  crosses the level  $-\delta_m$  when  $\varphi = m$ . It follows from the behaviour of the Hamiltonian (3.13) near the unstable orbit (cf. [14, Section 4]) that

$$|\log \delta_m| = m\lambda_+ T_+ + \mathcal{O}(e^{-m\lambda_+ T_+}) \quad (\text{A.15})$$

as  $m \rightarrow \infty$ , which shows that  $\delta_m \simeq e^{-m\lambda_+ T_+}$  for large  $m$ .

We wish to compute the rate function  $I_m(\varphi)$ , corresponding to sample paths reaching  $r = -\delta_m$  near a given value of  $\varphi$ .

**Proposition A.3.** *There exists a periodic function  $P(\varphi)$ , reaching its maximum if and only if  $\varphi \in \mathbb{Z}$ , such that*

$$I_m\left(\hat{\varphi} + \frac{|\log \delta_m|}{\lambda_+ T_+}\right) = I_\infty - P(\hat{\varphi}) \delta_m^2 + \mathcal{O}(\delta_m^4 + e^{-2\lambda_+ T_+ \hat{\varphi}}). \quad (\text{A.16})$$

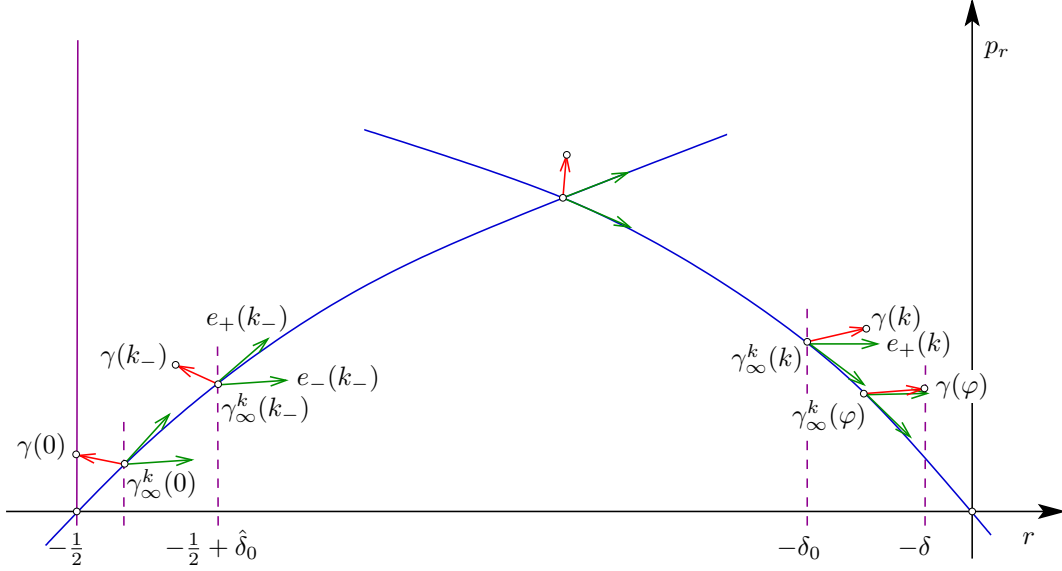


FIGURE 6. Construction of the optimal path  $\{\gamma(s)\}_{0 \leq s \leq \varphi}$  for a first exit at  $\varphi_{\tau_0} = \varphi$ . The path is constructed as a perturbation of the translate  $\gamma_{\infty}^k(s) = \gamma_{\infty}(s_k)$  of the path minimizing the rate function in arbitrary time, which lies at the intersection of the unstable and stable manifolds of the periodic orbits.

PROOF: The proof is based on similar considerations as in [14, Section 4]. We will construct a path  $\gamma$  minimising the rate function by perturbation of a translate of the optimal path  $\gamma_{\infty}$ . This is justified by our assumption that  $\gamma_{\infty}$  is a unique minimizer for transitions in arbitrary time, up to translations.

We fix a small constant  $\delta_0$ , such that  $\gamma_{\infty}$  reaches level  $-\delta_0$  for an integer value of  $\varphi$ . Without loss of generality, we may assume that  $\gamma_{\infty}(0) = -\delta_0$ . Then we can also find an integer  $\ell$  and a  $\hat{\delta}_0 \leq \delta_0$  such that  $\gamma_{\infty}(-\ell) = -1/2 + \hat{\delta}_0$ . The translate  $\gamma_{\infty}^k = \gamma_{\infty}(\cdot - k)$  crosses level  $-\delta_0$  at  $\varphi = k$  and level  $-1/2 + \hat{\delta}_0$  at  $\varphi = k_- := k - \ell$ .

The Hamiltonian flow of (3.13) can be viewed as a time-dependent flow for  $(r, p_r)$  in which  $\varphi$  plays the rôle of time [14, Section 2.2]. We denote by  $\pm\mu(s)$  the eigenvalues of the linearised flow at  $\gamma_{\infty}^k(s)$ . Let

$$\alpha(s, s_0) = \int_{s_0}^s \mu(u) du. \quad (\text{A.17})$$

The principal solution  $U(s, s_0)$  of the linearised flow has eigenvalues  $e^{\pm\alpha(s, s_0)}$ . We denote by  $e_{\pm}(s)$  the associated eigenvectors, which satisfy  $U(s, s_0)e_{\pm}(s_0) = e^{\pm\alpha(s, s_0)}e_{\pm}(s)$ . Consider a perturbed path given by

$$\gamma(s) = \gamma_{\infty}^k(s) + a_s e_+(s) + b_s e_-(s) \quad (\text{A.18})$$

(Figure 6). Then we have

$$\begin{aligned} a_s &= a_k e^{\alpha(s, k)} + \mathcal{O}(\|a^2 + b^2\|_{\infty}), \\ b_s &= b_k e^{-\alpha(s, k)} + \mathcal{O}(\|a^2 + b^2\|_{\infty}), \end{aligned} \quad (\text{A.19})$$

where  $\|\cdot\|_{\infty}$  denotes the supremum over  $[0, \varphi]$ .

Given an interval  $[u, s]$ , let  $I^0(u, s)$  denote the contribution of  $[u, s]$  to the integral defining the rate function of  $\gamma_{\infty}^k$  (cf. (3.9)). Let  $I^1(u, s)$  denote its analogue for the rate

function of  $\gamma$ . Then a computation similar to the one in [14, Proposition 4.1] shows that, up to a multiplicative error  $1 + \mathcal{O}(\delta_0)$ ,

$$I^1(k, \varphi) - I^0(k, \infty) = -\frac{\delta_0^2}{2} \frac{h^{\text{per}}(\varphi)}{h^{\text{per}}(0)^2} x_+^2 - \delta_0 b_k + \mathcal{O}(b_k^2, b_k x_+^2), \quad (\text{A.20})$$

where  $x_+ = e^{-\alpha(\varphi, k)}$ . In addition, the boundary condition  $\gamma(\varphi) \in \{r = -\delta_m\}$  yields

$$a_k = \delta_0 \frac{h^{\text{per}}(\varphi)}{h^{\text{per}}(0)} x_+^2 - \delta_m x_+ + \mathcal{O}(b_k x_+^2). \quad (\text{A.21})$$

A similar analysis can be made in the vicinity of the stable periodic orbit, and yields

$$I^1(0, k_-) - I^0(-\infty, k_-) = -\frac{\hat{\delta}_0^2}{2} \frac{1}{h_-^{\text{per}}(0)} x_-^2 + \hat{\delta}_0 a_k e^{-\alpha(k, k_-)} + \mathcal{O}(a_k^2, a_k x_-^2), \quad (\text{A.22})$$

where  $x_- = e^{-\alpha(k_-, 0)}$ , and  $h_-^{\text{per}}$  is a periodic function related to the linearisation at the stable orbit. Note that  $\alpha(k, k_-) =: \alpha_0$  does not depend on  $k$ , but only on the time  $\ell$  it takes for the optimal path  $\gamma_\infty$  to go from  $-1/2 + \hat{\delta}_0$  to  $-\delta_0$ . The boundary condition  $\gamma(0) \in \{r = -1/2\}$  yields the condition

$$e^{\alpha_0} b_k = -\hat{\delta}_0 x_-^2 + \mathcal{O}(a_k x_-^2). \quad (\text{A.23})$$

Finally, the transition between times  $k_-$  and  $k$  yields

$$I^1(k, k_-) - I^0(k, k_-) = a_k c_+ + b_k c_- + \mathcal{O}(a_k^2 + b_k^2), \quad (\text{A.24})$$

where the constants  $c_\pm$  depend only on the behaviour of  $\gamma_\infty^k$  on  $[k_-, k]$ , and are thus independent of  $k$ . Adding the estimates (A.20), (A.22) and (A.24), and using the boundary conditions, we obtain

$$I^1(\varphi, 0) - I_\infty = Ah_+^{\text{per}}(\varphi)x_+^2 + Bx_-^2 - C\delta_m x_+ + \mathcal{O}(x_+^4 + x_-^4 + \delta_m^4) \quad (\text{A.25})$$

for some positive constants  $A, B, C$ . The optimal rate function is obtained by minimizing (A.25) under the constraint  $x_- x_+ e^{-\alpha_0} = e^{-\alpha(\varphi, 0)}$ . The result is that the minimum is reached when

$$x_+ = \frac{C}{2Ah^{\text{per}}(\varphi)} \delta_m [1 + \mathcal{O}(\delta_m^2) + \mathcal{O}(\delta_m^{-4} e^{-\alpha(\varphi, 0)})], \quad (\text{A.26})$$

and has value

$$I_m(\varphi) - I_\infty = -\frac{C^2}{4Ah^{\text{per}}(\varphi)} \delta_m^2 + \mathcal{O}(\delta_m^4 + \delta_m^{-2} e^{-2\alpha(\varphi, 0)}). \quad (\text{A.27})$$

Noting that for large  $m$ ,  $\alpha(\varphi, 0) - \lambda_+ T_+ \varphi$  is bounded below, which implies  $e^{-\alpha(\varphi, 0)} = \mathcal{O}(\delta_m e^{-\lambda_+ T_+ \varphi})$ , finishes the proof.  $\square$

### A.3 Final steps of the proof

Consider the random variable

$$\hat{\varphi}_m = \varphi_{\tau_{-\delta_m}} - \frac{|\log \delta_m|}{\lambda_+ T_+}. \quad (\text{A.28})$$



Note that (A.15) implies  $\hat{\varphi}_m - \varphi_{\tau_{-\delta_m}} + m \rightarrow 0$  as  $m \rightarrow \infty$ , meaning that asymptotically,  $\hat{\varphi}_m$  is just an integer shift of  $\varphi_{\tau_{-\delta_m}}$ . We introduce the integer-valued random variable

$$Y_m^\sigma = \left\lfloor \hat{\varphi}_m + \frac{1}{2} \right\rfloor, \quad (\text{A.29})$$

which has the property

$$\mathbb{P}\{Y_m^\sigma = n\} = \mathbb{P}\left\{\hat{\varphi}_m \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right)\right\}, \quad (\text{A.30})$$

that is,  $Y_m^\sigma$  counts the number of periods until the process reaches the unstable orbit. The same argument as the one yielding (3.20) shows that the random variable  $Y_m^\sigma$  is asymptotically geometric. The large-deviation principle implies that the success probability is of order  $e^{-I_m/\sigma^2}$ , with  $I_m = I_\infty - P(0)\delta_m^2 + \mathcal{O}(\delta_m^4)$ .

**Proposition A.4.** *We have*

$$\lim_{m \rightarrow \infty} \left( \lim_{\sigma \rightarrow 0} \mathbb{P}\{|\hat{\varphi}_m - Y_m^\sigma| > \eta\} \right) = 0, \quad (\text{A.31})$$

for all  $\eta > 0$ . Therefore,  $\hat{\varphi}_m - Y_m^\sigma$  converges in distribution to  $\delta_0$ , the Dirac mass at 0.

PROOF: Proposition A.3 implies that the rate function  $I_m(\hat{\varphi})$  has at most one minimum in  $[n - 1/2, n + 1/2)$ , satisfying  $\hat{\varphi} = n + \mathcal{O}(\delta_m^2) + \mathcal{O}(e^{-2n\lambda_+ T_+})$ . We decompose

$$\mathbb{P}\{|\hat{\varphi}_m - Y_m^\sigma| > \eta\} \leq \mathbb{P}\{|\hat{\varphi}_m - Y_m^\sigma| > \eta, Y_m^\sigma > 2m\} + \mathbb{P}\{Y_m^\sigma \leq 2m\}. \quad (\text{A.32})$$

If  $Y_m^\sigma > 2m$ , then  $e^{-Y_m^\sigma \lambda_+ T_+} = \mathcal{O}(\delta_m^2)$ . Thus as  $\sigma \rightarrow 0$ , the first term on the right-hand side becomes bounded by  $1_{\{\eta < \mathcal{O}(\delta_m^2)\}}$ , which converges to 0 as  $m \rightarrow \infty$ . By the large-deviation principle, the second term on the right-hand side has order  $2m e^{-c/\sigma^2}$  for some  $c > 0$ , which goes to zero as  $\sigma \rightarrow 0$ .  $\square$

By (A.28) and the definition (4.5) of  $\theta_\delta$  we have

$$\theta(\varphi_{\tau_0}) - |\log \sigma| - \lambda_+ T_+ Y_m^\sigma = [\theta_{\delta_m}(\varphi_{\tau_0}) - |\log \sigma| - \lambda_+ T_+ \varphi_{\tau_{-\delta_m}}] + \lambda_+ T_+ (\hat{\varphi}_m - Y_m^\sigma). \quad (\text{A.33})$$

By Lévy's continuity theorem, Proposition A.2 and periodicity, conditionally on  $\tau_0 < \tau_{-2\delta_m}$ , the term in square brackets converges in distribution to  $(Z - \log 2)/2$  as  $\sigma \rightarrow 0$  and  $m \rightarrow \infty$  (in that order). The second term on the right-hand side converges to 0, as we have just seen. This proves the result conditionally on  $\tau_0 < \tau_{-2\delta_m}$ . The result remains true unconditionally because if  $\hat{\tau}_{-2\delta_m} = \inf\{t > \tau_{-\delta_m} : r_t = -2\delta_m\}$ , then

$$\mathbb{P}\{\hat{\tau}_{-2\delta_m} < \tau_0 \mid \tau_0 \in B\} = \mathcal{O}(e^{-c\delta_m^2/\sigma^2}) \quad (\text{A.34})$$

for some constant  $c > 0$ , as a consequence of [14, Proposition 4.2]. This reflects the fact that it is more expensive, in terms of rate function, to move back and forth between levels  $-\delta_m$  and  $-2\delta_m$  before reaching the unstable orbit, than to go directly from level  $-\delta_m$  to 0 (see also the renewal equation in [11]).  $\square$

## B Duration of phase slips

### B.1 Proof of Theorem 6.2

We will start by characterising the exit distribution from a small strip of width of order  $h_0 = \sigma^\gamma$  around the unstable orbit. We set

$$\tilde{\tau}_{\pm h_0} = \inf\{t > 0: r_t = \pm h_0 \sqrt{2\lambda_+ T_+ h^{\text{per}}(\varphi_t)}\}. \quad (\text{B.1})$$

The following result is an adaptation of [23, Theorem 2.1] to the nonlinear,  $\varphi$ -dependent situation.

**Lemma B.1.** *Fix an initial condition  $(\varphi_0, 0)$  and a constant  $h_0 = \sigma^\gamma$  for  $\gamma \in (1/2, 1)$ . Then the solution of (3.3) satisfies*

$$\lim_{\sigma \rightarrow 0} \text{Law}\left(\theta(\varphi_{\tilde{\tau}_{h_0}}) - \theta(\varphi_0) - \log\left(\frac{h_0}{\sigma}\right) \mid \tilde{\tau}_{h_0} < \tilde{\tau}_{-h_0}\right) = \text{Law}\left(\Theta + \frac{1}{2} \log(2\lambda_+)\right), \quad (\text{B.2})$$

where  $\Theta = -\log|N|$  and  $N$  is a standard normal random variable.

PROOF: Let  $\hat{\tau} = \tau_{h_0} \wedge \tau_{-h_0}$ , and  $\nu = \text{sign}(r_{\hat{\tau}})$ . We will introduce several events that have probabilities going to 1 as  $\sigma \rightarrow 0$ . The first event is

$$\Omega_1 = \left\{ \left| \varphi_t - \varphi_0 - \frac{t}{T_+} \right| \leq M(h_0^2 t + h_0) \quad \forall t \leq \hat{\tau} \wedge \frac{1}{h_0} \right\}, \quad (\text{B.3})$$

where  $M$  is such that  $|b_\varphi(r, \varphi)| \leq Mr^2$  for all  $(r, \varphi)$ . Then [14, Proposition 6.3] shows that

$$\mathbb{P}(\Omega_1^c) \leq e^{-\kappa_1 h_0 / \sigma^2} \quad (\text{B.4})$$

for a constant  $\kappa_1 > 0$ . On  $\Omega_1$ , the phase  $\varphi_t$  remains  $h_0$ -close to  $\varphi_0 + t/T_+$ . Hence the equation for  $r_t$  can be written as

$$dr_t = [\lambda_+ r_t + b_r(r_t, \varphi_t)] dt + [g_0(t) + g_1(r_t, \varphi_t, t)] dW_t \quad (\text{B.5})$$

where  $g_0(t) = g_r(0, \varphi_0 + t/T_+)$ , and  $g_1 = \mathcal{O}(|r| + h_0)$  on  $\Omega_1$ . The solution can be represented as

$$r_t = e^{\lambda_+ t} \left[ \sigma \int_0^t e^{-\lambda_+ s} g_0(s) dW_s + \sigma \int_0^t e^{-\lambda_+ s} g_1(r_s, \varphi_s, s) dW_s + \int_0^t e^{-\lambda_+ s} b_r(r_s, \varphi_s) ds \right]. \quad (\text{B.6})$$

Let  $Y_t$  denote the second integral in (B.6), and define

$$\Omega_2(t) = \left\{ \sup_{0 \leq s \leq \hat{\tau} \wedge t} |Y_s| > H \right\}. \quad (\text{B.7})$$

Then the Bernstein-type estimate [51, Theorem 37.8] yields

$$\mathbb{P}(\Omega_1 \cap \Omega_2(t)^c) \leq e^{-\kappa_2 H^2 / h_0^2} \quad (\text{B.8})$$

for a constant  $\kappa_2 > 0$ , uniformly in  $t$ . Evaluating (B.6) in  $\hat{\tau}$ , we obtain that on  $\Omega_1 \cap \Omega_2(t)$ ,

$$\nu h_0 \sqrt{2\lambda_+ T_+ h^{\text{per}}(\varphi_{\hat{\tau}})} = e^{\lambda_+ \hat{\tau}} \left[ \sigma \int_0^{\hat{\tau}} e^{-\lambda_+ s} g_0(s) dW_s + \mathcal{O}(\sigma H + h_0^2) \right]. \quad (\text{B.9})$$

Now we decompose

$$\int_0^{\hat{\tau}} e^{-\lambda+s} g_0(s) dW_s = \int_0^\infty e^{-\lambda+s} g_0(s) dW_s - \int_{\hat{\tau}}^\infty e^{-\lambda+s} g_0(s) dW_s = \sqrt{v_\infty} N - R, \quad (\text{B.10})$$

where  $N$  is a standard normal random variable, and (cf. [14, (2.29)])

$$v_\infty = \int_0^\infty e^{-2\lambda+s} D_{rr}(\varphi_0 + s/T_+) ds = T_+ h^{\text{per}}(\varphi_0). \quad (\text{B.11})$$

The remainder  $R$  satisfies

$$\mathbb{E}\{R^2\} = \frac{1}{2\lambda_+} \mathbb{E}\{e^{-2\lambda+\hat{\tau}}\} = \mathcal{O}\left(\frac{\sigma^2}{h_0^2}\right), \quad (\text{B.12})$$

since (B.9) implies that  $e^{-\lambda+\hat{\tau}} h_0/\sigma$  is bounded (see also [44]). This shows that if we set  $\Omega_3 = \{|R| > H\}$ , then by Markov's inequality there is a constant  $\kappa_3 > 0$  such that

$$\mathbb{P}(\Omega_3^c) \leq \kappa_3 \frac{\sigma^2}{h_0^2 H^2}. \quad (\text{B.13})$$

Taking the logarithm of the absolute value of (B.9), we find that on  $\Omega_1 \cap \Omega_2 \cap \Omega_3$ ,

$$\log h_0 + \frac{1}{2} \log(2\lambda_+ T_+ h^{\text{per}}(\varphi_{\hat{\tau}})) = \lambda_+ \hat{\tau} + \log \sigma + \frac{1}{2} \log(T_+ h^{\text{per}}(\varphi_0)) + \log |N| + \mathcal{O}\left(H + \frac{h_0^2}{\sigma}\right). \quad (\text{B.14})$$

Noting that on  $\Omega_1$ ,  $\hat{\tau} = T_+(\varphi_{\hat{\tau}} - \varphi_0) + \mathcal{O}(h_0)$ , and recalling the definition (4.4) of  $\theta$ , we obtain

$$\lim_{\sigma \rightarrow 0} \text{Law}\left(\theta(\varphi_{\hat{\tau}}) - \theta(\varphi_0) - \log\left(\frac{h_0}{\sigma}\right)\right) = \text{Law}\left(\Theta + \frac{1}{2} \log(2\lambda_+)\right), \quad (\text{B.15})$$

by choosing  $H = \sigma^{(1-\gamma)/2}$  and taking the limit  $\sigma \rightarrow 0$ . Finally, (B.9) implies that  $\nu = \text{sign}(r_{\hat{\tau}})$  converges to  $\text{sign}(N)$  as  $\sigma \rightarrow 0$ , so that in this limit  $\mathbb{P}\{\nu = 1\} = \mathbb{P}\{\nu = -1\} = 1/2$ . The result follows.  $\square$

The following result shows that after time  $\tau_{h_0}$ , the system essentially follows the deterministic dynamics.

**Lemma B.2.** *Let  $h_0$  be as in the previous lemma. Fix an initial condition  $(r_1, \varphi_1)$  such that  $r_1 = h_0 \sqrt{h^{\text{per}}(\varphi_1)}$ , and a curve  $\{r = \delta\rho(\varphi)\}$ , where  $\rho$  is a continuous function such that  $0 < \rho_0 \leq \rho(\varphi) \leq 1$  for all  $\varphi$ . Denote by*

- $\tau_\delta$  the first-passage time at  $r = \delta\rho(\varphi)$  of the solution  $(r_t, \varphi_t)$  starting in  $(r_1, \varphi_1)$ ,
- $\tau_\delta^{\text{det}}$  the first-passage time at  $r = \delta\rho(\varphi)$  of the deterministic solution  $(r_t^{\text{det}}, \varphi_t^{\text{det}})$  starting in  $(r_1, \varphi_1)$ .

Then

$$\lim_{\sigma \rightarrow 0} \varphi_{\tau_\delta} = \varphi_{\tau_\delta^{\text{det}}}^{\text{det}} \quad (\text{B.16})$$

for sufficiently small  $\delta$ , where the convergence is in probability.

PROOF: The difference  $\zeta_t = (r_t - r_t^{\text{det}}, \varphi_t - \varphi_t^{\text{det}})$  satisfies an equation of the form

$$d\zeta_t = A(t)\zeta_t dt + \sigma g(\zeta_t, t) dW_t + b(\zeta_t, t) dt, \quad (\text{B.17})$$

where  $A(t)$  is a matrix with top left entry  $a(t) = \lambda_+ + \mathcal{O}(r_t^{\text{det}})$ , and the other entries of order  $r_t^{\text{det}}$  or  $(r_t^{\text{det}})^2$ , while  $b$  has order  $\|\zeta\|^2$ . Then

$$\zeta_t = \zeta_t^0 + \zeta_t^1 = \sigma \int_0^t U(t, s) g(\zeta_s, s) dW_s + \int_0^t U(t, s) b(\zeta_s, s) ds, \quad (\text{B.18})$$

where  $U(t, s)$  is the fundamental solution of  $\dot{\zeta} = A(t)\zeta$ . Denote by  $\alpha(t, s)$  the integral of  $a(u)$  between  $s$  and  $t$ . One can show that there exist constants  $M, c > 0$  such that

$$\|U(t, s)\| \leq M e^{\alpha(t, s) + c\delta(t-s)} \quad \forall t > s > 0. \quad (\text{B.19})$$

We want to show that with probability going to 1 as  $\sigma \rightarrow 0$ ,  $\zeta_t$  remains of order  $h_1 e^{\alpha(t, 0)}$  up to time  $\tau_\delta \vee \tau_\delta^{\text{det}}$ , where  $h_1 \rightarrow 0$  as  $\sigma \rightarrow 0$ . Set

$$\tau_1 = \inf\{t: \|\zeta_t\| > h_1 e^{\alpha(t, 0)}\}. \quad (\text{B.20})$$

A Bernstein-type estimate shows that

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} e^{-\alpha(s, 0)} \|\zeta_s^0\| > h_1\right\} \leq \exp\left\{-\kappa \frac{h_1^2}{\sigma^2} e^{-2\alpha(t, 0) - 2ct}\right\}. \quad (\text{B.21})$$

Furthermore, there exists a constant  $M > 0$  such that

$$\|\zeta_{t \wedge \tau_1}^1\| \leq M h_1^2 e^{2\alpha(t, 0) + 2ct}. \quad (\text{B.22})$$

Since  $e^{\alpha(\tau_\delta^{\text{det}}, 0)} = \mathcal{O}(\delta/h_0)$ , we can find, for sufficiently small  $\delta$ , an  $h_1$  such that

$$\sigma e^{\alpha(t, 0) + ct} \ll h_1 \ll e^{-\alpha(t, 0) - ct} \quad (\text{B.23})$$

for all  $t \leq \tau_\delta \vee \tau_\delta^{\text{det}}$ . This shows that with probability going to 1 as  $\sigma \rightarrow 0$ ,  $\|\zeta_t\| \leq (h_1/2) e^{\alpha(t, 0)}$  up to time  $\tau_1 \wedge (\tau_\delta \vee \tau_\delta^{\text{det}})$ , and thus  $\tau_1 > \tau_\delta \vee \tau_\delta^{\text{det}}$ . This proves the claim.  $\square$

To finish the proof of Theorem 6.2, we consider the deterministic dynamics of (3.3). Using  $\varphi$  as new time variable, the dynamics can be written as  $dr/d\varphi = f_r(r, \varphi)/f_\varphi(r, \varphi)$ . If we set  $y = r/\sqrt{h^{\text{per}}(\varphi)}$ , it follows from (4.3) and (4.4) that this ODE is equivalent to

$$\frac{dy}{d\theta} = y + \tilde{b}(y, \theta), \quad (\text{B.24})$$

where  $\tilde{b}(y, \theta) = \mathcal{O}(y^2)$ . Standard perturbation theory shows that when starting with a small initial condition  $y_0 = h_0$ , solutions of this equation are of the form

$$y(\theta) = h_0 e^{\theta - \theta_0} [1 + \mathcal{O}(h_0 e^{\theta - \theta_0})]. \quad (\text{B.25})$$

Thus  $y(\theta)$  reaches  $\delta$  when  $\theta - \theta_0 = \log(\delta/h_0) + \mathcal{O}(\delta)$ . Combining this with (B.2) yields the result.  $\square$

## B.2 Proof of Theorem 6.4

Fix a small value of  $\delta$ , and consider an orbit starting at time  $\tilde{\tau}_\delta$  on the curve  $\{y = \delta\}$ . For  $u \geq 0$ , let  $\Gamma_+^u(\delta)$  be the image after time  $u$  of  $\{y = \delta\}$  under the deterministic flow (B.24). Theorem 6.2 and Lemma B.2 imply that the sample path will hit  $\Gamma_+^u(\delta)$  at a time  $\varphi_{\tau_+(\delta)}$  such that

$$\theta(\varphi_{\tau_+(\delta)}) - \theta(\varphi_{\tau_0}) - |\log \sigma| \rightarrow \Theta + \frac{\log(2\lambda_+\delta^2)}{2} + u + \mathcal{O}(\delta) \quad (\text{B.26})$$

as  $\sigma \rightarrow 0$ . Setting  $u = s - \log(\delta)$  and taking the limit  $\delta \rightarrow 0$ , we obtain (6.10), where

$$\Gamma_+^s = \lim_{\delta \rightarrow 0} \Gamma_+^{s - \log \delta}(\delta). \quad (\text{B.27})$$

Since  $y_u \simeq y_0 e^u = \delta e^u$  for small  $\delta e^u$ , this limit is indeed well-defined. This is the equivalent of the regularization procedure used in (5.14).

Now (6.9) is obtained in an analogous way, using Proposition A.2, and (6.11) follows from the two previous results, using Lemma 6.3.  $\square$

We note that more explicit forms of the curves  $\Gamma_\pm^s$  can be provided if we can find a change of variables  $y \mapsto \hat{y}$  such that

$$\frac{d\hat{y}}{d\theta} = F(\hat{y}) = \hat{y} + \hat{b}(\hat{y}), \quad (\text{B.28})$$

in which the right-hand side does not depend on  $\theta$ . Such a construction can be performed, for instance, in the averaging regime. In these variables, the curves  $\Gamma_\pm^s$  are simply horizontal lines of constant  $\hat{y}$ , and their dependence on  $s$  can be determined by solving a one-dimensional ODE, yielding expressions similar to (5.14).

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